Chapter 5
Multi-valued Logic for a Point-Free Foundation of Geometry

Cristina Coppola and Giangiacomo Gerla

5.1 Introduction

Łukasiewicz's many-valued logic (see Hájek 1998), Chang and Keisler's continuous logic (1966) and Pavelka's fuzzy logic (1979) define very interesting chapters of formal logic. Recently, under the name 'continuous logic', these researches were reconsidered to extend the powerful tools of model theory to classes of structures which are not definable in classical first order logic. This since these structures assume as primitive a real-valued function. Examples are the metric spaces, the normed spaces, the probabilities (see for example Yaacov and Usvyatsov 2010).

The basic ideas of point-free geometry were firstly formulated by A. N. Whitehead in *An Inquiry Concerning the Principles of Natural Knowledge* (Whitehead 1919) and in *The concept of Nature* (Whitehead 1920), where he proposed as primitive notions *events* and *extension relation* between events (in geometrical terms, regions and inclusion relation). Later, in *Process and Reality* (Whitehead 1929), Whitehead proposed a different treatment, inspired by De Laguna (1922), in which the topological notion of 'connection' between two regions was assumed as primitive and the inclusion was defined (see Gerla 1994). Successively, in a series of papers, metric-based approaches to point-free geometry were proposed in which, apart the inclusion relation, distances and diameters are also considered (see Di Concilio and Gerla 2006; Gerla 1990; Gerla and Miranda 2004). The resulting notion of point-free pseudo-metric space is a promising base for a metric foundation of geometry in accordance with the ideas of L.M. Blumenthal (1970). Indeed, it is possible to associate every point-free pseudo-metric space with a metric space via a natural definition of point and distance.

C. Coppola (✉) • G. Gerla
Dipartimento di Matematica, Università degli Studi di Salerno, Salerno, Italy
e-mail: ccoppola@unisa.it; gerla@unisa.it

C. Calosi and P. Graziani (eds.), *Mereology and the Sciences*, Synthese Library 371,
DOI 10.1007/978-3-319-05356-1_5,

In this exploratory paper we suggest the possibility of applying the ideas of continuous logic to point-free geometry. This is done by assuming as primitives predicates, geometrical in nature, as ‘to be included’, ‘to be small’, ‘to be close’. Indeed, since these predicates apply at different grades, we have to interpret them as fuzzy relations and therefore we have to refer to a first order multi-valued logic. Perhaps we can look the resulting formalisms as a way to modelize the passage from the original, naive, predicate based description of the geometrical space, qualitative in nature, to the modern real-number based approach to geometry, quantitative in nature.

More precisely, in Sect. 5.2 we propose the notion of inclusion space corresponding to some of the geometrical properties of the inclusion analyzed in Whitehead (1919, 1920). In Sect. 5.3 we give the notion of connection space corresponding to the analysis given in Whitehead (1929). Taking in account of the difficulties of the inclusion-based proposal in defining the notion of point, in Sects. 5.4 and 5.5 we reformulate it in the framework of multi-valued logic. This means that the inclusion is intended as a graded notion. We show that this enables us to overcome the observed difficulties. Finally, in Sect. 5.6 we reformulate the metric-based theory of point-free geometry into a theory in a multi-valued logic involving the graded predicates ‘to be close’ and ‘to be small’.

5.2 Inclusion Spaces

We isolate the main properties considered by Whitehead (1919) and we transform them into a system of axioms. Indeed, we consider the following first order theory in a language $L_{\leq}$ containing only the binary predicate $\leq$. As usual, we write $x < y$ to denote the formula $(x \leq y) \wedge (\neg(x = y))$.

Definition 1. An *inclusion space* is a structure satisfying the following axioms:

I1 $\forall x\,(x \leq x)$ (reflexivity)
I2 $\forall x \forall y \forall z\,((x \leq z \wedge z \leq y) \Rightarrow x \leq y)$ (transitivity)
I3 $\forall x \forall y\,(x \leq y \wedge y \leq x \Rightarrow x = y)$ (anti-symmetry)
I4 $\forall z \exists x\,(x < z)$ (there is no minimal region)
I5 $\forall x \forall y\,(x < y \Rightarrow \exists z\,(x < z < y))$ (density)
I6 $\forall x \forall y\,(\forall x'(x' < x \Rightarrow x' < y) \Rightarrow x \leq y)$ (below approximation)
I7 $\forall x \forall y\, \exists z(x \leq z \wedge y \leq z)$ (upward-directed).

We call *regions* the elements of an inclusion space and *inclusion relation* the relation $\leq$. Then an inclusion space is an ordered set $(S, \leq)$ such that $\leq$ has not a minimum, it is dense and upward-directed. Moreover, in this set it is possible to approximate every region from below. To find a model for this theory, we refer to the notion of bounded closed regular subset of the Euclidean space $\mathbb{R}^n$. This is a natural candidate to represent the idea of region which is usually accepted in literature.

Definition 2. Given a topological space we call *closed regular* a subset which is a fixed point of the operator $creg$ defined by setting

$$creg(X) = cl(int(X))$$

where we denote by cl and int the closure and the interior operators, respectively.

We denote by $RC(\mathbb{R}^n)$ the class of all the closed regular subsets of $\mathbb{R}^n$. $RC(\mathbb{R}^n)$ is a very interesting example of complete atomic-free Boolean algebra. We are interested to the class $\mathscr{R}$ of the nonempty bounded elements of $RC(\mathbb{R}^n)$. It is easy to prove the following theorem.

Theorem 1. *The structure* $(\mathscr{R}, \subseteq)$ *is an inclusion space.*

We call *canonical inclusion space* the structure $(\mathscr{R}, \subseteq)$. Whitehead (1919) defines the points by the following basic notion.

Definition 3. Given an inclusion space $(S, \leq)$, we call *abstractive sequence* any sequence $(r_n)_{n\in\mathbb{N}}$ of regions such that

(i) $(r_n)_{n\in\mathbb{N}}$ is order-reversing with respect to the inclusion
(ii) There is no region which is contained in all the regions in $(r_n)_{n\in\mathbb{N}}$.

We denote by AS the class of abstractive sequences.

Whitehead's idea is that an abstractive sequence $(r_n)_{n\in\mathbb{N}}$ represents an 'abstract object' which is obtained as a 'limit' of $(r_n)_{n\in\mathbb{N}}$. On the other hand, it is possible that two different abstractive sequences define the same abstract object. Then, we introduce the following equivalence relation.

Definition 4. The *covering* relation $\leq_c$ is the relation in AS defined by setting, for every $(r_n)_{n\in\mathbb{N}}$ and $(s_n)_{n\in\mathbb{N}}$,

$$(r_n)_{n\in\mathbb{N}} \leq_c (s_n)_{n\in\mathbb{N}} \Leftrightarrow \forall n \exists m \; r_n \leq s_n.$$

The relation $\equiv$ is defined by setting

$$(r_n)_{n\in\mathbb{N}} \equiv (s_n)_{n\in\mathbb{N}} \Leftrightarrow (r_n)_{n\in\mathbb{N}} \leq_c (s_n)_{n\in\mathbb{N}} \text{ and } (s_n)_{n\in\mathbb{N}} \leq_c (r_n)_{n\in\mathbb{N}}.$$

It is possible to prove that $\leq_c$ is a pre-order and therefore that $\equiv$ is an equivalence. Then we can consider the quotient $AS/\equiv$ and an order relation in $AS/\equiv$ by setting

$$[(r_n)_{n\in\mathbb{N}}] \leq_c [(s_n)_{n\in\mathbb{N}}] \Leftrightarrow (r_n)_{n\in\mathbb{N}} \leq_c (s_n)_{n\in\mathbb{N}}.$$

The following definition remembers Euclid's definition of point.

Definition 5. We call *geometrical element* any element of the quotient $AS/\equiv$, i.e. any class of equivalence $[(r_n)_{n\in\mathbb{N}}]$ modulo $\equiv$. A *point* is a geometrical element which *has no part*, i.e. which is minimal in $(AS/\equiv, \leq_c)$.

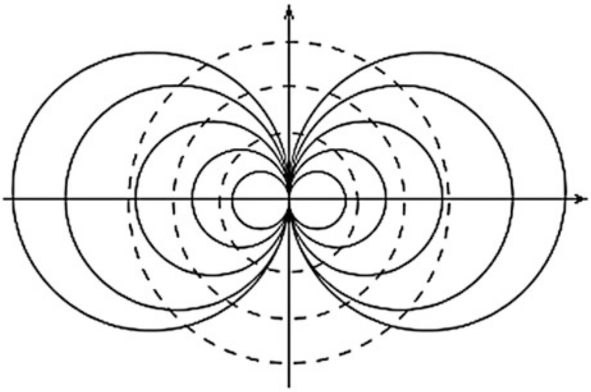

Fig. 5.1 Three different "points" in the origin

Unfortunately, in spite of the evident elegance of this definition of point, it is possible to prove the following theorem (see Gerla and Miranda 2004, 2008).

Theorem 2. *In a canonical inclusion space the definition of point is empty, i.e. there is no minimal element in* $(AS/\equiv, \leq_c)$.

Instead of an precise exposition of the proof of this theorem, we prefer to illustrate the idea which is on its basis by the following example. Consider in the Euclidean plane the abstractive sequence G defined by the sequence (B_n) of closed balls with center in the origin (0,0) and radius $r_n = 1/n$. From an intuitive point of view such an abstractive sequence represents a point. Unfortunately, we can consider the sequences G_1 and G_2 defined by the closed balls with radius r_n and centre in $(-1/n, 0)$ and $(1/n, 0)$, respectively (see Fig. 5.1). It is immediate that $[G] > G_1$, $[G] > G_2$ and that $[G]$, $[G_1]$ and $[G_2]$ are three different geometrical elements. This means that $[G]$ is not minimal and therefore that $[G]$ is not a point. Such an argument holds true if we start from any abstractive sequence. Perhaps Whitehead's passage from the inclusion-based approach to the connection-based approach was done to avoid such a counterintuitive behaviour. This theorem shows that the proposal of Whitehead of assuming as a primitive only the mereological notion of inclusion is unsatisfactory.

5.3 Connection Structures

Some years later the publication of Whitehead (1919, 1920), Whitehead (1929), proposed a different idea based on the primitive notion of *connection relation*. By isolating the main properties of the connection relation considered by Whitehead, we obtain the following theory. The considered language L_C has only a binary relation symbol C.

Definition 6. Denote by $x \leq y$ the formula $\forall z(zCx \Rightarrow zCy)$. Then we call *connection space theory* the following list of axioms.

C1 $\forall x \forall y\, (xCy \Rightarrow yCx)$ (symmetry)
C2 $\forall z \exists x \exists y\, ((x \leq z) \wedge (y \leq z) \wedge (\neg xCy))$
C3 $\forall x \forall y \exists z\, (zCx \wedge zCy)$
C4 $\forall x\, (xCx)$
C5 $\forall x \forall y\, ((x \leq y \wedge y \leq x) \Rightarrow x = y)$.

The intended interpretation is that the connection is either a surface contact or an overlap. It is easy to prove that in any connection space the relation $\leq$ is an order relation. We denote by xOy the formula$\exists z(z \leq x \wedge z \leq y)$ and we call *overlapping relation* the corresponding relation. Again we use the class $\mathscr{R}$ to find a model of this theory. We denote again by C the interpretation of the relation symbol C in $\mathscr{R}$.

Theorem 3. *Define in $\mathscr{R} \subseteq RC(\mathbb{R}^n)$ the relation C by setting*

$$XCY \Leftrightarrow X \cap Y \neq \emptyset.$$

Then $(\mathscr{R}, C)$ is a connection space in $\mathbb{R}^n$, whose associated order coincides with the set-theoretical inclusion.

We call *canonical connection space* a connection space defined in such a way. The observation of a canonical connection space makes evident way the connection relation is different from the overlapping relation. Indeed, while XCY means that there is a point belonging in both the regions, XOY means that there is a region contained in both the regions. To obtain an adequate definition of point, we need the notion of nontangential inclusion.

Definition 7. Given a connection space (S, C), we say that two regions have a *tangential connection* when

(i) They are connected,
(ii) They do not overlap.

We say that x is *non-tangentially included* in y and we write $x \prec y$ provided that

(j) x is included in y,
(jj) There is no region having a tangential connection with both x and y.

The following is a simple characterization of the non-tangential inclusion.

Proposition 1. *The non-tangential inclusion is the relation defined by the formula*

$$\forall z(zCx \Rightarrow zOy). \tag{5.1}$$

It is possible to prove that in a canonical connection space

$$X \prec Y \Leftrightarrow X \subseteq int(Y).$$

Definition 8. An *abstractive sequence* in a connection space is a sequence $(r_n)_{n\in\mathbb{N}}$ of regions such that

(j) $(r_n)_{n\in\mathbb{N}}$ is order-reversing with respect to the non-tangential inclusion,
(jj) There is no region which is contained in all the regions of $(r_n)_{n\in\mathbb{N}}$.

The notions of covering, equivalence, geometrical element, point are defined as in Sect. 5.2. Differently from the case of the inclusion spaces, in the canonical connection space $(\mathscr{R}, C)$ Whitehead's definition of point works well. Indeed the following theorem holds (see Coppola et al. 2010).

Theorem 4. *Consider the canonical space $(\mathscr{R}, C)$ and denote by $B_n(p)$ the closed ball centered in p and with radius $1/n$. Then the map associating every point p in $\mathbb{R}^n$ with the geometrical element $[(B_n(p))_{n\in\mathbb{N}}]$ is a one-to-one map from $\mathbb{R}^n$ to the set of points in $(\mathscr{R}, C)$.*

This theorem shows that connection space theory gives to point-free geometry a more suitable framework than the one of inclusion space theory. A further reason in furnished by the following theorem.

Theorem 5. *While in a canonical connection space $(\mathscr{R}, C)$ we can define the inclusion relation, in a canonical inclusion space $(\mathscr{R}, \subseteq)$ it is not possible to define the connection relation.*

The proof of this theorem is based on the fact that if a relation is definable in a structure, then it is invariant with respect to all the automorphisms of this structure. So, it is sufficient to exhibit an one-to-one map preserving the inclusion and not preserving the connection (for a complete proof see Gerla and Miranda 2004).

5.4 Multi-valued Logic for an Inclusion-Based Point-Free Geometry

As we have seen, there are some troubles in the inclusion-based point-free geometry. Indeed in rather natural models Whitehead's definition of point is empty, moreover the topological notion of connection cannot be defined. In spite of that, we claim that an inclusion-based approach it is possible provided that we consider the inclusion as a graded notion and therefore provided that we shift from classical logic to multi-valued logic. Namely, we refer to the first order logic associated with a continuous triangular norm $\otimes : [0,1] \times [0,1] \to [0,1]$ (see for example Hájek 1998) and therefore to a first order language with:

- Two logical connectives $\wedge$ and $\Rightarrow$, interpreted by $\otimes$ and the related residuum $\to$,
- Two logical constant $\underline{0}$ and $\underline{1}$ to denote 0 and 1
- The quantifiers $\forall$ and $\exists$ interpreted by the operators inf and sup.

In addition, we consider a connective Ct we interpret by the function $ct : [0,1] \to [0,1]$ such that $ct(x) = 1$ if $x = 1$ and $ct(x) = 0$ otherwise. This means that the intended meaning of a formula as $Ct(\alpha)$ is '*α is completely true*'. To fix the ideas, we assume that $\otimes$ is the usual product and therefore that the implication is interpreted by the operation $\to$ such that $x \to y = 1$ if $x \leq y$ and $x \to y = y/x$ otherwise. Given a set D, an *n-ary fuzzy relation* in D is a map $r : D^n \to [0,1]$. We call *crisp* a fuzzy relation assuming only the values 0 and 1 and we identify a classical relation $R \subseteq D^n$ with the crisp relation $c_R : D^n \to [0,1]$ defined by setting $c_R(d_1, \ldots, d_n) = 1$ if $(d_1, \ldots, d_n) \in R$ and $c_R(d_1, \ldots, d_n) = 0$ otherwise. In other words, we identify R with its characteristic function c_R.

A *multi-valued interpretation* (D, I) is defined by a nonempty domain D and by a function I associating every constant c with an element $I(c) \in D$, every n-ary operation symbol with an n-ary operation in D and every n-ary relation symbol $\underline{r}$ with an n-ary fuzzy relation $r = I(\underline{r})$, i.e. a map $r : D^n \to [0,1]$. Given a multi-valued interpretation (D, I), the interpretation $I(t) : D^n \to D$ of a term t is defined as in classical logic. The valuation of the sentences is defined in a truth-functional way as follows.

Definition 9. Given a multi-valued interpretation (D, I), a formula α whose variables are among $x_1, \ldots, x_n$ and $d_1, \ldots, d_n$ in D, we define the value $Val(\alpha, d_1, \ldots, d_n)$ by recursion on the complexity of α, by the equations:

(i) $Val(\underline{r}(t_1, \ldots, t_p), d_1, \ldots, d_n) = I(\underline{r})(I(t_1)(d_1, \ldots, d_n), \ldots, I(t_p)(d_1, \ldots, d_n))$
(ii) $Val(\alpha_1 \underline{\diamond} \alpha_2, d_1, \ldots, d_n) = Val(\alpha_1, d_1, \ldots, d_n) \diamond Val(\alpha_q, d_1, \ldots, d_n)$
(iii) $Val(\underline{\bullet}\alpha, d_1, \ldots, d_n) = \bullet(Val(\alpha, d_1, \ldots, d_n))$
(iv) $Val(\forall x_h \beta, d_1, \ldots, d_n) = \inf(\{Val(\beta, d_1, \ldots, d_{h-1}, d, d_{h+1}, \ldots, d_n) : d \in D\})$
(v) $Val(\exists x_h \beta, d_1, \ldots, d_n) = \sup(\{Val(\beta, d_1, \ldots, d_{h-1}, d, d_{h+1}, \ldots, d_n) : d \in D\})$

where we denote by $\underline{\diamond}$ (by $\underline{\bullet}$) a binary (an unary) connective and by $\diamond$ (by $\bullet$) the corresponding interpretation.

We say that $d_1, \ldots, d_n$ *satisfy* α if $Val(\alpha, d_1, \ldots, d_n) = 1$. If α is a closed formula, then the value $Val(\alpha, d_1, \ldots, d_n)$ does not depend on the elements $d_1, .., d_n$ and we write $Val(\alpha)$ instead of $Val(\alpha, d_1, \ldots, d_n)$. In the case there are free variables in α, we write $Val(\alpha)$ to denote $Val(\forall x_1 \ldots \forall x_n(\alpha))$ where $\forall x_1 \ldots \forall x_n(\alpha)$ is the universal closure of α.

Definition 10. Given a theory T, we say that (D, I) is a *multi-valued model* of T if $Val(\alpha) = 1$ for every $\alpha \in T$.

The so defined multi-valued logic is rather expressive. For example, if $\underline{r}$ is an n-ary relation symbol then the formula

$$\forall x_1 \ldots \forall x_n (Ct(\underline{r}(x_1, \ldots, x_n)) \leftrightarrow \underline{r}(x_1, \ldots, x_n))$$

is satisfied if and only if $\underline{r}$ is interpreted by a crisp relation. Indeed it is sufficient to observe that this formula is satisfied if and only if $ct(r(d_1, \ldots, d_n)) = r(d_1, \ldots, d_n)$ for every $d_1, \ldots, d_n$ in D. In other words, 'to be crisp' is a first order property. This entails that all the classical notions which are definable in classical first order logic are definable in our multi-valued logic, too. In particular, the notion of order relation is definable.

Definition 11. Let (D, I) be a multi-valued interpretation and α be a formula whose free variables are among $x_1, \ldots, x_n$. Then the *extension* of α in (D, I) is the fuzzy relation $r_\alpha : D^n \to [0, 1]$ defined by setting $r_\alpha(d_1, \ldots, d_n) = Val(\alpha, d_1, \ldots, d_n)$ for every $d_1, \ldots, d_n$ in D. In such a case we say that r_α *is defined by* α. We call *crisp extension* of α the extension $r_{Ct(\alpha)}$ of $Ct(\alpha)$. In such a case we say that $r_{Ct(\alpha)}$ is the *crisp relation defined by* α.

Then the crisp relation defined by α is the (characteristic function of the) relation

$$\{(d_1, \ldots, d_n) \in D^n : \alpha \text{ is satisfied by } d_1, \ldots, d_n\}.$$

Coming back to point-free geometry, we consider the first order language with a binary relation symbol *Incl* and we write $x \leq y$ to denote the formula $Ct(Incl(x, y))$. An interpretation of such a language is defined by a pair $(S, incl)$ where S is a nonempty set and $incl : S \times S \to [0, 1]$ a fuzzy binary relation. The interpretation of $\leq$ is the (characteristic function of the) relation $\leq$ defined by setting

$$x \leq y \Leftrightarrow incl(x, y) = 1. \tag{5.2}$$

We call *the crisp inclusion* associated with *incl* this relation.

If $Sim(x, y)$ denotes the formula $Incl(x, y) \wedge Incl(y, x)$, then the interpretation of $Sim(x, y)$ is the fuzzy relation $sim : S \times S \to [0, 1]$ defined by setting

$$sim(x, y) = incl(x, y) \otimes incl(y, x). \tag{5.3}$$

We call *the graded identity associated with incl* this fuzzy relation.

Definition 12. A *graded preorder structure*, in brief *graded preorder*, is a multi-valued model $(S, incl)$ of the following theory:

A1 $\forall x(Incl(x, x))$
A2 $\forall x \forall y \forall z((Incl(x, z) \wedge Incl(z, y)) \to Incl(x, y))$.

Then a fuzzy relation *incl* is a graded preorder if and only if, for every $x, y, z \in S$,

a1 $incl(x, x) = 1$ (reflexivity)
a2 $incl(x, y) \otimes incl(y, z) \leq incl(x, z)$ (transitivity).

If the symmetry axiom is also satisfied then the fuzzy relation is called *fuzzy equivalence* or *similarity*. Then a similarity is a fuzzy relation $sim : S \times S \to [0, 1]$ such that

b1 $sim(x, x) = 1$ (reflexivity)
b2 $sim(x, y) \otimes sim(y, z) \leq sim(x, z)$ (transitivity)
b3 $sim(x, y) = sim(x, y)$ (symmetry).

This notion is a graded extension of the one of equivalence. It is easy to prove that the fuzzy relation sim defined by (5.3) is a similarity. A *fuzzy equality* is a similarity satisfying the following 'separation axiom'

b4 $sim(x, y) = 1 \Leftrightarrow x = y$.

To simulate Whitehead's definition of point, we define a notion of 'point-likeness' suggested by Euclid's definition of point as *minimal element*, i.e. an element x such that $x' \leq x$ entails $x' = x$.

Definition 13. We call *point-likeness* property the property expressed by the formula,

$$Pl(x) \equiv \forall x'(x' \leq x \rightarrow Sim(x, x')).$$

The extension of Pl is the fuzzy subset of regions pl defined by

$$pl(x) = \inf\{incl(x, x') : x' \leq x\}.$$

This means that all the regions are points at a suitable degree. The formula $Pl(x)$ enables us to express the next two axioms. The first axiom claims that if we apply the graded inclusion to regions which are (approximately) points, then such a relation is (approximately) symmetric .

A3 $Pl(x) \wedge Pl(y) \rightarrow (Incl(x, y) \rightarrow Incl(y, x))$.

This axiom is satisfied if and only if, for every x and y,

a3 $pl(x) \otimes pl(y) \leq (incl(x, y) \rightarrow incl(y, x))$.

The further axiom claims that every region x contains a point:

A4 $\forall x \exists x'((x' \leq x) \wedge Pl(x'))$.

Such an axiom is satisfied if and only if for every x,

a4 $\sup_{x' \leq x} pl(x') = 1$

i.e. if and only if for every x

$$\exists \epsilon > 0 \text{ there is } x' \leq x \text{ such that } pl(x') \geq 1 - \epsilon.$$

Definition 14. We call *graded inclusion space* every model of **A1–A4**.

The following notion enables us to emphasize the metrical nature of the graded inclusion spaces.

Definition 15. A *hemimetric space* is a structure (S, d) such that S is a nonempty set and $d : S \times S \to [0, \infty]$ is a mapping such that, for all $x, y, z \in S$,

d1 $d(x, x) = 0$
d2 $d(x, y) \leq d(x, z) + d(z, y)$.

Then, a metric space is a hemimetric space which is symmetric, i.e. such that $d(x, y) = d(y, x)$ for every $x, y \in S$, and such that $d(x, y) = 0$ only if $x = y$. Every hemimetric space is associated with a pre-order in the following way.

Proposition 2. *Let (S, d) be a hemimetric space, then the relation $\leq$ defined by setting:*

$$x \leq y \Leftrightarrow d(x, y) = 0$$

is a pre-order such that d is order-preserving with respect to the first variable and order-reversing with respect to the second variable.

In the case d is a metric, $\leq$ coincides with the identity relation. Given $x \in S$, we call *diameter* of x the number

$$\delta(x) = \sup\{d(x_1, x_2) : x_1 \leq x, x_2 \leq x\}.$$

Observe that this definition entails that

$$\delta(x) \geq d(y, x) \text{ for every } y \leq x. \tag{5.4}$$

In the case d is a metric, all the diameters are equal to 0.

The following proposition shows that the notion of hemimetric distance is 'dual' of the one of graded preorder. As usual, we put $10^{-\infty} = 0$ and $Log(0) = -\infty$.

Proposition 3. *Given a hemimetric space (S, d), the fuzzy relation incl defined by setting*

$$incl(x, y) = 10^{-d(x,y)}$$

is a graded preorder such that $pl(x) = 10^{-\delta(x)}$. Conversely, let incl $: S \times S \to [0, 1]$ be a graded preorder and let d be defined by setting

$$d(x, y) = -Log(incl(x, y)).$$

Then d is a hemimetric such that $\delta(x) = -Log(pl(x))$.

In the case d is a pseudo-metric the associated fuzzy relation *incl* is a similarity, obviously. We consider the following class of hemimetrics.

Definition 16. A *hemimetric space of regions* is a hemimetric space (S, d) such that for every x and y,

d3 $|d(x, y) - d(y, x)| \leq \delta(x) + \delta(y)$
d4 $\forall \epsilon > 0 \exists x' \leq x, \delta(x') \leq \epsilon.$

The following theorem shows a duality between the class of hemimetric spaces of regions and the one of the graded inclusion spaces (see also Di Concilio and Gerla 2006).

Theorem 6. *For every hemimetric space of regions (S, d), the fuzzy relation incl defined by setting*

$$incl(x, y) = 10^{-d(x,y)}$$

defines a graded inclusion space of regions. Conversely, let $(S, incl)$ be a graded inclusion space of regions and let $d : S \times S \to [0, +\infty]$ be defined by setting

$$d(x, y) = -Log(incl(x, y)).$$

Then (S, d) is a hemimetric space of regions.

5.5 Canonical Graded Inclusion Spaces, Connection and Points

The most famous hemimetric is the excess measure used to define the Hausdorff distance.

Definition 17. Given a metric space (M, d) the *excess measur*e is the map $e : \mathscr{P}(M) \times \mathscr{P}(M) \to [0, \infty]$ defined, for every pair X and Y of subsets of M, by setting

$$e(X, Y) = \sup_{p \in X} \inf_{q \in Y} d(p, q).$$

The following proposition is proved in Di Concilio and Gerla (2006).

Proposition 4. *Let $\mathscr{R}$ be the class nonempty, bounded, closed regular subsets of (M, d). Then the excess measure defines in $\mathscr{R}$ a hemimetric space of regions. Consequently, by setting*

$$incl(X, Y) = 10^{-e(X,Y)} = inf_{p \in X} sup_{q \in Y} 10^{-d(p,q)}$$

we obtain a graded inclusion space. The induced order is the usual set theoretical inclusion and the point-likeness property is defined by

$$pl(X) = 10^{-|X|},$$

where $|x|$ is the usual diameter in a metric space.

We call *canonical graded inclusion space* the inclusion space obtained by such a proposition. Observe that if we consider a fuzzy equality $eq : M \times M \to [0,1]$, then by setting $d(x, y) = -Log(eq(x, y))$ we obtain a metric. Indeed, it is evident that $d(x, y) = 0$ if and only if $eq(x, y) = 1$ if and only if $x = y$. By applying Proposition 4, we obtain that

$$incl(X, Y) = inf_{p\in X} sup_{q\in Y} eq(p, q).$$

Assume that in the language there is a name Eq to denote eq. Then, in accordance with the usual interpretation of the quantifiers in multi-valued logic, we can interpret the value $incl(X, Y)$ as the interpretation of the formula $\forall p \in X \exists q \in Y(Eq(p, q))$, i.e. of the claim 'every point in X is (approximately) equal with a point in Y'.

We will show that, differently from Whitehead's inclusion spaces, in a graded inclusion space we can define the connection relation as the crisp extension of the formula expressing the overlapping relation in an inclusion space.

Theorem 7. *Consider in a canonical graded inclusion space $(\mathcal{R}, incl)$ the formula $O(x, y) \equiv \exists z(Incl(z, x) \wedge Incl(z, y))$. Then the connection relation C in the canonical connection space $(\mathcal{R}, C)$ is defined by the formula $Ct(O(x, y))$.*

In other words, we can define the connection between two regions by saying that the two regions completely overlaps (at degree 1).

The second question is to show that in a graded inclusion space it is possible to give a nonempty notion of point.

Definition 18. Given a graded inclusion space, we call *abstraction process* any sequence $< p_n >_{n\in\mathbb{N}}$ of regions which are order-reversing with respect to the order associated with the graded inclusion. We extend the point-likeness property to the abstraction processes by setting

$$pl(< p_n >_{n\in\mathbb{N}}) = lim_{n\to\infty} pl(p_n) = sup_n pl(p_n).$$

We say that $< p_n >_{n\in\mathbb{N}}$ *represents a point* if $pl(< p_n >_{n\in\mathbb{N}}) = 1$ and we denote by Pr the class of abstraction processes representing a point.

Observe that **A4** enables us to prove that every region of a graded inclusion space 'contains' an abstraction process representing a point and therefore that $Pr \neq \emptyset$. Indeed, in accordance with **a4**, for every region x there is $x' \leq x$ such that $pl(x') \geq 1 - 1/n$. Then we can consider the sequence $< p_n >_{n\in\mathbb{N}}$ defined by setting $p_1 = x$

and p_n equal to a region such that $p_n \leq p_{n-1}$ and $pl(p_n) \geq 1 - 1/n$. Obviously, $pl(< p_n >_{n\in\mathbb{N}}) = 1$.

The following theorem shows that in the class of abstraction processes representing points it is possible to define a pseudo-metric d. We give the proof in order to emphasize the role of **A3** and therefore of **d3** in proving the symmetry of d.

Theorem 8. *Let* $(S, incl)$ *be a graded inclusion space and* d' *be the associated hemimetric. Then the map* $d : Pr \times Pr \to [0, \infty]$ *obtained by setting*

$$d(< p_n >_{n\in\mathbb{N}}, < q_n >_{n\in\mathbb{N}}) = lim_{n\to\infty} d'(p_n, q_n),$$

defines a pseudo-metric space (Pr, d).

Proof. To prove the convergence of the sequence $< d'(p_n, q_n) >_{n\in\mathbb{N}}$, let n and k be natural numbers and assume that $n \geq k$. Then, since $d'(p_n, p_k) = 0$ and, by (5.4), $d'(q_k, q_n) \leq \delta(q_k)$ we have that,

$$d'(p_n, q_n) \leq d'(p_n, p_k) + d'(p_k, q_k) + d'(q_k, q_n) \leq d'(p_k, q_k) + \delta(q_k)$$

and therefore,

$$d'(p_n, q_n) - d'(p_k, q_k) \leq \delta(q_k).$$

Likewise, since $d'(q_n, q_k) = 0$ and $d'(p_k, p_n) \leq \delta(p_k)$,

$$d'(p_k, q_k) \leq d'(p_k, p_n) + d'(p_n, q_n) + d'(q_n, q_k) \leq d'(p_n, q_n) + \delta(p_k)$$

and therefore

$$d'(p_k, q_k) - d'(p_n, q_n) \leq \delta(p_k).$$

This entails

$$|d'(p_n, q_n) - d'(p_k, q_k)| \leq \max\{\delta(q_k), \delta(p_k)\}.$$

Obviously, in the case $n \leq k$

$$|d'(p_n, q_n) - d'(p_k, q_k)| \leq \max\{\delta(q_n), \delta(p_n)\}.$$

Thus

$$|d'(p_n, q_n) - d'(p_k, q_k)| \leq \max\{\delta(q_n), \delta(p_n), \delta(q_k), \delta(p_k)\}.$$

The convergence of the diameters entails that $< d'(p_n, q_n) >_{n\in\mathbb{N}}$ is a Cauchy sequence.

It is evident that $d(<p_n>_{n\in\mathbb{N}}, <p_n>_{n\in\mathbb{N}}) = 0$ and that d satisfies the triangular inequality.

To prove the symmetry, observe that, by **d3**, $|d'(p_n, q_n) - d'(q_n, p_n)| \leq \delta(p_n) + \delta(q_n)$ and therefore, since $lim_{n\to\infty}\delta(p_n) + \delta(q_n) = 0$, $lim_{n\to\infty}|d'(p_n, q_n) - d'(q_n, p_n)| = 0$. Since the sequences $< d'(p_n, q_n) >_{n\in\mathbb{N}}$ and $< d'(q_n, p_n) >_{n\in\mathbb{N}}$ are both convergent, $lim_{n\to\infty}d'(p_n, q_n) = lim_{n\to\infty}d'(q_n, p_n)$. Thus

$$d(<p_n> n \in \mathbb{N}, <q_n> n \in \mathbb{N}) = lim_{n\to\infty}d'(p_n, q_n) = lim_{n\to\infty}d'(q_n, p_n)$$
$$= d(<q_n>_{n\in\mathbb{N}}, <p_n>_{n\in\mathbb{N}}).\square$$

Such a proposition enables us to associate any graded inclusion space with a metric space. Indeed, recall that the *quotient* of a pseudo-metric space (X, d) is the metric space $(\underline{X}, \underline{d})$ defined by assuming that

- $\underline{X}$ is the quotient of X modulo the relation $\equiv$ defined by setting $x \equiv x'$ if and only if $d(x, x') = 0$,
- $\underline{d}([x], [y]) = d(x, y)$ for every $[x], [y] \in X'$.

Definition 19. We call *metric space associated with a graded inclusion space* $(S, incl)$ the quotient $(\underline{Pr}, \underline{d})$ of the pseudo-metric space (Pr, d). We call *point* any element in $\underline{Pr}$.

Then, the metric space $(\underline{Pr}, \underline{d})$ associated with a graded inclusion space $(S, incl)$ is obtained

- By starting from the class Pr of abstraction processes,
- By setting $\underline{Pr}$ equal to the quotient of Pr modulo the equivalence relation $\equiv$ defined by

$$<p_n>_{n\in\mathbb{N}} \equiv <q_n>_{n\in\mathbb{N}} \Leftrightarrow lim_{n\to\infty}incl(p_n, q_n) = 1,$$

- By defining $\underline{d} : \underline{Pr} \times \underline{Pr} \to [0, \infty]$ by the equation

$$\underline{d}(P, Q) = lim_{n\to\infty} - Log(incl(p_n, q_n))$$

  where $P = [<p_n>_{n\in\mathbb{N}}]$ and $Q = [<q_n>_{n\in\mathbb{N}}]$ are elements in $\underline{Pr}$.

5.6 To Be Closed and To Be Small

In a series of papers a metric approach to point-free geometry is proposed in which, in addition to the inclusion relation, the notions of diameter of a region and distance between two regions are assumed as primitives (see Gerla 1990).

Definition 20. A *point-free pseudo-metric space*, in short a *ppm-space*, is a structure $(S, \leq, \Delta, \delta)$, where $(S, \leq)$ is an ordered set, $\Delta : S \times S \rightarrow [0, \infty)$ is order-reversing, $\delta : S \rightarrow [0, \infty]$ is order-preserving and, for every $x, y, z \in S$:

D1 $\Delta(x, x) = 0$
D2 $\Delta(x, y) = \Delta(y, x)$
D3 $\Delta(x, y) \leq \Delta(x, z) + \Delta(z, y) + \delta(z)$.

The elements in S are called *regions*, the order $\leq$ *inclusion relation*, $\Delta(x, y)$ *distance between* x *and* y, $\delta(x)$ the *diameter* of x. Inequality $D3$ is a weak form of the triangular inequality taking in account the diameters of the regions. The notion of *ppm*-space extends the one of pseudo-metric space (and therefore of metric space). Indeed, if all the diameters are equal to zero, then $D3$ coincides with the triangular inequality and the *ppm*-space is a pseudo-metric space. More precisely, we can identify the pseudo-metric spaces with the *ppm*-spaces in which $\leq$ is the identity and all the diameters are equal to zero. We identify the metric spaces with the *ppm*-spaces satisfying these conditions and such that Δ is finite and $\Delta(x, y) = 0$ entails $x = y$.

Notice that we can also assume as primitive a function Δ satisfying $D1$ and $D2$ and define a diameter by setting $\delta(z) = \sup\{\Delta(x, y) - \Delta(x, z) - \Delta(z, y) : x, y \in S\}$. Indeed, to prove that the resulting structure $(S, \leq, \Delta, \delta)$ is a pmm-space, we observe that by setting $x = y = z$ we obtain that $\delta(z)$ is greater or equal to 0. It is evident that δ is order-preserving and that $D3$ holds true by definition. It is also possible to assume as primitive only the diameter function (see Gerla and Paolillo 2010; Pultr 1988).

The following proposition gives prototypic examples of *ppm*-space (see Gerla 1990).

Proposition 5. *Let* (M, d) *be a pseudo-metric space and let* C *be a nonempty class of bounded and nonempty subsets of* M. *Define* Δ *and* δ *by setting*

$$\Delta(X, Y) = \inf\{d(x, y) : x \in X, y \in Y\}$$

$$\delta(X) = \sup\{d(x, y) : x, y \in X\},$$

respectively. Then $(C, \subseteq, \Delta, \delta)$ *is a ppm-space.*

In particular, we call *canonical ppm-space* the space $(\mathscr{R}, \subseteq, \Delta, \delta)$. By referring to the just defined class of *ppm*-spaces, the meaning of the proposed axioms becomes evident. For example, the meaning of $D3$ is given by Fig. 5.2: Indeed, it is evident that in this case $\Delta(X, Y) > \Delta(X, Z) + \Delta(Z, Y)$ and therefore that the usual triangular inequality cannot be assumed. Instead, it is matter of routine to prove that $\Delta(X, Y) \leq \Delta(X, Z) + \Delta(Z, Y) + \delta(Z)$.

The notion of point is defined as in Sect. 5.5. Indeed, we call *abstraction process* any sequence $< p_n >_{n \in \mathbb{N}}$ of regions which are order-reversing and we call *distance* between two abstraction processes $< p_n >_{n \in \mathbb{N}}$ and $< q_n >_{n \in \mathbb{N}}$ the number:

$$d(< p_n >_{n \in \mathbb{N}}, < q_n >_{n \in \mathbb{N}}) = lim_{n \rightarrow \infty} \Delta(p_n, q_n)$$

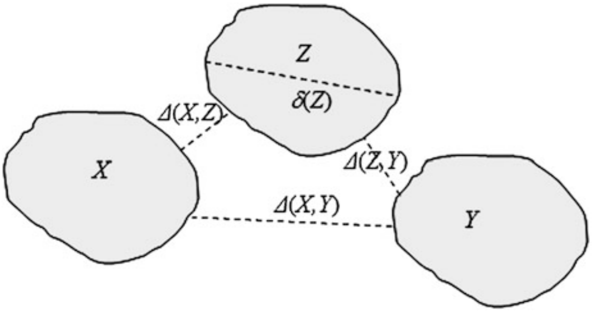

Fig. 5.2 Approximate triangular inequality

and *diameter* of an abstraction process $< p_n >_{n\in\mathbb{N}}$ the number

$$\delta(< p_n >_{n\in\mathbb{N}}) = lim_{n\to\infty}\delta(p_n).$$

Definition 21. We say that $< p_n >_{n\in\mathbb{N}}$ *represents a point* if its diameter is zero and we denote by Pr the class of abstraction processes representing a point.

It is matter of routine to prove that (Pr, d) is a pseudo-metric space.

Definition 22. A *point* is an element of the metric space $(\underline{Pr}, \underline{d})$ associated with (Pr, d).

The logical counterpart of the *ppm*-space is defined as follows. We refer to a first order language L_{CS} with the predicate symbols $\leq$, Cl and Sm. The intended meaning of $Cl(x, y)$ is 'x and y are close', the intended meaning of $Sm(x)$ is 'x is small'. We denote by $(S, \leq, cl, sm)$ an interpretation of L_{CS}.

Definition 23. A CS*-space* is an interpretation $(S, \leq, cl, sm)$ of L_{CS} such that $\leq$ is a crisp order relation and such that the following axioms are satisfied:

CS1 $\forall x\, Cl(x, x)$
CS2 $\forall x \forall y\, (Cl(x, y) \Rightarrow Cl(y, x))$
CS3 $\forall x \forall y \forall z\, (Cl(x, z) \wedge Cl(y, z) \wedge Sm(z) \Rightarrow Cl(x, y))$
CS4 $\forall y \forall x \forall x'\, (x \leq x' \Rightarrow (Cl(x, y) \Rightarrow Cl(x', y)))$
CS5 $\forall x \forall x'\, (x \leq x' \Rightarrow (Sm(x') \Rightarrow Sm(x)))$.

Observe that, as observed in Sect. 5.4, the logical connective Ct enables us to express the condition '$\leq$ is a crisp relation' by a first order formula in the multi-valued logic. Notice also that in Gerla (2008) the structures satisfying *CS1*, *CS2*, *CS3* are called *approximate similarity structures* and that they are proposed for a solution of Poincaré paradox.

The proof of the following proposition is obvious.

Proposition 6. *An interpretation* $(S, \leq, cl, sm)$ *of* L_{CS} *is a CS-space if and only if* $\leq$ *is an order relation, cl is order-preserving, sm is order-reversing and*

(i) $cl(x, x) = 1$,

(ii) $cl(x, y) = cl(y, x)$,
(iii) $cl(x, z) \otimes (y, z) \otimes (z) \leq cl(x, y)$.

The following theorem shows that there is a duality between the notions of *ppm*-space and the one of *CS*-space.

Theorem 9. *Let $(S, \leq, \Delta, \delta)$ be a ppm-space and define cl and sm by setting*

$$cl(x, y) = 10^{-\Delta(x,y)} \quad ; \quad sm(x) = 10^{-\delta(x)}.$$

Then $(S, \leq, cl, sm)$ is a CS-space. Conversely, let $(S, \leq, cl, sm)$ be an approximate CS-space and set

$$\Delta(x, y) = -Log(cl(x, y)) \quad ; \quad \delta(x) = -Log(sm(x)).$$

Then $(S, \leq, \Delta, \delta)$ is a ppm-space.

In particular, by starting from the canonical *ppm*-space $(\mathscr{R}, \subseteq, \Delta, \delta)$, we define the *canonical CS-space* $(\mathscr{R}, \subseteq, cl, sm)$ by setting $cl(X, Y) = 10^{-\Delta(X,Y)}$ and $sm(X) = 10^{-\delta(X)}$.

Theorem 10. *In the canonical CS space we can define the connection relation by the formula $Ct(Cl(x, y))$.*

Proof. It is sufficient to observe that $cl(X, Y) = 1$ if and only if $\Delta(X, Y) = 0$ if and only if $X \cap Y \neq \emptyset$.

References

Blumenthal, L. M. (1970). *Theory and applications of distance geometry*. New York: Chelsea.

Chang, C. C., & Keisler, H. J. (1966). *Continuous model theory*. Princeton: Princeton University Press.

Coppola, C., Gerla, G., & Miranda, A. (2010). Point-free foundation of geometry and multi-valued logic. *Notre Dame Journal of Formal Logic, 51*, 383–405.

De Laguna, T. (1922). Point, line and surface as sets of solids. *The Journal of Philosophy, 19*, 449–461.

Di Concilio, A., & Gerla, G. (2006). Quasi-metric spaces and point-free geometry. *Mathematical Structures in Computer Science, 16*, 115–137.

Gerla, G. (1990). Pointless metric spaces. *Journal of Symbolic Logic, 55*, 207–219.

Gerla, G. (1994). Pointless geometries. In F. Buekenhout (Ed.), *Handbook of incidence geometry* (pp. 1015–1031). Amsterdam/New York: Elsevier.

Gerla, G. (2008). Approximates similarities and Poincaré paradox. *Notre Dame Journal of Formal Logic, 49*, 203–226.

Gerla, G., & Miranda, A. (2004). Graded inclusion and point-free geometry. *International Journal of Pure and Applied Mathematics, 11*, 63–81.

Gerla, G., & Miranda, A. (2008). Mathematical features of Whitehead's pointfree geometry. In M. Weber & W. Desmond (Eds.), *Handbook of Whiteheadian process thought* (pp. 507–533). Frankfurt: Ontos.

Gerla, G., & Paolillo, B. (2010). Whitehead's point-free geometry and diametric posets. *Logic and Logical Philosophy, 19*, 289–308.

Hájek, P. (1998). *Metamathematics of fuzzy logic*. Dordrecht: Kluwer.

Pavelka, J. (1979). On fuzzy logic I many-valued rules of inference. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik, 25*, 45–52.

Pultr, A. (1988). Diameters in locales: How bad they can be? *Commentationes Mathematicae Universitatis Carolinae, 4*, 731–742.

Whitehead, A. N. (1919). *An inquiry concerning the principles of natural knowledge*. Cambrige: Cambrige University Press.

Whitehead, A. N. (1920). *The concept of nature*. Cambrige: Cambrige University Press.

Whitehead, A. N. (1929). *Process and reality*. New York: Macmillan.

Yaacov, I. B., & Usvyatsov, A. (2010). Continuous first order logic and local stability. *Transactions of the American Mathematical Society, 362*, 5213–5259.