Minimum Energy Control of Fractional Discrete-Time Linear Systems with Delays in State and Control

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Abstract. In the paper the problem of minimum energy control of fractional discrete-time linear system with multiple delays in state and control are addressed. General form of solution of the state equation of the system is given and conditions for reachability and minimum energy control are established. The considerations are illustrated by numerical example.

Keywords: fractional, linear systems, discrete-time, time-delay, minimum energy control.

1 Introduction

Dynamical systems described by fractional order differential or difference equations have been investigated in several areas such as viscoelasticity, electrochemistry, diffusion processes, automatic control, etc. (see [4, 8, 19, 20, 22], for example). The problem of controllability and reachability of dynamical systems without delays or with delays for standard or fractional order systems have been considered in [2, 3, 5, 15, 18, 21, 23]. The problem of minimum energy control for standard systems has been firstly introduced and solved in [12]. This problem has been investigated in [6, 13, 14] for standard systems and in [3, 9, 10, 16, 17] for fractional order systems. The problem of minimum energy control with bounded inputs has been recently examined in [3, 11].

The main purpose of the paper is to give the general form of solution of the state equation of fractional discrete-time linear system with multiple delays in state and control and solution of the minimum energy control problem for this systems.

2 Problem Formulation

Let us consider the discrete-time linear system with delays described by the state equation

$$\Delta^{\alpha} x_{i+1} = A_0 x_i + \sum_{k=1}^{h} A_k x_{i-k} + B_0 u_i + \sum_{j=1}^{q} B_j u_{i-j}, \tag{1}$$

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with the initial conditions

$$x_{-k} \in \Re^n,\ k = 0,1,...,h;\quad u_{-j} \in \Re^m,\ j = 1,2,...,q \tag{2}$$

where h and q are a positive integers (number of delays), $x_i \in \Re^n$, $u_i \in \Re^m$ are the state and input vectors, respectively, $A_k \in \Re^{n\times n}$ $(k = 0,1,...,h)$, $B_j \in \Re^{n\times m}$ $(j = 0,1,...,q)$,

$$\Delta^\alpha x_i = \sum_{k=0}^{i} (-1)^k \binom{\alpha}{k} x_{i-k} \tag{3}$$

is the fractional difference of order $\alpha \in \Re$ of the discrete-time function x_i and

$$\binom{\alpha}{k} = \frac{\alpha!}{k!(\alpha-k)!} \tag{4}$$

Substituting of (3) for $i+1$ into (1) we obtain

$$x_{i+1} = F_0 x_i + \sum_{k=1}^{h} A_k x_{i-k} + \sum_{k=1}^{i} c_k(\alpha) x_{i-k} + \sum_{j=0}^{q} B_j u_{i-j}, \tag{5}$$

where

$$F_0 = A_0 + I_n \alpha,\quad (I_n \text{ is the } n\times n \text{ identity matrix}) \tag{6}$$

and

$$c_k(\alpha) = (-1)^k \binom{\alpha}{k+1},\quad k = 1,2,... \tag{7}$$

The coefficients (7) we can compute by the use of the formulas [3]:

$$c_{k+1}(\alpha) = c_k(\alpha)\frac{k+1-\alpha}{k+2},\quad k = 1,2,...\quad c_1(\alpha) = 0.5\alpha(1-\alpha). \tag{8}$$

Now we shall formulate the fundamental definitions for the reachability of the system (1) which is necessary to the further considerations about the minimum energy control problem of the fractional system (1).

Definition 1. A state $x_f \in \Re^n$ is called reachable in *N* steps if there exists a sequence of inputs $u_i \in \Re^m$, $i = 0,1,...,N-1$, that transfers the fractional system with delays (1) from zero initial conditions (2) to the state x_f.

If every state $x_f \in \Re^n$ is reachable in *N* steps, according to the above definition, then we can say that the fractional system (1) is reachable in *N* steps.

Definition 2. If for every state $x_f \in \Re^n$ there exists a natural number *N* such that the state x_f is reachable in *N* steps then the system is called reachable.

The general problem of minimum energy control of the fractional system (1) we can formulate in the same manner as for the fractional discrete-time systems without delays. This problem can be stated as follows:

Find a control sequence $u_i \in \Re^m$, $i = 0,1,...,N-1$, *which transfers the fractional system* (*1*) *from zero initial conditions* (2) *to the desired final state* $x_f \in \Re^n$ *and minimizes the performance index*

$$I(u) = \sum_{i=0}^{N-1} u_i^T Q u_i \tag{9}$$

where $Q \in \Re^{m\times n}$ *is a symmetric positive definite weighting matrix.*

The control sequence $u_i \in \Re^m$, $i = 0,1,...,N-1$ that minimizes the performance index (9) is called minimal one.

The aim of this paper is to give the general form of solution of the state equation (1) of the fractional discrete-time linear system with delays, the condition of reachability, and in consequence the solution of the minimum energy problem of the fractional system with delays in state and control (1).

3 Problem solution

Taking the $\mathcal{Z}$-transform (similarly as in [1]) to both sides of the equation (5) with (2) we obtain

$$\begin{aligned} zX(z) - zx_0 = F_0 X(z) + \sum_{k=1}^{h} A_k z^{-k} [X(z) + \sum_{r=-k}^{-1} x_r z^{-r}] + \sum_{k=1}^{i} c_k z^{-k} X(z) + \\ + B_0 U(z) + \sum_{k=1}^{q} B_k z^{-k} [U(z) + \sum_{i=-k}^{-1} u_j z^{-j}] \end{aligned} \tag{10}$$

where

$$X(z) = \mathcal{Z}\{x_i\}, \quad U(z) = \mathcal{Z}\{u_i\}. \tag{11}$$

The equation (10) can be written in the form

$$\Delta(z)X(z) = zx_0 + \sum_{k=1}^{h} A_k z^{-k} \sum_{r=-k}^{-1} x_r z^{-r} + B_0 U(z) + \sum_{k=1}^{q} B_i z^{-i} [U(z) + \sum_{i=-k}^{-1} u_j z^{-j}] \tag{12}$$

where $\Delta(z)$ is the characteristic matrix and has the form [3]

$$\Delta(z) = zI_n - F_0 - \sum_{k=1}^{h} A_k z^{-k} - \sum_{k=1}^{i} I_n c_k(\alpha) z^{-k} \tag{13}$$

Solving the equation (13) for $X(z)$ we obtain

$$\begin{aligned} X(z) = {} & [\Delta^{-1}(z)z]x_0 + [\Delta^{-1}(z)z]\sum_{k=1}^{h} A_k \sum_{r=0}^{k-1} x_{r-k} z^{-r-1} + \\ & + [\Delta^{-1}(z)z]B_0 U(z) z^{-1} + [\Delta^{-1}(z)z]\sum_{k=1}^{q} B_i \sum_{r=0}^{k-1} z^{-r-1} u_{r-k} + [\Delta^{-1}(z)z]\sum_{k=1}^{q} B_i z^{-r-1} U(z) \end{aligned} \tag{14}$$

Taking the inverse $\mathcal{Z}$-transform to (14) gives the solution of the equation (5) (and the state equation (1)) in the form

$$x_i = \Phi_i x_0 + \sum_{k=1}^{h} \sum_{r=0}^{k-1} \Phi_{i-r-1} A_k x_{r-k} + \sum_{k=1}^{q} \sum_{r=0}^{k-1} \Phi_{i-r-1} B_k u_{r-k} + \sum_{j=0}^{i-1} \sum_{k=0}^{q} \Phi_{i-1-k-j} B_k u_j \tag{15}$$

where

$$\Phi_i = \mathcal{Z}^{-1}\{z\Delta^{-1}(z)\} \tag{16}$$

is the state-transition matrix for the equation (5). From (16) and (13) it follows that the state-transition matrix satisfies the equation

$$\Phi_{i+1} = F_0 \Phi_i + \sum_{k=1}^{h} A_k \Phi_{i-k} + \sum_{k=1}^{i} c_k(\alpha) \Phi_{i-k} \tag{17}$$

with the initial conditions

$$\Phi_0 = I_n, \quad \Phi_i = 0 \text{ for } i < 0. \tag{18}$$

From (16) it follows that the solution of the equation (5) for $i = N$ with the zero conditions (2) has the form

$$x_N = R_N u^N \tag{19}$$

where

$$R_N = [\Psi_{N-1} \quad \Psi_{N-2} \quad \dots \quad \Psi_1 \quad \Psi_0] \tag{20}$$

is called the reachability matrix, and

$$\Psi_i = \sum_{k=0}^{q} \Phi_{i-k} B_k, \quad i = 0,1,...,N-1 \tag{21}$$

$$u^N = \begin{bmatrix} u_{N-1} \\ u_{N-2} \\ \vdots \\ u_0 \end{bmatrix} \in \Re^{Nm}. \tag{22}$$

The matrix Φ_i in (21) has the form (17).

The following condition for the fractional system (1) can be proved in the same manner as for the positive system with $\alpha = 1$ (see [18], for example).

Theorem 1. The fractional system with delays (1) is reachable in N steps if and only if there exists integer number N such that rank of the reachability matrix (20) is equal to n. If this holds, then control u^N which transfers the fractional system (1) from zero initial conditions (2) to the desired final state $x_f \in \Re^n$, can be computed from the formula

$$u^N = R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} x_f + \left(I_{Nm} - R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} R_N\right)K = \bar{x}_f + HK, \tag{23}$$

where $\bar{x}_f = R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} x_f$, $H = I_{Nm} - R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} R_N$ and $K \in \Re^{Nm \times n}$ is an arbitrary matrix, but such that $\det[R_N K] \neq 0$. ■

The formula (23) is based on the right-inverse of the reachability matrix (20). Using another forms of the right-inverse of the rectangular matrix (see [7], for example) we can write the following formulas

$$u^N = K_1 [R_N K_1]^{-1} x_f, \tag{24}$$

or

$$u_0^N = G \begin{bmatrix} -\dfrac{1}{\bar{a}_0}(\bar{A}_1^{m-1} \bar{a}_{m-1} \bar{A}_1^{m-2} + ... + a_1 I_n)(I_n - \bar{A}_2 K_2) \\ K_2 \end{bmatrix} x_f \tag{25}$$

where $K_1 \in \Re^{Nm}$ ($K_1 \in \Re^{Nm \times n}$) is an arbitrary vector (matrix), $K_2 \in \Re^{(Nm-n)\times n}$ is also an arbitrary matrix, G is a permutation matrix of columns of matrix (21) such that

$$R_N G = \begin{bmatrix} \bar{A}_1 & \bar{A}_2 \end{bmatrix}, \quad \bar{A}_1 \in \Re^{n \times n}, \quad \bar{A}_2 \in \Re^{n \times (Nm-n)} \tag{26}$$

and $\bar{a}_0, ..., \bar{a}_{n-1}$ are coefficients of polynomial of matrix $\bar{A}_1$ in the form

$$\det\left[I_n z - \overline{A}_1\right] = z^n + \overline{a}_{m-1} s^{m-1} + \ldots + \overline{a}_1 z + \overline{a}_0 \tag{27}$$

Optimal control which minimizes performance index (9) depends on the weighting matrix Q ($Q = Q^T \succ 0$ – symmetric and positive defined). If we assume that

$$Q = diag(q_1, \ldots, q_m), \quad q_1 = \ldots = q_m = v^2 \tag{28}$$

then we can write the following condition.

Theorem 2. Let the fractional system (1) be reachable in N steps. The control sequence $\hat{u}_0^N$ that minimizes the performance index

$$\boldsymbol{I}(u) = v^2 \sum_{i=0}^{N-1} u_i^T u_i, \quad v > 0 \tag{29}$$

which steers the state of the system (1) from zero initial conditions to any desired final state x_f has the form

$$\hat{u}_0^N = R_N^{\mathrm{T}} [R_N R_N^{\mathrm{T}}]^{-1} x_f . \tag{30}$$

Proof. The performance index (30) for (23) we can write in the form

$$\begin{aligned} \boldsymbol{I}(u) &= v^2 (u_0^N)^T u_0^N = v^2 \left\{u^N = R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} x_f + \left(I_{Nm} - R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} R_N\right) K\right\}^T \times \\ &\times \left\{R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} x_f + \left(I_{Nm} - R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} R_N\right) K\right\} = \\ &= v^2 \left(\overline{x}_f^T \overline{x}_f + K^T H^T H K + K^T H^T \overline{x}_f + \overline{x}_f^T H K\right) \end{aligned} \tag{31}$$

Let us notice that

$$K^T H^T \overline{x}_f = K^T \left(I_{Nm} - R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} R_N\right)^T R_N^{\mathrm{T}}[R_N R_N^{\mathrm{T}}]^{-1} x_f = 0, \quad \overline{x}_f^T H K = 0 \tag{32}$$

Taking into account the above formulas we obtain the performance index in the form

$$\boldsymbol{I}(u_0^N) = v^T \left(\overline{x}_f^T \overline{x}_f + K^T H^T H K\right) \tag{33}$$

The performance index (33) achieves minimum for $K = 0$, therefore for the sequence of controls (30). ■

Let us notice that if $K_1 = R_N^T$ then from (24) we obtain (30).

The minimum energy problem can be solved by the use of the following procedure.

Procedure

Step1. Knowing $A_k \in \Re^{n\times n}$, $(k = 0,1,...,h)$, $B_j \in \Re^{n\times m}$ $(j = 0,1,...,q)$, α, N and using (20) find the matrix R_N (20).

Step 2. Knowing the matrix R_N (20), and using (30) compute sequence of inputs $\hat{u}_0,...,\hat{u}_{N-1}$.

Step 3. Knowing v and using (29) compute the value of the index $\boldsymbol{I}(\hat{u})$.

4 Example

Consider fractional system (1) with $h = q = 2$ delays, $\alpha = 0.8$ and the matrices

$$A_0 = \begin{bmatrix} 0.1 & 0 & 0 \\ -0.1 & 0 & 0.2 \\ 0.1 & 0 & -0.2 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0.1 \\ -0.3 & 0.1 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ -0.2 & 0 & 0 \\ -0.1 & 0 & 0.2 \end{bmatrix}$$

$$B_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{34}$$

Find an optimal control (sequence of inputs) that steers the state of the system from zero initial condition (2) to the state $x_f = [9 \quad 8 \quad 9]^T$ in $N = 8$ steps and minimizes the performance index (29) for $v = 0.4$.

Using Procedure we obtain the following.

Step 1. The condition of reachability in $N = 8$ is satisfied because the reachability matrix R_N (20)

$$R_8 = [\Psi_7 \quad \Psi_6 \quad \Psi_5 \quad \Psi_4 \quad \Psi_3 \quad \Psi_2 \quad \Psi_1 \quad \Psi_0] =$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.02 & 0 & 0 & 0 & 0.2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0.01 & -0.01 & 0 & 0 & 0.1 & -0.08 & 0 & -0.25 & 1.0 & -0.14 & 0 & 0.2 & 0 & 0 \\ 0 & 0.01 & 0 & -0.01 & 0 & 0.01 & 0 & 0.03 & 0 & -0.05 & 0 & 0.24 & 0 & -0.2 & 0 & 1 \end{bmatrix} \tag{35}$$

has rank equal to 3. The matrices $\Psi_0 ... \Psi_7$ we compute from (21), where $\Phi_1 ... \Phi_7$ are computed from recursive formula (17).

Step 2. The optimal sequence $\hat{u}_0,...,\hat{u}_7$ computed from (30) with (35) has the form

$$\hat{u}_0 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad \hat{u}_1 = \begin{bmatrix} 0.06 \\ -0.14 \end{bmatrix}, \quad \hat{u}_2 = \begin{bmatrix} 0 \\ 0.06 \end{bmatrix}, \quad \hat{u}_3 = \begin{bmatrix} 0.62 \\ 0.03 \end{bmatrix}$$
$$\hat{u}_4 = \begin{bmatrix} 0 \\ -2.04 \end{bmatrix}, \quad \hat{u}_5 = \begin{bmatrix} 6.22 \\ 3.29 \end{bmatrix}, \quad \hat{u}_6 = \begin{bmatrix} 0 \\ 8.34 \end{bmatrix}, \quad \hat{u}_7 = \begin{bmatrix} 0 \\ 9.78 \end{bmatrix} \tag{36}$$

Step 3. The minimal value of the performance index is equal to

$$\boldsymbol{I}(\hat{u}) = (0.4)^2 [\hat{u}_0^T \hat{u}_0^T + \hat{u}_1^T \hat{u}_1 + \hat{u}_2^T \hat{u}_2 + \hat{u}_3^T \hat{u}_3 + \hat{u}_4^T \hat{u}_4 + \hat{u}_5^T \hat{u}_5 + \hat{u}_6^T \hat{u}_6 + \hat{u}_7^T \hat{u}_7] = 35.08 \tag{37}$$

Let us check obtained results. Computing of the solution (17) for $i = 0,1,\ldots,7$ with (34) and zero initial conditions (2) we obtain

$$x_1 = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 0.1 \\ 0.02 \\ -0.16 \end{bmatrix}, \quad x_3 = \begin{bmatrix} -0.1 \\ -0.02 \\ 0.12 \end{bmatrix}, \quad x_4 = \begin{bmatrix} 0.04 \\ 0.05 \\ 0 \end{bmatrix},$$
$$x_5 = \begin{bmatrix} 0.03 \\ 0.05 \\ -2.05 \end{bmatrix}, \quad x_6 = \begin{bmatrix} -2.01 \\ 0.15 \\ 3.64 \end{bmatrix}, \quad x_7 = \begin{bmatrix} 2.85 \\ 0.95 \\ 7.12 \end{bmatrix}, \quad x_8 = \begin{bmatrix} 9 \\ 8 \\ 9 \end{bmatrix} \tag{38}$$

The optimal control sequence was computed correctly, because $x_f = x_8$. The trajectory of the considered fractional system is shown on Fig 1, where x^1, x^2, x^3 are components of the vectors (38) ($x_k = [x^1 \; x^2 \; x^3]^T, \quad k = 1,2,\ldots,8$).

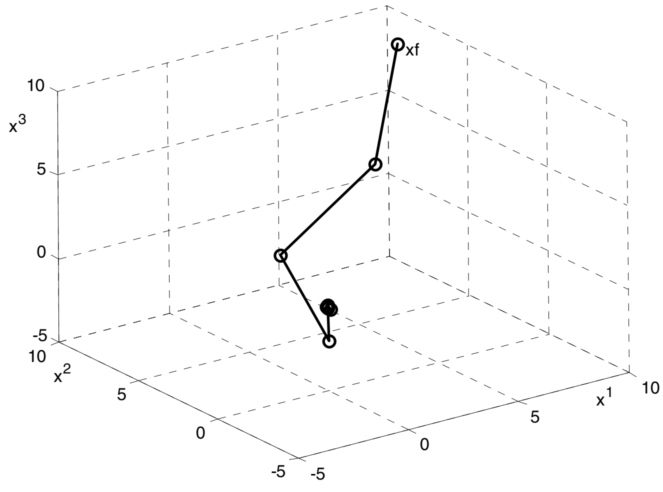

Fig. 1. Trajectory of the considered fractional system with order $\alpha = 0.8$, and the matrices (34) and delays ($h = q = 2$) ('o' are the states (38))

5 Concluding Remarks

The problem reachability and minimum energy control of fractional discrete-time system with delays (1) have been addressed. The general form of solution of state equation (1) is given. Necessary and sufficient conditions for reachability and minimum energy control have been established and illustrated by numerical example.

The considerations can be extended to fractional positive discrete-time linear systems with delays and for the minimum energy control for that class of dynamical systems with bounded controls.

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