Chapter 12
Disequilibrium Trade and the Dynamics of Stock Markets

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Abstract The present work considers pricing and trade dynamics for stock commodity markets, which, unlike flow commodity markets have been little studied, if at all. Concepts and tools in economics are shaped to deal with flow markets, where commodities disappear in each period and then reemerge. This allows one to define unique demand and supply functions and their equilibria. A durable commodity, a stock, in contrast, remains on the market to the next period and may just change owner through exchange. This, however, changes demand and supply functions, and hence the equilibrium state to which a dynamic process may be heading. Dynamic processes are provided with memory of the actual exchange history. We also need to state how disequilibrium trade in stock markets takes place. This is another neglected issue, though a fact of reality. Using a case with only two traders of two stock commodities, and focusing pure trade, it is possible to specify the exact conditions for disequilibrium trade in each step of the dynamic process. In the end any of an infinity of equilibria can be reached, or trade can stick in some disequilibrium point while complex, even chaotic, price dynamics goes on.

Keywords Disequilibrium trade · Durable commodity markets · Complex dynamics · Multiple equilibria · Path dependence

Introduction

The present work focuses two different but related theoretical issues of economics, both concerning the markets for stocks or assets.

These concepts usually connote financial claims, such as shares, bonds, or cash, so, to avoid misunderstanding, it should be said from the outset that we presently

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intend something much more general: We just do not refer to flow commodities, such that disappear during one time period through consumption or as inputs in the productive process, but to any class of commodities that remain on the market period after period and that may change ownership through exchange.

Obviously, there are many such commodities, in fact any durable commodity that has a user value, either for the owner or a potential buyer. To fix ideas, just think of the housing market.

Digression on Real Estate Value Instability

This market is of special interest as recent turmoil in the western economy has been related to instabilities on housing markets. The explanations offered have referred to speculative bubbles. No doubt there exists some speculation through the intervention of real estate dealers, but, there are other issues at work which may explain at least some of the phenomena without any reference to speculation, which is difficult to model convincingly.

The present author tends to lean on Lord Keynes's pessimistic view that it is impossible, even if he, more than anyone, admitted that it is a regrettable fact of life. See Keynes (1936). For this reason, and for the reason that so (too?) much has already been written on so called heterogenous (chartist/fundamentalist) agent models, the present work will not touch upon this topic, nor on speculative behaviour at all.

So, if we disregard speculation, what else could there remain to blame the housing market instabilities on? As a matter of fact, there is something very fundamental, even trivial, but totally overlooked: Due to mobility and demographic change, housing space always changes hands.

Consider, on one hand, an old family living in a large villa with the children moved out, who would like to move to an inner city apartment, and, on the other, a newly married couple planning to have children and wanting some larger suburban living space.

As we do not live in a barter economy, prices drift up and down depending on how the real estate dealers conceive of the market. Suppose prices are on the move up; then people in city apartments get well paid, and can offer more for a suburban villa, and the villa owners too can afford to move to an attractive city space.

Everybody gets nominally richer, and the upgoing price trend is sustained. Of course, the bank owns most of the wealth, but as long as there are no constraints imposed on the banking system, it functions as a passive credit multiplier, and need not be taken in explicit account.

And, what about speculation? The present author would dare the claim that most people just want to live in their accommodations without too much regard to second hand value.

So why is this never mentioned? The most obvious reason is that economists simply do not have tools to analyze stock markets. The durability of stocks (unlike flows) makes demand and supply functions *change every time actual trade takes place*, so unique market equilibria, the dearest tools to the economist, can never be defined.

Related to this is the lack of a theory for trade in disequilibrium which no doubt is a fact of reality. The housing market always has a shortage or a superfluity of accommodation, but yet exchange does take place, without waiting for Kingdom come when the tatonnement has reached its equilibrium.

It is these two issues we are going to focus in what follows.

Stocks and Flows

Economic quantities can be classified as either stocks or flows, depending on whether they refer to time *points* or time *periods*. The stock of productive capital, the quantity of money, the labour force, inventories, etc. are all examples of stocks. Conceptually they are easier than flows, which assume a periodized background of evolving facts, and hence *imply some process in time*. Such quantities depend on the *length of period* chosen, and if one models evolution in continuous time, then they become something rather abstract, i.e., time derivatives. Examples of flows are commodities traded, hours of labour worked, periodized incomes, savings, investments, etc. The distinction holds on the micro as well as on the macro level.

The tools of economic analysis relate almost exclusively to phenomena for flows. To see this, consider any elementary microeconomics textbook. In one of the opening chapters there is a picture of consumer's preferences displayed as an indifference map. There is a budget line representing the possible choices, whose axis intercepts are the income divided by the respective price. The consumers seek maximum satisfaction, which, given the proper curvature, is obtained at the point of tangency of the budget line with an indifference curve. This explains how the consumers make their choices.

Next step is to see what happens if one price changes. The axes intercept that represents the other price remains fixed, but the budget line changes slope and so rotates around this fixed point, and this rotating movement sweeps out a curve of new points of tangency. To each price corresponds a (unique) quantity demanded, no matter how we rotate the budget line. By associating the demand quantity to the price that represents the slope, one derives a *unique* demand curve which can be aggregated over consumers, and used in the market setting to establish price and quantity traded at equilibrium, where demand equals supply.

As a further step consider a case where the consumer does not have an income in money (for instance a university professor of ancient times getting his salary in terms of quantities of firewood, salted pork, and the like). No problem! The fixed point is not on the axis, but somewhere out in the positive quadrant. If, given the current market prices, he finds he would be better off changing some firewood for more pork, he can do this on the market. Again, considering different relative prices, the budget line rotates and sweeps out a *unique* demand/supply function.

Irreversibility and Hysteresis

However, it is taken for granted that all firewood is burnt and all pork is consumed during the period. So, when the same situation arises anew next period, everything is just repeated. This is the case for flows.

But what if the commodities were durable, remaining from one period to the next? Then, if at some stage the consumers find it profitable to make an actual exchange, this will influence the future. After any such exchange, the budget line would rotate through a *new* point, so sweeping out a *different* demand/supply curve. The process would thus be provided with a memory of the actual trading, and neither demand nor supply functions would be unique. We encounter phenomena of irreversibility and hysteresis, which never show up in elementary economics textbooks; a sign as good as any that main steam economic theory is shaped for flows, not for stocks.

In the sequel, to the purpose of pinpointing the issue, we even take the total supply for given. In the case of the housing market, we thus disregard new production and scrapping due to wear, so total supply is fixed, and what appears as demand and supply on the market is just what individual owners want to sell or buy of different habitation types within given totals. The reader will forgive me for taking the discussion down to such elementary trivialities, but it was necessary for making the point.

Tools developed for flow markets were, however, gladly transferred to markets for stocks, for instance, the demand for money, such an important issue in the monetarism controversy. What if we *cannot define* a unique demand function? This question was never posed, not even by Lord Keynes (1936) in his liquidity demand function.

Disequilibrium Trade

When we are considering apartments and houses changing owners *within* a given total, we enter the other main issue; *disequilibrium trade*. As was noted above, trade as a rule *does* take place even when there is no equilibrium. In general, it is quite tricky to specify how such takes place, i.e., which consumers get fully satisfied and which do not, or only get partial satisfaction from exchange.

However, there is one case that provides us with a clear setting—the old Edgeworth box (Edgeworth 1894), provided we consider only two commodities and two agents. This box was never used in such connection, but its use provides a simple, even visual, start with an otherwise as messy as neglected issue.

The Edgeworth Box

So, imagine the story told above about indifference maps and rotating budget lines, and consider two agents, one's indifference map in normal position, the other rotated 180° and translated to a position such that the horizontal and vertical distances between the two origins equal the (fixed) totals of the commodities available for exchange. Now, the diagram is exactly the Edgeworth box, familiar to economics students from its application in international trade theory. One indifference set is concave, the other convex, and if we consider the intersection of two curves, they form a lens-shaped region in which every point is better for *both* agents than the intersection itself.

The lens collapses to a single point along the curve of points where the indifference curves, one from each set, touch. These are the Pareto efficient optimal points, all candidates for *equilibria*. The relative price ratio, or slope of the budget line, would then have to take the slope of the two touching indifference curves at that point.

But, what if we dealt with stocks? Take any disequilibrium point *not* on this curve of contact, and any announced price ratio. Would trade then be possible even if we do not end up at equilibrium? The answer is yes, quite as in real life, and in the simple setting chosen we can even state the precise conditions for this and for how trade takes place.

Any point in the box can represent an initial distribution of the total wealth of the two agents (equal to the sides of the box). A line through this point with some slope, corresponding to the price ratio, would be the *common* budget line for *both* agents.

The question then is how far on this line the agents would want to move. Sooner or later the budget line would touch an indifference curve from either map. These are the optimal points for the two agents. Any of them might be outside the box, but it does not matter as the process designed will never go outside the box. If the announced price ratio does not correspond to equilibrium, then there are three possibilities.

The optimal points are on the budget line to the same side of the initial point, to the right or left of it. Then one agent would like to exchange more than the other. That agent has no means to force the other to move further than she/he wants, but profits from proceeding even part of the way. Thus, disequilibrium exchange is possible, and profitable for both, but the *limit* is set by the agent wanting to exchange *least*. This results in two cases. In neither case, however, the exchange results in equilibrium.

As a third possibility, the optimum points may be on either side of the initial point. Obviously, then no trade is possible as both agents want to change the same good for another. When setting up the formal model we will go through all this in tedious detail (as six logically possible cases) .

Just give a thought to how much more complicated everything would be if we had three traders or three commodities. We are fortunate that the disequilibrium trade condition were so intuitively easy to set up for this two by two case.

Price Adjustment

If we add some price adjustment mechanism that, for instance, generates relative prices due to excess demand/supply, then the dynamic model is closed. As we will see, we can end up at infinitely many equilibrium points, or at infinitely many *dis*equilibrium points where no more trade is possible, but complicated price dynamics, periodic, quasiperiodic, or chaotic, goes on for ever in a vain search for an equilibrium.

Digression on Ex Ante and Ex Post

These issues go through all of economics. In the wake of the Keynesian macroeconomics, people started collecting data for the actual calculation of national income, which before had been a theoretical construct, such as periodic interest on the national wealth (see Lindahl 1939), and its components—consumption, saving, investment.

Soon the question was posed whether the equality of saving and investment was an equilibrium condition or an accounting identity.

The Stockholm School

The so called "Stockholm School" set out to clear these things up. Obviously, individual agents have plans, and in a modern economy saving (abstaining from consuming) and investment in capital for productive service are different actions dependent on different decisions of the individual agents. If their planned decisions (ex ante) match, then we are in a lucky and rare state of equilibrium.

However, if they do not, then in the national accounts they still balance (ex post). The trick is played by unintended saving by consumers who due to shortage were not able to buy the goods they wanted, and unintended investments in inventories of goods that could not be sold.

The Stockholm School never got further than coining the concepts ex ante and ex post, now absorbed by the entire economics profession. They did not manage to explain how these unintended savings and investments came about.

For the interested reader Palander's critique of all this (Palander 1941, 1953) cannot be too highly recommended. There are about five or six key works in this Stockholm School, but we only cite Myrdal's book (Myrdal 1939), as Palander's extensive article formally is a book review, though it is also a thorough critique of the confusion between stocks an flows and the total lack of any even rudimentary treatment of disequilibrium trade. This digression served to show how deep these issues cut even in macroeconomics. The merit of the Stockholm school was to shed some little light on these issues, especially in view of the fact that we afterwards used flow theory for stock markets and concentrated on equilibria to such an extent that disequilibrium trade was never dealt with.

The Model

Notation

Denote the commodity quantities for the first agent (x, y). As it makes no harm in a pure exchange model, let us normalize the totals of both commodities available on the market to unity. Hence, the corresponding quantities for the second agent are denoted $(1 - x, 1 - y)$. The Edgeworth box thus becomes a unit square. Any actual distribution of assets in denoted in upper case, (X, Y) for the first agent, $(1 - X, 1 - Y)$ for the second.

We only deal with the relative price, so normalize the price of the first commodity to unity, which just becomes a numéraire. The price of the second commodity is denoted p, which hence is a relative price.

Budget Constraints

The budget constraint for the first agent reads

$$x + py = X + pY \tag{12.1}$$

and accordingly for the second agent,

$$(1 - x) + p(1 - y) = (1 - X) + p(1 - Y) \tag{12.2}$$

However the latter is identical with (12.1), which we see if we subtract the expression $(1 + p)$ from both sides of (12.2).

This almost is all notation we need. Let us just call the optimal points to which the agents would like to move (x_1, y_1) and (x_2, y_2) respectively, and we are finished. These two points obviously have to lie on the budget line (12.1) or (12.2), quite as the actual wealth distribution point (X, Y). The points (x_1, y_1) and (x_2, y_2) touch an indifference curve each from the preference map of either agent. All this is illustrated in Fig. 12.1. Obviously the system cannot move from (X, Y) to both optimal points. Below we will discuss to which of them the new wealth distribution point $\left(X', Y'\right)$ actually moves through trade.

Utility Functions

As for utility, take a Cobb-Douglas form for the utility function. Then, for the first agent the utility function is

$$U = x^{\alpha} y^{1-\alpha} \tag{12.3}$$

and for the second

$$V = (1 - x)^{\beta} (1 - y)^{1-\beta} \tag{12.4}$$

Obviously, the Cobb-Douglas exponents must be in the interval $0 < \alpha, \beta < 1$.

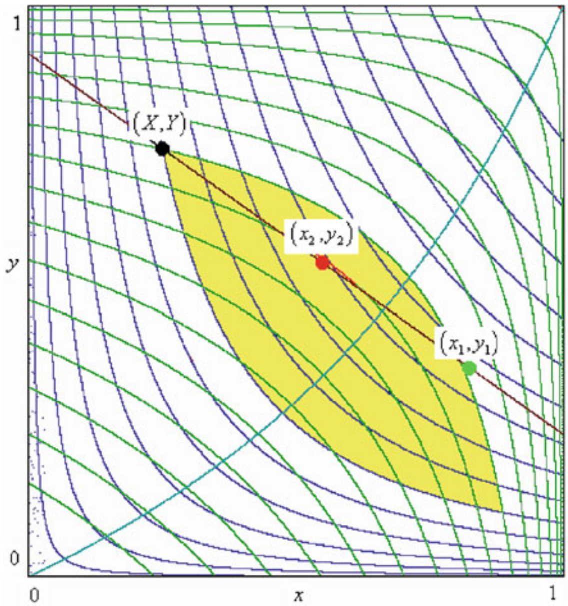

Fig. 12.1 Indifference *curve maps* for $U = x^{\alpha} y^{1-\alpha}$, $V = (1-x)^{\beta}(1-y)^{1-\beta}$, and the *budget line* $x + py = X + pY$. Initial point (X, Y), and the points (x_1, y_1) and (x_2, y_2), to which the agents would like to move, are displayed. Further on display is the *curve* for equilibria, along which the indifference curves touch, and the lense shaped area, where both agents are better off than in the initial point. *Parameter values* $\alpha = 0.6$, $\beta = 0.4$, $X = 0.25$, $Y = 0.75$, $p = 1.5$. Given these, from (12.5) $x_1 = 0.825$, $y_1 = 0.3\overline{66}$, and from (12.6) $x_2 = 0.55$, $y_2 = 0.55$

Individual Optima

The results of maximizing Cobb-Douglas functions, such as (12.3)–(12.4) subject to linear budget constraints, such as (12.1) or (12.2), are well known: Fixed shares of the budget, equal to α, $(1-\alpha)$ for the first agent and β, $(1-\beta)$ for the second are spent on the commodities x, y. The budgets are given by the right hand sides of (12.1) and (12.2) respectively. We only need to recall that for x the price is unity, so for that commodity quantity and value are identical. For commodity y one has to divide the budget share by price p in order to get the quantity demanded.

Hence the desired optima are

$$x_1 = \alpha\,(X + pY)$$
$$y_1 = (1-\alpha)\left(\frac{X}{p} + Y\right) \tag{12.5}$$

for the first agent, and

$$1 - x_2 = \beta\left((1 - X) + p\,(1 - Y)\right)$$
$$1 - y_2 = (1 - \beta)\left(\frac{1 - X}{p} + 1 - Y\right) \tag{12.6}$$

for the second. Note that, in addition to the exponents of the utility functions, (12.5) and (12.6) depend on relative price p and on the actual asset distribution X, Y, or $(1 - X), (1 - Y)$.

As a rule (x_1, y_1) and (x_2, y_2) are different; from each other, and from the actual asset distribution point (X, Y), as we see in Fig. 12.1. Only in one situation are they equal, as stated in the introduction, i.e. on the equilibrium curve which is also illustrated in Fig. 12.1.

This curve can be obtained in different ways. We can put $x_1 = x_2$ and $y_1 = y_2$ in (12.5), (12.6) and eliminate p; or we can calculate the locus of points where the indifference curves, one from each utility map, touch (through calculating and equating the implicit derivatives). The latter is the usual way way used in international trade theory where the equilibrium curve is called "cont(r)act curve". Whatever the procedure chosen, the formula reads

$$Y = \frac{(1 - \alpha)\,\beta X}{\alpha\,(1 - \beta) + (\beta - \alpha)\,X} \tag{12.7}$$

which too is shown in Fig. 12.1. Note that if $\alpha = \beta$ then from (12.7) is a straight line (the diagonal).

The equilibrium relative price p in each point of (12.7) is well defined. It just equals the slope of the touching indifference curves in that point. Also note that along (12.7) there is a nondenumerable infinity of different possible equilibrium points.

Trade

Above it was stated that trade is limited to the *smallest* change that any agent wants to make, because the agent wanting to change more would then also benefit from moving part of the way towards his optimum, whereas he has no means to force the other agent to move further than he wants. Referring to Fig. 12.1, we see that, in the case portrayed, this means moving from (X, Y) to the new wealth distribution point $\left(X', Y'\right) = (x_2, y_2)$, distinguished by a dash.

Once this move has taken place, the budget line would pivot through this *new* point at any further change of the relative price p., thus changing the demand/supply functions.

Supposing price changes from p to p' and then back to p again, the original asset distribution point (X, Y) would *not* be retrieved, because after trading this point would not even be on the new budget line. For this reason one can neither speak of unique demand or supply functions in the case of assets, nor of unique equilibria.

Any asymptotically approached equilibrium state depends on the intervening trading process. As mentioned, there are infinitely many equilibria, and any of these can be the asymptotic state, depending on where the process starts and how the "tâtonnement" for pricing works.

Even worse, as we will see, there are also infinitely many *disequilibrium* fixed points off the curve of equilibria, where the process may stop, as further trade is not possible. This is the nucleus of the present argument.

The trade possibilities can now be classified in six distinct categories, based on how the points (X, Y), (x_1, y_1), and $(x_2.y_2)$ are ordered on the budget line, from left to right. Actually, we can just use either the x or the y coordinate for this ordering, so we go for the first option. This means that the trading map too can be set up in terms of x alone; a deceptive simplicity as the map (12.5)–(12.6) uses both, and so in one additional step brings both coordinates back in.

Six Cases

(i) $x_1 \le x_2 < X$: The first agent wants to sell $(X - x_1)$ but the second is only willing to buy $(X - x_2)$. There can be trade, but it is limited by the buyer's willingness. As a consequence one gets the new trade point $\left(X', Y'\right) = (x_2, y_2)$. There is left an excess supply amounting to $(x_2 - x_1)$, and a corresponding excess demand for the other commodity. Note for future use that for this case $(x_2 - x_1)(x_2 - X) \le 0$.

(ii) $x_2 \le x_1 < X$: The first agent wants to sell $(X - x_1)$ and the second wants to buy $(X - x_2)$, which is more Trade is now limited by the seller's, the first agent's willingness. The new trade point becomes $\left(X', Y'\right) = (x_1, y_1)$. There is left an excess demand amounting to $(x_1 - x_2)$, and a corresponding excess supply of the other commodity. Note that for this case $(x_1 - x_2)(x_1 - X) \le 0$.

(iii) $X < x_1 \le x_2$: The first agent wants to buy $(x_1 - X)$ and the second wants to sell $(x_2 - X)$, which is more. Trade is limited by the buyer's, i.e., the first agent's offer. The new trade point becomes $\left(X', Y'\right) = (x_1, y_1)$. There is left an excess supply amounting to $(x_2 - x_1)$. For this case $(x_1 - x_2)(x_1 - X) \le 0$.

(iv) $X < x_2 \le x_1$: The first agent wants to buy $(x_1 - X)$ and the second wants to sell $(x_2 - X)$, which is less. Trade is limited by the seller's, i.e., the second agent's offer. The new trade point is $\left(X', Y'\right) = (x_2, y_2)$. There is left an excess demand amounting to $(x_1 - x_2)$. For this case $(x_2 - x_1)(x_2 - X) \le 0$.

(v) $x_1 < X < x_2$: The first agent wants to sell $(X - x_1)$ and the second as well wants to sell $(x_2 - X)$. As both want to sell the same commodity, no trade is possible, so $\left(X', Y'\right) = (X, Y)$. There is left an excess supply amounting to $(x_2 - x_1)$. For this case $(X - x_1)(X - x_2) < 0$.

(vi) $x_2 < X < x_1$: The first agent wants to buy $(x_1 - X)$ and the second as well wants to buy $(X - x_2)$. As both want to buy the same commodity, no trade is possible. Again $\left(X', Y'\right) = (X, Y)$. There is left an excess demand amounting to $(x_1 - x_2)$. For this case too $(X - x_1)(X - x_2) < 0$.

The very simple argument is that if x_1 and x_2 are on either side of X, then trade is impossible (cases v and vi), because both agents want to buy/sell the same commodity. If x_1 and x_2 are on the same side of X, then there is one potential buyer and one seller, but the change is limited by the agent who wants to buy or sell least, agent 1 in cases ii and iii, or agent 2 in cases i and iv. The map formulated below thus boils down to three cases $\left(X', Y'\right) = (x_1, y_1)$, $\left(X', Y'\right) = (x_2, y_2)$, and $\left(X', Y'\right) = (X, Y)$, quite as suggested in the introduction.

The Trade Map

It just reads

$$\left(X', Y'\right) = \begin{cases} (x_1, y_1) & \text{if} \quad (x_1 - x_2)(x_1 - X) \le 0 \\ (x_2, y_2) & \text{if} \quad (x_2 - x_1)(x_2 - X) \le 0 \\ (X, Y) & \text{if} \quad (X - x_1)(X - x_2) < 0 \end{cases} \tag{12.8}$$

The application clauses exhaust all logical possibilities and are mutually exclusive as can be easily established. The two first rows of (12.8) represent trade corresponding to the limits set by agent 1 and 2 respectively, whereas the last row represents blocked trade because the agents want to buy and sell the same commodity. The weak inequality signs in the first two branches let us include the equilibria in the map.

Excess Demand

Let us just restate the definitions for the desired optimal points to be used in (12.8), (x_1, y_1) and (x_2, y_2). They were given in (12.5) and (12.6), though it is nicer to solve for y_1 and y_2 in explicit form. Hence

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \alpha (X + pY) \\ (1 - \alpha)\left(\frac{X}{p} + Y\right) \end{pmatrix} \tag{12.9}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 - \beta ((1 - X) + p (1 - Y)) \\ 1 - (1 - \beta)\left(\frac{1-X}{p} + 1 - Y\right) \end{pmatrix} \tag{12.10}$$

This almost completes the iterative map; only one item is missing, the setting of relative price p.

Price Adjustment

The case of two traders may seem to set the stage for bilateral monopoly, but we intend the model as a first stepping stone for generalization to more traders, so it is better to use some excess demand dependent price adjustment function of the type Samuelson suggests (Samuelson 1947). It formalizes Walrasian tâtonnement (Walras 1874–1877). Actually, Walras seems to deal with testing out equilibrium prices which come in effect only when equilibrium is reached, but Samuelson implies a dynamic process where excess demand/supply drives prices or down, with a force dependent on the size of the excess. When price increases due to excess demand, the latter is reduced. These prices seem to be conceived as real transaction prices in a transitory process As the demand and supply functions remain unchanged, we must conclude that a market for flows is intended. However, there seems to be no harm in choosing this most widely used type of price adjustment mechanism also for stock markets.

As we deal with relative price p, we can choose either excess supply on the market for x, or excess demand on the market for y, to trigger rises for the only variable price p, as there is a simple reciprocity between the two in the model. We go for the first alternative. From the six cases listed above we find that excess supply, or if negative, excess demand for x always equals $x_2 - x_1$.

In discrete time a linear price adjustment function could easily lead to negative prices which we want to avoid, so we choose the semilogarithmic,

$$p' = p \exp\left(\delta\left(x_2 - x_1\right)\right) \tag{12.11}$$

where δ denotes an adjustment step length. The choice of the map (12.11) has the advantage of symmetry with respect to the other relative price $1/p$, as the exponent then just changes sign.

The dynamic model we propose consists of (12.8) and (12.11), where (x_1, y_1) and (x_2, y_2) are as defined in (12.9)–(12.10). Despite its simple look, the model seems to be too complicated for further closed form analysis. We can, however, obtain much information through numerical experiment.

Numerical Analysis and Graphics

The Phase Plane

Trade Equilibria and Disequilibria

It is easy to run the map (12.8) and (12.11) with definitions (12.9)–(12.10) on the computer and display the results in a phase plane such as Fig. 12.1. One just needs to set the *parameters*, which are the exponents of the utility functions α, β, and the price adjustment step length δ, and further choose the *initial values* for the asset

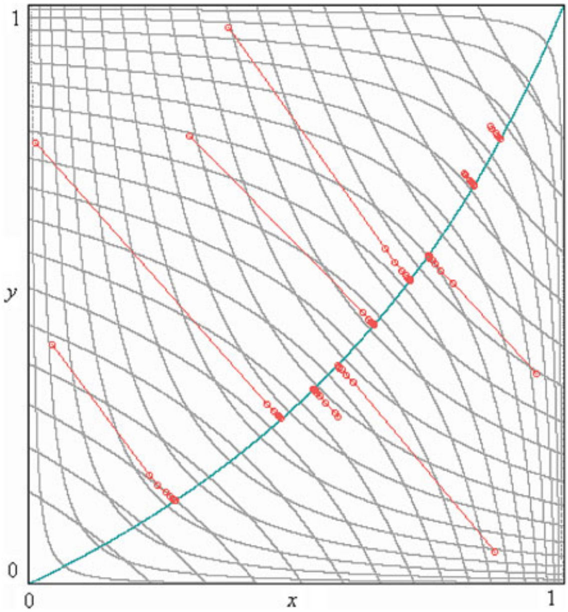

Fig. 12.2 *Orbits* of 10 randomly generated initial points (X, Y) using 100,000 iterations. *Parameters* $\alpha = 0.6, \beta = 0.4, \delta = 1$. Notably, the *orbits* converge very fast to the equilibrium *curve*, though to different points. Initial relative price is set at $p = 1.5$ in all iterations. Though p is adjusted in each iterate according to (12.11), it cannot be excluded that initial p like initial (X, Y) influence the *orbit* and the final equilibrium

distribution (X, Y) as well as for relative price p. Figures 12.2 and 12.3 show the orbits generated from ten randomly chosen initial points (X, Y) in the unit square. As in Fig. 12.1 we keep the parameter values $\alpha = 0.6$, and $\beta = 0.4$, and fix the initial relative price at $p = 1.5$. The system was run in 100,000 iterations in each case. The iterates are indicated by circles joined by line segments representing the jumps. Obviously, the first steps are giant, and the orbits in Fig. 12.2 converge rather fast on the final positions.

The difference between Figs. 12.2 and 12.3 is due to the step size, $\delta = 1$ in Fig. 12.2, $\delta = 5$ in Fig. 12.3. With the smaller step size all orbits converge to the equilibrium curve as displayed in Fig. 12.1, though to different points, thereby illustrating what was said about the dependence of equilibrium upon the dynamic adjustment process.

In Fig. 12.3, some orbits still converge to the equilibrium curve, but some stop at a distance from it. Visually this stopping in disequilibrium fixed points can occur in just few steps; the large number of iterations suggests that the process indeed does not leave these final disequilibrium fixed points

From the discussion above we know what these disequilibria signify—cases where both agents want to buy/sell the same commodity so that no further trading

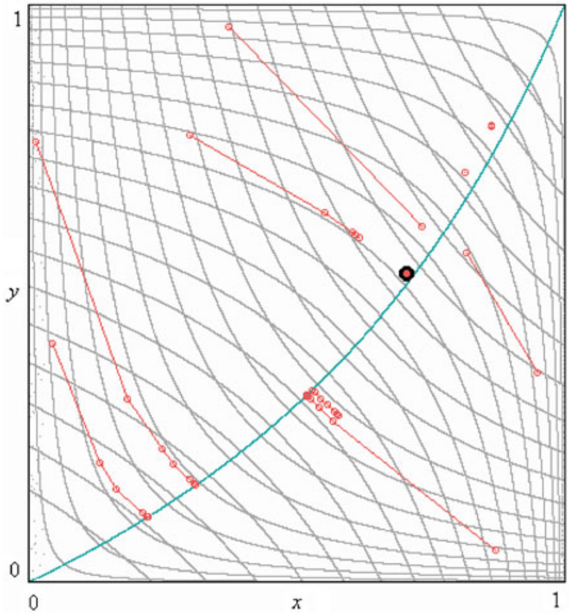

Fig. 12.3 *Orbits* of the same 10 initial points (X, Y) as in Fig. 12.2. Initial relative price is $p = 1.5$. *Parameters* $\alpha = 0.6$, $\beta = 0.4$, but now the step size is increased to $\delta = 5$. Again the *orbits* converge very fast, though to *disequilibrium* fixed points of the dynamic system where no further trade is posssible. One *orbit* which will be studied closer is marked with a *large dot*; this one never leaves the initial state. *Note* that it is not missing in Fig. 12.2, it just merges with another track

is possible. The excess demand triggered price adjustment process simply fails to reach an equilibrium point. Numerical experiment also indicates that in cases where the process seems to end up at the equilibrium curve, it only reaches a disequilibrium point in the close neighbourhood of an unstable equilibrium point.

Note that one of the disequilibrium fixed points in Fig. 12.3 is indicated by a larger dot. It represents a point where the price dynamics will be studied more closely in the sequel.

To get some more information about disequilibrium fixed points, instead of just generating a few initial phase points, as in Figs. 12.3 and 12.4, we next run the process from all initial phase points, packed as close as the resolution admits and mark just the final wealth distribution plane fixed points. As we see, they cover curves and areas in the phase plane. To make the computation manageable the number of iterations for each orbit was reduced to 5,000. The area of fixed points in Fig. 12.4 seems to gather around the equilibrium curve known from Figs. 12.1, 12.2 and 12.3, sometimes thin as a curve, sometimes swelling out to structures with nonzero area measure.

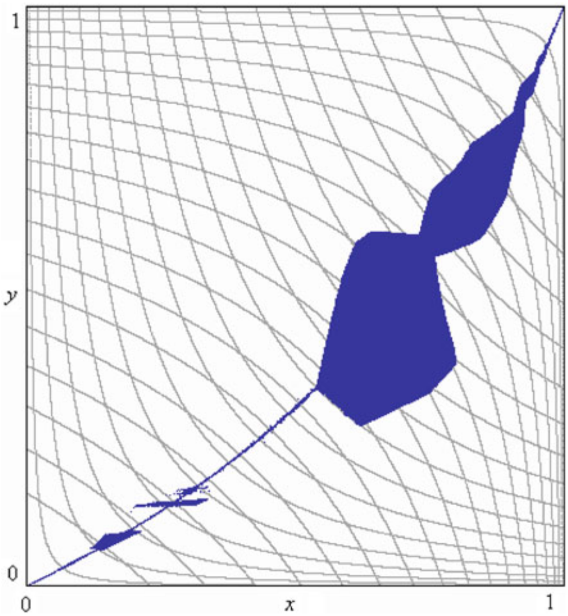

Fig. 12.4 Final fixed points of the *orbits* from all initial points (X, Y) in the *square*. The number of iterations from each initial point was reduced to 5,000. Initial relative price was again $p = 1.5$. *Parameters* $\alpha = 0.6$, $\beta = 0.4$, $\delta = 5$ as in Fig. 12.3. The *disequilibrium* fixed points agglomerate to the neighbourhood of the equilibrium *curve*, but occasionally swell out over considerable areas of the *square*

Price Oscillations

An interesting feature of the model is that the price adjustment process in a disequilibrium fixed point produces continued price dynamics, periodic or aperiodic. This is illustrated in Figs. 12.5, 12.6, 12.7 and 12.8, produced for the parameter combinations $\alpha = 0.6$, $\beta = 0.4$ quite as in Figs. 12.1, 12.2, 12.3 and 12.4. The initial price was again taken as $p = 1.5$, and the initial asset distribution as $(X, Y) \approx (0.74, 0.55)$, corresponding to the large dot indicated in Fig. 12.3, actually one of the randomly generated initial points in Figs. 12.2 and 12.3, from which the process does not take one single step.

Note that from (12.9)–(12.10) $x_2 - x_1 = (1 - \beta) + (\beta - \alpha) X + p (1 - (1 - \alpha) Y)$, which substituted in (12.11) $p' = p \exp(\delta (x_2 - x_1))$, gives an autonomous iterative map $p \to p'$, whenever (X, Y) is fixed, as it is in any disequilibrium fixed point.

The parameter that takes on different values in this series of illustrations is the step size parameter; $\delta = 5$ in Fig. 12.5, $\delta = 5.1$ in Fig. 12.6, $\delta = 5.2$ in Fig. 12.7, and $\delta = 5.25$ in Fig. 12.8. These pictures display the indifference maps and equilibrium curve in the phase plane; further the disequilibrium fixed point (X, Y) and a number

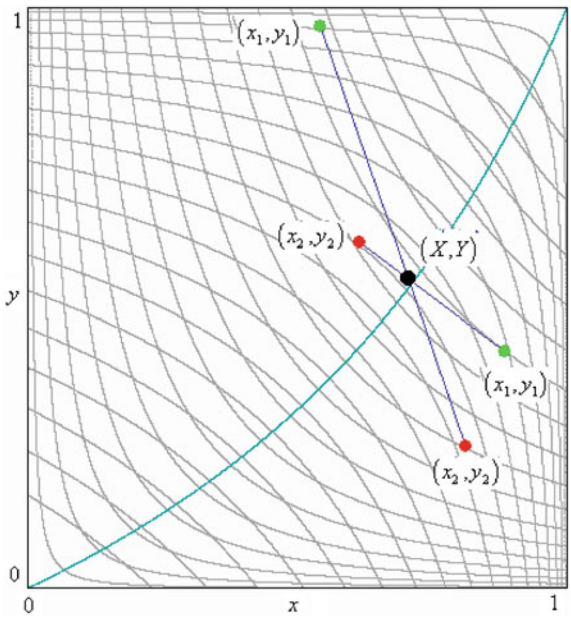

Fig. 12.5 2-period relative price oscillation. Fixed disequilibrium point (X, Y) as indicated by the *large dot*in Fig. 12.3, with initial relative price $p = 1.5$. *Parameters* $\alpha = 0.6$, $\beta = 0.4$, and $\delta = 5$. Shown are flipping *budget line* segments with endpoints (x_1, y_1) and (x_2, y_2). *Note* that they swap their positions relative to (X, Y)

of optimum points for the agents, (x_1, y_1) and (x_2, y_2). These come in pairs and are joined by line segments, two in Fig. 12.5, four in Fig. 12.6, and six in Fig. 12.7. Note that all line segments pass the point (X, Y). These line segments are actually segments of the budget lines.

In Fig. 12.8 the line segments are deleted and the end point pairs crowd dense along curves. These curves can be obtained in closed form through eliminating p in (12.5) and (12.6) respectively,

$$y_1 = \frac{(1-\alpha)\, Y x_1}{x_1 - \alpha X}, \tag{12.12}$$

$$y_2 = \frac{(1-\beta)\,(1-Y)\, x_2}{x_2 - \beta\,(1-X)} \tag{12.13}$$

These curves have been superposed on the numerically calculated trains of budget segment endpoints in order to show that this indeed is so.

The mechanism can be explained referring to Fig. 12.5. The relative price p oscillates between two different values and so the budget line flips between two different slopes. The endpoints (x_1, y_1) and (x_2, y_2), optima for the agents, are always on

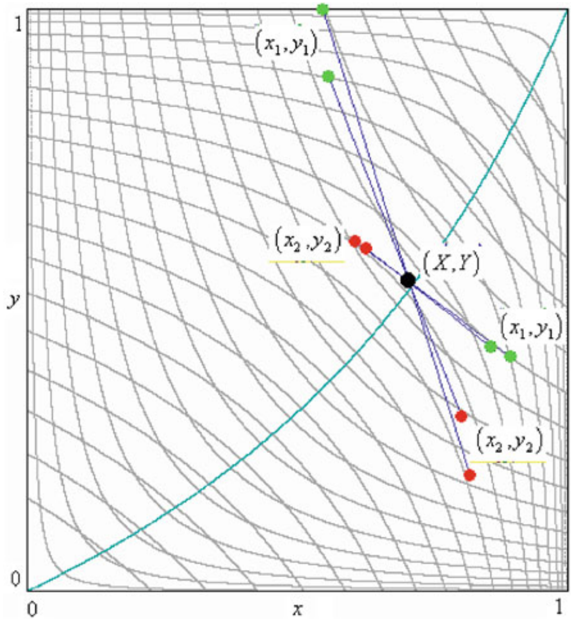

Fig. 12.6 The same case as in Fig. 12.5 but with a 4-period relative price oscillation when $\delta = 5.1$

either side of the fixed point (X, Y), but they switch positions when relative price oscillates; at one value both want to sell, at the other both want to buy the same commodity. Excess demand and excess supply alternate and the adjustment process for pricing always overshoots equilibrium. In Fig. 12.6 this 2-period oscillation has changed to a 4-period, in Fig. 12.7 to a 6-period, and in Fig. 12.8 to something aperiodic. Again the budget line flips between the different positions, and the endpoints swap their positions so that there is always excess demand or supply of the same commodity.

Bifurcation Diagrams

Step Size Bifurcations

The relative price dynamics displayed in Figs. 12.5, 12.6, 12.7 and 12.8 can be summarized by the bifurcation diagram shown in Fig. 12.9. We now display p versus δ. The initial asset distribution point (X, Y) as well as the initial relative price p were kept to the fixed values used in Figs. 12.5, 12.6, 12.7 and 12.8, as were the parameters α, β. At each value of $\delta \in [4, 6]$ the system was run for 10,000 iterations. The first 9,000 were trashed in order to get rid of transients, and the last 1,000 were then plotted. If there is a fixed point then the same p will eventually be hit over and over.

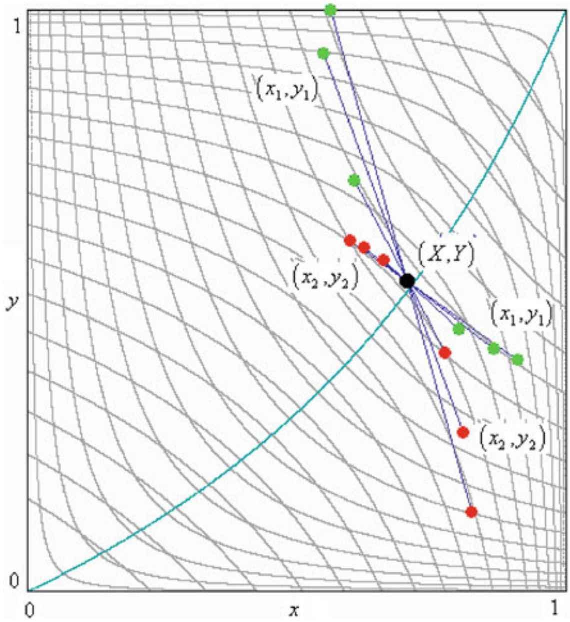

Fig. 12.7 6-period *orbit* when step size $\delta = 5.2$

We just see a point, or, considering different adjacent δ producing fixed points, a line or curve. Once the fixed point bifurcates to a 2-period cycle we see the curve split in 2 branches, and so on in a cascade, eventually seeming to cover entire areas.

We see that the case of $\delta = 5$ shown in Fig. 12.5 fits into the 2-branch region, whereas at $\delta = 5.1$ shown in Fig. 12.6, there has been a further period doubling to 4. Then, after a stretch of (possibly chaotic) intervals, at $\delta = 6.2$ there are clearly 6 curve branches complying with Fig. 12.7. For $\delta = 5.25$ a dense vertical stretch is shown in accordance with Fig. 12.8.

Bifurcations in the Utility Coefficient Plane

A different bifurcation diagram in parameter plane is produced in Fig. 12.10. Again we deal with the unit square, but now it is parameter space α, β and not phase space that is concerned. As we see the dominant shade is labelled 1, indicating fixed points. In the lower left corner there appears an irregularly concentric structure of periodicity "tongues" of a period adding appearance; 1, 2, 3, 4, 5, 6, with large gaps between indicating more complex dynamics.

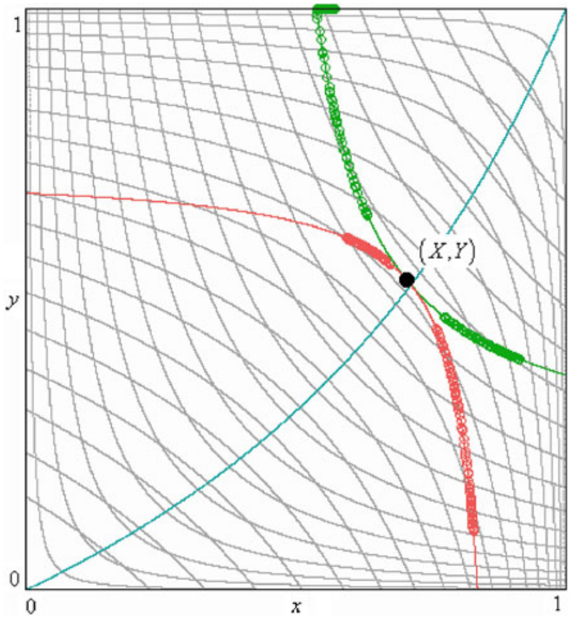

Fig. 12.8 When step size is increased to $\delta = 5.25$, the simple periodicity of relative price oscillation disappears. The points (x_1, y_1), (x_2, y_2) crowd densely on the *curves* (12.12) and (12.13), or on the edges of the box

Summary

To sum up, we suggested a unified model of stock market dynamics, the clue to which was the simple fact that stock commodities, unlike flow commodities, remain on the market from period to period. Through trade these are redistributed among the agents, which, however, changes the basis for future plans and actions. Due to this, unique demand/supply functions and market equilibria do not exist as they do in the case of flow commodities. *If* the system goes to an equilibrium, there are infinitely many to choose from, and the one on which it converges depends on the dynamic process itself. The system can also stick in *dis*equilibrium states from which it cannot move because the agents always want to sell or buy the same commodities.

Notable is that trade occurs in *dis*equilibrium states; the agents move towards higher satisfaction, but not all can reach their desired optima. In the simple two agent two commodities model, trade was limited to what the agent wanting to trade least in an actual asset distribution was willing to exchange. In this way the agent wanting to trade more could get part of the way to higher utility, lacking possibilities to force the other to exchange more than she/he wants.

Prices were assumed to be excess demand driven, and could overshoot unstable equilibrium points, resulting in complex price dynamics, periodic or aperiodic.

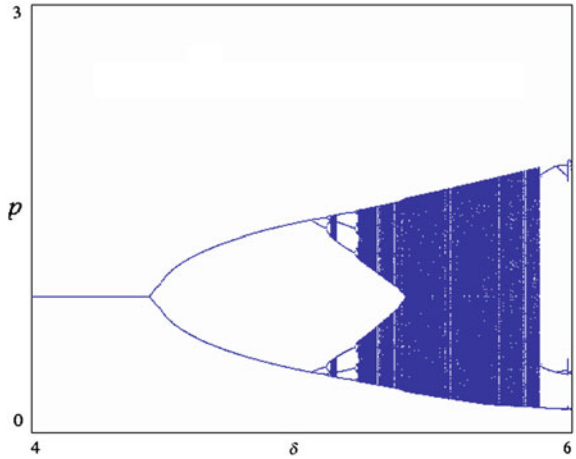

Fig. 12.9 Bifurcation diagram showing eventual relative price oscillations, periodic and aperiodic, as dependent on the step size parameter δ

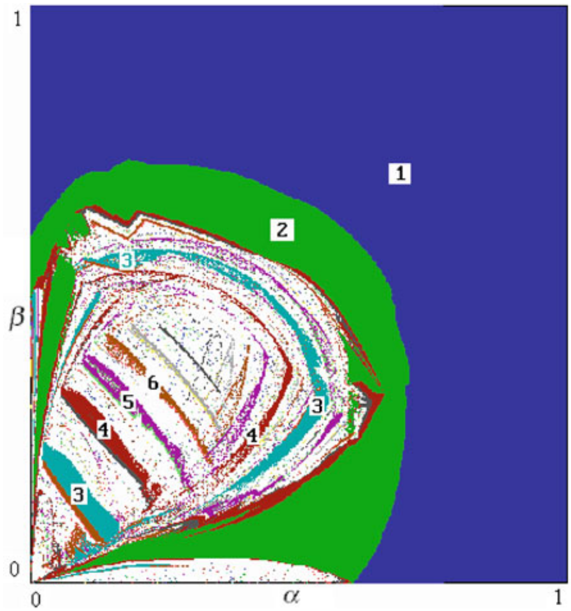

Fig. 12.10 This *picture* shows the bifurcation diagram in α, β parameter plane. The remaining parameter was fixed at $\delta = 5$, and an initial point in phase space $X = 0.25$, $Y = 0.75$ was chosen. For each combination of α, β, the system was run for 5,000 itertions after which the program checked for periodicities 1–15

A challenge to economists would be to set up a model with three (or more) agents, and three (or more) commodities.

The case for three commodities could be dealt with in a solid Edgeworth cube, with budget planes involving two relative prices. Trade possibilities could be studied in terms of the indifference surface projections on such budget planes, though it is no longer so obvious how to specify the conditions for trade. Likewise, things are much more complicated with three agents, as we need further assumptions on the success of the competitors. If there is excess demand and only one supplier, then it is clear that the supplier gets what she/he wants, but as for the demanders we must state who will come out more successful, and likewise for the other (now more than six) cases.

It seems to be important to make some advance on this neglected issues. It also is important to check the price generating process, which, after all, is responsible for the complex dynamic with overshooting. We took the traditional case of prices automatically dependent on excess demand/supply, but more realistic hypotheses concerning price formation would be highly desirable. This is as much neglected in economic theory as is disequilibrium trade.

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