Chapter 4
Harmonic Maps Relative to α-Connections

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Abstract In this paper, we study harmonic maps relative to α-connections, but not necessarily relative to Levi-Civita connections, on Hessian domains. For the purpose, we review the standard harmonic map and affine harmonic maps, and describe the conditions for harmonicity of maps between level surfaces of a Hessian domain in terms of the parameter α and the dimension n. To illustrate the theory, we describe harmonic maps between the level surfaces of convex cones.

4.1 Introduction

Harmonic maps are important objects in certain branches of geometry and physics. Geodesics on Riemannian manifolds and holomorphic maps between Kähler manifolds are typical examples of harmonic maps. In addition a harmonic map has a variational characterization by the energy of smooth maps between Riemannian manifolds and several existence theorems for harmonic maps are already known. On the other hand the notion of a Hermitian harmonic map from a Hermitian manifold to a Riemannian manifold was introduced and investigated by [4, 8, 10]. It is not necessary a harmonic map if the domain Hermitian manifold is non-Kähler. The similar results are pointed out for affine harmonic maps, which is analogy to Hermitian harmonic maps [7].

Statistical manifolds have mainly been studied in terms of their affine geometry, information geometry, and statistical mechanics [1]. For example, Shima established conditions for harmonicity of gradient mappings of level surfaces on a Hessian domain, which is a typical example of a dually flat statistical manifold [14]. Level surfaces on a Hessian domain are known as 1- and (-1)-conformally flat statistical

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manifolds for primal and dual connections, respectively [17, 19]. The gradient mappings are then considered to be harmonic maps relative to the dual connection, i.e., the (-1)-connection [13].

In this paper, we review the notions of harmonic maps, affine harmonic maps and α-affine harmonic maps, and investigate different kinds of harmonic maps relative to α-connections. In Sect. 4.2, we give definitions of an affine harmonic map, a harmonic map and the standard Laplacian. In Sect. 4.3, we explain the generalized Laplacian which defines a harmonic map relative to an affine connection. In Sect. 4.4, we present the Laplacian of a gradient mapping on a Hessian domain, as an example of the generalized Laplacian. Moreover, we compare the harmonic map defined by Shima with an affine harmonic map defined in Sect. 4.2. In Sect. 4.5, α-connections of statistical manifolds are explained. In Sect. 4.6, we define α-affine harmonic maps which are generalization of affine harmonic maps and also a generalization of harmonic maps defined by Shima. In Sect. 4.7, we describe the α-conformal equivalence of statistical manifolds and a harmonic map relative to two α-connections. In Sect. 4.8, we review α-conformal equivalence of level surfaces of a Hessian domain. In Sect. 4.9, we study harmonic maps of level surfaces relative to two α-connections, for examples of a harmonic map in Sect. 4.7, and provide examples on level surfaces of regular convex cones.

Shima [13] investigated harmonic maps of n-dimensional level surfaces into an $(n+1)$-dimensional dual affine space, rather than onto other level surfaces. Although Nomizu and Sasaki calculated the Laplacian of centro-affine immersions into an affine space, which generate projectively flat statistical manifolds (i.e. (-1)-conformally flat statistical manifolds), they did not discuss any harmonic maps between two centro-affine hypersurfaces [12]. Then, we study harmonic maps between hypersurfaces with the same dimension relative to general α-connections that may not satisfy $\alpha = -1$ or 0 (where the 0-connection implies the Levi-Civita connection). In particular, we demonstrate the existence of non-trivial harmonic maps between level surfaces of a Hessian domain with α-parameters and the dimension n.

4.2 Affine Harmonic Maps and Harmonic Maps

First, we recall definitions of an affine harmonic map and a harmonic map.

Let M an m-dimensional affine manifold and $\{x^1, \ldots, x^m\}$ a local affine coordinate system of M. If there exist a symmetric tensor field of degree 2

$$g = g_{ij} dx^i dx^j$$

on M satisfying locally

$$g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j} \tag{4.1}$$

for a convex function φ, M is said to be a *Kähler affine manifold* [2, 7]. A matrix $[g_{ij}]$ is positive definite and defines a Riemannian metric. Then for the Kähler affine manifold M, (M, D, g) is a Hessian manifold, where D is a canonical flat affine connection for $\{x^1, \ldots, x^m\}$. We will mention details of Hessian manifolds and Hessian domains in later sections of this paper.

The *Kähler affine structure* (4.1) defines an affinely invariant operator L by

$$L = \sum_{i,j=1}^{m} g_{ij} \frac{\partial^2}{\partial x^i \partial x^j}. \tag{4.2}$$

A smooth function $f : M \to \mathbf{R}$ is said to be *affine harmonic* if

$$Lf = 0.$$

For a Kähler affine manifold (M, g) and a Riemannian manifold (N, h), a smooth map $\phi : M \to N$ is said to be *affine harmonic* if

$$\sum_{i,j=1}^{m} g^{ij} \left(\frac{\partial^2 \phi^\gamma}{\partial x^i \partial x^j} + \sum_{\delta,\beta=1}^{n} \hat{\Gamma}^{\gamma}_{\delta\beta} \frac{\partial \phi^\delta}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} \right) = 0, \quad \gamma = 1, \ldots, n, \tag{4.3}$$

where $\hat{\Gamma}$ is the Christoffel symbol of the Levi-Civita connection for a Riemannian metric h, and $n = \dim N$.

Let us compare an affine harmonic map with a harmonic map. For this purpose, we give a definition of a harmonic function at first. For a Riemannian manifold (M, g), a smooth function $f : M \to \mathbf{R}$ is said to be a *harmonic function* if

$$\Delta f = 0,$$

where Δ is the standard *Laplacian*, i.e.,

$$\Delta f = \operatorname{div} \operatorname{grad} f = \frac{1}{\sqrt{g}} \sum_{i,j=1}^{m} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j} \right) \tag{4.4}$$

$$= \sum_{i,j=1}^{m} g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{k=1}^{m} \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \right) \tag{4.5}$$

$$= \sum_{i=1}^{m} \{ e_i(e_i f) - (\nabla^{LC}_{e_i} e_i) f \},$$

$$g = \det[g_{ij}],$$

$\{e_1, \ldots, e_m\}$ is a local orthogonal frame on a neighborhood of $x \in M$, and ∇^{LC}, Γ are the Levi-Civita connection, the Christoffel symbol of ∇^{LC}, respectively. Remark that the sign of definition (4.4) is inverse to the sign of the Laplacian in [3, 21].

For Riemannian manifolds (M, g), (N, h), a smooth map $\phi : M \to N$ is said to be a *harmonic map* if

$$\tau(\phi) \equiv 0; \quad \text{the } \textit{Euler-Lagrange equation},$$

where $\tau(\phi) \in \Gamma(\phi^{-1}TN)$ is the standard tension field of ϕ defined by

$$\tau(\phi)(x) = \sum_{i=1}^{m} (\tilde{\nabla}^{LC}_{e_i} \phi_* e_i - \phi_* \nabla^{LC}_{e_i} e_i)(x), \quad x \in M, \tag{4.6}$$
$$\tilde{\nabla}^{LC}_{e_i} \phi_* e_i = \hat{\nabla}^{LC}_{\phi_* e_i} \phi_* e_i; \quad \text{the pull-back connection,}$$

and ∇^{LC}, $\hat{\nabla}^{LC}$ are the Levi-Civita connections for g, h, respectively. For local coordinate systems $\{x^1, \ldots, x^m\}$ and $\{y^1, \ldots, y^n\}$ on M and N, the γ-th component of $\tau(\phi)$ at $x \in M$ is described by

$$\tau(\phi)^\gamma(x) = \sum_{i,j=1}^{m} g^{ij} \left\{ \frac{\partial^2 \phi^\gamma}{\partial x^i \partial x^j} - \sum_{k=1}^{m} \Gamma^k_{ij}(x) \frac{\partial \phi^\gamma}{\partial x^k} + \sum_{\delta,\beta=1}^{n} \hat{\Gamma}^\gamma_{\delta\beta}(\phi(x)) \frac{\partial \phi^\delta}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} \right\} \tag{4.7}$$
$$= \Delta \phi^\gamma + \sum_{i,j=1}^{m} \sum_{\delta,\beta=1}^{n} g^{ij} \hat{\Gamma}^\gamma_{\delta\beta}(\phi(x)) \frac{\partial \phi^\delta}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j},$$
$$\phi^\delta = y^\delta \circ \phi, \quad \gamma = 1, \ldots, n,$$

where

$$\tau(\phi)(x) = \sum_{\gamma=1}^{n} \tau(\phi)^\gamma(x) \frac{\partial}{\partial y^\gamma},$$

and Γ^k_{ij}, $\hat{\Gamma}^\gamma_{\delta\beta}$ are the Christoffel symbols of ∇^{LC}, $\hat{\nabla}^{LC}$, respectively. The original definition of a harmonic map is described in [3, 21], and so on.

Remark 1 Term (4.5) is not equal to the definition (4.2). Hence an affine harmonic function is not necessary a harmonic function.

Remark 2 Term (4.7) is not equal to the definition (4.3). Hence an affine harmonic map is not necessary a harmonic map.

4.3 Affine Harmonic Maps and Generalized Laplacians

In Sect. 4.2, the Laplacian is defined for a function on a Riemannian manifold. In this section, we treat Laplacians for maps between Riemannian manifolds.

For Riemannian manifolds (M, g) and (N, h), a tension field of a smooth map $\phi : M \to N$ is defined by

$$\begin{aligned}\tau(\phi) &= \sum_{i=1}^{m} (\hat{\nabla}_{e_i}(\phi_* e_i) - \phi_*(\nabla^{LC}_{e_i} e_i)) \quad \in \Gamma(\phi^{-1}TN) \qquad (4.8)\\ &= \sum_{i,j=1}^{m} g^{ij} \left\{ \hat{\nabla}_{\frac{\partial}{\partial x^i}} \left(\phi_* \frac{\partial}{\partial x^j} \right) - \phi_* \left(\nabla^{LC}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right) \right\},\end{aligned}$$

where $\{e_1, \ldots, e_m\}$ is a local orthonormal frame for g, $\{x^1, \ldots, x^m\}$ is a local coordinate system on M, ∇^{LC} is the Levi-Civita connection of g, and $\hat{\nabla}$ is a torsion free affine connection on N [12]. The affine connection $\hat{\nabla}$ does not need to be the Levi-Civita connection. We also denote by $\hat{\nabla}$ the pull-back connection of $\hat{\nabla}$ to M. Then ϕ is said to be a *harmonic map relative to* $(g, \hat{\nabla})$ if

$$\tau(\phi) = \sum_{i=1}^{m} (\hat{\nabla}_{e_i}(\phi_* e_i) - \phi_*(\nabla^{LC}_{e_i} e_i)) \equiv 0.$$

If a Riemannian manifold N is an finite dimensional real vector space V, the tension field $\tau(\phi)$ is said to be a *Laplacian of a map* $\phi : M \to V$. Then a notation Δ for the standard Laplacian is often used for the Laplacian of a map as the following;

$$\Delta\phi = \Delta_{(g, \hat{\nabla})}\phi = \tau(\phi) : M \to V. \qquad (4.9)$$

For $V = \mathbf{R}$, $\Delta\phi$ defined by Eqs. (4.8) and (4.9) coincides with the standard Laplacian for a function defined by (4.4).

See in [12] for an affine immersion and the Laplacian of a map, and see in [13, 14] for the gradient mapping and the Laplacian on a Hessian domain.

4.4 Gradient Mappings and Affine Harmonic Maps

In this section, we investigate the Laplacian of a gradient mapping in view of geometry of affine harmonic maps.

Let D be the canonical flat affine connection on an $(n+1)$-dimensional real affine space $\mathbf{A}^{n+1}$ and let $\{x^1, \ldots, x^{n+1}\}$ be the canonical affine coordinate system on $\mathbf{A}^{n+1}$, i.e., $Ddx^i = 0$. If the Hessian $Dd\varphi = \sum_{i,j=1}^{n+1} (\partial^2 \varphi / \partial x^i \partial x^j) dx^i dx^j$ of a function φ is non-degenerate on a domain Ω in $\mathbf{A}^{n+1}$, then $(\Omega, D, g = Dd\varphi)$ is a *Hessian domain* [14].

For the dual affine space $\mathbf{A}^*_{n+1}$ and the dual affine coordinate system $\{x^*_1, \ldots, x^*_{n+1}\}$ of $\mathbf{A}^{n+1}$, the *gradient mapping* ι from a Hessian domain $(\Omega, D, g = Dd\varphi)$ into $(\mathbf{A}^*_{n+1}, D^*)$ is defined by

$$x^*_i \circ \iota = -\frac{\partial \varphi}{\partial x^i}.$$

The dually flat affine connection D' on Ω is given by

$$\iota_*(D'_X Y) = D^*_X \iota_*(Y) \quad \text{for } X, Y \in \Gamma(T\Omega), \tag{4.10}$$

where $D^*_X \iota_*(Y)$ denotes the covariant derivative along ι induced by the canonical flat affine connection D^* on $\mathbf{A}^*_{n+1}$.

The Laplacian of ι with respect to (g, D^*) is given by

$$\begin{aligned}
\Delta_{(g,D^*)}\iota &= \sum_{i,j} g^{ij} \left\{ D^*_{\frac{\partial}{\partial x^i}} \left(\iota_* \frac{\partial}{\partial x^j} - \iota_* \left(\nabla^{LC}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right) \right) \right\} \\
&= \iota_* \left\{ \sum_{i,j} g^{ij} (D' - \nabla^{LC})_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right\} \qquad (4.11) \\
&= \iota_* \left\{ \sum_{i,j} g^{ij} (\nabla^{LC} - D)_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right\} \qquad (4.12) \\
&= \iota_* \left(\sum_i \alpha^{Ki} \frac{\partial}{\partial x^i} \right),
\end{aligned}$$

where ∇^{LC} is the Levi-Civita connection for g and Γ is the Christoffel symbol of ∇^{LC}, and where α^K is the Koszul form, i.e.,

$$\alpha^K = d \log |det[g_{ij}]|^{\frac{1}{2}}, \quad \alpha^K_i = \sum_r \Gamma^r_{ri}, \ \alpha^{Ki} = \sum_j g^{ij} \alpha^K_j$$

([13], p. 93 in [14]). We have the deformation from (4.11) to (4.12) by

$$\frac{D + D'}{2} = \nabla^{LC}.$$

Details of dual affine connections are described in later sections.

Let ∇^{LC*} be the Levi-Civita connection for the Hessian metric $g^* = D^* d\varphi^*$ on the dual domain $\Omega^* = \iota(\Omega)$, where $\varphi^* = \sum_i x^i (\partial\varphi/\partial x^i) - \varphi$ is the *Legendre transform* of φ. Then the term (4.12) is described as the follows:

$$\Delta_{(g,D^*)}\iota = \sum_{i,j} g^{ij}\{\nabla^{LC*}_{\frac{\partial}{\partial x^i}}(\iota_* \frac{\partial}{\partial x^j}) - \iota_*(D_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j})\}.$$

Moreover the γ-th component of $\Delta_{(g,D^*)}\iota$ is

$$(\Delta_{(g,D^*)}\iota)^\gamma = \sum_{i,j} g^{ij} \left(\frac{\partial^2 \iota^\gamma}{\partial x^i \partial x^j} + \sum_{\delta,\beta} \Gamma^\gamma_{\delta\beta} \frac{\partial \iota^\delta}{\partial x^i} \frac{\partial \iota^\beta}{\partial x^j} \right), \quad \gamma = 1, \ldots, n+1$$

where $\iota^i(x) = x_i^* \circ \iota(x)$. Therefore, if the gradient mapping ι is a harmonic map with respect to (g, D^*), i.e., if $\Delta_{(g,D^*)}\iota \equiv 0$, we have

$$\sum_{i,j} g^{ij} \left(\frac{\partial^2 \iota^\gamma}{\partial x^i \partial x^j} + \sum_{\delta,\beta} \Gamma^\gamma_{\delta\beta} \frac{\partial \iota^\delta}{\partial x^i} \frac{\partial \iota^\beta}{\partial x^j} \right) = 0, \quad \gamma = 1, \ldots, n+1. \tag{4.13}$$

Equation (4.13) is obtained by putting ι on ϕ of Eq. (4.3). Thus considering a Hessian domain Ω as a Kähler affine manifold, we have the next proposition.

Proposition 1 *The followings are equivalent:*

(i) *the gradient mapping ι is a harmonic map with respect to (g, D^*);*
(ii) *the gradient mapping $\iota : (\Omega, D) \to (\mathbf{A}^*_{n+1}, \nabla^{LC*})$ is an affine harmonic map.*

In [13, 14], Shima studied an affine harmonic map with the restriction of the gradient mapping ι to a level surface of a convex function φ.

The author does not clearly distinguish a phrase “relative to something” with a phrase “with respect to something”.

4.5 α-Connections of Statistical Manifolds

We recall some definitions that are essential to the theory of statistical manifolds and relate α-connections to Hessian domains.

Given a torsion-free affine connection ∇ and a pseudo-Riemannian metric h on a manifold N, the triple (N, ∇, h) is said to be a *statistical manifold* if ∇h is symmetric. If the curvature tensor R of ∇ vanishes, (N, ∇, h) is said to be *flat*.

Let (N, ∇, h) be a statistical manifold and let ∇' be an affine connection on N such that

$$Xh(Y, Z) = h(\nabla_X Y, Z) + h(Y, \nabla'_X Z) \ \text{ for } X, Y \text{ and } Z \in \Gamma(TN),$$

where $\Gamma(TN)$ is the set of smooth tangent vector fields on N. The affine connection ∇' is torsion free and $\nabla' h$ is symmetric. Then ∇' is called the *dual connection* of

∇. The triple (N, ∇', h) is the *dual statistical manifold* of (N, ∇, h), and (∇, ∇', h) defines the *dualistic structure* on N. The curvature tensor of ∇' vanishes if and only if the curvature tensor of ∇ also vanishes. Under these conditions, (∇, ∇', h) becomes a *dually flat structure*.

Let N be a manifold with a dualistic structure (∇, ∇', h). For any $\alpha \in \mathbf{R}$, an affine connection defined by

$$\nabla^{(\alpha)} := \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla' \tag{4.14}$$

is called an α-*connection* of (N, ∇, h). The triple $(N, \nabla^{(\alpha)}, h)$ is also a statistical manifold, and $\nabla^{(-\alpha)}$ is the dual connection of $\nabla^{(\alpha)}$. The 1-connection $\nabla^{(1)}$, the (-1)-connection $\nabla^{(-1)}$, and the 0-connection $\nabla^{(0)}$ correspond to the ∇, ∇', and the Levi-Civita connection of (N, h), respectively. An α-connection does not need to be flat.

A Hessian domain is a flat statistical manifold. Conversely, a local region of a flat statistical manifold is a Hessian domain. For the dual connection D' defined by (4.10), (Ω, D', g) is the dual statistical manifold of (Ω, D, g) if a Hessian domain (Ω, D', g) is a statistical manifold [1, 13, 14].

4.6 α-Affine Harmonic Maps

In this section, we give a generalization of an affine harmonic map.

Considering a Hessian domain (Ω, D, g) as a statistical manifold, we have the α-connection of (Ω, D, g) by

$$D^{(\alpha)} = \frac{1+\alpha}{2}D + \frac{1-\alpha}{2}D'$$

for each $\alpha \in \mathbf{R}$. Let $D^{(\alpha)*}$ be an α-connection of (Ω^*, D^*, g^*) which is the dual statistical manifold of (Ω, D, g). Then the *Laplacian of the gradient mapping* ι with respect to $(g, D^{(\alpha)*})$ is given by

$$\begin{aligned}
\Delta_{(g,D^{(\alpha)*})}\iota &= \sum_{i,j} g^{ij}\left\{D^{(\alpha)*}_{\frac{\partial}{\partial x^i}}\left(\iota_*\frac{\partial}{\partial x^j}\right) - \iota_*\left(\nabla^{LC}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right)\right\} \\
&= \iota_*\left\{\sum_{i,j} g^{ij}(D^{(-\alpha)} - \nabla^{LC})_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right\} \\
&= \iota_*\left\{\sum_{i,j} g^{ij}(\nabla^{LC} - D^{(\alpha)})_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right\}
\end{aligned}$$

$$= \sum_{i,j} g^{ij} \left\{ \nabla^{LC*}_{\frac{\partial}{\partial x^i}} (\iota_* \frac{\partial}{\partial x^j}) - \iota_* \left(D^{(\alpha)}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right) \right\}.$$

If $\Delta_{(g,D^{(\alpha)*})}\iota \equiv 0$, we have

$$\sum_{i,j} g^{ij} \left\{ \frac{\partial^2 \iota^\gamma}{\partial x^i \partial x^j} - (1-\alpha) \sum_k \Gamma^k_{ij} \frac{\partial \iota^\gamma}{\partial x^k} + \sum_{\delta,\beta} \hat{\Gamma}^\gamma_{\delta\beta} \frac{\partial \iota^\delta}{\partial x^i} \frac{\partial \iota^\beta}{\partial x^j} \right\} = 0,$$

$$\gamma = 1, \ldots, n+1.$$

In general, we define the notion of α-affine harmonic maps as follows:

Definition 1 For a Kähler affine manifold (M, g) and a Riemannian manifold (N, h), a map $\phi : M \to N$ is said to be an α-affine harmonic map if

$$\sum_{i,j} g^{ij} \left(\frac{\partial^2 \phi^\gamma}{\partial x^i \partial x^j} - (1-\alpha) \sum_k \Gamma^k_{ij} \frac{\partial \phi^\gamma}{\partial x^k} + \sum_{\delta,\beta} \hat{\Gamma}^\gamma_{\delta\beta} \frac{\partial \phi^\delta}{\partial x^i} \frac{\partial \phi^\beta}{\partial x^j} \right) = 0, \qquad (4.15)$$

$$\gamma = 1, \ldots, \dim\ N.$$

Then we obtain that the gradient mapping ι is a harmonic map with respect to $(g, D^{(\alpha)*})$ if and only if the map $\iota : (\Omega, D^{(\alpha)}) \to (\mathbf{A}^*_{n+1}, \nabla^*)$ is an α-affine harmonic map.

Remark 3 For $\alpha = 1$, a 1-affine harmonic map is an affine harmonic map.

Remark 4 For $\alpha = 0$, a 0-affine harmonic map is a harmonic map in the standard sense.

They are problems to find applications of α-affine harmonic maps and to investigate them.

4.7 Harmonic Maps for α-Conformal Equivalence

In this section, we describe harmonic maps with respect to α-conformal equivalence of statistical manifolds.

For a real number α, statistical manifolds (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ are regarded as *α-conformally equivalent* if there exists a function ϕ on N such that

$$\bar{h}(X, Y) = e^{\phi} h(X, Y), \tag{4.16}$$

$$h(\bar{\nabla}_X Y, Z) = h(\nabla_X Y, Z) - \frac{1+\alpha}{2} d\phi(Z) h(X, Y) + \frac{1-\alpha}{2}\{d\phi(X) h(Y, Z) + d\phi(Y) h(X, Z)\} \tag{4.17}$$

for X, Y and $Z \in \Gamma(TN)$. Two statistical manifolds (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ are α-conformally equivalent if and only if the dual statistical manifolds (N, ∇', h) and $(N, \bar{\nabla}', \bar{h})$ are $(-\alpha)$-conformally equivalent. A statistical manifold (N, ∇, h) is said to be *α-conformally flat* if (N, ∇, h) is locally α-conformally equivalent to a flat statistical manifold [19].

Let (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ be α-conformally equivalent statistical manifolds of $\dim n \geq 2$, and $\{x^1, \ldots x^n\}$ a local coordinate system on N. Suppose that h and $\bar{h}$ are Riemannian metrices. We set $h_{ij} = h(\partial/\partial x^i, \partial/\partial x^j)$ and $[h^{ij}] = [h_{ij}]^{-1}$. Let $\pi_{id} : (N, \nabla, h) \to (N, \bar{\nabla}, \bar{h})$ be the identity map, i.e., $\pi_{id}(x) = x$ for $x \in N$, and π_{id*} the differential of π_{id}.

We define a harmonic map relative to $(h, \nabla, \bar{\nabla})$ as follows:

Definition 2 ([16, 18]) If a tension field $\tau_{(h,\nabla,\bar{\nabla})}(\pi_{id})$ vanishes on N, i.e.,

$$\tau_{(h,\nabla,\bar{\nabla})}(\pi_{id}) \equiv 0,$$

the map $\pi_{id} : (N, \nabla, h) \to (N, \bar{\nabla}, \bar{h})$ is said to be a *harmonic map relative to* $(h, \nabla, \bar{\nabla})$, where the tension field is defined by

$$\tau_{(h,\nabla,\bar{\nabla})}(\pi_{id}) := \sum_{i,j=1}^{n} h^{ij} \left\{ \bar{\nabla}_{\frac{\partial}{\partial x^i}} \left(\pi_{id*}(\frac{\partial}{\partial x^j}) \right) - \pi_{id*} \left(\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right) \right\} \in \Gamma(\pi_{id}^{-1} TN)$$
$$= \sum_{i,j=1}^{n} h^{ij} (\bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}) \in \Gamma(TN). \tag{4.18}$$

Then the next theorem holds.

Theorem 1 ([16, 18]) *For α-conformally equivalent statistical manifolds (N, ∇, h) and $(N, \bar{\nabla}, \bar{h})$ of $\dim N \geq 2$ satisfying Eqs. (4.16) and (4.17), if $\alpha = -(n-2)/(n+2)$ or ϕ is a constant function on N, the identity map $\pi_{id} : (N, \nabla, h) \to (N, \bar{\nabla}, \bar{h})$ is a harmonic map relative to $(h, \nabla, \bar{\nabla})$.*

Proof By Eqs. (4.17) and (4.18), for $k \in \{1, \ldots, n\}$ we have

$$h\left(\tau_{(h,\nabla,\bar{\nabla})}(\pi_{id}), \frac{\partial}{\partial x^k} \right)$$
$$= h\left(\sum_{i,j=1}^{n} h^{ij} \left(\bar{\nabla}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} - \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} \right), \frac{\partial}{\partial x^k} \right)$$

$$
\begin{aligned}
&= \sum_{i,j=1}^{n} h^{ij} \left\{ -\frac{1+\alpha}{2} d\phi \left(\frac{\partial}{\partial x^k} \right) h \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \right. \\
&\quad + \frac{1-\alpha}{2} \left\{ d\phi \left(\frac{\partial}{\partial x^i} \right) h \left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \right) \right. \\
&\quad \left. \left. + d\phi \left(\frac{\partial}{\partial x^j} \right) h \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k} \right) \right\} \right\} \\
&= \sum_{i,j=1}^{n} h^{ij} \left\{ -\frac{1+\alpha}{2} \frac{\partial \phi}{\partial x^k} h_{ij} + \frac{1-\alpha}{2} \left(\frac{\partial \phi}{\partial x^i} h_{jk} + \frac{\partial \phi}{\partial x^j} h_{ik} \right) \right\} \\
&= \left\{ -\frac{1+\alpha}{2} \cdot n \cdot \frac{\partial \phi}{\partial x^k} + \frac{1-\alpha}{2} \left(\sum_{i=1}^{n} \frac{\partial \phi}{\partial x^i} \delta_{ik} + \sum_{j=1}^{n} \frac{\partial \phi}{\partial x^j} \delta_{jk} \right) \right\} \\
&= \left(-\frac{1+\alpha}{2} \cdot n + \frac{1-\alpha}{2} \cdot 2 \right) \frac{\partial \phi}{\partial x^k} \\
&= -\frac{1}{2} \left\{ (n+2)\,\alpha + (n-2) \right\} \frac{\partial \phi}{\partial x^k},
\end{aligned}
$$

where δ_{ij} is the Kronecker's delta. Therefore, if $\tau_{(h,\nabla,\bar{\nabla})}(\pi_{id}) \equiv 0$, it holds that $(n+2)\alpha + (n-2) = 0$ or $\partial\phi/\partial x^k = 0$ for all $k \in \{1, \ldots, n\}$ at each point in N. Thus we obtain Theorem 1. □

4.8 α-Conformal Equivalence of Level Surfaces

We show our previous results of α-conformal equivalence of level surfaces.

The next theorem holds for a 1-conformally flat statistical submanifold.

Theorem 2 ([19]) *Let M be a simply connected n-dimensional level surface of φ on an $(n+1)$-dimensional Hessian domain $(\Omega, D, g = Dd\varphi)$ with a Riemannian metric g, and suppose that $n \geq 2$. If (Ω, D, g) is a flat statistical manifold, then (M, D^M, g^M) is a 1-conformally flat statistical submanifold of (Ω, D, g), where D^M and g^M are the connection and the Riemannian metric on M induced by D and g, respectively.*

See in [15, 17–19] for realization problems related with α-conformal equivalence.

We now consider two simply connected level surfaces of dim $n \geq 2$ (M, D, g) and $(\hat{M}, \hat{D}, \hat{g})$, which are 1-conformally flat statistical submanifolds of (Ω, D, g). Let λ be a function on M such that $e^{\lambda(p)}\iota(p) \in \hat{\iota}(\hat{M})$ for $p \in M$, where $\hat{\iota}$ is the restriction of the gradient mapping ι to $\hat{M}$, and set $(e^{\lambda})(p) = e^{\lambda(p)}$. Note that the function e^{λ} projects M to $\hat{M}$ with respect to the dual affine coordinate system on Ω.

We define a mapping $\pi : M \to \hat{M}$ by

$$
\hat{\iota} \circ \pi = e^{\lambda} \iota,
$$

where ι (as denoted above) is the restriction of the gradient mapping ι to M. Let $\bar{D}'$ be an affine connection on M defined by

$$\pi_*(\bar{D}'_X Y) = \hat{D}'_{\pi_*(X)}\pi_*(Y) \quad \text{for } X, Y \in \Gamma(TM),$$

and $\bar{g}$ be a Riemannian metric on M such that

$$\bar{g}(X, Y) = e^{\lambda} g(X, Y) = \hat{g}(\pi_*(X), \pi_*(Y)).$$

The following theorem has been proposed elsewhere (cf. [9, 11]).

Theorem 3 ([20]) *For affine connections D' and $\bar{D}'$ on M, the following are true:*

(i) *D' and $\bar{D}'$ are projectively equivalent.*
(ii) *(M, D', g) and $(M, \bar{D}', \bar{g})$ are (-1)-conformally equivalent.*

Let $\bar{D}$ be an affine connection on M defined by

$$\pi_*(\bar{D}_X Y) = \hat{D}_{\pi_*(X)}\pi_*(Y) \quad \text{for } X, Y \in \Gamma(TM).$$

From the duality of $\hat{D}$ and $\hat{D}'$, $\bar{D}$ is the dual connection of $\bar{D}'$ on M. Then the next theorem holds (cf. [6, 9]).

Theorem 4 ([20]) *For affine connections D and $\bar{D}$ on M, we have that*

(i) *D and $\bar{D}$ are dual-projectively equivalent.*
(ii) *(M, D, g) and $(M, \bar{D}, \bar{g})$ are 1-conformally equivalent.*

For α-connections $D^{(\alpha)}$ and $\bar{D}^{(\alpha)} = D^{(-\alpha)}$ defined similarly to (4.14), we obtain the following corollary by Theorem 3, Theorem 4, and Eq. (4.17) with $\phi = \lambda$ [15].

Corollary 1 *For affine connections $D^{(\alpha)}$ and $\bar{D}^{(\alpha)}$ on M, $(M, D^{(\alpha)}, g)$ and $(M, \bar{D}^{(\alpha)}, \bar{g})$ are α-conformally equivalent.*

4.9 Harmonic Maps Relative to α-Connections on Level Surfaces

We denote $\hat{D}^{(\alpha)}_{\pi_*(X)}\pi_*(Y)$ by $\hat{D}^{(\alpha)}_X\pi_*(Y)$, considering it in the inverse-mapped section $\Gamma(\pi^{-1}T\hat{M})$. Let $\{x^1, \ldots, x^n\}$ be a local coordinate system on M. The notion of a harmonic map between two level surfaces $(M, D^{(\alpha)}, g)$ and $(\hat{M}, \hat{D}^{(\alpha)}, \hat{g})$ is defined as follows:

Definition 3 ([16, 18]) If a tension field $\tau_{(g, D^{(\alpha)}, \hat{D}^{(\alpha)})}(\pi)$ vanishes on M, i.e.,

$$\tau_{(g, D^{(\alpha)}, \hat{D}^{(\alpha)})}(\pi) \equiv 0,$$

the map $\pi : (M, D^{(\alpha)}, g) \to (\hat{M}, \hat{D}^{(\alpha)}, \hat{g})$ is said to be a harmonic map relative to $(g, D^{(\alpha)}, \hat{D}^{(\alpha)})$, where the tension field is defined by

$$\tau_{(g,D^{(\alpha)},\hat{D}^{(\alpha)})}(\pi) := \sum_{i,j=1}^{n} g^{ij}\left\{\hat{D}^{(\alpha)}_{\frac{\partial}{\partial x^i}}\left(\pi_*(\frac{\partial}{\partial x^j})\right) - \pi_*\left(D^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right)\right\} \in \Gamma(\pi^{-1}T\hat{M}). \tag{4.19}$$

We now specify the conditions for harmonicity of a map $\pi : M \to \hat{M}$ relative to $(g, D^{(\alpha)}, \hat{D}^{(\alpha)})$.

Theorem 5 ([16, 18]) *Let $(M, D^{(\alpha)}, g)$ and $(\hat{M}, \hat{D}^{(\alpha)}, \hat{g})$ be simply connected n-dimensional level surfaces of an $(n+1)$-dimensional Hessian domain (Ω, D, g) with $n \geq 2$. If $\alpha = -(n-2)/(n+2)$ or λ is a constant function on M, a map $\pi : (M, D^{(\alpha)}, g) \to (\hat{M}, \hat{D}^{(\alpha)}, \hat{g})$ is a harmonic map relative to $(g, D^{(\alpha)}, \hat{D}^{(\alpha)})$, where*

$$\hat{\iota} \circ \pi = e^{\lambda}\iota, \ (e^{\lambda})(p) = e^{\lambda(p)}, \quad e^{\lambda(p)}\iota(p) \in \hat{\iota}(\hat{M}), \ p \in M,$$

and ι, $\hat{\iota}$ are the restrictions of the gradient mappings on Ω to M and $\hat{M}$, respectively.

Proof The tension field of the map π relative to $(g, D^{(\alpha)}, \hat{D}^{(\alpha)})$ is described by the pull-back of $(\hat{M}, \hat{D}^{(\alpha)}, \hat{g})$, namely $(M, \bar{D}^{(\alpha)}, \bar{g})$, as follows:

$$\begin{aligned}
\tau_{(g,D^{(\alpha)},\hat{D}^{(\alpha)})}(\pi) &= \sum_{i,j=1}^{n} g^{ij}\left\{\hat{D}^{(\alpha)}_{\frac{\partial}{\partial x^i}}\left(\pi_*\left(\frac{\partial}{\partial x^j}\right)\right) - \pi_*\left(D^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right)\right\} \\
&= \sum_{i,j=1}^{n} g^{ij}\left\{\pi_*\left(\bar{D}^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right) - \pi_*\left(D^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right)\right\} \\
&= \pi_*\left(\sum_{i,j=1}^{n} g^{ij}\left(\bar{D}^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - D^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right)\right)
\end{aligned}$$

Identifying $T_{\pi(x)}M$ with T_xM and considering the definition of π, we obtain

$$\tau_{(g,D^{(\alpha)},\hat{D}^{(\alpha)})}(\pi) = e^{\lambda}\sum_{i,j=1}^{n} g^{ij}\left(\bar{D}^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - D^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right).$$

By Corollary 1, $(M, D^{(\alpha)}, g)$ and $(M, \bar{D}^{(\alpha)}, \bar{g})$ are α-conformally equivalent, so that Eq. (4.17) holds with $\phi = \lambda$, $h = g$, $\nabla = D^{(\alpha)}$, and $\bar{\nabla} = \bar{D}^{(\alpha)}$ for X, Y and $Z \in \Gamma(TM)$. Thus, for all $k \in \{1, \ldots, n\}$,

$$g\left(\tau_{(g,D^{(\alpha)},\hat{D}^{(\alpha)})}(\pi), \frac{\partial}{\partial x^k}\right)$$

$$
\begin{aligned}
&= g\left(e^{\lambda}\sum_{i,j=1}^{n} g^{ij}\left(\bar{D}^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j} - D^{(\alpha)}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\right), \frac{\partial}{\partial x^k}\right)\\
&= e^{\lambda}\sum_{i,j=1}^{n} g^{ij}\left\{-\frac{1+\alpha}{2}d\lambda\left(\frac{\partial}{\partial x^k}\right)g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\right.\\
&\qquad + \frac{1-\alpha}{2}\left\{d\lambda\left(\frac{\partial}{\partial x^i}\right)g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right)\right.\\
&\qquad\quad \left.\left. + d\lambda\left(\frac{\partial}{\partial x^j}\right)g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^k}\right)\right\}\right\}\\
&= e^{\lambda}\sum_{i,j=1}^{n} g^{ij}\left\{-\frac{1+\alpha}{2}\frac{\partial\lambda}{\partial x^k}g_{ij} + \frac{1-\alpha}{2}\left(\frac{\partial\lambda}{\partial x^i}g_{jk} + \frac{\partial\lambda}{\partial x^j}g_{ik}\right)\right\}\\
&= e^{\lambda}\left\{-\frac{1+\alpha}{2}\cdot n\cdot\frac{\partial\lambda}{\partial x^k} + \frac{1-\alpha}{2}\left(\sum_{i=1}^{n}\frac{\partial\lambda}{\partial x^i}\delta_{ik} + \sum_{j=1}^{n}\frac{\partial\lambda}{\partial x^j}\delta_{jk}\right)\right\}\\
&= \left(-\frac{1+\alpha}{2}\cdot n + \frac{1-\alpha}{2}\cdot 2\right)\, e^{\lambda}\frac{\partial\lambda}{\partial x^k}\\
&= -\frac{1}{2}\left\{(n+2)\,\alpha + (n-2)\right\}\, e^{\lambda}\frac{\partial\lambda}{\partial x^k}.
\end{aligned}
$$

Therefore, if $\tau_{(g,D^{(\alpha)},\hat{D}^{(\alpha)})}(\pi) \equiv 0$, then $(n+2)\alpha + (n-2) = 0$ or $\partial\lambda/\partial x^k = 0$ for all $k \in \{1, \ldots, n\}$ at each point in N. Thus we obtain Theorem 5. □

Remark 5 If $n = 2$, harmonic maps π with non-constant functions λ exist if and only if $\alpha = 0$.

Remark 6 If $n \geq 3$, and a map π is a harmonic map with a non-constant function λ, then $-1 < \alpha < 0$.

Remark 7 For $\alpha \leq -1$ and $\alpha > 0$, harmonic maps π with non-constant functions λ do not exist.

Definition 3 and Theorem 5 are special cases of harmonic maps between α-conformally equivalent statistical manifolds discussed in our previous study [16].

We now provide specific examples of harmonic maps between level surfaces relative to α-connections.

Example 1 (Regular convex cone) Let Ω and ψ be a regular convex cone and its characteristic function, respectively. On the Hessian domain $(\Omega, D, g = Dd\log\psi)$, $d\log\psi$ is invariant under a 1-parameter group of dilations at the vertex p of Ω, i.e., $x \longrightarrow e^t(x-p)+p$, $t \in \mathbf{R}$ [5, 14]. Then, under these dilations, each map between level surfaces of $\log\psi$ is also a dilated map in the dual coordinate system. Hence, each dilated map between level surfaces of $\log\psi$ in the primal coordinate system is a harmonic map relative to an α-connection for any $\alpha \in \mathbf{R}$.

Example 2 (Symmetric cone) Let Ω and $\psi = Det$ be a symmetric cone and its characteristic function, respectively, where Det is the determinant of the Jordan algebra that generates the symmetric cone. Then, similar to Example 1, each dilated map at the origin between level surfaces of $\log\psi$ on the Hessian domain $(\Omega, D, g = Dd\log\psi)$ is a harmonic map relative to an α-connection for any $\alpha \in \mathbf{R}$

It is an important problem to find applications of non-trivial harmonic maps relative to α-connections.

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