Chapter 20
Recent Advances in Nonlinear Potential Theory

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Abstract Recent developments in Nonlinear Potential Theory show the possibility of estimating solutions to nonlinear degenerate equations via potentials. We give a brief description of these results.

20.1 Pointwise Estimates

The classical potential theory deals with the fine properties – including regularity – of harmonic functions and, more in general, of solutions to linear elliptic equations. In this case, a central tool is given by Riesz potentials, defined for $\alpha > 0$ as

$$I_\alpha(\mu)(x) := \int_{\mathbb{R}^n} \frac{d\mu(y)}{|x-y|^{n-\alpha}} \ ,$$

where μ is a Borel (signed) measure defined on $\mathbb{R}^n$, whenever $x \in \mathbb{R}^n$. When dealing with estimates in bounded domains, it is also useful to deal with their "truncated versions", which are defined by

$$\mathbf{I}^{\mu}_{\beta}(x,R) := \int_0^R \frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta}} \frac{d\varrho}{\varrho} \,, \qquad \beta > 0\,,$$

whenever $x \in \mathbb{R}^n$ and $0 < R \leq \infty$. The classical representation formulas via convolution with the fundamental solution

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$$u(x) = \int_{\mathbb{R}^n} G(x,y)\, d\mu(y)\,, \qquad G(x,y) := |x-y|^{2-n} \tag{20.1}$$

allow to reconstruct pointwise properties of solutions to equations as

$$-\triangle u = \mu \qquad \text{in } \mathbb{R}^n \tag{20.2}$$

via potentials; here we confine to the case $n > 2$ for simplicity. For instance the following inequalities hold:

$$|u(x)| \lesssim I_2(|\mu|)(x)| \qquad \text{and} \qquad |Du(x)| \lesssim I_1(|\mu|)(x)\,. \tag{20.3}$$

The formulas in (20.3) together with (20.1), allow to address issues as

- Study of the optimal regularity properties of solutions to (20.2) with respect to the regularity of the datum μ; for example in various function spaces.
- Study of the fine properties of solutions: sets of Lebesgue points of solutions and gradients, removability of singularities etc.
- Various convergence properties of sequences of solutions

The main advantage of this approach is that, ultimately, it allows to unrelate the solution u from the equation, and to perform the analysis only looking at the potential of μ. Needless to say, via suitable localisation arguments, the same approach extends to other, more general linear equations for a formula as (20.1) holds. Now, although all this seems at a first sight to be peculiar of the linear situation, being linked to the existence of fundamental solutions, recent developments from the last years have shown that a similar approach can be pursued in several nonlinear situations too. This is the main object of this note, which is an extended written version of the lecture delivered by the author and Indam Day on June 7, 2012, in Genoa. A larger and more comprehensive presentation, with proofs, can be found in the Guide [32].

20.1.1 Some Notation

Constants generically denoted by c are always larger or equal than one; relevant dependence on parameters is indicated using parenthesis. We denote by $B(x,r) \equiv B_r(x) := \{\tilde{x} \in \mathbb{R}^n : |\tilde{x} - x| < r\}$ the open ball with center x and radius $r > 0$. When not important we shall omit denoting the center as follows: $B_r \equiv B(x,r)$. Moreover, with B being a generic ball with radius r, we will denote by σB the ball concentric to B having radius σr, $\sigma > 0$. With $\mathcal{O} \subset \mathbb{R}^n$ being a measurable subset with positive measure, and with $g\colon \mathcal{O} \to \mathbb{R}^k$, $k \geq 1$, being a measurable map, we shall denote by

$$(g)_{\mathcal{O}} \equiv \fint_{\mathcal{O}} g\, d\tilde{x} := \frac{1}{|\mathcal{O}|}\int_{\mathcal{O}} g(x)\, dx$$

its integral average; here $|\mathscr{O}|$ denotes the Lebesgue measure of $\mathscr{O}$. In the following, μ will always denote a Borel measure with finite total mass, which is initially defined on a certain open subset $\Omega \subset \mathbb{R}^n$. Since this will not affect the rest, with no loss of generality, all such measures will be considered as defined in the whole $\mathbb{R}^n$ so that $|\mu|(\mathbb{R}^n) < \infty$.

20.2 Nonlinear Potential Estimates

We shall treat nonlinear elliptic equations of divergence form of the type

$$-\operatorname{div} a(Du) = \mu \qquad \text{in } \Omega \tag{20.4}$$

and sometimes also having measurable coefficients as

$$-\operatorname{div} a(x, Du) = \mu\,. \tag{20.5}$$

Here $\Omega \subset \mathbb{R}^n$ denotes an open subset, and we shall always consider the case $n \geq 2$. When considering equations as in (20.4) we shall assume the following growth and ellipticity assumptions on the C^1-regular vector field $a\colon \mathbb{R}^n \to \mathbb{R}^n$:

$$\begin{cases} |a(z)| + |\partial a(z)|(|z|^2+s^2)^{1/2} \leq L(|z|^2+s^2)^{(p-1)/2} \\ \nu(|z|^2+s^2)^{(p-2)/2}|\lambda|^2 \leq \langle \partial a(z)\lambda, \lambda\rangle \\ p \geq 2 \end{cases} \tag{20.6}$$

whenever $z, \lambda \in \mathbb{R}^n$, where $s \geq 0$ and $0 < \nu < L$. The restriction to the case $p \geq 2$ is done for sake of simplicity; we shall make a few remarks on the case $p < 2$ later. When instead dealing with equations as in (20.5) we assume the weaker growth and monotonicity conditions on the Carathéodory vector field $a\colon \Omega \times \mathbb{R}^n \to \mathbb{R}^n$:

$$\begin{cases} |a(x,z)| \leq L(|z|^2+s^2)^{(p-1)/2} \\ \nu(|z_1|^2+|z_2|^2+s^2)^{(p-2)/2}|z_1-z_2|^2 \leq \langle a(x,z_1)-a(x,z_2), z_1-z_2\rangle \\ p \geq 2 \end{cases} \tag{20.7}$$

satisfied whenever $z_1, z_2, x \in \Omega$. By taking $s = 0$, both assumptions (20.6) and (20.7) are satisfied by the p-Laplacean equation

$$-\triangle_p u := -\operatorname{div}(|Du|^{p-2}Du) = \mu \tag{20.8}$$

which is, when $p > 2$, the most prominent model example for us; see [35].

The assumptions in (20.6) and (20.7) settle the Sobolev space $W^{1,p}(\Omega)$ as the natural one in which considering our equations; the situation is anyway not exactly so. It is indeed easy to see that if $u \in W^{1,p}(\Omega)$ is a local solution to (20.4) then μ belongs to the dual space of $W^{1,p}(\Omega)$, while on the other hand it is possible to consider distributional solutions to (20.4) also in the case μ is a more general Borel measure. These solutions do not in general belong to $W^{1,p}(\Omega)$ and are therefore called very weak solutions. In turn, very weak solutions are too general to be considered, and therefore, also keeping in mind the various methods for solving the existence problems, one is led to consider SOLA (Solutions Obtained as Limits of Approximations), according to the work of [4]. Here are the precise definitions, referring to [39] for more details.

Definition 20.1. A function $u \in W^{1,1}_{\text{loc}}(\Omega)$ is called a very weak solution to Eq. (20.5) in Ω if solves (20.5) in the sense of distributions, and if $a(x, Du) \in L^1_{\text{loc}}(\Omega, \mathbb{R}^n)$. A function $u \in W^{1,1}_{\text{loc}}(\Omega)$ is a SOLA to Eq. (20.5) under assumptions (20.7), iff is a very weak solution and there exists a sequence of local energy solutions $\{u_k\} \subset W^{1,p}_{\text{loc}}(\Omega)$ to the equations $-\text{div}\, a(x, Du_k) = \mu_k$, such that $u_k \to u$ locally in $W^{1,p-1}(\Omega)$.

Our aim here is to present sharp nonlinear analogs of the estimates (20.3). These lie at the core of what is nowadays called Nonlinear Potential Theory, a field whose name stems from the seminal papers of Havin and Maz'ya [16, 17]. In these papers the authors laid the fundamentals of this field and, in particular, introduced and studied two nonlinear potentials (see also [2]). The first is called Havin-Maz'ya potential and is defined via a nonlinear iteration of Riesz potentials:

$$\mathbf{V}_{\beta,p}(|\mu|)(x) := I_\beta \left\{ \left[I_\beta(|\mu|) \right]^{1/(p-1)} \right\} (x) .$$

The second potential, which is nowadays called Wolff potential, is in fact connected to the 1970 breakthrough paper of Maz'ya [36] about the validity of the Wiener criterion for the p-Laplacean; it is defined as follows:

$$\mathbf{W}^{\mu}_{\beta,p}(x, R) := \int_0^R \left(\frac{|\mu|(B(x,\varrho))}{\varrho^{n-\beta p}} \right)^{1/(p-1)} \frac{d\varrho}{\varrho}, \qquad \beta > 0$$

whenever $x \in \mathbb{R}^n$ and $0 < R \leq \infty$. Notice that Wolff potentials have been obtained from Riesz potentials by inserting the scaling exponent of Eq. (20.8) under the sign of integral. The three classes of potentials – Riesz, Havin-Maz'ya and Wolff – coincide when $p = 2$. A fundamental result of Havin and Maz'ya [17] asserts that Wolff potentials can be controlled by Riesz potentials $\mathbf{W}^{\mu}_{\beta,p}(x, \infty) \lesssim \mathbf{V}_{\beta,p}(|\mu|)(x)$, in the case $p > 2 - 1/n$ and $p\beta < n$; a related integral inequality for the case $1 < p \leq 2 - 1/n$ is contained in the paper by Hedberg and Wolff [18]. The result of Havin and Maz'ya allows to reduce the study of Wolff potentials to that of Riesz potentials in many important situations. For instance, when studying the mapping properties of Wolff potentials in various rearrangement invariant function spaces [6]. Wolff potentials are nowadays a common tool in order to face several basic questions of Nonlinear Potential Theory; see for instance [19, 40].

The first result towards a nonlinear extension of formulas (20.3) appeared in the fundamental papers of Kilpeläinen and Malý [20, 21], where the authors proved a pointwise potential estimate for solutions in terms of Wolff potentials. Subsequently, different proofs and approaches have been found. We summarise some of the results available in the following:

Theorem 20.1. *Let $u \in W^{1,p-1}(\Omega)$ be a SOLA to the equation with measurable coefficients* (20.5)*, under the assumptions* (20.7)*. Then*

- [14, 21, 22, 42] *There exists a constant $c \equiv c(n, p, \nu, L)$ such that the inequality*

$$|u(x)| \leq c\mathbf{W}^{\mu}_{1,p}(x, R) + c \fint_{B(x,R)} (|u| + Rs)\, d\tilde{x} \tag{20.9}$$

  holds whenever $B(x, R) \subset \Omega$ and the right hand side is finite.
- [14, 21, 22, 42] *Moreover, if*

$$\lim_{R\to 0} \mathbf{W}^{\mu}_{1,p}(x, R) = 0 \ \text{ locally uniformly in } \Omega \text{ w.r.t. } x$$

  then u is continuous in Ω.
- [32] *Finally, the condition $\mathbf{W}^{\mu}_{1,p}(x, R) < \infty$ implies that the following limit exists and therefore defines the precise representative of u at the point x:*

$$\lim_{\varrho\to 0} (u)_{B(x,\varrho)} =: u(x)\,. \tag{20.10}$$

  In particular, the set of non-Lebesgue points of u has p-capacity zero.

Notice that in Theorem 20.1, by writing $u(x)$ for the precise representative of u at x, the set of points x for which (20.9) holds in the sense that $u(x)$ exists is in fact the one for which $\mathbf{W}^{\mu}_{1,p}(x, R) < \infty$ and (20.9) itself becomes nontrivial. This follows from (20.10). We explicitly remark that the full strength of the previous result is already in the case $p = 2$. Indeed, the real point here is passing from linear to nonlinear equations, because fundamental solutions and representation formulas are not available in the nonlinear setting. We also remark that estimate (20.9) is optimal in that when $\mu, u \geq 0$ then it holds that (see [21])

$$\mathbf{W}^{\mu}_{1,p}(x, R) \lesssim u(x) \lesssim \mathbf{W}^{\mu}_{1,p}(x, 2R) + \inf_{B(x,R)} u\,.$$

The validity of a nonlinear analog of the second inequality in (20.3) has remained a controversial open problem since the paper of Kilpeläinen and Malý [21]. The first result has been obtained in [38]. We report a global version aimed at highlighting the similarities with the linear case.

Theorem 20.2 ([38]). *Let $u \in W^{1,1}(\mathbb{R}^n)$ be a SOLA to Eq. (20.4) considered in $\mathbb{R}^n$, under the assumptions (20.6) with $p = 2$. Then there exists a constant $c \equiv c(n,\nu,L)$ such that the following estimate holds for a.e. $x \in \mathbb{R}^n$:*

$$|Du(x)| \leq c \int_{\mathbb{R}^n} \frac{d|\mu|(y)}{|x-y|^{n-1}} .$$

The degenerate case $p > 2$ is intriguing and reserves surprises. The standard orthodoxy in Nonlinear Potential Theory prescribes to replace Riesz potentials with Wolff potentials whenever one is dealing with the p-Laplacean, and indeed a first gradient potential estimate using Wolff potentials has been obtained in [14]. On the other hand let us observe that Eq. (20.8) can be formally decomposed as

$$\begin{cases} -\operatorname{div} v = 0 \\ v = |Du|^{p-2}Du . \end{cases}$$

In other words, Eq. (20.8) can be read both as a nonlinear equation in the gradient and as a linear equation with respect to the nonlinear vector field of the gradient v. This paves the way to an estimate that involves linear Riesz potentials, and that improves the ones involving Wolff potentials, as those in [14]. It indeed holds the following, surprising:

Theorem 20.3. *Let $u \in W^{1,p-1}(\Omega)$ be a SOLA to Eq. (20.4) under the assumptions (20.6). Then*

- [23,27] *There exists a constant $c \equiv c(n,p,\nu,L)$ such that the inequality*

$$|Du(x)|^{p-1} \leq c\mathbf{I}_1^{\mu}(x,R) + c\left(⨍_{B(x,R)} (|Du|+s)\, d\tilde{x}\right)^{p-1} \tag{20.11}$$

  holds whenever $B(x,R) \subset \Omega$ and the right hand side is finite.
- [23,27] *Moreover, if*

$$\lim_{R\to 0} \mathbf{I}_1^{\mu}(x,R) = 0 \ \textit{ locally uniformly in } \Omega \textit{ w.r.t. } x \tag{20.12}$$

  then Du is continuous in Ω.
- [32] *Finally, the condition $\mathbf{I}_1^{\mu}(x,R) < \infty$ implies that the following limit exists and therefore defines the precise representative of Du at the point x:*

$$\lim_{\varrho\to 0} (Du)_{B(x,\varrho)} =: Du(x) .$$

  In particular, the Hausdorff dimension of the set of non-Lebesgue points of Du does not exceed $n-1$.

Estimate (20.11) still holds when $2 - 1/n < p < 2$ [13], but in this case it does not improve the analogous estimates via Wolff potentials previously proved in [14]. The main outcome of Theorem 20.3 is that the gradient theory of general quasilinear, possibly degenerate equations with measure data, is now reduced to the one of the Poisson equation, up to the C^0-level. In particular, essentially all the classical estimates known for the model equation (20.8), plus more difficult borderline cases, now follow with a unified approach via Riesz potentials as in the case of Poisson equation (20.2). It is important to observe that Theorem 20.3 cannot hold for solutions to equations with measurable coefficients as in (20.5), since in this case the gradients of solutions are only known to be in L^q for some $q > p$ (Gerhing's lemma). In this situation it is possible to prove a few nonlocal estimates for the level sets of the gradient using the level sets of the Riesz potential, as shown in [37]. In particular, in this last paper we have given a nonlinear analog of the basic theorems of Adams [1] valid for the Poisson equation. Remarkable extensions of estimate (20.11) have been recently provided by Baroni [3].

It is worth mentioning a few relevant corollaries of the continuity criterion in (20.12). It is possible to obtain a full nonlinear analog of a famous theorem of Stein [41] which is in fact the limit case of Sobolev embedding theorem. This claims that if $v \in W^{1,1}$ is a Sobolev function defined in $\mathbb{R}^n$ with $n \geq 2$, then $Dv \in L(n,1)$ implies that v is continuous. We recall that the Lorentz space $L(n,1)$ (over a subset Ω) is defined as the set of measurable maps $g\colon \Omega \to \mathbb{R}^k$ such that

$$\int_0^\infty |\{x \in \Omega \,:\, |g(x)| > \lambda\}|^{1/n}\, d\lambda < \infty\,.$$

Another way to state Stein's theorem concerns the regularity of solutions $u\colon \Omega \to \mathbb{R}^m$ to the Laplacean system and amounts to observe that $\triangle u \in L(n,1)$ implies the continuity of Du. This follows by the previous result and classical Calderón-Zygmund theory. The condition $\mu \in L(n,1)$ allows to satisfy condition (20.12) and therefore we conclude with the following:

Theorem 20.4 (Nonlinear Stein theorem [27]). *Let $u \in W^{1,p}(\Omega)$ be a solution to Eq.* (20.4)*, under the assumptions* (20.6) *and such that $\mu \in L(n,1)$ locally in Ω. Then Du is continuous in Ω.*

Without appealing to potentials, but by using different means, the result of the previous theorem also holds for systems.

Theorem 20.5 (Vectorial nonlinear Stein theorem [33]). *Let $u \in W^{1,p}(\Omega,\mathbb{R}^m)$, $m \geq 1$, be a vector valued solution to the p-Laplacean system $-\triangle_p u = F$, with $p > 1$. Assume that the components of the vector field $F\colon \Omega \to \mathbb{R}^m$ locally belong to the space $L(n,1)$. Then Du is continuous in Ω.*

We refer to [7, 8] for a global Lipschitz continuity result involving the Lorentz space $L(n,1)$ and the p-Laplacean system, while a Lipschitz local result has been obtained in [12].

20.3 Universal Potential Estimates

Theorems 20.1–20.3 allow to estimate the size of solutions and their gradients via potentials. The idea is now to use potentials to estimate also derivatives of intermediate order i.e. fractional derivatives. This will in turn allow to give bounds for the oscillation of solutions via potentials. Before going on, we give a suitable definition of fractional derivatives, using the ideas of DeVore and Sharpley [9].

Definition 20.2 (Calderón spaces [9]). Let $\alpha \in (0,1]$, $q \geq 1$, and let $\Omega \subset \mathbb{R}^n$ be an open subset. A measurable function v, finite a.e. in Ω, belongs to the Calderón space $C^\alpha_q(\Omega)$ if and only if there exists a nonnegative function $m \in L^q(\Omega)$ such that

$$|v(x) - v(y)| \leq [m(x) + m(y)]|x-y|^\alpha \tag{20.13}$$

holds for almost every couple $(x, y) \in \Omega \times \Omega$.

Calderón spaces C^α_q are strictly related to classical fractional Sobolev spaces $W^{\alpha,q}$, and in this setting $m(\cdot)$ represents a α-fractional derivative, in L^q-sense, of v. The advantage is that the typical nonlocal character of fractional derivatives is reduced to a minimal status: only two points are considered in (20.13). There is always a canonical choice for the function $m(\cdot)$ in (20.13); for this we need another definition.

Definition 20.3 (Fractional sharp maximal operator). Let $\beta \in [0,1]$, $x \in \Omega$ and $R \leq \operatorname{dist}(x, \partial\Omega)$, and let $f \in L^1(\Omega)$; the function defined by

$$M^\#_{\beta,R}(f)(x) := \sup_{0<r\leq R} r^{-\beta} \fint_{B(x,r)} |f - (f)_{B(x,r)}|\, d\tilde{x}$$

is called the restricted (centered) sharp fractional maximal function of f.

The connection with the fractional derivatives is then given by the following result, that relies on the original methods of Campanato [5]:

Proposition 20.1 ([9, 32]). *Let $f \in L^1(B_{8R/5})$; for every $\alpha \in (0,1]$ the inequality*

$$|f(x) - f(y)| \leq \frac{c}{\alpha}\left[M^\#_{\alpha,R}(f)(x) + M^\#_{\alpha,R}(f)(y)\right]|x-y|^\alpha \tag{20.14}$$

holds whenever $x, y \in B_{2R/5}$, for a constant c depending only on n. More precisely, x and y are Lebesgue points of f whenever $M^\#_{\alpha,R}(f)(x)$ and $M^\#_{\alpha,R}(f)(y)$ are finite, respectively. Therefore, whenever the right hand side in (20.14) is finite, the values of f are defined as follows:

$$f(x) := \lim_{\varrho\to 0}(f)_{B(x,\varrho)} \qquad \text{and} \qquad f(y) := \lim_{\varrho\to 0}(f)_{B(y,\varrho)}\,.$$

The previous result tells that, in order to give a bound for the fractional derivatives of a function v – in the sense of Definition 20.2 – it is sufficient to give a bound for the fractional maximal operator of v. When considering solutions to nonlinear equations we indeed have

Theorem 20.6 (Uniform maximal-potential estimates [32]). *Let $u \in W^{1,p-1}(\Omega)$ be a SOLA to Eq.* (20.4) *under assumptions* (20.6)*. Then for every ball $B(x,R) \subset \Omega$ the following estimate:*

$$M^{\#}_{\alpha,R}(u)(x) \leq c\left[\mathbf{I}^{\mu}_{p-\alpha(p-1)}(x,R)\right]^{1/(p-1)} + cR^{1-\alpha}\fint_{B_R}(|Du|+s)\,d\tilde{x} \tag{20.15}$$

holds uniformly in $\alpha \in [0,1]$, with $c \equiv c(n,p,\nu,L)$.

Applying estimate (20.14) together with (20.15) yields a pointwise estimate on the oscillations of solutions to (20.4), that is

Theorem 20.7 (Uniform Riesz potential estimate [32]). *Let $u \in W^{1,p-1}(\Omega)$ be a SOLA to Eq.* (20.4) *under assumptions* (20.6)*. Let $B_R \subset \Omega$ be such that $x, y \in B_{R/4}$; then*

$$\begin{aligned}|u(x)-u(y)| \leq c\left[\mathbf{I}^{\mu}_{p-\alpha(p-1)}(x,R)+\mathbf{I}^{\mu}_{p-\alpha(p-1)}(y,R)\right]^{1/(p-1)}|x-y|^{\alpha}\\ +c\fint_{B_R}(|u|+Rs)\,d\tilde{x}\cdot\left(\frac{|x-y|}{R}\right)^{\alpha}\end{aligned} \tag{20.16}$$

holds provided the right hand side is finite and $0<\alpha\leq 1$. Moreover, whenever $\tilde{\alpha} \in (0,1]$ is fixed, the dependence of the constant c is uniform for $\alpha \in [\tilde{\alpha},1]$ as c depends only n,p,ν,L and $\tilde{\alpha}$.

We note that the previous estimate gives back (20.11) when $\alpha = 1$, and extends it in the whole range of differentiability $\alpha \in (0,1]$. In view of Definition 20.2 estimate (20.16) can be interpreted, with a strong abuse of notation, as

$$|\partial^{\alpha}u(x)|^{p-1} \lesssim I_{p-\alpha(p-1)}(|\mu|)(x)\,, \qquad 0<\alpha\leq 1\,,$$

a formula that, needless to say, has only a symbolic meaning. The case $\alpha = 0$ is not included in Theorem 20.7, and it cannot, as when $\alpha = 0$ Wolff potentials come into the play. As a matter of fact the validity of (20.16) would ultimately contradict the optimality of (20.9). On the other hand it is also possible to produce a formula to estimate oscillations of solutions via Wolff potentials when α is not very large. Actually, we are going to give a nonlinear potentials formulation of the classical DeGiorgi's theory for solutions to (20.5) in the homogeneous case $\mu = 0$. This theory provides the existence of a *universal Hölder continuity exponent* $\alpha_m \in (0,1)$, depending only on n,p,ν,L, but not on the solutions or of the vector field $a(\cdot)$, such that

$$u \in C^{0,\alpha}_{\mathrm{loc}}(\Omega) \qquad \text{for every } \alpha < \alpha_m$$

and

$$|u(x)-u(y)| \leq c \fint_{B_R} (|u|+Rs)\, d\tilde{x} \cdot \left(\frac{|x-y|}{R}\right)^{\alpha} .$$

The previous estimate holds whenever $x, y \in B_{R/2}$ and $B_R \subset \Omega$, for a constant depending only on n, p, ν, L and α. For the case $\mu \neq 0$ we then have the following:

Theorem 20.8 (De Giorgi's theory via potentials [25]). *Let $u \in W^{1,p-1}(\Omega)$ be a SOLA to the equation with measurable coefficients (20.5) under assumptions (20.7). Fix $\tilde{\alpha} < \alpha_m$, then the inequality*

$$\begin{aligned} |u(x)-u(y)| \leq c &\left[\mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(x,R) + \mathbf{W}^{\mu}_{1-\alpha(p-1)/p,p}(y,R)\right] |x-y|^{\alpha} \\ &+c \fint_{B_R} (|u|+Rs)\, d\tilde{x} \cdot \left(\frac{|x-y|}{R}\right)^{\alpha} \end{aligned} \tag{20.17}$$

holds whenever $B_R \subset \Omega$ and $x, y \in B_{R/2}$ and $\alpha \in [0, \tilde{\alpha}]$, provided the right hand side is finite. The constant c depends only on n, p, ν, L and $\tilde{\alpha}$.

In view of the available regularity theory for the gradient of solutions to equations as (20.4) we wonder if a similar result holds for the oscillations of the gradient. For this let us recall the basic information about the maximal regularity of solutions to homogeneous equations as in (20.4). This theory goes back to the fundamental contribution of Ural'tseva [43], who proved the Hölder continuity of the gradient of solutions to (20.8) with $\mu = 0$; subsequently, different proofs have been given in [10, 15, 34]. The outcome is the existence of another positive exponent $\alpha_M \in (0,1)$, depending only on n, p, ν and L, such that

$$Du \in C^{0,\alpha}_{\mathrm{loc}}(\Omega) \qquad \text{for every } \alpha < \alpha_M$$

holds and

$$|Du(x)-Du(y)| \leq c \fint_{B_R} (|Du|+s)\, d\tilde{x} \cdot \left(\frac{|x-y|}{R}\right)^{\alpha} .$$

We then have

Theorem 20.9 (Ural'tseva theory via potentials [25]). *Let $u \in W^{1,p-1}(\Omega)$ be a SOLA to Eq. (20.4) under assumptions (20.6). Fix $\tilde{\alpha} < \min\{1/(p-1), \alpha_M\}$, then the inequality*

$$\begin{aligned} |Du(x)-Du(y)| \leq c &\left[\mathbf{W}^{\mu}_{1-\frac{(1+\alpha)(p-1)}{p},p}(x,R) + \mathbf{W}^{\mu}_{1-\frac{(1+\alpha)(p-1)}{p},p}(y,R)\right] |x-y|^{\alpha} \\ &+c \fint_{B_R} |Du-(Du)_{B_R}|\, d\tilde{x} \cdot \left(\frac{|x-y|}{R}\right)^{\alpha} \end{aligned} \tag{20.18}$$

holds whenever $B_R \subset \Omega$ *and* $x, y \in B_{R/2}$*, and* $\alpha \in [0, \tilde{\alpha}]$*, provided the right hand side is finite. The constant* c *depends only on* n, p, ν, L *and* $\tilde{\alpha}$*.*

Remark 20.1 (Comparison with the linear case). When $p = 2$, Wolff and Riesz potentials coincide and we have results that hold uniformly in the range $\alpha \in [0, 1]$. In this case Theorems 20.7 and 20.8 provide a complete analog of the estimates available in the linear case (20.2) via fundamental solutions (20.1). Indeed, by using the elementary inequality

$$\left||x-\tilde{x}|^{2-n} - |y-\tilde{x}|^{2-n}\right| \leq c\left||x-\tilde{x}|^{2-n-\alpha} + |y-\tilde{x}|^{2-n-\alpha}\right||x-y|^{\alpha}, \quad (20.19)$$

which is valid whenever $x, y, \tilde{x} \in \mathbb{R}^n$, and (20.1), we get

$$|u(x) - u(y)| \leq c\,[I_{2-\alpha}(|\mu|)(x) + I_{2-\alpha}(|\mu|)(y)]\,|x-y|^{\alpha}, \quad 0 \leq \alpha \leq 1.$$

This is exactly the global version of estimates (20.16) and (20.17) when $p = 2$ (it is sufficient to let $R \to \infty$ there and to assume global $W^{1,1}$-regularity on u). Differentiating (20.1) under the sign of integral and again applying (20.19) yields

$$|Du(x) - Du(y)| \leq c\,[I_{1-\alpha}(|\mu|)(x) + I_{1-\alpha}(|\mu|)(y)]\,|x-y|^{\alpha}, \quad 0 \leq \alpha < 1$$

which is again the global analog of (20.18) for $p = 2$ (but for the upper bound on α).

20.4 Nonlinear Parabolic Equations

Here we briefly summarise the nonlinear potential estimates that are available in the parabolic setting; again we restrict to the case $p \geq 2$ for simplicity, while we refer to [28] for the subquadratic case. Again for ease of presentation, we deal with local energy solutions and a priori estimates, without treating SOLA of parabolic equations. The case of general measure data problems is treated in [30, 31] to which we refer the interested readers.

Dealing with parabolic problems is more difficult, and requires additional new ideas. In particular, the potential theoretic approach developed for the elliptic case has to meet the fundamental concept of intrinsic geometry introduced by DiBenedetto [11]. The basic idea is to rebalance the lack of scaling (for $p \neq 2$) of equations as

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = 0 \qquad \text{in } \Omega \times (-T, 0) \subset \mathbb{R}^{n+1} \quad (20.20)$$

by using certain special cylinders adapted to the solution itself, indeed called *intrinsic parabolic cylinders*. This means that, instead of using standard parabolic cylinders $Q_r(x_0, t_0) := B(x_0, r) \times (t_0 - r^2, t_0)$, one uses *cylinders whose time-length is stretched accordingly to the size of the gradient on the cylinder itself*. These are of type

$$Q_r^\lambda(x_0, t_0) := B(x_0, r) \times (t_0 - \lambda^{2-p} r^2, t_0)\,, \qquad \lambda > 0\,, \tag{20.21}$$

on which it simultaneously happens that a condition of the type

$$\fint_{Q_r^\lambda(x_0,t_0)} |Du|\, dx\, dt \lesssim \lambda \tag{20.22}$$

is satisfied. The terminology "intrinsic geometry" stems exactly from this point. When considered on intrinsic cylinders, estimates become homogeneous and therefore they become suitable to be used in the iteration procedures which are typical of regularity theory. To see this fact, let us make a comparison. If we consider standard parabolic cylinders, then a priori estimates exhibit an anisotropicity linked to the one of the equation, that is

$$\sup_{Q_{r/2}(x_0,t_0)} |Du| \leq c(n, p) \fint_{Q_r(x_0,t_0)} (|Du| + s + 1)^{p-1}\, dx\, dt\,. \tag{20.23}$$

When instead considering intrinsic cylinders with (20.21) and (20.22) being in force, *estimates become dimensionally homogeneous*:

$$\left(\fint_{Q_r^\lambda(x_0,t_0)} (|Du| + s)^{p-1}\, dx\, dt\right)^{1/(p-1)} \lesssim \lambda \Longrightarrow |Du(x_0, t_0)| \leq \lambda\,. \tag{20.24}$$

Both (20.23) and (20.24) are basic results of DiBenedetto and Friedman, for which we refer to [11]. The basic idea introduced in [28,30,31] is now to consider "intrinsic potentials" linked to the local intrinsic geometry, i.e. caloric Riesz potentials of the type

$$\mathbf{I}^{\mu}_{\beta,\lambda}(x_0, t_0; r) := \int_0^r \frac{|\mu|(Q_\varrho^\lambda(x_0, t_0))}{\varrho^{N-\beta}}\, \frac{d\varrho}{\varrho}\,. \tag{20.25}$$

Here $N := n + 2$ is the usual parabolic dimension and $\lambda > 0$ is a parameter to be chosen in the formulation of the results. Note that for $\lambda = 1$ the one in (20.25) gives back the usual caloric Riesz potential. We now come to the results. Our emphasis will be on a priori estimates and for simplicity we shall treat the case of energy distributional solutions to (20.20), that is, functions u such that

$$u \in C^0(-T, 0; L^2(\Omega)) \cap L^p(-T, 0; W^{1,p}(\Omega))\,.$$

The results hold for general equations of the type

$$u_t - \operatorname{div} a(Du) = 0 \qquad \text{in } \Omega \times (-T,0) \subset \mathbb{R}^{n+1} \tag{20.26}$$

under assumptions (20.6) on the vector field $a(\cdot)$.

Theorem 20.10 (Intrinsic Riesz potential bound [31]). *Let u be a solution to (20.26) with $p \geq 2$. There exist a constant $c > 1$ depending only on n, p, ν, L, such that the following implication holds:*

$$c\mathbf{I}^{\mu}_{1,\lambda}(x_0,t_0;r) + c\left(\fint_{Q^{\lambda}_r(x_0,t_0)} (|Du|+s)^{p-1}\,dx\,dt\right)^{1/(p-1)} \leq \lambda \\ \Longrightarrow |Du(x_0,t_0)| \leq \lambda \tag{20.27}$$

whenever $Q^{\lambda}_r(x_0,t_0) \subset \Omega_T$ and (x_0,t_0) is Lebesgue point of Du.

As expected, (20.27) extends (20.24) to the case $\mu \neq 0$. As a matter of fact Theorem 20.10 implies a gradient linear potential estimate involving standard Riesz potentials that again reduces to (20.23) when $\mu = 0$. We indeed have the following:

Theorem 20.11 (Riesz potential bound in classic form [31]). *Let u be a solution to (20.26) with $p \geq 2$. There exists a constant c, depending only on n, p, ν, L, such that*

$$|Du(x_0,t_0)| \leq c\mathbf{I}^{\mu}_1(x_0,t_0;r) + c\fint_{Q_r(x_0,t_0)} (|Du|+s+1)^{p-1}\,dx\,dt$$

holds whenever $Q_r(x_0,t_0) \subset \Omega_T$ is a standard parabolic cylinder and (x_0,t_0) is Lebesgue point of Du.

The proof of Theorem 20.10 opens the way to an optimal continuity criterion for the gradient that involves classical (caloric) Riesz potentials and that, as such, is again independent of p.

Theorem 20.12 (Gradient continuity via linear potentials [31]). *Let u be a solution to (20.26) with $p \geq 2$. If*

$$\lim_{r\to 0} \mathbf{I}^{\mu}_1(x,t;r) = 0$$

locally uniformly w.r.t. (x,t), then Du is continuous in Ω_T.

An important corollary involves Lorentz spaces; as in the elliptic case this result can be proved both for general equations as in (20.26) and for the evolutionary p-Laplacean system. We give the formulation directly in this case; when the right hand side is time-independent then we recover the full content of Theorem 20.5.

Theorem 20.13 (Parabolic nonlinear Stein theorem [29]). *Let* $u \in W^{1,p}(\Omega, \mathbb{R}^m)$, $m \geq 1$, *be a vector valued solution to the evolutionary* p*-Laplacean system* $u_t - \triangle_p u = F$ *with* $p > 2n/(n+2)$. *Assume that one of the two conditions are satisfied:*

- *The components of the vector field* $F\colon \Omega \to \mathbb{R}^m$ *locally belong to the space* $L(N, 1)$.
- *The components of the vector field* $F\colon \Omega \to \mathbb{R}^m$ *are time independent and locally belong to the space* $L(n, 1)$.

Then Du *is continuous in* Ω.

In the previous theorem, the lower bound $p > 2n/(n+2)$ is not a technical assumption, but is essential and cannot be avoided. A related Lipschitz regularity result is in [24].

The results above are based on a very delicate interplay between the new approaches necessary to derive potential estimates, and the classical approaches to prove regularity in the parabolic case when $\mu \equiv 0$. In particular, we make a new presentation of the basic regularity results of DiBenedetto [11], making them suitable to be applied in this new context. As a sample, we propose Theorem 20.14 below. It deals with homogeneous equations as

$$v_t - \operatorname{div} a(Dv) = 0 \tag{20.28}$$

but it works for the evolutionary p-Laplacean system as well, as in fact shown in [26]. It has basically two important features: on one hand it incorporates the basic elements of the gradient regularity theory of equations as (20.28). On the other one, it allows to derive a completely homogeneous decay estimate, which is totally similar to the standard elliptic one, provided certain intrinsic geometry conditions are satisfied.

Theorem 20.14 (Elliptic type excess decay [30]). *Let* v *be a solution to* (20.28) *in a cylinder* $Q \equiv Q_r^\lambda(x_0, t_0)$ *of the type in* (20.21). *Consider numbers*

$$A, B \geq 1 \qquad \text{and} \qquad \varepsilon \in (0,1)\,.$$

Then there exists a constant $\sigma \in (0, 1/4)$ *depending only on* $n, p, \nu, L, A, B, \varepsilon$ *such that if*

$$\frac{\lambda}{B} \leq \sup_{\sigma Q} |Dv| \leq s + \sup_{\frac{1}{4}Q} |Dv| \leq A\lambda$$

holds, then

$$\fint_{\tau Q} |Dv - (Dv)_{\tau Q}|\, dx\, dt \leq \varepsilon \fint_{\frac{1}{4}Q} |Dv - (Dv)_{\frac{1}{4}Q}|\, dx\, dt$$

holds too, whenever $\tau \in (0, \sigma]$. *Here we are denoting*

$$\tau Q_r^\lambda(x_0, t_0) := B(x_0, \tau r) \times (t_0 - \lambda^{2-p}(\tau r)^2, t_0)\,.$$

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