Chapter 4
Gaps, Gluts and Paraconsistency

4.1 Chapter Overview

This chapter will look at some other semantic theories and the logics they generate. Mainly, these logics come about by allowing truth value *gaps* and truth value *gluts*. If a semantic theory allows for statements that are neither true nor false, then it allows for gaps. If, on the other hand, it makes room for statements that are *both* true and false, then it allows for truth value gluts.[1]

An important theme, closely connected to truth value gluts, will be *paraconsistency*. Any[2] logic that does not allow inferences of the form $A, \neg A \vDash B$ (or, if it is syntactically defined, of the form $A, \neg A \vdash B$) is paraconsistent. This rejected rule, which we have already met under the name *Ex Contradictione Quodlibet*, is called by paraconsistent logicians, more dramatically, *Explosion*. It is valid, of course, in classical logic, but many non-classical logics are *explosive* in this sense as well. In particular, we have seen that intuitionistic logic is explosive (cf. Axiom 10 of Heyting's axiomatization).

Paraconsistency, to this day, is often confused with *dialetheism*. While the main idea behind paraconsistency is simply that a contradiction should not entail everything whatsoever, dialetheism is a much more ambitious and contentious metaphysical stance: It is the view that there actually are contradictions that are true, statements that are both true or false. I will try to clear up the relation between dialetheism and paraconsistency. Also, a lesser known alternative to dialetheism, *analetheism*, will be presented.

A further theme that will start to emerge here and continue throughout the book is the phenomenon of *duality*. Many of the concepts are mirror images of each other, but often it will matter from which direction one looks into the mirror.

[1] Sometimes I will talk about "gappy" and "glutty" theories. The first kind are also known as "partial" theories.

[2] One common misconception is that paraconsistent logic is one particular logical system. In fact, there are a lot of them, cf. Priest (2003).

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Finally, I will ask how these semantic theories fit into the Dummettian setup outlined in the first chapter.

Talking about gaps will give me occasion to present the distinction he draws between assertoric content and ingredient sense.

All of this material, even if this might not yet be apparent in some cases, will be important background knowledge for the remaining chapters. As most of the discussion of gaps and gluts takes place outside of expressly constructive contexts, I will mostly talk about truth and say little about assertibility in this chapter. The question how the ideas we will meet here will come to connect up with correct assertibility and verifiability later on is discussed in the last section of this chapter.

I will start the story with a simple *relevant* logic called First Degree Entailment (FDE). This logic contains all the main ideas I just mentioned, and it is easy to modify it to obtain other paradigmatic logics, such as Strong Kleene (K_3) and Priest's Logic of Paradox (LP).

Even though the logics I will eventually propose will not be relevant logics (I will explain what that means in a minute), many of the ideas that I will draw on come out of the literature on relevant logic.

4.2 Relevant Logic

What then is the concern of the relevant logician? She is unhappy about classical logic, because it allows inferences that seem to be blatant *non sequiturs*. Take the inference we just mentioned above, *Explosion*.

Isn't there something very strange about the inference "It is raining and it is not raining, therefore bananas can be used as a substitute for onions in most recipes" (an instance of *Explosion*)? The diagnosis of the relevant logician is that in an inference, the premises should have something to *do* with the conclusion and that in particular whether or not it is raining is *irrelevant* for determining the possible culinary uses of bananas. Relevant logic rejects such irrelevant inferences. So, every relevant logic is paraconsistent, because it will reject *Explosion*.

However, Explosion is not the only kind of inference that is irrelevant according to the relevant logician.[3] Consider a form of the LEM: $B \vDash A \vee \neg A$. An instance would be "The onions in this dish taste a bit mushy, therefore it either rains or it doesn't." This is obviously just as irrelevant as the first example.

It is interesting to note that both relevant logicians and intuitionistic logicians want to reject this inference, but for completely different reasons.

The intuitionist is, as we have seen, concerned about the possibility of undecidable statements.

[3] That is, relevant logics form a proper subset of the set of paraconsistent logics. This shows that the not uncommon perception that paraconsistency is a more radical doctrine than relevance is completely unfounded. It rests, again, on the common confusion between paraconsistency and dialetheism.

On the other hand, the relevantists usually do not care that much about decidability, nor do they necessarily take issue with the logical validity of the LEM. In fact, some relevant logicians even endorse the LEM in the form $\models A \vee \neg A$. What they dislike is the claim that such a tautology should follow from something completely unrelated, as in the example above, $B \models A \vee \neg A$.

What they want to argue is this: The meaning of the turnstile is not exhausted by "technical device that guarantees the transmission of truth from premises to conclusions." No, it is meant to be an analysis of "therefore" and similar locutions, and as such, it needs to do more than merely guarantee truth preservation.

However, if there are no premises, "therefore" makes no sense; "Therefore, it is raining or it is not raining" is only an admissible thing to say if it follows another sentence that specified some premises. If such premises are truly lacking, then some other locution has to be found to pronounce the turnstile, maybe "it is logically true" or some such.

Thus, the relevantist claims, "Blue is a color, therefore the president will be re-elected or not" is objectionable, while "It is logically true that the president will be re-elected or not" might be acceptable.

Let us see how the relevant ideas are put into practice by way of a concrete example.

4.3 First Degree Entailment

N. Belnap introduced FDE in two influential papers in the 1970s. One is entitled "A useful four-valued logic" Belnap 1977a), the other "How a computer should think" (Belnap 1977b). The two papers have been merged into Chap. 81 of Anderson et al. (1992), and this is the most accessible source.[4]

The main concerns driving this logic are first to give a basis for relevant entailment and second, as is obvious from the title of the second paper, to give computers some rules for processing information.

To see how FDE manages to get rid of irrelevant inferences, let me first remind you why classical logicians endorse *Explosion* and the LEM. Logical consequence is defined in terms of truth preservation in all models: If the premises are true, then so is the conclusion. Since contradictions have no classical models, the condition for the validity of $A \wedge \neg A \models B$ is vacuously satisfied. A good way to expulse this inference, then, is to provide models in which both A and $\neg A$ are satisfied. Likewise, the trick to invalidate the inference $B \models A \vee \neg A$ is to provide models in which neither A nor $\neg A$ is satisfied.

FDE achieves this by introducing two new truth values in addition to the classical $\mathcal{T}$ (True) and $\mathcal{F}$ (False). These values are called $\mathcal{B}$ (Both) and $\mathcal{N}$ (Neither).

[4] A scan of the chapter is available for download on Belnap's homepage, http://www.pitt.edu/belnap/papers.html.

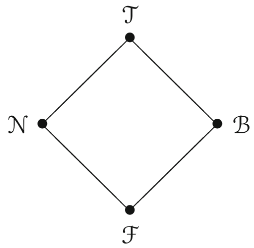
Logically complex statements get their values according to the following truth tables for negation, conjunction, and disjunction.[5]

$\neg$	
$\mathcal{T}$	$\mathcal{F}$
$\mathcal{B}$	$\mathcal{B}$
$\mathcal{N}$	$\mathcal{N}$
$\mathcal{F}$	$\mathcal{T}$

$\wedge$	$\mathcal{T}$	$\mathcal{B}$	$\mathcal{N}$	$\mathcal{F}$
$\mathcal{T}$	$\mathcal{T}$	$\mathcal{B}$	$\mathcal{N}$	$\mathcal{F}$
$\mathcal{B}$	$\mathcal{B}$	$\mathcal{B}$	$\mathcal{F}$	$\mathcal{F}$
$\mathcal{N}$	$\mathcal{N}$	$\mathcal{F}$	$\mathcal{N}$	$\mathcal{F}$
$\mathcal{F}$	$\mathcal{F}$	$\mathcal{F}$	$\mathcal{F}$	$\mathcal{F}$

$\vee$	$\mathcal{T}$	$\mathcal{B}$	$\mathcal{N}$	$\mathcal{F}$
$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
$\mathcal{B}$	$\mathcal{T}$	$\mathcal{B}$	$\mathcal{T}$	$\mathcal{B}$
$\mathcal{N}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{N}$	$\mathcal{N}$
$\mathcal{F}$	$\mathcal{T}$	$\mathcal{B}$	$\mathcal{N}$	$\mathcal{F}$

Algebraically, these four values can be arranged nicely in the following lattice.[6]



We will view conjunction as the *meet* (the greatest lower bound), disjunction as the *join* (the least upper bound), and negation as an operator that flips $\mathcal{T}$ and $\mathcal{F}$ but is a fixed-point operator for $\mathcal{B}$ and $\mathcal{N}$. That is to say, applying the negation operator to $\mathcal{B}$ will give $\mathcal{B}$ again and likewise for $\mathcal{N}$.

Here is what that means: Note that the four values in the diagram are connected by lines that represent an ordering relation. One value is said to be greater than the other if it is higher up on the page, and there is an ascending line from the lower value to the higher one. To compute disjunctions, one looks for the lowest value that is greater or equal than either value of the two disjuncts. For example, the value of $T \vee \mathcal{B}$ is $\mathcal{T}$ because it is greater or equal than either $\mathcal{B}$ or $\mathcal{T}$, and there is no smaller value that meets this description. How about $\mathcal{N} \vee \mathcal{B}$? There is only one value that is greater than or equal to those two values, and that is $\mathcal{T}$.

[5] There is no table for the conditional, simply because FDE does not have a conditional. This is actually the feature that gives First Degree Entailment its name: An entailment of the first degree is one in which the turnstile ($\vdash$) is the only entailment or conditional-like item. We will later see an example for how a conditional can be added to FDE.

[6] It is not essential to know what a lattice is to understand the following. Those who know what this means might benefit from knowing that the lattice is one of the de Morgan lattice variety. It also is the paradigmatic example of a *bilattice*, which in essence means that you can find two distinct lattice structures on the elements (although there are some more requirements, cf. Fitting (2002)). Additionally to the ordering that goes from bottom to top, you find the ordering that goes from left to right, that is, $\mathcal{N}$ is the lowest element and $\mathcal{B}$ the highest. The first is called the truth ordering (ascending the ordering means either gaining in truth or waning in falsity), the second the information order. Moving from left to right means to increase the amount of information available. This will become clearer in the discussion of the intuitive interpretation of the values below.

Conjunction looks for the greatest value that is less than or equal to the values of both conjuncts. For example, $\mathcal{N} \wedge \mathcal{B}$ is $\mathcal{F}$, because no other value is less than or equal to either $\mathcal{N}$ or $\mathcal{B}$. It should now be clear how the truth tables above correspond to the lattice operations.

As for negation, imagine a horizontal axis going through the values $\mathcal{N}$ and $\mathcal{B}$. Negation then is an operation that flips the value over this axis. In other words, it takes $\mathcal{T}$ to $\mathcal{F}$ and vice versa, but leaves $\mathcal{B}$ and $\mathcal{N}$ as they are, just as is recorded in the truth table.

4.3.1 Designated Values

I have told you what semantic values FDE ascribes to statements ($\mathcal{T}$, $\mathcal{N}$, $\mathcal{B}$, and $\mathcal{F}$) and how the value of a complex statement depends on the values of its parts. According to Dummett's plan, the next step in the development is to explain how the values relate to truth, that is, how we can move from the knowledge of the semantic value of a statement to the knowledge whether it is true or not. Then, we can finally find out what the consequence relation will look like, because we can define it as the relation that transmits truth from the premises to the conclusions in all models.

However, in the study of many-valued logics such as FDE, the normal procedure is often slightly different. The detour through the concept of truth is either left implicit or not intended at all. Instead, consequence is defined by singling out *designated values*, that is, values that the consequence relation is required to preserve. Whether or not these values jointly make up truth or some other desirable property is then either left open or addressed as an afterthought.

Which values did Belnap designate? Actually, he gave two equivalent choices: One may designate either $\mathcal{T}$ and $\mathcal{B}$ or one may choose $\mathcal{T}$ and $\mathcal{N}$. I will discuss the different intuitive ideas that back these alternatives below. Let us for now go with the more usual pair, $\mathcal{T}$ and $\mathcal{B}$.

Now, given these two truth values as designated, we can see how both LEM and Explosion can be avoided. The inference from C to $A \vee \neg A$ is not valid, because a counter model can be defined thus: Take C to be $\mathcal{T}$ and A to be $\mathcal{N}$. Then, the premise has received a designated value, but the conclusion has not (as will be easy to check with either the aid of the tables or the lattice diagram). On the other hand, a counter model to $A \wedge \neg A \vDash C$ can be given by assigning A the value $\mathcal{B}$ and C the value $\mathcal{F}$.

Indeed, FDE manages to avoid any inference that does not meet the *parameter sharing* requirement. This requirement is one way of making the somewhat vague notion of relevance more precise. An inference of propositional logic meets this requirement if at least one of the propositional parameters (these are the atomic statements) occurs both in the conclusion and in one of the premises. Obviously, both

LEM and Explosion fail to meet this requirement.[7] To see why all other inferences that fail to meet the requirement will have a counter model in FDE, observe that it will be possible to assign the value $\mathcal{B}$ to every atomic statement in the premises and the value $\mathcal{N}$ to every atomic statement that occurs in the conclusion. This will be an admissible valuation, because, by assumption, no atomic statement occurs both in the premises and in the conclusion, so no conflict can arise.

The next step is to note, by inspecting the truth tables, that every complex statement made up solely of atomic statements with the value $\mathcal{B}$ will receive the value $\mathcal{B}$ as well. The same goes for the value $\mathcal{N}$. This means that, no matter their logical form, the premises will receive the value $\mathcal{B}$ under our interpretation and the conclusion the value $\mathcal{N}$. But that means that we have constructed a counter model, as the valuation is one under which all premises are assigned a designated value, while the conclusion has a value that is not designated. In particular, this also implies that there are no logical truths in FDE. Any purported tautology will be seen to receive value $\mathcal{N}$ if all the atomic statements occurring in it are assigned value $\mathcal{N}$.[8] Likewise, there are no logical falsehoods in the sense of formulas which imply every other formula. It is easy to give, for any particular formula, a valuation that assigns to it the designated value $\mathcal{B}$. Again, this is achieved by setting all the atomic statements to $\mathcal{B}$. But a formula that takes a designated value in some model will not imply everything.

Finally, it should be clear that the same kinds of arguments as in the last paragraphs can be run if we go with Belnap's second suggestion regarding designated values. That is, if, instead of $\mathcal{T}$ and $\mathcal{B}$, we choose $\mathcal{T}$ and $\mathcal{N}$ as designated values. To see this, simply switch all occurrences of "$\mathcal{B}$" and "$\mathcal{N}$" in these arguments. Indeed, all inferences that are valid under one choice of designated values are valid under the other as well.

4.3.2 Thinking Computers

However, for all this to be more than a mere mathematical trick to fulfill the parameter sharing requirement, it would be nice to be given an informal interpretation of what those four values are supposed to *mean*. (Of course, the way they were named gives a pretty good hint already.)

This is the point where the second motivation for Belnap's logic comes into play. He thinks of the valuations as recording what information a computer has received about different statements. The computer is given input in the form of statements that are designated as true or false. As many people are supposed to be building up the database of the computer, it is not impossible that one person might enter a statement as true, while another enters it as false. We then end up with four possibilities for each statement:

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[7] Inferences such as $A \wedge \neg A \vdash A \wedge B$ or $C \vdash C \wedge (A \vee \neg A)$ show that parameter sharing is necessary, but not sufficient for a relevant consequence relation. Note, however, that these inferences are invalid in FDE as well.

[8] That means that FDE will not suit those relevantists who would like to keep some tautologies, but only reject that these tautologies follow from an arbitrary premise.

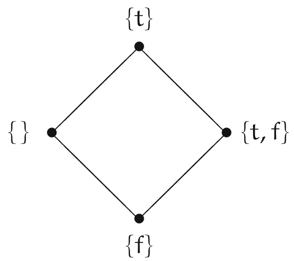
$\mathcal{N}$: The computer has received no information pertaining to the statement.
$\mathcal{F}$: The computer has received the information that the statement is false.
$\mathcal{T}$: The computer has received the information that the statement is true.
$\mathcal{B}$: The computer has received the information that the statement is true and the information that it is false.

The job of the computer now is to compute the values of complex statements and draw suitable inferences. If you prefer a less gadgety example, you might instead want to think about a criminal trial (this is not Belnap's example). Let us suppose that only testimonial evidence is available, then the transition is completely straightforward; just substitute "court" for "computer." I will come back to the example of legal trials repeatedly in the remainder of the book.

In either case, it is not clear that these four values need to have much to do with any substantial notion of *truth* at all.[9] The computer might have been fed false data, and the court might have been lied to. Belnap sometimes calls his semantic values "epistemic" values and marks the distinction between them and what he calls "ontological" truth.

Given these interpretations for the four values, how can Belnap justify the choice of designated values as being either $\mathcal{T}$ and $\mathcal{B}$ or, alternatively, $\mathcal{T}$ and $\mathcal{N}$?

To make his argument more conspicuous, it is convenient to relabel the four values in a manner suggested by J.M. Dunn. The four corners will be furnished by members of the power set of the two basic semantic values t and f. These are what Belnap calls "told-truth" values, "told-true," and "told-false." $\mathcal{N}$ corresponds to the empty set { }, $\mathcal{F}$ to {f}, $\mathcal{T}$ to {t}, and $\mathcal{B}$ to {t,f}.



The idea is not so much that there are again four semantic values that merely have different names. Rather, this way of looking at the logic suggests that there are only two basic semantic values, just like in classical logic (modulo the "told-" prefix). However, valuations are not, as in classical semantics, total functions, but rather *relations*. That is, statements will not be assigned one and only one value, but may receive none, one or two of the values on offer (told-true and told-false).

Whether the semantics is a four-valued functional one or a two-valued relational one does not make too much of a difference when it comes to formal properties. The consequence relation that will be induced will be the same one, provided the

[9] For this reason, I try to avoid calling them "truth values."

corresponding values are chosen. In a sense, only the relational semantics really has gaps and gluts, because in the functional semantics, every statement receives a value. But the values $\mathcal{N}$ and $\mathcal{B}$ are, at the end of the day, just designations of gaps and gluts as well. However, the two-valued semantics is better suited to Belnap's choice of designated values, while the four-valued version makes a different choice that I want to suggest below more comfortable to discuss.

Belnap has two distinct stories to tell about logical consequence. First and most common is the request that the consequence relation preserve *truth*, just what we have seen in the discussion of Dummett's theory. What might truth be in this setup, though? Of the two basic values, t and f, only t even comes close to any notion of truth, even if it is not "ontological" truth. As in the set formulation there are two sets that contain t, namely $\{t\}$ and $\{t, f\}$, it seems quite natural to take these as designated values ($\mathcal{T}$ and $\mathcal{B}$ in the four-valued version).

However, there is also a second property that Belnap suggests we might want to preserve, and this is something that we have not seen in Dummett (yet). This alternative is that we should like the consequence relation to transmit *non-falsity*: If the premises are not false, then neither is the conclusion. Belnap gives relatively little by way of motivation why we should call such a relation logical consequence, but we will come to discuss this topic at length in the following chapters.

Looking up the truth values that would get designated under this requirement, we find that $\{t\}$ and $\{\}$ are the only ones that do not contain value f. These values, of course, correspond to $\mathcal{T}$ and $\mathcal{N}$, and we have seen above that this choice of designated values leads to the same logic as the choice of $\mathcal{T}$ and $\mathcal{B}$.

4.4 Exactly True Logic

Very well. Either of the two features that one might wish to preserve, told-truth or non-told-falsity,[10] leads to the same logic. Is there anything more one could ask for?

That this question should be answered positively is suggested in Pietz and Rivieccio (2013).[11] Their paper argues that we should ask for a consequence relation that preserves *truth-and-non-falsity*. This property is held together with hyphens because at one point Belnap describes FDE as preserving truth *and* non-falsity,[12] which is completely correct in the following sense: If all premises are true, so is the conclusion, and if all premises are non-false, so is the conclusion.

The requirement Pietz and Rivieccio propose, however, is as follows: If all premises are true and not false, then so is the conclusion. Simply put, "told-truth"

[10] I will leave off the "told-" prefixes for the rest of this section to increase readability.

[11] The first author of that paper and the author of the present study are the same person, different last names notwithstanding.

[12] "Now for an account which is close to the informal considerations underlying our understanding of the four values as keeping track of markings with told True and told False: say that the inference from A to B is valid, or that A entails B, if the inference never leads us from told True to the absence of told True (preserves Truth), and also never leads us from the absence of told False to told False (preserves non-Falsity). Given our system of markings, this is hardly to ask too much." Anderson et al. (1992), p. 519.

is good, “told-false” is bad[13]; the prudent computer should choose those pieces of information that are univocally supported and infer conclusions that are similarly univocally supported. In searching material to draw inferences from, it should stay away from those inputs that it has been told are half-false (those with the value {t, f}). Of course, the only truth value that both contains t and does not contain f is {t}.

Talking “across” the two variants of semantics (the four- and the two-valued one), the difference between Belnap and the new proposal is that he wants to designate t, while Pietz and Rivieccio want to designate $\mathcal{T}$. On page 512 of Anderson et al. (1992), Belnap discusses the difference between the two values and suggests to read t as “told at least true” and $\mathcal{T}$ as “told exactly true” in circumstances where confusion between the two threatens. In view of this, Pietz and Rivieccio call the new logic “Exactly True Logic” (ETL)[14].

What happens to logical consequence if we designate only $\mathcal{T}$? For one thing, a contradiction will now never take a designated value. Therefore, the new logic validates Explosion (and thus fails to fulfill the needs of relevant logicians).

Indeed, one might well think that the new logic will coincide with a known logic. This, however, is not the case. Even though ETL validates Explosion, $A \wedge \neg A \vDash C$, just as strong Kleene does, the inference $(A \wedge \neg A) \vee (D \wedge \neg D) \vDash C$ fails. For a counterexample, take $v(A) = \mathcal{B}$, $v(D) = \mathcal{N}$, and $v(C) = \mathcal{F}$. It is easy to check that under this valuation, the premise will be assigned value $\mathcal{T}$.[15]

This is a most unusual feature. For one thing, it allows for theories that contain disjunctions, but cannot consistently be expanded by *either* disjunct, a property Pietz and Rivieccio dub *anti-primeness*.

Pietz and Rivieccio make no pretensions that these uncommon features are particularly *desirable* in a logic. However, the paper argues that the cause of the problem here is not so much the choice of designated value, but rather the logical lattice itself:

> [T]hese are quite counterintuitive features. However, when it comes to a direct comparison between FDE and the new logic, we believe that this should not weigh too heavily against the latter. This is because what we see here is merely a slight exacerbation of an unintuitive feature that has been with FDE ever since it was proposed. The lattice will give out the value $\mathcal{T}$ for a disjunction of two statements with the values $\mathcal{B}$ and $\mathcal{N}$ (...). In particular, the fact that a contradiction with the value $\mathcal{B}$ and a contradiction with the value $\mathcal{N}$ will receive value $\mathcal{T}$ when disjoined is a feature of the logical lattice, not of ETL in particular. (Pietz & Rivieccio 2013, p. 134)

[13] Bad for the proposition in question or for you, if you are reluctant to give up your belief in it. Belnap writes: “We note that in the logical lattice, each of the values None and Both is intermediate between $\mathcal{F}$ and $\mathcal{T}$, and this is as it should be, for the worst thing to be told is that something you cling to is false, simpliciter. You are better off (it is one of your hopes) either being told nothing about it or being told both that it is true and also that it is false; while of course best of all is to be told that it is true with no muddying the waters.” Anderson et al. (1992), p. 516.

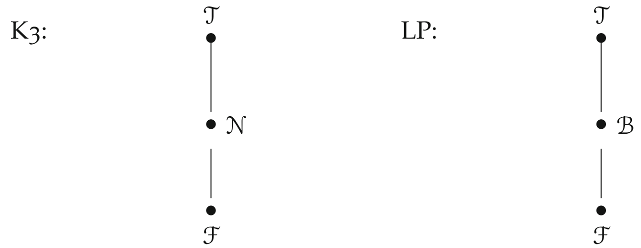
[14] The logic has independently been described in Marcos (2011).

[15] This shows that the rule of proof “If $A \vDash C$ and $D \vDash C$, then $A \vee D \vDash C$” fails, which seems to stand in the way of a natural sequent calculus for this logic; the paper gives a Hilbert-style proof system instead.

In a sense, the symptoms are more visible with the new logic, but the root cause of the problem is shared by both FDE and ETL. The contrast between FDE and ETL will come to play a role in Chap. 8 when I discuss the possibility of verification–falsification gluts and the best way to handle them.

4.5 LP and K_3

Let us now move from the logics FDE and ETL with their gaps and gluts to logics that have only one of those features. The logics we will be looking at are easily obtained from FDE by tightening the requirements on the valuations. To get our gappy logic, which is known as strong Kleene logic (K_3), we simply disallow the value $\mathcal{B}$. In terms of the alternative two-valued account, we require a valuation function again, not a relation. A function is different from a relation in that it will assign at most one value to each argument. That is, no statement will be assigned more than one value, even though we are open to the possibility that it should receive none. On the other hand, our paradigmatic paraconsistent logic, G. Priest's Logic of Paradox (LP), comes about by dropping the value $\mathcal{N}$ or alternatively by requiring the valuation relation to assign at least one of t or f to each statement.[16] Algebraically put, we end up with two lattices that look quite alike and give rise to truth tables that look very similar as well:



K3:

¬	
$\mathcal{T}$	$\mathcal{F}$
$\mathcal{N}$	$\mathcal{N}$
$\mathcal{F}$	$\mathcal{T}$

∧	$\mathcal{T}$	$\mathcal{N}$	$\mathcal{F}$
$\mathcal{T}$	$\mathcal{T}$	$\mathcal{N}$	$\mathcal{F}$
$\mathcal{N}$	$\mathcal{N}$	$\mathcal{N}$	$\mathcal{F}$
$\mathcal{F}$	$\mathcal{F}$	$\mathcal{F}$	$\mathcal{F}$

∨	$\mathcal{T}$	$\mathcal{N}$	$\mathcal{F}$
$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$	$\mathcal{T}$
$\mathcal{N}$	$\mathcal{T}$	$\mathcal{N}$	$\mathcal{N}$
$\mathcal{F}$	$\mathcal{T}$	$\mathcal{N}$	$\mathcal{F}$

[16] Here, there is a relevant difference between the two ways of giving the semantics. For we are dealing with a logic that, on the first interpretation, is a three-valued logic and therefore is not bivalent. However, one might argue that bivalence holds on the second interpretation, as there are only two truth values and every statement is either true or false. The only difference to classical logic is that the "either true or false" is an inclusive disjunction. Of course, one could hold that part of the idea of bivalence is that there shall be no gluts, that is, that the disjunction is an exclusive one. It is hard to guess what Dummett would have said, at least it does not clearly emerge from his extended discussion of terminology in the preface of TOE (p. xix).

On the other hand, K_3 is not bivalent, no matter how we choose to give the semantics.

Indeed, the only thing that makes a real difference for logical consequence is that in the first case, the middle value is standardly not designated, while in the second case, it is.

Unlike FDE, neither K_3 nor LP has the parameter sharing property. As is easy to see, K_3 validates Explosion, while LP validates the LEM. Indeed, LP validates *all* classical tautologies,[17] while K_3 agrees with classical logic on *all* logical falsehoods (understood as statements that imply everything).

That is to say that as the basis for a relevant logic, neither K_3 nor LP will do. There have to be other arguments for adopting such gap-only or glut-only logics. I will go through some of them quickly, focusing on those issues that will prove important to the further unfolding of the book. Note that in most of these proposals, the semantic values are not taken to be mere "told-truth" values but aspire to be genuine truth values (in one sense or other).

4.6 Uses of Gaps

Even if K_3 is not a relevant logic, there have been several applications suggested for it and similar gappy (three-valued, partial) logics.[18] Here are some of them:

- Kleene originally introduced it to deal with functions that were not everywhere defined.
- Łukasiewicz had much earlier proposed a logic with slightly different truth tables to account for future contingents.
- The semantic paradoxes and the puzzling phenomenon of linguistic vagueness have been treated (but hardly cured) with partial logics.
- Non-referring singular terms have been argued to give rise to truth value gaps.

Let us examine the last item a bit more closely, because here Dummett's attitude toward gappy theories comes out relatively clearly and because this gives me occasion to introduce his distinction between assertoric content and ingredient sense.

4.6.1 Presupposition Failure

The problem of presupposition failure is epitomized by B. Russell's classic example:

The present King of France is bald.

[17] Including $(A \wedge \neg A) \rightarrow B$, if the arrow is interpreted as the material conditional. This makes it quite obvious that modus ponens is not a valid rule for the material conditional in LP, and usually, LP is thought of as having no conditional (not even a defined one) at all.

[18] Cf. for example Blamey (1986).

This sentence seems to say of someone that he is bald. But who is it talking about? The non-existent King of France? And which truth value should we think it has? It surely is not true, so, under the classical assumption of bivalence, it must be false. But then, shouldn't its negation be true? That is, shouldn't

The present King of France is not bald.

be true? But this seems as untrue as the first sentence!

Russell suggested that the actual logical form of such sentences is quite a bit more complicated than meets the eye, and he gave a very influential but fairly elaborate analysis that allowed the ascription of the value "False" to both of the examples without any breach of classical doctrine. The regimented but not yet formalized versions of the two statements he proposed are "There is exactly one person who is presently King of France and that person is bald" and "There is exactly one person who is presently King of France and that person is not bald."

P. Strawson, on the other hand, held that these sentences have just the logical form that you would expect [viz., Bald(King of France)] and that they are not false at all.[19] Of course, he did not suggest they are true, either; they are *neither true nor false*, that is, prime examples of truth value gaps.

Strawson argues that there is a *presupposition* that has to be met for a statement of this form to be true or false: the presupposition that the singular term in it actually has a reference. If that presupposition fails, the statement will be neither true nor false; it will either have no truth value at all or have a third truth value, neither-true-nor-false; again, the difference is relatively insubstantial. On either reading, bivalence does not hold any more.

Now, Dummett argued that this rejection of bivalence, as opposed to his own, is not a *deep*[20] one. Attributing a third truth value or a truth value gap to a statement does in and of itself not make a difference to our linguistic usage of this statement itself. What possible difference to our use of a statement would it make to call it neither-true-nor-false, instead of just plain false? Would not the speaker who asserted the statement be equally wrong in both cases?

It would of course make a difference to how we use the negation of that statement: In the case the statement that is negated is false, this negation will be true, and a speaker would be right to assert it. In the other case, where the statement is deemed neither-true-nor-false, it would be just as wrong to assert the negation of the statement.[21] A rejection of bivalence in order to "give a smooth account of the internal structure of our sentences"[22] is what he at one point thought to be a shallow one.

An interesting difference between "shallow" and "deep" (i.e., intuitionistically motivated) rejections of bivalence is that in the first case, bivalence is actually *denied*. That is, we can point to specific cases in which bivalence fails, such as statements

[19] Strawson (1950).

[20] TOE, p. 23.

[21] TOE, p. 12.

[22] TOE, p. xviii.

with existential presupposition failure. On the other hand, the deep reasons to reject bivalence outlined in the first chapter will not allow such counterexamples. This is because on the intuitionistic understanding of negation, being able to ascertain that a sentence is not true is tantamount to ascertaining that it is false. Thus, no statement can be known to be neither true nor false. In Dummett's terminology, his is an attack on the principle of bivalence (every statement is either true or false), while the presupposition theorists also attack tertium non datur, which says that no statement is neither true nor false.

In TRUTH, Dummett tried to defend tertium non datur by suggesting to distinguish, not between false statements and statements that are neither true nor false, but rather between different *ways* in which a statement can be false. The middle value in the truth tables for K_3 (or some other three-valued logic) would then be interpreted as a special kind of being false. Statements that are false in this particular way are such that their negations will also be false in this particular way.[23] That is, the way in which "The present King of France is bald" is false is such that "The present King of France is not bald" is false as well.

4.7 Designated Values, Assertoric Content, and Ingredient Sense

According to Dummett, then, truth and falsity *simpliciter* correspond to the class of designated and undesignated values. He asks us to appreciate the following points:

> (i) The sense of a sentence is determined wholly by knowing the case in which it has a designated value and the cases in which it has an undesignated one.
>
> (ii) Finer distinctions between different designated values or different undesignated ones, however naturally they come to us, are justified only if they are needed in order to give a truth-functional account of the formation of complex statements by means of operators,
>
> (iii) In most philosophical discussions of truth and falsity, what we really have in mind is the distinction between a designated and an undesignated value, and hence choosing the names 'truth' and 'falsity' for particular [ones amongst the] designated and undesignated values respectively will only obscure the issue. (TOE, p. 14)

Later,[24] he came to express the idea in terms of a new distinction, the distinction between the *assertoric content* and the *ingredient sense* of a statement. The first of these refers to the content of the statement on its own. It will only need to delineate those states of affairs in which an assertion of it would be correct, that is, where the asserted statement receives a designated value, no matter which.

The ingredient sense contains all that a statement can contribute to the assertoric content of complex statements containing it (see (ii) above). This might be much

[23] TOE, p. 14.

[24] For example in LBM, pp. 47–49.

more intricate than the assertoric content.[25] We need to know the ingredient sense of a statement to judge whether a complex statement of which it is a part is correctly assertible.

And here now is the connection between the designated values and assertoric content/ingredient sense:

> One way to understand the traditional semantics for many-valued logics, with its distinction between designated and undesignated values, is to take the assertoric content of a sentence to be given by the condition for it to have a designated truth-value, while the distinctions among different undesignated values, and those (if any) among different designated ones, serve to explain the ingredient senses of sentences. (Dummett 2004, p. 34)

By associating truth with the possession of a designated value and falsity with the possession of an undesignated one, he hopes to be able to acknowledge Strawson's point without having to give up his contention that the positing of gaps is not a good, deep reason to give up bivalence. What Strawson calls a gap, Dummett calls a form of falsity.

However, it is not really clear how one should incorporate the idea into the semantical account of intuitionistic logic. The Kripke semantics we have seen in the last chapter leaves no room for sentences that are unassertible because they have an existential presupposition that fails. Such statements should never receive value 1, but their negations should never receive value 1 either. However, we know that the negation will receive value 1 immediately as soon as we realize that the negated statement will never receive value 1.[26] It might be possible to augment the semantical theory to accommodate the idea, but Dummett does not develop a concrete proposal.

But why does he think that such a strategy to explain away truth value gaps is necessary anyway? Why not reject bivalence for whatever kind of reason there might be, whether deep (his) or shallow (Strawson's)? K. Green gives an interesting interpretation here (Green 2005). According to her, Dummett sees his rejection as deep because it has a deep metaphysical consequence: the rejection of realism.

The truth value gaps, on the other hand, might not lead to such a deep result. It seems to be a completely tenable option to hold that truth value gaps exist but have nothing at all to do with our cognitive abilities. However, Green argues, Dummett wanted the deep implications of the failure of bivalence, and giving up bivalence for the wrong (shallow) reasons would threaten those implications. Therefore, the shallow rejection of bivalence could not be allowed to stand.

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[25] While I do not know of an a priori argument why assertoric content and ingredient sense might not turn out to be completely distinct, I would guess that in most theories, we will find the assertoric content somehow subsumed under the ingredient sense. The ingredient sense will have to answer; for example, how conjunctions are decided to be assertible, and it seems hard to answer that if we do not know the conditions under which the conjuncts were assertible on their own.

[26] However, we can see a different example in the Kripke semantics in which the assertoric content and the ingredient sense come apart: Assume that a statement receives value 0 at a world. Then, it is already settled that it is, at that world, not assertible. However, more is needed by way of information to decide whether the negation of that statement is assertible, namely the future development of our investigation.

Green goes on in her essay to observe how Dummett grew more and more lenient toward truth value gaps over time. This is because he came to see *any* kind of rejection of bivalence as a form of anti-realism (cf. LBM, p. 325). Thus, the connection between bivalence and realism is, according to Dummett's later view, not threatened by truth value gaps, and thus, there is no real dialectical need for him to oppose them.

4.8 Motivations for Paraconsistency

Let us now come to truth value gluts and paraconsistency. Just as there is a great diversity of incentives for gaps, there are many motivations for paraconsistency apart from the considerations involving relevance we saw earlier. Again, a paraconsistent logic is one in which the inference, known as *ex contradictione quodlibet* or Explosion, from a contradiction to an arbitrary statement is rejected. Indeed, we have already seen such a rejection in the last chapter, when minimal logic was mentioned (4.3.6.1).

However, there is some reluctance to classify minimal logic as a genuine paraconsistent logic. While not every statement can be derived from a contradiction, a contradiction will entail every *negated* statement. It is easy to see why this is so, if one remembers that $\sim A$ is actually short for $A \supset \bot$. A contradiction will then entail $\bot$ simply by modus ponens, and because minimal logic, just as intuitionistic and classical logic, supports the inference from C to $B \supset C$, we can draw an inference from $\bot$ to $B \supset \bot$ (i.e., $\sim B$) for any B.

For most mainstream paraconsistent purposes, this is too much; little is gained if not all statement, but all negated statements are derivable from a contradiction.

Here is a partial list of reasons for turning to a (truly) paraconsistent system[27]:

- The recognition that there are interesting but inconsistent scientific theories and the perceived need to treat them without inferring everything whatsoever in them.
- As a special case, there is the research conducted in inconsistent mathematics (Mortensen 1995).
- Paraconsistent logics have been proposed to reason about inconsistent fictions.
- There are inconsistencies in most bodies of law that need to be dealt with.
- Finally, the semantic paradoxes like the Liar and others have been a main concern of paraconsistent logic.

In the majority of cases, paraconsistency is achieved by allowing gluts in the semantics. However, the connection is not a necessary one. Below, I will introduce a view called Analetheism according to which it is actually *gaps* that induce paraconsistency.

As I said before, paraconsistency is not to be confused with *dialetheism*, the view that some contradictions are actually true. The next section gives a very quick introduction to the fascinating world of dialetheism.

[27] See, for example, Berto (2007) for more.

4.9 Dialetheism

The main reason to be a dialetheist has always been the deep puzzle posed by semantic paradoxes like the famous Liar sentence:

This sentence is false.

These paradoxes have been around for the longest time without any plausible consistent solution in sight. This is not due to a lack of trying; some of the smartest philosophers have tried hard to explain away the obvious contradictions such statements give rise to, to little effect.

The dialetheist[28] argues that this is because there is nothing to explain away: These statements are true and false, just like they appear to be.

This view has some important advantages. Here is one of them: I listed the semantic paradoxes under the motivations for partial logics as well. Here, the view would be that the Liar sentence and its kin are neither true nor false. But this view, unlike dialetheism, is open to a relatively straightforward counterargument. It might do away with the Liar, but what about the following "Revenge Liar":

This sentence is either false or it has no truth value.

It is easy to see that if one holds that this sentence has no truth value, it will be true, contradicting the view on display. Dialetheism is quite immune to such attacks.

On the other hand, dialetheism has a big disadvantage as well: It seems utterly unbelievable. It contradicts what many have seen as the most basic insight of all, the law of non-contradiction[29]:

LAW OF NON- CONTRADICTION: No statement can be true if its negation is, and no statement is both true and false.

As we see, the law of non-contradiction incorporates two very closely related ideas. Often you will find the law of non-contradiction defined as only one of these ideas. Many of these authors will understand the other principle to be entailed by the one they make explicit. I will use the above definition, and when there is occasion to discuss the two principles separately, I will adopt the following terminology:

NO GLUTS: No statement is both true and false.
and
NEGATION INCOMPATIBILITY: No statement can be true if its negation is.

The dialetheist takes both of these aspects of the law of non-contradiction to be unfounded prejudices and tries to give cogent counterarguments. It is fair to say that it took the philosophical world some time to take dialetheism seriously, but by now, it has become a major position in the discussion of the paradoxes.

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[28] The most important exposition and defense is Priest (2006a).

[29] Unfortunately, received terminology works a bit against clarity in this case: The law of non-contradiction is a semantic principle, like bivalence, and not a logical principle like the Law of Excluded Middle. As said above, I write semantic principles in lowercase letters in the hope to avert confusion.

As far as the requirements on logic that dialetheism entails are concerned, again it is clear that paraconsistency is indispensable. Else the dialetheists would have to infer everything from the Liar sentence, which they take at face value. LP was the first suggestion that Priest made in the 1970s, and even though it went through some modifications,[30] it is still the basis of the logic he advocates today.

4.10 Expressing the Law of Non-Contradiction

A striking fact that follows from LP's validating all classical truths is that it also validates $\neg(A \wedge \neg A)$. This must come as a surprise to anyone who is told that the law of non-contradiction is supposed to fail in LP. What else but an expression of that law could the validity of $\neg(A \wedge \neg A)$ be?

LP is not the only paraconsistent logic that sports this remarkable feature. In fact, I would guess that *most* paraconsistent systems share it.[31] As long as no connection between paraconsistency and the law of non-contradiction is claimed, there is not even any particular tension here. We have no reason to view a paraconsistent logic as flawed because of its validating $\neg(A \wedge \neg A)$, because we have defined paraconsistency not in terms of this schema, but rather in terms of the failure of Explosion, $A, \neg A \vdash B$. If this specific feature fails to capture the rejection of the law of non-contradiction, then so be it.

That being said, most people in the debate nowadays *do* seem to think that it is by being paraconsistent rather than by invalidating $\neg(A \wedge \neg A)$ that a logic flaunts the law of non-contradiction, or rather that by sanctioning Explosion rather than by validating $\neg(A \wedge \neg A)$ that a logic makes its allegiance to the law of non-contradiction known. This view can be witnessed in many chapters of a relatively recent volume on the law of non-contradiction (Priest et al. 2004) (cf. especially Brady's, Restall's, and Grim's contributions).

This is a very modern view of the matter. Not too long ago, the role of logic was seen as delineating a specific set of formulas, the tautologies. On this view, there could be no doubt that the question whether a logic satisfies the law of non-contradiction is the question whether $\neg(A \wedge \neg A)$ is derivable; which other tautology should be better suited to say that there are no contradictions?

However, as more and more non-classical logics came into view, the idea of logic as only concerned with tautologies had to give way. As we noted, LP and classical logic are indistinguishable if we only look at their tautologies. But of course, they are different logics. Similarly, both strong Kleene and FDE have the same set of tautologies, namely the empty set. This obviously does not make them the same logic.

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[30] Cf. Chap. 16 of Priest (2006a).

[31] This is impressionistic. Counting is, as so often, difficult, as there is an infinite number of different paraconsistent systems.

Logic then has to be about something more than just tautologies if we want to be able to differentiate between the above logics. A more comprehensive view is that logic is about consequence relations. The job of a logic is to separate the valid inferences from the invalid ones. This view allows us to distinguish between classical logic and LP, and similarly between strong Kleene and FDE, by noting again that (for example) $A \wedge \neg A \vdash B$ is a valid inference of classical logic and strong Kleene, while it is not in LP and FDE.

While this view of logic now makes it *possible* to say that a logic like LP, while it may allow the derivation of $\neg(A \wedge \neg A)$, is nonetheless commendable to someone who rejects the law of non-contradiction because it is paraconsistent, it is far from clear that one *should* say that. The argument normally given at this point is that if you accept contradictions, then the further contradiction between the contradiction you accept and the negation of it that you accept on account of its validity does nothing to weaken your position. That is, if you accept A and $\neg A$, then you accept a contradiction, and the further news that this compels you to accept another pair of contradictory statements, $A \wedge \neg A$ and $\neg(A \wedge \neg A)$, should not concern you. Of course, it is not just two instead of one contradiction you will have to accept, because from the second contradiction, a third is easily generated and so on. However, even in view of this staggering number of contradictions that you are committing yourself to, you still can go on reasoning as long as your logic is paraconsistent. So, the law of non-contradiction can not be captured by the validity of $\neg(A \wedge \neg A)$, or else it should prevent such ongoing use of the logic when it is applied to inconsistent premises.

This might be an argument against seeing the law of non-contradiction embodied in the validity of $\neg(A \wedge \neg A)$, but it gives no grounds yet why a friend of contradictions should want these formulas valid. Indeed, I do not think there is an argument here over and above the fact that these validities come bundled up with other logical commitments. For example, they follow from the following principles that Priest wants to endorse: the validity of Excluded Middle,[32] Double Negation Elimination, and the de Morgan laws [from $\vdash A \vee \neg A$ infer by de Morgan $\vdash \neg(\neg A \wedge \neg\neg A)$ and by Double Negation Elimination and permutation $\vdash \neg(A \wedge \neg A)$].

That is to say, surely the dialetheist should not object to, say, FDE because it does not validate $\neg(A \wedge \neg A)$, whereas he may or may not want to object to it on the grounds that it does not validate Excluded Middle (for reasons that are probably unrelated to her dialetheism). Indeed, there still seems much to be said for trying to keep the number of contradictions down and not have a single contradiction mushroom up to an infinity of contradictions in no time. But granted, a dialetheist can not be forced to renounce his dialetheias by force of the validity of $\neg(A \wedge \neg A)$ alone.

Now, what about the relation between Explosion and the law of non-contradiction? If a logic has any means at all to fend off contradictions, then it is by blowing up into the trivial consequence relation in an heroic act of suicide bombing. This is what

[32] He argues against truth value gaps and intuitionism alike in Chap. 4 of Priest (2006a), though in the auto-commentary to that chapter that is supplied in the second edition of the book, he takes a slightly more lenient approach.

one gets for bringing in contradictions into such a system, so one should better not do it.[33]

If this is the only way how the law of non-contradiction can be enforced by a logic, then giving up the law will mean giving up this defense mechanism. It need not be the other way around, though. If you have arrived at a preference for paraconsistent logics by way of concerns about relevance, for example, you have no apparent need to allow for the existence of true contradictions.

This last section is not at all an exhaustive account of the state of the discussion about the very thorny issue of the correct definition of the law of non-contradiction. It is only meant to flag the issue and warn against the very plausible, but false assumption that a paraconsistent logic could not count $\neg(A \wedge \neg A)$ among its theorems, because most paraconsistent logics that will come up in the remainder of this book will do just that.

4.11 The Law of Non-Contradiction, Bivalence, and Duality

Before leaving the subject, note the similarity between the dialetheic rejection of the law of non-contradiction and the intuitionistic rejection of bivalence and how they influence the logical principles. The law of non-contradiction and bivalence are *dual* principles: One forbids gluts, the other gaps. To those, semantic principles correspond Explosion and the LEM, which are in an important sense dual as well, although that sense is not perfectly obvious.

Intuitively, to dualize something is to "flip it over" in some way, such as when I dualize my face by looking into a mirror, or by taking a picture of it and then inverting the color spectrum in Photoshop, or maybe by pressing my head into a bowl of plaster. All of these actions leave me with some sort of dual of my face, but of course these duals look quite different.

Likewise, in logic and mathematics, the meaning of "dual" is quite context sensitive, and often one does not know what an author who uses the term means by it until one sees some examples. It can refer to the switching of polarly opposite connectives, such as conjunctions and disjunctions, necessity and possibility operators, or universal and existential quantifiers. It might involve the deletion or the insertion of negations. On the algebraic or semantical side, it can refer to the inversion on some algebraic order, the substitution of open sets for closed ones, a switch in designated values such as from true to non-false ones, or the switch from a underdetermined valuation function (one with gaps) to an overdetermined valuation relation (one with gluts). The correspondence of these switches in the semantics to the valid inference patterns is most often far from obvious.

[33] If even the prospect of being committed to trivialism, the view that everything is true, cannot scare you off, then even Explosion cannot compel you to keep your reasoning contradiction free. Priest has tried to argue against an imaginary trivialist in Priest (2006b), and this turns out not to be an easy task at all.

P. Halmos and S. Givant describe the potential confusion the mention of duality can cause in logic (they are writing about classical propositional logic in its algebraic guise as a Boolean algebra):

> If an experienced Boolean algebraist is asked for the dual of a Boolean polynomial, such as say $p \vee q$, the answer might be $p \wedge q$ one day and $\neg p \vee \neg q$ another day; the answer $\neg p \wedge \neg q$ is less likely but not impossible. (Halmos & Givant 1998, p. 47)

If we widen our scope and consider single premise entailments of the form $A \vDash B$, there are even more possibilities of duality, as there is the further option of a switch between the left- and right-hand side of the turnstile. The inferences we are concerned with, $B \vDash A \vee \neg A$ and $A \wedge \neg A \vDash B$, are then dual in the sense that premise and conclusion are switched, as well as conjunctions and disjunctions.

So, we observe that the rejection of the semantically dual principles of bivalence and the law of non-contradiction results in the rejection of the two inferences LEM and Explosion, which are dual in the sense just mentioned. However, had one been asked to guess the dual of $\vDash A \vee \neg A$, another plausible answer would surely also have been $\vDash \neg(A \wedge \neg A)$, which we have just seen to be valid in LP.

Likewise, a dual of $A \wedge \neg A \vDash B$ could also have been $\neg(A \vee \neg A) \vDash B$. This pattern is valid in intuitionistic logic, so here is yet another aspect in which there is a duality between LP and intuitionistic logic: The startling fact that giving up the law of non-contradiction does not entail giving up $B \vDash \neg(A \wedge \neg A)$ is mirrored by the equally surprising fact that giving up bivalence does not force us to give up $\neg(A \vee \neg A) \vDash B$.

There is a point where this kind of duality breaks down, though: Intuitionistic logic validates $\vDash \neg\neg(A \vee \neg A)$, while $\neg\neg(A \wedge \neg A) \vDash C$ is not a valid inference of LP. Not surprisingly, the duality of LP and K_3 is stronger than the duality of LP and intuitionistic logic. We will get to know a logic that has a better claim to being "the" dual of intuitionistic logic soon. It is named, aptly enough, *dual intuitionistic logic*.

4.12 Analetheism

A recent discussion that also exemplifies a form of duality is that between dialetheism and *analetheism*, a new philosophical position canvassed by J.C. Beall and D. Ripley.[34] Dialetheism, as we have seen, encompasses the idea that there are truth value gluts, that is, sentences that are both true and false, and furthermore stipulates that such sentences are assertible, since all that is true is assertible. Analetheism takes another route: In the authors' own words: "Analetheism, for us, is the thesis that some sentences lack truth-value, coupled with the willingness to assert such sentences."[35] Analetheism, thus, has assertible gaps as opposed to the assertible gluts of dialetheism. A different way of phrasing the credo of analetheism is thus: "Assert

[34] Beall and Ripley (2004).

[35] Beall and Ripley (2004), p. 30.

only that which is not false," rather than the dialetheist's "Assert only that which is true." Thus, one could argue that analetheism was foreshadowed by Belnap's idea of logical consequence as transmission of non-falsity.

As to which sentences they have in mind and how to treat them logically, Beall and Ripley follow the dialetheist's arguments closely. They principally want to address the semantic paradoxes, and they want to suggest LP as the appropriate logic, at least in terms of the consequence relation. However, they use the truth tables of K_3, that is, they regard the middle value as a gap rather than a glut. But in contrast to K_3, they take this gappy middle value as designated. The resulting logic, of course, coincides with LP. Analetheists and dialetheist are in complete agreement which statements are assertible and which are not.

Thus, analetheists and dialetheists in particular agree that there are designatedly valued (that is, assertible) statements of the form $A \wedge \neg A$, but they disagree both about the question whether these statements are true and about the question whether they are false. Thus, as Beall and Ripley observe, "each position runs counter to one traditional dogma while accepting another,"[36] the two dogmas being the unassertibility of non-truths and the untruth of contradictions. A decision between dialetheism and analetheism, to them, seems to have to be based on a decision on which of these two dogmas should be retained and which one given up, as the rest of the two theories are so similar in motivation, virtues, and vices.[37]

As they make clear, given this close proximity, they are not able to come to a conclusion which theory should be preferred and thus are not in the business of advocating analetheism over dialetheism. Rather, they point out the apparent stalemate and challenge the dialetheist to explain why she is not an analetheist. The choice seems quite arbitrary indeed, but in later parts, I will present a paraconsistent view that, if anything, is a form of constructive analetheism.

4.13 Chapter Summary

In this chapter, I have introduced the ideas of truth value gaps and truth value gluts. The first occurs if some statements fail to receive a truth value, the second if some statements receive more than one value. I gave example logics exhibiting these phenomena, the gappy K_3, the glutty LP, and the logic FDE, which has both gaps and gluts. Moreover, I gave some idea of the philosophical motivations that have been offered for these logics, and gaps and gluts more generally.

Furthermore, we have seen a number of dualities in this chapter: the duality between gaps and gluts, the duality between truth preservation and non-falsity preservation, and the duality between the LEM and Explosion. I also stressed that it is not

[36] Beall and Ripley (2004), p. 34.

[37] There is, quite obviously, also a view on which there are gluts but no gaps and according to which only non-falsities should be asserted. It should not come as a surprise that this view (which I do not think has a dedicated name) would give rise to K_3's consequence relation.

always clear what the dual of a concept, a formula, or a position is supposed to be. Is *the* dual of LP the logic K_3? Or is it the logic that uses the truth tables of LP but transmits non-falsity? Or is it the logic the analetheists prefer, which uses the truth tables of K_3 and transmits non-falsity?

This shows the difficulty of the concept of duality, a difficulty often played over in formal texts, where the reader often gets the mistaken impression that "dualization" is a clearly defined term and that its definition is common knowledge. This is not the case though, and if the definition is not given explicitly, one has to be careful to pick up the intended meaning. Duality will feature heavily in what is to come, and I shall try to be clear about what I mean when I use the word.

Having given this promise let me end this chapter by asking the following: How do the various semantic theories with their gaps and gluts fit into the role laid out for such theories by Dummett?

Again, Dummett's idea was that a semantic theory should spell out how complex statements receive their semantic values, given the values of the constituent statements. Once that was accomplished, it should show how the semantic value of a statement determines its being assertible/true or not. Given that, logical consequence can be defined in terms of preservation of correct assertibility or truth.

There was one idea in this chapter that seems not to fit particularly well into Dummett's scheme. That idea was that logic need not be defined as truth preservation, but that it might rather be defined as non-falsity preservation. This came up in Belnap's discussion and is the leading idea of analetheism. I pointed out that there is yet a third alternative here that suggests itself naturally, namely truth-and-non-falsity preservation, and showed that at least in the case of Belnap's lattice, this choice is different from the other two.[38]

The incompatibility with Dummett's program of these ideas is not too grave, though, and we will soon see that Dummett at times suggests something like non-falsity preservation as the base of logical consequence as well.

There is another way of looking at the differences here, and that is to say that the question is not how consequence is defined, but how semantic values are wired to truth. This might be less plausible in the case of the interpretations we have seen the dialetheists and analetheists give, who clearly single out some of the values as true and others as untrue. On the other hand, it might be a sensible way to view the differences between FDE and ETL. The way to report this difference would then be that FDE takes the value t to correspond to truth, while the new proposal would take truth to coincide with value T.

Be that as it may, one may well wonder what the semantics of FDE are supposed to have to do with Dummett's project at all. The semantic values are given in terms of told-truth and told-falsity. Even if there are different ways of matching these up with a concept of truth simpliciter that can then be used to define logical consequence, this concept will surely end up being one variety or other of uncertain, hearsay "truth."

[38] If the underlying lattice is either of the three-valued ones, the consequence relation defined in terms of truth-and-non-falsity preservation will coincide with K_3.

Certainly, this would qualify as a notion of truth that is very “anti-realistic,” in the sense that it is far from the realistic notion of truth.[39] But Dummett, it would seem, had something a little more weighty in mind than just pieces of unchecked information. A proof of p, for example, certainly seems to imply something more solid than just a piece of information claiming that p. Even if we are not after truth at all, but content ourselves with correct assertibility, something more conclusive seems called for. Simply being told that p by someone seems not enough to make a correct assertion that p.

However, as I mentioned several times before, the projected transformation of the intuitionistic account of mathematical statements to an account of empirical statements will involve an important exchange of central concepts: Whereas mathematics can talk of proofs, in the empirical realm, there are *verifications* and *falsifications* to build on. The conceptual leap from *told-true/false* to *is verified/falsified* is not too great, but it will require some discussion whether we want to allow gaps and/or gluts between the latter pair.

The move from proofs to verifications and falsifications is the concern of the next chapter.

[39] Indeed, Wansing (Wansing 2012) offers a notion of “non-inferentialist, anti-realistic truth” based on told-truth values.