Chapter 4
Arrow and the Aggregation of Fuzzy Preferences

Abstract. This chapter builds off of chapter 3 by examining the aggregation of fuzzy weak preference relations in order to determine how a social preference relation emerges. Specifically, this chapter focuses on Arrow's theorem which employs a deductive analysis of aggregation rules and establishes five necessary conditions for an ideal aggregation rule. When Arrow's theorem is applied with fuzzy preferences, not only do serious complications arise when conceiving the fuzzy definitions of an ideal aggregation rule, but there exist specific combinations of conditions that allow for a fuzzy aggregation rule to satisfy all of the fuzzy counterparts of Arrow's conditions. Moreover, this chapter shows that a fuzzy aggregation rule exists which satisfies all five Arrowian conditions including non-dictatorship.

Introduction

Chapter 3 detailed the underlying structure of FWPRs and the complications that arise when trying to incorporate the logic of exact preferences into the fuzzy framework. Essentially, there is no obvious one-to-one procedure that fuzzifies the underlying assumptions of a rational preference relation. Among these complications, there exist several methods for extracting a fuzzy choice set, and there is little guarantee that these methods will return equivalent results. However, a proper specification of the fuzzy maximal set, along with other characteristics of an FWPR, identifies obvious best outcomes that should emerge given a preference relation of an individual or a collective body. Yet in the case of social preference relations, it is very unlikely that one will be specified *a priori,* without the use of some social welfare function relating individual preferences, i.e. those belonging to voters, committee members or legislators, to those of a social relation. Even if such an example exists, the applications of the various fuzzy maximal sets can be done without complication. Thus, it is worthwhile to consider situations where individual FWPRs are aggregated to form a fuzzy social preference relation.

The goal of this chapter is to examine aggregation of FWPRs in order to determine how a social preference relation emerges. In doing so, it focuses on a classic result in social choice theory: Arrow's Theorem (1951). Because the number

M.B. Gibilisco et al., *Fuzzy Social Choice Theory*,
Studies in Fuzziness and Soft Computing 315,
DOI: 10.1007/978-3-319-05176-5_4,

of aggregation rules is quite large and considering each aggregation individually can become quite tedious, Arrow employs a deductive analysis of aggregation rules and establishes five requiste conditions of an ideal rule that possess inherit trade offs. More simply, if an aggregation rule posssesses four of the five coniditions, it must violate the fifth, thereby demonstating the impossibility of an ideal aggregation rule. Nonetheless, these traditional results rely on exact preferences. When the formal logic of Arrow's theorem is extended into the fuzzy framework, not only do serious complications arise when conceiving the fuzzy definitions of an ideal aggregation rule, but there exist specific combinations of conditions that allow for a fuzzy aggregation rule to satisfy all of the fuzzy counterparts of Arrow's conditions.

The chapter is organized as follows. The first section introduces the classic results of Arrow's theorem and then proposes several fuzzifications of the original five conditions. Next, Section 2 presents the formal proof of fuzzy Arrow's theorem and demonstrates under what conditions a fuzzy aggregation rule will satisfy the five criteria proposed in Section 1. Finally, Section 3 concludes the chapter with a discussion on the empirical applications of fuzzy aggregation.

4.1 Fuzzifying Arrow's Conditions

This section lays out the preliminary definitions used in Arrow's formal considerations of aggregation rules. To do so, we use the following notation. Let $N = \{1,\dots,n\}$ be a finite set of individuals where $n \geq 2$. As in Chapter 3, X is a finite set of alternatives such that $3 \leq |X|$. Throughout the chapter, each individual i is assumed to possess an FWPR, $\rho_i \in \mathscr{F}(X^2)$, such that ρ_i is reflexive and complete. In this case, we call ρ_i a *fuzzy weak order*.[1]

Let $\mathscr{FR}$ denote the set of all fuzzy weak orders on X. Then a *preference profile* is an n-tuple of fuzzy weak orders, $\bar{\rho} = (\rho_1,\dots,\rho_n) \in \mathscr{FR}^n$ and describes the fuzzy preferences of all individuals. Throughout, we manipulate the consistency conditions concerning the weak orders of individuals. When doing so, we will write "assume $\bar{\rho}$ satisfies a particular consistency condition" or "suppose ρ_i is max-$*$ transitive for all $i \in N$." Finally, our definitions related to FPAR's are written generally (that is, with domain $\mathscr{FR}^n$), but our results often assume that these definitions reflect the transitivity restrictions when appropriate.

For any non-empty $S \subseteq X$, let $\bar{\rho}\rceil_S = (\rho_1|_{S\times S},\dots,\rho_n|_{S\times S})$. In words, $\bar{\rho}\rceil_S$ denotes the restriction of the preference profile to the subset $S \times S$ and, accordingly, $\bar{\rho}\rceil_S$ describes only $\rho(x,y)$ and $\rho(y,x)$ for $x,y \in S$ and every $i \in N$. In addition, for any FWPR ρ and all $\alpha \in [0,1]$, $\rho^\alpha = \{(x,y) \in X \times X \mid \rho(x,y) \geq \alpha\}$. Often, ρ^α is called the *α-cut* of ρ.

Finally, for all $\bar{\rho} \in \mathscr{FR}^n$ and $x,y \in X$,

$$R(x,y;\bar{\rho}) = \{i \in N \mid \rho_i(x,y) > 0\}$$

[1] Fuzzy weak orders usually possess some consistency or transitivity condition. However, throughout this chapter, we vary these types of assumptions. The more general definition given here permits us to do so.

and

$$P(x,y;\bar{\rho}) = \{i \in N \mid \pi_i(x,y) > 0\}.$$

In words, $R(x,y;\bar{\rho})$ denotes the collection of individuals who view x as at least as good as y to some degree and $P(x,y;\bar{\rho})$ the collection of individuals who strictly prefer x to y to some degree.

Definition 4.1. A function $\tilde{f} : \mathscr{FR}^n \to \mathscr{FR}$ is called a *fuzzy preference aggregation rule*.

Hence, a fuzzy preference aggregation rule (FPAR) relates a $\bar{\rho} \in \mathscr{FR}^n$ to a social preference relation $\tilde{f}(\bar{\rho}) \in \mathscr{FR}$. When this occurs, $\tilde{f}(\bar{\rho})(x,y)$ represents the degree to which society, or more specifically the set of N actors, views x as at least as good as y. Obviously, this encompasses the exact case where $\tilde{f}(\bar{\rho})(x,y) \in \{0,1\}$. At times, we suppress the $\tilde{f}(\bar{\rho})$ and let ρ denote the social preference relation. In this manner, we can derive ρ's components ι and π, which correspond to the social fuzzy indifference and social fuzzy strict preference relations, respectively. Furthermore, we will at times restrict FPAR's to particular domains of fuzzy weak orders that satisfy consistency conditions. For example, we may assume ρ_i is weakly transitive for all $i \in N$. Then we analyze $\tilde{f} : D_w^n \to \mathscr{FR}$, where D_w is the set of all weakly transitive fuzzy weak orders. While this may appear to be an unnecessary technical complication, the intent is to illustrate the consequences of various types of consistency conditions without needless notation to redefine FPAR's in every case. With this in mind, we assume that any FPAR has an *unrestricted domain*. That is, an FPAR must assign a social preference relation to every fuzzy preference profile with the consistency condition under consideration regardless of the specific combination of the indvidual ρ_is. Unrestricted domain is fairly innocuous because the assumption allows individuals to choose any fuzzy weak order in $\mathscr{FR}$. In democratic terms, the aggregation rule does not require individuals to possess certain types of opinions about the possible alternatives. The understanding of an FPAR in Definition 4.1 allows for a greater variety of aggregation rules than that of exact rules.

Example 4.2. Let $\bar{\rho} \in \mathscr{FR}^n$. Then the following are examples of fuzzy preference aggregation rules:

(1) For all $x,y \in X$,

$$\rho(x,y) = \frac{1}{n}\sum_{i=1}^{n} \rho_i(x,y),$$

(2) For all $x,y \in X$ and any $\beta \in (0,1)$,

$$\rho(x,y) = \begin{cases} 1 & \text{if } \rho_i(x,y) \geq \rho_i(y,x),\ \forall i \in N, \\ \beta & \text{otherwise,} \end{cases}$$

(3) For all $x,y \in X$,

$$\rho(x,y) = \max_{i \in N}\{\rho_i(x,y)\}.$$

It is easily verified that ρ is complete and reflexive in all three cases.[2]

Arrow's seminal work lays out five requisite and incompatible conditions for preference aggregation. The original conditions are

- universal admissibility,
- non-negative monotonicity,
- independence of irrelevant alternatives,
- non-imposition and
- non-dictatorship.

Efforts to dismiss the relevance of the theorem outright (Little, 1952) were followed by attempts to replace certain original conditions. For example, some studies eliminated non-negative monotonicity (Blau, 1972; Inada, 1955) while others replaced it with positive responsiveness (Black, 1969; Fishburn, 1974; May, 1952). The ultimate result of these reinterpreations was a simpler form of Arrow's theorm by Blau (1972) that is generally accepted by contemporary scholars (Austen-Smith and Banks, 1999). In this form, any preference aggregation rule that is transitive, weakly Paretian and independent of irrelevant alternatives must be dictatorial. In the remainder of this section we discuss these terms further and provide several definitions of their fuzzy counterparts.

4.1.1 Transitivity

There are several fuzzy consistency conditions that correspond to transitivity in the traditional sense of determing how FWPRs behave across pairwise comparisons. In the fuzzy Arrow literature, the most pervasive approach is the use of some specific form of max-star transitivity. The definition can be used to derive an infinite number of transitivity conditions and few tudies consider the general condition of max-star transitivity (Duddy et al., 2011; Fono and Andjiga, 2005; Fono et al., 2009). However, the most common definitions make use of the Gödel (minimum) and Łukasiewicz t-norm (Banerjee, 1994; Dutta, 1987; Ovchinnikov, 1991; Richardson, 1998). In these two cases, for all $x,y,z \in X$ and $\rho \in \mathscr{F}(X^2)$, $\rho(x,z) \geq \min\{\rho(x,y), \rho(y,z)\}$ or $\rho(x,z) \geq \rho(x,y) + \rho(y,z) - 1$, respectively. As Duddy, Perote-Peña and Piggins (2007) demonstrate, designating a specific t-norm for max-star transitivity has important consequences on whether Arrow's conclusions hold in the fuzzy frame work. Hence, it is important to consider a variety of consistency defintions. In one of the first applications of fuzzy sets to Arrow's theorem, Barrett, Pattanaik and Salles (1992) propose the following for asymmetric preferences.

Definition 4.3. Let ρ be a complete and reflexive FWPR and let π be its asymmetric component.

[2] Example 4.2(1) was first proposed by Skala (1978) and is now standard in the fuzzy social choice literature, 4.2(2) comes from Dutta (1987), and 4.2(3)'s first application to Arrow's theorem can be found in Fung and Fu (1975).

(1) **(partially transitive)** ρ is said to be *partially transitive* if, for all $x,y,z \in X$, $\rho(x,y) > 0$ and $\rho(y,z) > 0$ implies $\rho(x,z) > 0$,
(2) **(partially quasi-transitive)** ρ is said to be *partially quasi-transitive* if, for all $x,y,z \in X$, $\pi(x,y) > 0$ and $\pi(y,z) > 0$ implies $\pi(x,z) > 0$.

The relationship in Definition 4.3(1) relates to a special case of max-$*$ transitivitiy where $*$ has no zero divisors. In this case, for three alternatives $x,y,z \in X$, $\rho(x,y) > 0$ and $\rho(y,z) > 0$ implies $\rho(x,y) * \rho(y,z) > 0$. In addition, Definition 4.3(2) strengthens the condition of acyclicity in Definition (3). Specifically, partial quasi-transitivity requires not only $\pi(z,x) = 0$, as in acyclicity, but also $\pi(x,z) > 0$ when $\pi(x,y) > 0$ and $\pi(y,z) > 0$ for all $x,y,z \in X$. An application of partial quasi-transitivity can also be found in Dasgupta and Deb (1999). In a similar manner, we can define consistency conditions of fuzzy aggregation rules, which, like FWPRs, possess more or less strictness.

Definition 4.4. Let $\tilde{f}$ be an FPAR.

(1) **(max-star transitive)** $\tilde{f}$ is said to be *max-$*$ transitive* if, for all $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$, $\tilde{f}(\bar{\rho})$ is max-$*$ transitive,
(2) **(weakly transitive)** $\tilde{f}$ is said to be *weakly transitive* if, for all $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$, $\tilde{f}(\bar{\rho})$ is weakly transitive,
(3) **(partially quasi-transitive)** $\tilde{f}$ is said to be *partially quasi-transitive* if, for all $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$, $\tilde{f}(\bar{\rho})$ is partially quasi-transitive,
(4) **(partially acyclic)** $\tilde{f}$ is said to be *partially acyclic* if, for all $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$, $\tilde{f}(\bar{\rho})$ is partially acyclic.

Definition 4.4 presents the consistency conditions of fuzzy aggregations rules used in this text, but there are other consistency conditions previously explored in the fuzzy Arrow literature, which we will not focus on because they have already been explicated in the existing literature. These other conditions include *minimal transitivity*, i.e. $\min\{\rho(x,y), \rho(y,x)\} = 1$ implies $\rho(x,z) = 1$ for all $x,y,z \in X$, and *negative transitivity*, i.e. $\pi(x,y) > 0$ implies $\max\{\pi(x,z), \pi(z,y)\} > 0$ for all $x,y,z \in X$, the contrapositive of which is called *positive transitivity* Fono et al. (2009); Fung and Fu (1975); Richardson (1998).

4.1.2 Weak Paretianism

Weak Paretianism, as the name suggests, determines how an FPAR will behave when every actor in society holds a certain preference between two alternatives. In the exact case, an aggregation rule is weakly Paretian if, for two possible alternatives x and y, every $i \in N$ strictly prefers x to y then the social preference must prefer x to y (Austen-Smith and Banks, 1999; Blau, 1972). In this sense, weak Paretianism has little to say about the final social preferences if all actors possess the same *weak* preferences between two alternatives or if all actors in $N \backslash \{i\}$ strictly prefer x to y, but individual i is indifferent between the two. Weak Paretianism in the fuzzy context, often called the "Pareto Condition", has a fairly uniform definition across the fuzzy literature (Banerjee, 1994; Barrett et al., 1992; Dasgupta and Deb, 1999; Dutta, 1987; Fono et al., 2009; Fung and Fu, 1975; Richardson, 1998).

Definition 4.5 (Pareto Condition). Let $\tilde{f}$ be an FPAR. Then $\tilde{f}$ is said to satisfy the *Pareto Condition* if, for all $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$ and $x, y \in X$, $\pi(x,y) \geq \min_{i \in N}\{\pi_i(x,y)\}$.

Of course, derivations from Definition 4.5 exist in the fuzzy literature. Examples include the *strict Pareto Condition* where $\min_{i \in N}\{\pi_i(x,y)\} = 1$ implies $\pi(x,y) = 1$ for all $x, y \in X$ (Ovchinnikov, 1991) and *unanimity*, which, for all $x, y \in X$ and $t \in [0,1]$, requires $\rho(x,y) = t$ if $\rho_i(x,y) = t$ for all $i \in N$ (Duddy et al., 2011; García-Lapresta and Llamazares, 2000). In addition, when formal arguments do not require constructing a fuzzy strict preference relation, Definition 4.5 can be applied to FWPRs (Duddy et al., 2011; Perote-Peña and Piggins, 2007). To better explicate fuzzy Arrow's theorem, we also consider a weaker assumption than the Pareto Condition that was first proposed by Mordeson and Clark (2009).

Definition 4.6 (weakly Paretian). Let $\tilde{f}$ be an FPAR. Then $\tilde{f}$ is said to be *weakly Paretian* if, for all $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$ and $x, y \in X$, $\min_{i \in N}\{\pi_i(x,y)\} > 0$ implies $\pi(x,y) > 0$.

Obviously, Definition 4.6 relaxes Definition 4.5 because Definition 4.6 no longer restricts the social strict preference between the two alternatives to a more specific alpha level. Nonetheless, both definitions correspond to weak Paretianism in the exact case because $\min\{\pi_i(x,y)\} > 0$ implies $\min\{\pi_i(x,y)\} = 1$, which, under a weakly Paretian aggregation rule, implies $\pi(x,y) = 1 \geq \min\{\pi_i(x,y)\}$, $i \in N$. It is still important to distinguish between these two definitions because, as discussed in a subsequent section, there is an important relationship between these conditions and the types of FPARs that satisfy all Arrowian conditions. Example 4.7 illustrates some basic differences between the conditions.

Example 4.7. Let $\tilde{f}$ be an FPAR and $X = \{a, b\}$. Suppose $\bar{\rho}$ is reflexive and defined as follows:

$$\rho_i(a,b) = .5$$
$$\rho_i(b,a) = .3$$
$$\pi_i(a,b) = .2$$

for all $i \in N$. If $\tilde{f}$ is unanimous, then the social weak preference, ρ, will be $\rho(a,b) = .5$ and $\rho(b,a) = .3$. If $\tilde{f}$ satisfies the Pareto Condition, the social strict preference relation, π, will be $\pi(a,b) \geq .2$. Finally, if $\tilde{f}$ is weakly Paretian, $\pi(a,b) > 0$. Notice the Pareto Condition and weak Paretianism do not guarantee any specific value of $\rho(a,b)$ or $\rho(b,a)$; however, assuming that the social strict preference relation is regular, all three conditions ensure that $\rho(a,b) > \rho(b,a)$.

4.1.3 Independence of Irrelevant Alternatives

Unlike some of the other Arrowian conditions, independence of irrelevant alternatives is less normatively democratic, i.e. where the FPAR responds to some conditions of the preference profile, and more technically desirable. In theory, an aggregation rule satisfies the independence of irrelevant alternatives conditions if the social preference between x and y is solely determined by individuals' preferences between x and y. According to Austen-Smith and Banks (1999), the traditional independence criterion implies two requirements:

(1) the social preference between two alternatives is specifically determined by individual preferences between two alternatives and
(2) cardinal and relative information contained in indivdual preferences is unrelated to the societal preference.

In other words, these requirements stipulate that each individual can produce a ranked list of the alternatives, including ties, and that the aggregation rule only considers the ordinal relationship between x and y when determining the social preference. Information such as x is four alternatives higher in the preference ranking than y or x is 2.5 times more preferred than y becomes trivial. In the fuzzy framework, the literature has most frequently relied on one definition for independence of irrelevant alternatives Banerjee (1994); Barrett et al. (1992); Duddy et al. (2011); Fono and Andjiga (2005); Fono et al. (2009); García-Lapresta and Llamazares (2000); Ovchinnikov (1991); Richardson (1998).

Definition 4.8 (IIA-1). Let $\tilde{f}$ be an FPAR. Then $\tilde{f}$ is said to be *independent of irrelevant alternatives, type 1* (IIA-1), if for all $\bar{\rho}, \bar{\rho}' \in \mathscr{FR}^n$ and all $x, y \in X$, $\rho_i(x,y) = \rho_i'(x,y)$ for all $i \in N$ implies $\tilde{f}(\bar{\rho})(x,y) = \tilde{f}(\bar{\rho}')(x,y)$.

In terms of the two previously discussed criteria, Definition 4.8 certainly satisfies the first condition where $\tilde{f}(\bar{\rho})(x,y)$ is only related to $\bar{\rho}\rceil_{\{x,y\}}$ because the values of $\rho(w,z)$ are left undefined for all $w \neq x$ and $z \neq y$. However, IIA-1 does not faithfully reproduce the second condition of ordinality where the strength of an actor's preference for one alternative over another becomes arbitrary.

One recent effort to reconsider a fuzzy version of the independence condition appears in Mordeson and Clark (2009) where the support of fuzzy preference relations is used.

Definition 4.9 (IIA-2). Let $\tilde{f}$ be an FPAR. Then $\tilde{f}$ is said to be *independent of irrelevant alternatives, type 2* (IIA-2), if for all $\bar{\rho}, \bar{\rho}' \in \mathscr{FR}^n$ and $x, y \in X$, $\text{Supp}(\rho_i\rceil_{\{x,y\}}) = \text{Supp}(\rho_i'\rceil_{\{x,y\}})$ for all $i \in N$ imples $\text{Supp}(\tilde{f}(\rho_i)\rceil_{\{x,y\}}) = \text{Supp}(\tilde{f}(\rho_i')\rceil_{\{x,y\}})$.

Definition 4.9 certainly captures some aspects of the ordinal quality of the crisp independence condition. In words, if there exist two profiles $\bar{\rho}, \bar{\rho}' \in \mathscr{FR}^n$ such that, when restricted to two alternatives x and y, the supports of the individual fuzzy weak orders in $\bar{\rho}$ are identical to those in $\bar{\rho}'$, then the support of the two social preference relations generated by an IIA-2 FPAR should be identical as well, regardless of the relationship between the other alternatives and regardless of the specific values for $\rho_i(x,y)$ and $\rho_i(y,x)$. However, constructing the independence condition in this manner offers no guarantee that the relationship between $\tilde{f}(\bar{\rho})(x,y)$ and $\tilde{f}(\bar{\rho})(y,x)$ will be preserved in the fuzzy social preference relation generated by $\tilde{f}(\bar{\rho}')$. This can have important consequences when constructing a social strict preference relation as the following example demonstrates.

Example 4.10. Let $X = \{x,y\}$ and let $\bar{\rho}, \bar{\rho}' \in \mathscr{FR}^n$. Suppose the fuzzy social preference relations derived from $\bar{\rho}$ and $\bar{\rho}'$, denoted ρ and ρ', respectively, are derived as follows:

$$\rho(x,y) = \rho'(y,x) = .5,$$
$$\rho(y,x) = \rho'(x,y) = .2.$$

Obviously, $\text{Supp}(\rho) = \text{Supp}(\rho')$. However, if we were to construct a fuzzy social strict preference relation by assuming that social strict preference relations, π and π', are regular, then $\pi(x,y) > 0$ and $\pi'(y,x) > 0$.

Example 4.10 begs the question: How truly similar are two preferences relations when their supports are identical? If we are also interested in creating a social strict preference, then we may want to consider an independence condition that maintains the ordinal relationships between two FWPRs. Such a definition is proposed by Billot (1992), which has remained largely overlooked in the literature. Before proceeding, we need the following definition.

Definition 4.11 (equivalent). Let $\rho,\rho' \in \mathscr{F}(X^2)$ and let $Im(\rho) = \{s_1,\ldots,s_m\}$ and $Im(\rho') = \{t_1,\ldots,t_n\}$ be such that $s_1 < \ldots < s_m$ and $t_1 < \ldots < t_n$. We then say ρ and ρ' are *equivalent*, written $\rho \sim \rho'$, if and only if

(1) $s_1 = 0 \iff t_1 = 0$,
(2) $n = m$,
(3) $\rho^{s_i} = \rho'^{t_i}$, for all $i = 1,\ldots,m$.

Using this concept of analogous preference relations, we can model a third variant of the independence condition in the manner of Billot (1992).

Definition 4.12 (IIA-3). Let $\tilde{f}$ be an FPAR. Then $\tilde{f}$ is said to be *independent of irrelevant alternatives, type 3* (IIA-3), if for all $\bar{\rho}, \bar{\rho}' \in \mathscr{FR}^n$ and $x,y \in X$, $\rho_i\rceil_{\{x,y\}} \sim \rho_i'\rceil_{\{x,y\}}$ for all $i \in N$ implies $\tilde{f}(\bar{\rho})\rceil_{\{x,y\}} \sim \tilde{f}(\bar{\rho}')\rceil_{\{x,y\}}$.

Proposition 4.13 demonstrates that the binary relation $\sim$ preserves the ordinal relationship between $\rho(x,y)$ and $\rho(y,x)$ across analogous preference relations.

Proposition 4.13. *Let ρ and ρ' be FWPRs on X where $x,y \in X$. Suppose $\rho \sim \rho'$. Then $\rho(x,y) > \rho(y,x) \iff \rho'(x,y) > \rho'(y,x)$.*

Proof. Suppose $\rho(x,y) > \rho(y,x)$ and $\rho(x,y) = s_i$. Then $s_i > \rho(y,x)$ and $\rho(y,x) \notin \rho^{s_i}$. Thus, $\rho(y,x) \notin \rho'^{t_i}$. Now $(x,y) \in \rho^{s_i}$ implies $(x,y) \in \rho'^{t_i}$. Hence $\rho'(x,y) \geq t_i > \rho'(y,x)$. □

Proposition 4.13 helps us to interpret IIA-3. For some fuzzy preference profile $\bar{\rho}$, suppose there exists another profile $\bar{\rho}'$ such that $\rho_i(x,y) > \rho(y,x)$ if and only if $\rho_i'(x,y) > \rho_i'(y,x)$ for all $i \in N$. Then an IIA-3 FPAR will associate equivalent social preferences over x and y to $\bar{\rho}$ and $\bar{\rho}'$, where $\tilde{f}(\bar{\rho})(x,y) > \tilde{f}(\bar{\rho})(y,x)$ if and only if $\tilde{f}(\bar{\rho}')(x,y) > \tilde{f}(\bar{\rho}')(y,x)$. Hence, IIA-3 preserves the ordinal relationship between the social preference over (x,y) and over (y,x) without considering the specific values of social preference, thereby satifying the conditions presented earlier in this subsection.

4.1.4 Dictatorship

In contrast to the other fuzzy Arrow conditions, dictatorship or a dictatorial aggregation rule exhibits very little variation over definitions throughout the literature (Banerjee, 1994; Barrett et al., 1992; Duddy et al., 2011; Fono and Andjiga, 2005; Fono et al., 2009; Mordeson and Clark, 2009; Richardson, 1998; Salles, 1998).

Definition 4.14 (dictatorial). Let $\tilde{f}$ be an FPAR. Then $\tilde{f}$ is said to be *dictatorial* if there exists an $i \in N$ such that for all $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$ and $x,y \in X$, $\pi_i(x,y) > 0$ implies $\pi(x,y) > 0$.

Definition 4.14 is standard in the literature. Obviously a dictatorship over an FPAR corresponds neatly to a dictatorship in the case of exact preferences, where society striclty prefers one alternative to another if the dictator does as well. As discussed previously, some scholars have chosen to avoid fuzzy strict preference relation and rely on another definition of dictatorship (Billot, 1992; Duddy et al., 2011).

Definition 4.15 (strongly dictatorial). Let $\tilde{f}$ be an FPAR. Then $\tilde{f}$ is said to be *strongly dictatorial* if there exists an $i \in N$ such that for all $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$ and $x, y \in X$

$$\rho_i(x,y) = \tilde{f}(\bar{\rho})(x,y) \ .$$

A strong dictatorship implies a dictatorship assuming that π is regular on both the individual and social levels.

4.2 Making and Breaking Arrow's Theorem

The traditional proofs of Arrow's theorem use exact preference relations. This section demonstrates the conditions under which Arrow's conclusion holds in the fuzzy framework discussed in the previous section. Further, we also detail under what conditions there exists an FPAR that satisfies certain combinations of fuzzy Arrowian conditions. To prove our main results, we make use of the following definition.

Definition 4.16. Let $\tilde{f}$ be an FPAR, let $(x,y) \in X \times X$ and let λ be a fuzzy subset of N.

(1) **(semidecisive)** λ is called *semidecisive for x against y*, written $x\tilde{D}_\lambda y$, if for every $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$,

$$\pi_i(x,y) > 0 \text{ for all } i \in \text{Supp}(\lambda) \text{ and } \pi_j(y,x) > 0 \text{ for all } j \notin \text{Supp}(\lambda)$$

implies $\pi(x,y) > 0$.

(2) **(decisive)** λ is called *decisive for x against y*, written $xD_\lambda y$, if for every $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$,

$$\pi_i(x,y) > 0 \text{ for all } i \in \text{Supp}(\lambda)$$

implies $\pi(x,y) > 0$.

In words, we call λ a *fuzzy coalition* when $|\text{Supp}(\lambda)| \geq 1$. In addition we say a coalition λ is *semidecisive* or *decisive* if it is *semidecisive* or *decisive* for all ordered pairs of alternatives.

There are two comments worth making about Definition 4.16 before proceeding to the formal arguments of fuzzy Arrow's theorem. First, the fuzzy definition of (semi)decisiveness introduces another application of fuzzy sets to social choice theory. Here we use a fuzzy subset of the actors rather than a traditional crisp case. Such a nuance is necessary when actors possess varied levels of influences within a coalition. These situations can arise in informal committees where the preferences of a more senior member may have more influence on the group's final preferences than those of a more junior member. Second, it is important to emphasize how very little semidecisiveness implies about a specific coalition λ. Obviously, decisiveness implies semidecisiveness, but the converse does not hold because semidecisiveness incorporates the preferences of individuals not in $\text{Supp}(\lambda)$. Hence, if there exists a $j \in \text{Supp}(\lambda)$ such that $\pi_j(y,x) = 0$, we cannot conclude that λ is semidecisive for x against y, and we know very little about the social preference between x and y. Given these restrictions on semidecisiveness, the following lemma is quite remarkable in the fact that additional structure on the FPAR implies a semidecisive coalition over an ordered pair is actually a decisive coalition over all pairs of alternatives.

Lemma 4.17. *Let λ be a fuzzy subset of N. Let $\tilde{f}$ be a partially quasi-transitive FPAR that is weakly Paretian and IIA-3 where π is regular. If λ is semidecisive for x against y, then for all $(v,w) \in X \times X$, λ is decisive for v against w.*

Proof. Suppose λ is semidecisive for x against y. Let $\bar{\rho}$ be a preference profile such that $\pi_i(x,z) > 0$, for all $i \in \text{Supp}(\lambda)$ and all $z \in X \backslash \{x,y\}$. Let $\bar{\rho}'$ be a fuzzy preference profile such that

$$\begin{aligned}
\rho'_i(x,z) &= \rho_i(x,z) \text{ and } \rho'_i(z,x) = \rho_i(z,x), \forall i \in N \quad (4.1)\\
\pi'_i(x,y) &> 0, \forall i \in \text{Supp}(\lambda)\\
\pi'_j(y,x) &> 0, \forall i \in N \backslash \text{Supp}(\lambda)\\
\pi'_i(y,z) &> 0, \forall i \in N.
\end{aligned}$$

Since $\pi_i(x,z) > 0$ for all $i \in \text{Supp}(\lambda)$, $\pi'_i(x,z) > 0$ for all $i \in \text{Supp}(\lambda)$ by the definition of $\bar{\rho}'$. Since $x\tilde{D}_\lambda y$, $\pi'(x,y) > 0$ by hypothesis. Since $\tilde{f}$ is weakly Paretian, $\pi'(y,z) > 0$. Since $\tilde{f}$ is partially quasi-transitive, $\pi'(x,z) > 0$ and $\rho'(x,z) > \rho'(z,x)$. Since $\rho_i\rceil_{\{x,z\}} = \rho'_i\rceil_{\{x,z\}}$ for all $i \in N$ and $\tilde{f}$ is IIA-3, $\rho\rceil_{\{x,z\}} \sim \rho'\rceil_{\{x,z\}}$ implies $\rho(x,z) > \rho(z,x)$. Hence $\pi(x,z) > 0$. Since $\bar{\rho}$ is arbitrary, $xD_\lambda z$. Since z was arbitrary in $X \backslash \{x,y\}$,

$$x\tilde{D}_\lambda y \implies xD_\lambda z, \forall z \in X \backslash \{x,y\}. \quad (4.2)$$

Since λ is decisive for x against z implies λ is semidecisive for x against z, interchanging y and z in Eq. (4.2) implies λ is decisive for x against y.

Now let $\bar{\rho}^*$ be another profile such that $\pi_i^*(y,z) > 0$ for all $i \in \text{Supp}(\lambda)$ and let $\bar{\rho}^+$ be such that

$$\begin{aligned}
&\rho_i^+(y,z) = \rho_i^*(y,z) \text{ and } \rho_i^+(z,y) = \rho_i^*(z,y), \forall i \in N \\
&\pi_i^+(y,x) > 0, \forall i \in N \\
&\pi_i^+(x,z) > 0, \forall i \in \text{Supp}(\lambda) \\
&\pi_j^+(z,x) > 0, \forall j \in N \backslash \text{Supp}(\lambda).
\end{aligned}$$

Then $\pi_i^+(y,z) > 0$ for all $i \in \text{Supp}(\lambda)$. Since $xD_\lambda z$, $\pi^+(x,z) > 0$. Since $\tilde{f}$ is weakly Paretian, $\pi^+(y,x) > 0$. Since $\tilde{f}$ is partially quasi-transitive, $\pi^+(y,z) > 0$. Since $\rho_i^*\rceil_{\{y,z\}} = \rho_i^+\rceil_{\{y,z\}}$ for all $i \in N$ and $\tilde{f}$ is IIA-3, $\rho^*\rceil_{\{y,z\}} \sim \rho^+\rceil_{\{y,z\}}$, and so $\rho^*(y,z) > \rho^*(z,y)$. Thus, $\pi^*(y,z) > 0$ and so $yD_\lambda z$ because $\bar{\rho}^*$ is arbitrary. Because z is arbitrary in $X \backslash \{x,y\}$,

$$x\tilde{D}y \implies yD_\lambda z, \forall z \notin \{x,y\}. \tag{4.3}$$

Now because λ is decisive for y against z, λ is semidecisive for y against z. Thus by 4.13, λ is decisive for y against x. To summarize, we have, for all $(v,w) \in X \times X$,

$$x\tilde{D}_\lambda y \implies xD_\lambda v \text{ (by 4.13)} \implies x\tilde{D}v \implies vD_\lambda w$$

by Eq. (4.3). □

Lemma 4.17 lays out the formal argument in the fuzzy framework of what Sen (1976) labels the "Paretian epidemic", where a coalition that is semidecisive over an ordered pair becomes globally decisive after adopting the Arrowian conditions. An important aspect of Lemma 4.17 is the generalization of strict preference to a regular fuzzy strict preference relation, which as Chapter 3 illustrated, imposes minimal assumptions on the structure of FWPRs. Nonetheless, the argument still holds for certain non-regular strict preference relations but requires a new specification of IIA. The following definition and proposition explores this relationship formally using the *cosupport* of a fuzzy subset U of X. That is, $\text{Cosupp}(U) = \{x \in X \mid U(x) < 1\}$.

Definition 4.18 (IIA-4). Let $\tilde{f}$ be an FPAR. Then $\tilde{f}$ is said to be *independent of irrelevant alternatives, type 4* (IIA-4), if for all $\bar{\rho}, \bar{\rho}' \in \mathscr{FR}^n$ and $x,y \in X$,

$$\text{Cosupp}(\bar{\rho}_i\rceil_{\{x,y\}}) = \text{Cosupp}(\bar{\rho}_i'\rceil_{\{x,y\}})$$

for all $i \in N$ implies

$$\text{Cosupp}(\tilde{f}(\bar{\rho}_i)\rceil_{\{x,y\}}) = \text{Cosupp}(\tilde{f}(\bar{\rho}_i')\rceil_{\{x,y\}}) \,.$$

It is easily verified that $\pi_{(2)}(x,y) = 1 - \rho(y,x)$ and

$$\pi_{(4)}(x,y) = \begin{cases} \rho(x,y) & \text{if } \rho(y,x) = 0, \\ 0 & \text{otherwise,} \end{cases}$$

are not regular when there is no further structure placed on ρ besides completeness and reflexivity.

Lemma 4.19. *Let λ be a fuzzy subset of N. Let $\tilde{f}$ be a partially quasi-transitive FPAR that is weakly Paretian and IIA-2 when $\pi = \pi_{(4)}$ and IIA-4 when $\pi = \pi_{(2)}$. If λ is semidecisive for x against y, then for all $(v,w) \in X \times X$, λ is decisive for v against w.*

Proof. Suppose λ is semidecisive for x against y. Consider a profile $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$ such that $\pi_i(x,z) > 0$, for all $i \in \text{Supp}(\lambda)$. Let $\bar{\rho}'$ be the fuzzy preference profile as defined in Lemma 4.17. By an identical argument, we know $\pi_i'(x,z) > 0$ for all $i \in \text{Supp}(\lambda)$ and $\pi'(x,y) > 0$. Likewise, $\pi'(y,z) > 0$, because $\tilde{f}$ is weakly Paretian. Since $\tilde{f}$ is partially quasi-transitive, $\pi'(x,z) > 0$.

For $\pi = \pi_{(4)}$, $\pi_{(4)}'(x,z) > 0$ implies $\rho'(x,z) > 0$ and $\rho'(z,x) = 0$. Since

$$\text{Supp}(\rho_i\rceil_{\{x,z\}}) = \text{Supp}(\rho_i'\rceil_{\{x,z\}})$$

for all $i \in N$ and $\tilde{f}$ is IIA-2,

$$\text{Supp}(\rho\rceil_{\{x,z\}}) = \text{Supp}(\rho'\rceil_{\{x,z\}}) .$$

Thus, $\rho(x,z) > 0$ and $\rho(z,x) = 0$, which implies $\pi_{(4)}(x,z) > 0$ by the definition of $\pi_{(4)}$.

For $\pi = \pi_{(2)}$, $\pi_{(2)}'(x,z) > 0$ implies $\rho'(z,x) < 1$. Since

$$\text{Cosupp}(\rho_i\rceil_{\{x,z\}}) = \text{Cosupp}(\rho_i'\rceil_{\{x,z\}})$$

for all $i \in N$ and $\tilde{f}$ is IIA-4,

$$\text{Cosupp}(\rho\rceil_{\{x,z\}}) = \text{Cosupp}(\rho'\rceil_{\{x,z\}}) .$$

Thus, $\rho'(z,x) < 1$, which implies $\pi_{(2)}(x,z) > 0$ by definition of $\pi_{(2)}$.

Because $\pi(x,z) > 0$ and $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$ and $z \in X \backslash \{x,y\}$ are arbitrary, we obtain the following result:

$$x\tilde{D}_\lambda y \implies xD_\lambda z, \forall z \in X \backslash \{x,y\}.$$

The remainder of the proof follows easily from a similar argument using Lemma 4.17.

Before presenting the main results, we prove the following proposition.

Proposition 4.20. *Let ρ be an FWPR on X. Then the following properties are equivalent:*

(1) ρ is weakly transitive.
(2) For all $x,y,z \in X$, $\rho(x,y) \geq \rho(y,x)$ and $\rho(y,z) \geq \rho(z,y)$ with a strict equality holding at least once, then $\rho(x,z) > \rho(z,x)$.

Proof. Suppose 4.20(1). Assume that $\rho(x,y) \geq \rho(y,x)$ and $\rho(y,z) > \rho(z,y)$. Then $\rho(x,z) \geq \rho(z,x)$. Suppose $\rho(z,x) \geq \rho(x,z)$. Then $\rho(z,y) \geq \rho(y,z)$ by 4.20(1), a contradiction. Hence, $\rho(x,z) > \rho(z,x)$. A similar argument shows that $\rho(x,y) > \rho(y,x)$ and $\rho(y,z) \geq \rho(z,y)$ implies $\rho(x,z) > \rho(z,x)$.

Suppose 4.20(2). Let $x,y,z \in X$. Suppose $\rho(x,y) \geq \rho(y,x)$ and $\rho(y,z) \geq \rho(z,y)$. Suppose $\rho(z,x) > \rho(x,z)$. Then by (2), $\rho(z,x) > \rho(x,z)$ and $\rho(x,y) \geq \rho(y,x)$ imply $\rho(z,y) > \rho(y,z)$, a contradiction. Hence, $\rho(x,z) \geq \rho(z,x)$. □

Corollary 4.21. *Let ρ be an FWPR on X. If ρ is weakly transitive, then ρ is partially quasi-transitive.*

As Proposition 4.20 and Corollary 4.21 show, weak transitivity is more restrictive than partial quasi-transitivity. This added assumption, when paired with the conditions of independence and weak Paretianism, implies a dictatorial FPAR. To illustrate this formally, the results in Lemmas 4.17 and 4.19 make it sufficient to show that $\text{Supp}(\lambda) = \{i\}$, where λ is any semidecisive coalition under the Arrowian conditions. In such a case, $\pi_i(x,y) > 0$ implies $\pi(x,y) > 0$ for all $\bar{\rho} \in \mathscr{FR}^n$ and $x,y \in X$, and λ is a dictator rather than a coalition.

Theorem 4.22 (Fuzzy Arrow's Theorem). *Let $\tilde{f} : D_w^n \to \mathscr{FR}$ be a fuzzy aggregation rule. Suppose π is regular, and $\tilde{f}$ is weakly Paretian, weakly transitive and IIA-3. Then $\tilde{f}$ is dictatorial.*

Proof. Since $\tilde{f}$ is weakly Paretian, there exists a decisive λ for any pair of alternatives, namely, $\text{Supp}(\lambda) = N$. For all $(u,v) \in X \times X$, let $m(u,v)$ denote the size of the smallest $|\text{Supp}(\lambda)|$ for a λ semidecisive for u against v. Let $m = \wedge\{m(u,v) \mid (u,v) \in X \times X\}$. Without loss of generality, suppose λ is semidecisive for x against y where $|\text{Supp}(\lambda)| = m$. If $m = 1$, the proof is complete. Suppose $m > 1$. Let $i \in \text{Supp}(\lambda)$, and let $z \in X \backslash \{x,y\}$. Consider any fuzzy profile $\bar{\rho}$ such that

$$\begin{aligned}
&\pi_i(x,y) > 0, \pi_i(y,z) > 0 \text{ and } \pi_i(x,z) > 0 \\
&\pi_j(z,x) > 0, \pi_j(x,y) > 0 \text{ and } \pi_j(z,y) > 0, \forall j \in \text{Supp}(\lambda) \backslash \{i\} \\
&\pi_k(z,x) > 0, \pi_k(x,y) > 0 \text{ and } \pi_k(z,y) > 0, \forall k \notin \text{Supp}(\lambda).
\end{aligned}$$

Since λ is semidecisive for x against y and $\pi_j(x,y) > 0$ for all $j \in \text{Supp}(\lambda)$, $\pi(x,y) > 0$. Since $|\text{Supp}(\lambda)| = m$, it is not the case that $\pi(z,y) > 0$, or otherwise λ' is semidecisive for z against y, where $\text{Supp}(\lambda') = \text{Supp}(\lambda) \backslash \{i\}$. However, this contradicts the minimality of m since $|\text{Supp}(\lambda')| = m - 1$. Because π is regular, $\pi(z,y) = 0$ implies $\rho(y,z) \geq \rho(z,y)$. Since $\rho(x,y) > \rho(y,z)$, $\rho(x,z) > \rho(z,x)$ by weak transitivity and Proposition 4.20. Hence $\pi(x,z) > 0$. By IIA-3, λ^* is semidecisive for x against z, where $\text{Supp}(\lambda^*) = \{i\}$. However, this contradicts the fact the $m > 1$. □

The added assumption of weak transitivity, rather than partial quasi-transitivity, in Theorem 4.22 allows Arrow's results to hold in the fuzzy framework with a general strict preference relation without putting added assumptions on individual preferences such as those in Fono and Andjiga (2005) and Mordeson and Clark (2009). However, we can relax the transitivity condition of the FPAR and still obtain similar results by specifying a strict preference relation. To do so, we make use of the following proposition.

Proposition 4.23. *Let ρ be an FWPR on X. If ρ is partially transitive, then ρ is partially quasi-transitive with respect to $\pi = \pi_{(4)}$.*

Proof. Let $x, y, z \in X$. Suppose $\pi(x,y) > 0$ and $\pi(y,z) > 0$. Then $\rho(x,y) > 0$, $\rho(y,x) = 0$, $\rho(y,z) > 0$, and $\rho(z,y) = 0$. Hence, $\rho(x,z) > 0$. Suppose $\pi(x,z) = 0$. Then $\rho(z,x) > 0$. However, $\rho(y,z) > 0$ and $\rho(z,x) > 0$ implies $\rho(y,x) > 0$, a contradiction. Hence $\pi(x,z) > 0$. □

Using this proposition, we can relax the transitivity condition on $\tilde{f}$ to partial transitivity when $\pi = \pi_{(4)}$.

Let $D_p$denote the set of all partially transitive fuzzy weak orders.

Theorem 4.24 (Fuzzy Arrow's Theorem 2). *Let $\tilde{f} : D_p^n \to \mathcal{FR}$ be an FPAR. Suppose $\pi = \pi_{(4)}$. Let $\tilde{f}$ be weakly Paretian, partially transitive, and IIA-2. Then $\tilde{f}$ is dictatorial.*

Proof. Since $\tilde{f}$ is partially transitive and $\pi = \pi_{(4)}$, $\tilde{f}$ is partially quasi-transitive by Proposition 4.23. Further, because $\tilde{f}$ is weakly Paretian, there exists a decisive λ for any pair of alternatives. Let $m(u,v)$ denote the size of the smallest $|\text{Supp}(\lambda)|$ for a λ semidecisive for u against v in X. Let $m = \wedge\{m(u,v) \mid (u,v) \in X \times X\}$. Likewise, suppose λ is semidecisive for x against y where $|\text{Supp}(\lambda)| = m$, and suppose $m > 1$. Now consider a $\bar{\rho} \in \mathcal{FR}^n$ such that $\bar{\rho}$ is identical to $\bar{\rho}$ in Theorem 4.24.

Then $\pi(x,y) > 0$ because λ is semidecisive for x against y and $\pi_j(x,y) > 0$ for all $j \in \text{Supp}(\lambda)\backslash\{i\}$. In addition, $\pi(z,y) = 0$, else λ' is semidecisive for z against y, a contradition of the minimality of m. Thus, $\rho(y,z) > 0$. Since $\tilde{f}$ is partially transitive, $\rho(x,y) > 0$ and $\rho(y,z) > 0$ imply $\rho(x,z) > 0$. Suppose $\pi(x,z) = 0$. Then $\rho(z,x) > 0$ by definition of $\pi_{(4)}$. However, $\rho(y,z) > 0$ and $\rho(z,x) > 0$ imply $\rho(y,x) > 0$ by the partial transitivity of $\tilde{f}$. This contradicts $\pi(x,y) > 0$. Hence $\pi(x,z) > 0$. By IIA-2, λ^* is semidecisive for x against y, where $\text{Supp}(\lambda^*) = \{i\}$. However, this contradicts $m > 1$. □

Theorems 4.22 and 4.24 lay out the consequences of two specific combinations of assumptions on fuzzy aggregation rules. Given an FPAR that satisfies these definitions of transitivity, weak Paretianism and independence of irrelevant alternatives, the FPAR must be dictatorial under a variety of social strict preference relations. However, the implication of dictatatorship cannot be generalized over all derivations of fuzzy Arrowian conditions. Thus, we now consider under what circumstances a nondictatorial FPAR can satisfy fuzzy Arrowian conditions. The key to these series of formal arguments lies in the concept of neutrality.

Definition 4.25 (neutral). Let $\tilde{f}$ be an FPAR. Then $\tilde{f}$ is said to be *neutral* if, for all $\bar{\rho}, \bar{\rho}' \in \mathcal{FR}^n$ and all $w, x, y, z \in X$, $\rho_i'(x,y) = \rho_i'(w,z)$, for all $i \in N$, implies $\tilde{f}(\bar{\rho})(x,y) = \tilde{f}(\bar{\rho}')(w,z)$.

In words, neutrality guarantees that an aggregation rule treats every pair of alternatives in a similar manner across preference profiles, i.e., the labeling of alternatives is arbitrary and does not affect the aggregation of preferences. In the exact case, neutrality has an important part in May's (1952) theorem characterizing the importance of majority rule as the only anonymous, neutral, and monotone choice function if there are two alternatives. In Arrowian context, Blau (1972) first noticed the logic of neutrality plays an important part in the formal arguments; however, he is unable to use neutrality to prove Arrow's theorem. Ubeda Ubeda (2003) first showed that IIA and weak Paretianism imply neutrality and that neutrality can be used in a more direct proof of Arrow's theorem. In the fuzzy case, this relationship no longer holds. This occurs because the concept of weak Paretianism is ordinal: for any two alternatives x and y, $\rho_i(x,y) > \rho_i(y,x)$ for all $i \in N$ implies $\rho(x,y) > \rho(y,x)$ in the social preference relation when π is regular. Yet neutrality, as defined in Definition 4.25, is cardinal in conception and weak Paretianism is insufficient to imply neutrality even when paired with IIA. Thus, we consider another characteristic of FPARs.

Definition 4.26 (unanimous in acceptance). Let $\tilde{f}$ be an FPAR. Then $\tilde{f}$ is said to be *unanimous in acceptance* if, for all $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$, $\rho_i(x,y) = 1$ for all $i \in N$ implies $\tilde{f}(\bar{\rho})(x,y) = 1$ Duddy et al. (2011).

Unanimity in acceptance is significantly less restrictive than unanimity (see Section 4.1.2) and requires the social preference to take a specific value only when all individuals definitely view one alternative as at least as good as another. Further, Definition 4.26 has no implications for a fuzzy aggregation rule when there exist some $x,y \in X$ and $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$ such that $\rho_i(x,y) = c$ for all $i \in N$ and $c \in [0,1)$. This seemingly insubstantial condition allows Duddy et al. (2011) to obtain the following relationship.

Let $\mathscr{F}\mathscr{R}^*$ denote the set of all max-$*$ transitive fuzzy weak orders.

Proposition 4.27. *Let $\tilde{f} : \mathscr{F}\mathscr{R}^{*n} \to \mathscr{F}\mathscr{R}$ be an FPAR. Suppose $\tilde{f}$ is max $-\, *$ transitive, IIA-1 and unanimous in acceptance. Then $\tilde{f}$ is neutral.*

Proof. The proof, which comes from Duddy et al. (2011), demonstrates that $\tilde{f}$ is neutral by considering all combinations of (x,y), $(w,z) \in X \times X$.

Case 1: $(x,y) = (w,z)$. The proof follows immediately from the IIA-1 definition.

Case 2: $(x,y),(x,z) \in X \times X$. Let $\bar{\rho} \in \mathscr{F}\mathscr{R}^{*n}$ be such that $\rho_i(y,z) = \rho_i(z,y) = 1$ for all $i \in N$. Then, by max-$*$ transitivity of all individual weak orders, $\rho_i(x,y) \geq \rho_i(x,z) * \rho_i(z,y) = \rho_i(x,z)$ and $\rho_i(x,z) \geq \rho_i(x,y) * \rho_i(y,z) = \rho_i(x,y)$. Next, $\rho_i(x,y) \geq \rho_i(x,z)$ and $\rho_i(x,z) \geq \rho_i(x,y)$ imply $\rho_i(x,y) = \rho_i(x,z)$ for all $i \in N$. Similarly, by max-$*$ transitivity, $\rho_i(y,x) \geq \rho_i(y,z) * \rho_i(z,x) = \rho_i(z,x)$ and $\rho_i(z,x) \geq \rho_i(z,y) * \rho_i(y,x) = \rho_i(y,x)$; and $\rho_i(y,x) = \rho_i(z,x)$, for all $i \in N$. Because $\rho_i(y,z) = \rho_i(z,y) = 1$ for all $i \in N$, $\rho(y,z) = \rho(z,y) = 1$. Hence, by the previous arguments, $\rho(x,y) \geq \rho(x,z)$, $\rho(x,z) \geq \rho(x,y)$, $\rho(y,x) \geq \rho(z,x)$, and $\rho(z,x) \geq \rho(y,x)$. Thus, $\rho(x,y) = \rho(x,z)$ and $\rho(y,x) = \rho(z,x)$ for the social preference as well.

The above arguments apply to all $\bar{\rho} \in \mathscr{FR}^{*n}$ such that $\rho_i(y,z) = \rho_i(z,y) = 1$ for all $i \in N$. Let $\mathscr{G}^n$ denote the set of all such profiles. Because the individual preferences between y and z are "irrelevant" so to speak, the proof now uses IIA-1 to prove the conclusion.

Case 3: Now for any profile $\bar{\rho} \in \mathscr{FR}^{*n}$ such that $\rho_i(x,y) = \rho_i(x,z)$ for all $i \in N$, there exists a $\bar{\rho}' \in \mathscr{G}^n$ such that $\rho_i(x,y) = \rho_i'(x,y) = \rho_i'(x,z) = \rho_i(z,x)$. IIA-1 implies $\rho(x,y) = \rho(y,x) = \rho'(y,x) = \rho'(x,y)$. For two distinct profiles $\bar{\rho}, \bar{\rho}' \in \mathscr{FR}^n$ such that $\rho_i(x,y) = \rho_i'(x,z)$ for all $i \in N$, there also exists a profie $\bar{\rho}^* \in \mathscr{G}^n$ such that $\rho_i(x,y) = \rho_i^*(x,y) = \rho_i^*(x,z) = \rho_i'(x,z)$. By IIA-1, $\rho(x,y) = \rho^*(x,y) = \rho^*(x,z) = \rho'(x,z)$.

Case 4: $(x,y),(w,y) \in X \times X$. The same conclusions can be proved using symmetric logic in Case 2. The first step is to assume $\rho_i(x,w) = \rho_i(w,x) = 1$ for all $i \in N$.

Case 5: $(x,y),(w,z) \in X \times X$. Let $\bar{\rho} \in \mathscr{FR}^{*n}$ such that $\rho_i(y,z) = \rho_i(z,y) = \rho_i(x,w) = \rho_i(w,x) = 1$ for all $i \in N$. Because ρ_i is max-$*$ transitive, $\rho_i(x,y) \geq \rho_i(x,z)$ and $\rho_i(x,z) \geq \rho_i(x,y)$, and thus $\rho_i(x,y) = \rho_i(x,z)$, for all $i \in N$. Because $\tilde{f}$ satisfies max-$*$ transitivity and unanimity in acceptance, $\rho(x,y) = \rho(x,z)$. Further, $\rho_i(x,z) \geq \rho_i(x,w) * \rho_i(w,z)$ and $\rho_i(w,z) \geq \rho_i(w,x) * \rho_i(x,z)$ imply $\rho_i(x,z) = \rho_i(w,z)$, for all $i \in N$. To summarize, $\rho_i(x,y) = \rho_i(x,z) = \rho_i(w,z)$ for all $i \in N$. And because the conditions of unanimity in acceptance and max-$*$ transitivity have been met, an identical argument applies to the social preference relation and $\rho(x,y) = \rho(x,z) = \rho(w,z)$.

The above arguments apply to all ρ_i that are max-$*$ transitive such that $\rho_i(y,z) = \rho_i(z,y) = \rho_i(x,w) = \rho_i(w,x) = 1$ for all $i \in N$. Let $\mathscr{G}^n$ denote the set of all such profiles. Because the individual preferences between y and z and between x and w are "irrelevant" so to speak, the proof now uses IIA-1 to prove the conclusion.

Case 6: Now for any profile $\bar{\rho} \in \mathscr{FR}^{*n}$ such that $\rho_i(x,y) = \rho_i(w,z)$ for all $i \in N$, there exists a $\bar{\rho}' \in \mathscr{G}^n$ such that $\rho_i(x,y) = \rho_i'(x,y) = \rho_i'(w,z) = \rho_i(w,z)$. IIA-1 implies $\rho(x,y) = \rho(w,z) = \rho'(w,z) = \rho'(x,y)$. For two distinct profiles $\bar{\rho}, \bar{\rho}' \in \mathscr{FR}^n$ such that $\rho_i(x,y) = \rho_i'(w,z)$ for all $i \in N$. Then there exists a profile $\bar{\rho}^* \in \mathscr{G}^n$ such that $\rho_i(x,y) = \rho_i^*(x,y) = \rho_i^*(w,z) = \rho_i'(w,z)$. By IIA-1, $\rho(x,y) = \rho^*(x,y) = \rho^*(w,z) = \rho'(w,z)$.

Case 7: $(x,y),(w,z) \in X \times X$ where $x = z$ or $y = w$. (This case is similar to Cases 2 and 3.) Let a denote an arbitrary alternative that is distinct from x and w. One exists because $|X| \geq 3$. Take any profile $\bar{\rho} \in \mathscr{FR}^{*n}$ where $\rho_i(a,y) = \rho_i(y,a) = \rho_i(x,w) = \rho_i(w,x) = \rho_i(z,a) = \rho_i(a,z) = 1$. Cases 2 and 3 imply $\rho_i(x,y) = \rho_i(x,a) = \rho_i(w,a) = \rho_i(w,z)$ and $\rho(x,y) = \rho(x,a) = \rho(w,a) = \rho(w,z)$ by unanimity in acceptance and max-$*$ transitivity.

Let W^ndenote the set of all such profiles. Let $(r_1,\ldots,r_n) \in \mathscr{FR}^{*n}$ be such that $r_j(x,y) = r_j(z,w)$ for all $j \in N$. Then there exists $(r_1^{'},\ldots,r_n^{'}) \in W^n$ such that

$r_j(x,y) = r_j(z,w) = r'_j(x,y,) = r'_j(z,w)$ for all $j \in N$. IIA-1 implies that $\tilde{f}(\bar{\rho})(x,y) = \tilde{f}(\bar{\rho})(z,w) = \tilde{f}(\bar{\rho}')(x,y) = \tilde{f}(\bar{\rho}')(z,w)$ where $\bar{\rho} = (r_1,\ldots,r_n)$ and $\bar{\rho}' = (r'_1,\ldots,r'_n)$. Take any pair of distinct profiles $\bar{\rho}'' = (r''_1,\ldots,r''_n)$ and $\bar{\rho}^* = (r^*_1,\ldots,r^*_n)$ in $\mathscr{F}\mathscr{R}^{*n}$ such that $r''_j(x,y) = r^*_j(z,w)$ for all $j \in N$. Then there exists $(r^{**}_1,\ldots,r^{**}_n) \in W^n$ such that $r''_j(x,y) = r''_j(z,w) = r^{**}_j(x,y,) = r^{**}_j(z,w)$ for all $j \in N$. IIA-1 implies $\tilde{f}(\bar{\rho}'')(x,y) = \tilde{f}(\bar{\rho}'')(z,w) = \tilde{f}(\bar{\rho}^{**}(x,y) = \tilde{f}(\bar{\rho}^{**})(z,w)$ □

While we do not use Proposition 4.25 to establish further results, it does illustrate that neutrality is not necessarily a strong restriction to place on an aggregation rule. As Proposition 4.27 demonstrates, neutrality arises naturally from the combination of max-$*$ transitivity, IIA-1, and unanimity in acceptance. We have already discussed the importance of max-$*$ transitivity and IIA-1; if one can justify Definition 4.26 and its application to fuzzy aggregation rules, neutrality is the natural conclusion. With a few more assumptions, we can use neutrality to derive a specific fuzzy aggregation rule.

In what follows, we show how neutrality can be used to classify a wide range of FPARs and determine whether these FPARs satisfy fuzzy Arrowian conditions. To do this, we need the following lemma.

Lemma 4.28. *Let $\tilde{f}$ be an FPAR. Then the following conditions are equivalent.*

(1) $\tilde{f}$ is neutral;
(2) There exists a unique function $f_n : [0,1]^n \to [0,1]$ such that, for all $x,y \in X$ and all $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$, $f_n(\rho_1(x,y),\ldots,\rho_n(x,y)) = \tilde{f}(\bar{\rho})(x,y)$.

Proof. (1) $\Longrightarrow$ (2): Let $x,y \in X$. Let $(a_1,\ldots,a_n) \in [0,1]^n$. Then there exists $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$ such that $\rho_i(x,y) = a_i$ for all $i = 1,\ldots,n$. Define $f_n : [0,1]^n \to [0,1]$ as follows:

$$f_n((a_1,\ldots,a_n)) = \tilde{f}(\bar{\rho})(x,y).$$

It remains to be shown that f_n is single-valued. Let $w,z \in X$. Then there exists a $\bar{\rho}' \in \mathscr{F}\mathscr{R}^n$ such that $\rho'_i(w,z) = a_i$ for all $i = 1,\ldots,n$. Thus, $\rho_i(x,y) = \rho'_i(w,z)$ for all $i \in N$. Since $\tilde{f}$ is neutral, $\tilde{f}(\bar{\rho})(x,y) = \tilde{f}(\bar{\rho}')(w,z)$. Thus, f_n is single-valued. In addition, uniqueness of f_n is guaranteed by construction.

(2) $\Longrightarrow$ (1): Let $\bar{\rho},\bar{\rho}' \in \mathscr{F}\mathscr{R}^n$ and $w,x,y,z \in X$. Suppose $\rho_i(x,y) = \rho'_i(w,z)$ for all $i \in N$. Then,

$$\begin{aligned}\tilde{f}(\bar{\rho})(x,y) &= f_n(\rho_1(x,y),\ldots,\rho_n(x,y)) = f_n(\rho'_1(w,z),\ldots,\rho'_n(w,z)) \\ &= \tilde{f}(\bar{\rho})(w,z)\ .\end{aligned}$$

Thus, $\tilde{f}$ is neutral. □

In words, f_n is the *auxillary function* associated with a specific FPAR $\tilde{f}$, and a_i can be interpreted as the *weak preference intensity* of player i for one alternative over another. By itself, Lemma 4.28 may seem unremarkable, but the lemma is an important step in examining the implications of neutrality on fuzzy aggregation rules. To derive a unique aggregation rule, we need one more definition.

Definition 4.29. Let $\tilde{f}$ be a neutral FPAR, and let f_n be an auxillary function associated with $\tilde{f}$. Then $\tilde{f}$ is said to be

(1) *linearly decomposable* if, for all $(a_1,\ldots,a_n) \in [0,1]^n$, $f_n(a_1,\ldots,a_n) = a_1 f_n(1,0,\ldots,0) + \ldots + a_n f_n(0,,\ldots,0,1)$;
(2) *additive* if, for all $(a_1,\ldots,a_n),(b_1,\ldots,b_n) \in [0,1]^n$ such that $a_i + b_i \in [0,1]$, $i = 1,\ldots,n$, $f_n((a_1,\ldots,a_n)+(b_1,\ldots,b_n)) = f_n((a_1,\ldots,a_n)) + f_n((b_1,\ldots,b_n))$.

Linear decomposability implies two criteria. First, the condition requires that the collective preference between two alternatives is the sum of the n collective preferences when only one individual preference is considered at a time by the FPAR. Second, the specific individual preference intensity (a_i) can be "removed" from the individual preference relation (ρ_i), and "reapplied" directly to the FPAR that only considers the preference of individual i. The stronger assumption of additivity requires that given a preference profile, the collective preference for one alternative over another can be created by first decomposing the preference intensities of the individuals, then applying the FPAR to those two profiles of preference intensities, and finally adding the two collective preferences.

The followinig lemma states the relationship between the conditions in Definition 4.29.

Lemma 4.30. *Let $\tilde{f}$ be a neutral FPAR. If $\tilde{f}$ is linearly decomposable, then $\tilde{f}$ is additive.*

Proof. Because $\tilde{f}$ is neutral, there exits an auxillary function f_n associated with $\tilde{f}$ such that $f_n((a_1,\ldots,a_n)) = \tilde{f}(\bar{\rho})(x,y)$ for all $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$, $x,y \in X$, and $(a_1,\ldots,a_n) \in [0,1]^n$. Let $(a_1,\ldots,a_n),(b_1,\ldots,b_n) \in [0,1]^n$ be such that $a_i + b_i \in [0,1]$ for all $i = 1,\ldots,n$. Then,

$$\begin{aligned}
f_n((a_1,\ldots,a_n)+(b_1,\ldots,b_n)) &= f_n((a_1+b_1,\ldots,a_n+b_n)) \\
&= (a_1+b_1)f_n((1,0,\ldots,0)) + \ldots \\
&\quad +(a_n+b_n)f_n(0,\ldots 0,1) \\
&= a_1 f_n((1,0,\ldots,0) + \ldots \\
&\quad +a_n f_n((1,0,\ldots,0)) + b_1 f_n((1,0,\ldots,0) \\
&\quad + \ldots + b_n f_n((1,0,\ldots,0)) \\
&= f_n((a_1,\ldots,a_2)) + f_n((b_1,\ldots,b_n)
\end{aligned}$$

as desired. □

Finally, Theorem 4.31 and Corollary 4.32, which are simplified generalizations of García-Lapresesta and Llamazares (2000), illustrate the effects of a neutral and linear decomposable FPAR. To do so, it introduces the concept of restricting an auxillary function between the interval [0,1], denoted $\hat{f}_n|_{[0,1]^n}$, because, under additivity, there is no guarantee that the sum of two n-tuples of preferences intensities will have components less than or equal to one. The restriction places no added assumptions on FPARs or individual preferences, but it allows us to obtain the following result.

Theorem 4.31. *Let $\tilde{f}$ be a neutral fuzzy aggregation rule. If $\tilde{f}$ is linearly decomposable, then there exists a unique linear transformation $\tilde{f}_n$ of $\mathbb{R}^n$ into $\mathbb{R}$ such that $\hat{f}|_{[0,1]^n} = f_n$.*

Proof. Because $\tilde{f}$ is neutral, there exits an auxillary function f_n associated with $\tilde{f}$ such that $f_n((a_1,\ldots,a_n)) = \tilde{f}(\bar{\rho})(x,y)$ for all $\bar{\rho} \in \mathscr{FR}^n$, $x,y \in X$, and $(a_1,\ldots,a_n) \in [0,1]^n$. For $i = 1,\ldots,n$, let $\bar{1}_i = (u_1,\ldots,u_n)$, where $u_i = 1$ and $u_j = 0$ for $j \neq i$. Then there exists a unique linear transformation $\hat{f}$ of $\mathbb{R}^n$ into $\mathbb{R}$ such that $\hat{f}(\bar{1}_i) = w_i$, where $w_i = f_n(\bar{1}_i)$ for all $i \in N$. Since f_n is additive by the previous lemma,

$$\begin{aligned}\sum_{i=1}^n w_i &= \sum_{i=1}^n f_n(\bar{1}_i)\\ &= f_n((1,\ldots,1)) \leq 1\ .\end{aligned}$$

Now,

$$\hat{f}_n(\sum_{i=1}^n c_i\bar{1}_i) = \sum_{i=1}^n c_i\hat{f}_n(\bar{1}_i)\ .$$

Thus if $c_i \in [0,1]$, for $i \in N$, then

$$f_n(\sum_{i=1}^n c_i\bar{1}_i) = \sum_{i=1}^n c_i f_n(\bar{1}_i) \in [0,1]$$

because $\sum_{i=1}^n w_i \leq 1$. Let $((a_1,\ldots,a_n)) \in [0,1]^n$. Then

$$\begin{aligned}\hat{f}_n|_{[0,1]^n}((a_1,\ldots,a_n)) &= \hat{f}_n((a_1,\ldots,a_n))\\ &= = \sum_{i=1}^n a_i\hat{f}_n(\bar{1}_i)\\ &= f_n((a_1,\ldots,a_n))\end{aligned}$$

since $\tilde{f}$ is linearly decomposable. □

Corollary 4.32. *Let $\tilde{f}$ be a neutral FPAR. If $\tilde{f}$ is linearly decomposable, then, for all $\bar{\rho} \in \mathscr{FR}^n$ and all $x,y \in X$,*

$$\tilde{f}(\bar{\rho})(x,y) = \sum_{i=1}^n w_i\rho_i(x,y)\ ,$$

$w_i = f_n(\bar{1}_i)$ for all $i \in N$.

According to Theorem 4.31 and Corollary 4.32, a neutral and linearly decomposable aggregation rule must be a weighted mean aggregation rule. A weighted mean FPAR is a generalization of Example 4.2(1). Such a generalization emphasizes two important distinctions between exact and fuzzy aggregation rules. First, there is a difference between the possible rules modeled under exact preferences and those modeled under the fuzzy framework. Corollary 4.32 allows scholars to consider committee or other voting bodies where individuals do not contribute equally to the social preference. In other words, some opinions are more relevant to the final collective preference than others. These situations can arise on any committee that produces a social fuzzy preference, which affected by seniority, professional rank, or any number of other social factors could influence the group's final decision. However, this rule is not necessarily anonymous, i.e. the labeling of the individuals does matter, because each individual has a preassigned weight to his or her preference. If the weighted mean is anonymous, then it is easily verified that $w_i = \frac{1}{n}$ for all $i \in N$.

Second and more importantly, neutrality does not imply a dictatorship. Unlike the findings in Ubeda (2003), fuzzy neutrality, when paired with linear decomposability, does not guarantee a non-dictatorial FPAR. This brings us one step closer to identifying conditions under which fuzzy social choice permits FPARs to satsify all Arrowian conditions.

Definition 4.33 (weighted mean rule). Let $\tilde{f}$ be an FPAR. Then $\tilde{f}$ is said to be the *weighted mean rule* if, for all $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$ and all $x, y \in X$,

$$\tilde{f}(\bar{\rho})(x,y) = \sum_{i=1}^{n} w_i \cdot \rho_i(x,y),$$

where $\sum_{i=1}^{n} w_i = 1$ and $w_i > 0$ for all $i \in N$.

Obviously, the weighted mean is non-dictatorial and independent of irrelevant alternatives under IIA-1. What remains to be shown is whether the FPAR satisfies weak Paretianism and max-$*$ transitivity, which we now consider.

Proposition 4.34. *Let $\tilde{f}$ be an FPAR as defined in Definition 4.33. If π is regular, then $\tilde{f}$ is weakly Paretian.*

Proof. Let $x, y \in X$. Suppose $\pi_i(x,y) > 0$ for all $i \in N$. Because π is assumed to be regular, $\pi_i(x,y) > 0$ implies $\rho_i(x,y) > \rho_i(y,x)$. Further, $w_i \cdot \rho_i(x,y) > w_i \cdot \rho(y,x)$ for all $i \in N$ because $w_i \in (0,1]$. Hence,

$$\sum_{i=1}^{n} w_i \cdot \rho_i(x,y) > \sum_{i=1}^{n} w_i \cdot \rho_i(y,x) \,.$$

Thus, $\tilde{f}(\bar{\rho})(x,y) > \tilde{f}(\bar{\rho})(y,x)$, and by regularity of the social strict preference, $\pi(x,y) > 0$. Hence, $\tilde{f}$ is weakly Paretian. □

As a result of Proposition 4.34, the weighted mean is weakly Paretian. Further, it satisfies stronger Paretian conceptualizations as well.

Definition 4.35 (positive responsiveness). Let $\tilde{f}$ be an FPAR and let π be the social strict preference with respect to $\tilde{f}(\bar{\rho})$, where $\bar{\rho} \in \mathscr{F}\mathscr{R}^n$. Then $\tilde{f}$ satisfies *positive responsiveness* with respect to π if, for all $\bar{\rho},\bar{\rho}' \in \mathscr{FR}^n$ and all $x,y \in X$, $\tilde{f}(\bar{\rho})(x,y) = \tilde{f}(\bar{\rho})(y,x)$ and there exists a $j \in N$ such that $\rho_i = \rho_i'$ for all $i \neq j$ and ($\pi_j(x,y) = 0$ and $\pi_j'(x,y) > 0$ or $\pi_j(y,x) > 0$ and $\pi_j'(y,x) = 0$) imply $\pi'(x,y) > 0$.

In other words, positive responsiveness requires that given a preference profile in which there is no social strict preference between two alternatives x and y, if one individual who has no strict preference for x over y acquires such a preference or who has a strict preference for y over x and loses such a preference, then the FPAR should "respond" and exhibit a social strict preference for x over y. To show that the weighted mean satisfies positive responsiveness, we make use of the following proposition.

Proposition 4.36. *Let $\rho \in \mathscr{FR}$ and let π and $\pi_{(*)}$ be two different types of strict preference with respect to ρ such that for all $x,y \in X$, $\pi(x,y) > 0$ if and only if $\pi_{(*)}(x,y) > 0$. Let $\tilde{f}$ be an FPAR and $\bar{\rho} \in \mathscr{FR}^n$. Then $\tilde{f}$ satisfies positive responsiveness with respect to π if and only if $\tilde{f}$ satisfies positive responsiveness with respect to $\pi_{(*)}$.*

Proof. Suppose $\tilde{f}$ satisfies positive responsiveness with respect to π. Suppose for all $\bar{\rho},\bar{\rho}' \in \mathscr{FR}^n$ and all $x,y \in X$, $\tilde{f}(\bar{\rho})(x,y) = \tilde{f}(\bar{\rho})(y,x)$ and there exists a $j \in N$ such that $\rho_i = \rho_i'$ for all $i \neq j$ and ($\pi_j(x,y) = 0$ and $\pi_j'(x,y) > 0$ or $\pi_j(y,x) > 0$ and $\pi_j'(y,x) = 0$). Because $\pi(x,y) > 0$ if and only if $\pi_{(*)}(x,y) > 0$ and $\pi'(x,y) > 0$ if and only if $\pi_{(*)}'(x,y) > 0$ for all $x,y \in X$, for all $\bar{\rho},\bar{\rho}' \in \mathscr{FR}^n$ and all $x,y \in X$, $\tilde{f}(\bar{\rho})(x,y) = \tilde{f}(\bar{\rho})(y,x)$ and there exists a $j \in N$ such that $\rho_i = \rho_i'$ for all $i \neq j$ and ($\pi_{(*)j}(x,y) = 0$ and $\pi_{(*)}'(x,y) > 0$ or $\pi_{(*)j}(y,x) > 0$ and $\pi_{(*)j}'(y,x) = 0$). Then $\pi(x,y) > 0$ since $\tilde{f}$ satisfies positive responsiveness with respect to π. Thus, $\pi_{(*)}(x,y) > 0$ by hypothesis. Hence, $\tilde{f}$ satisfies positive responsiveness with respect to $\pi_{(*)}$.

Using Proposition 4.36, we can characterize the weighted mean as satisfying positiveness responsiveness with respect to any regular π.

Proposition 4.37. *Let $\tilde{f}$be an FPAR as defined in Definition 4.35. Then $\tilde{f}$ satisfies positive responsiveness with respect to any regular π.*

Proof. By Proposition 4.36, it suffices to show that the weighted mean rule satisfies positive responsiveness with respect to $\pi_{(3)}$, where $\pi_{(3)}(x,y) = \max\{0,(\rho(x,y) - \rho(y,x))\}$. Let $\bar{\rho},\bar{\rho}' \in \mathscr{FR}^n$ and $x,y \in X$. Suppose $\tilde{f}(\bar{\rho})(x,y) = \tilde{f}(\bar{\rho})(y,x)$ and $\rho_i = \rho_i'$ for all $i \in N \backslash \{j\}$. In addition, suppose either

Proof. $\pi_j(x,y) = 0$ and $\pi_j'(x,y) > 0$ or
$\pi_j(y,x) > 0$ and $\pi_j'(y,x) = 0$, □

where strict reference is of type 3. Then $\pi'(x,y) = \max\{0,\rho'(x,y) - \rho'(y,x)\}$, and

$$\begin{aligned}
\rho'(x,y)-\rho'(y,x) &= \sum_{i=1}^{n}(w_i\cdot\rho_i'(x,y)-w_i\cdot\rho_i'(y,x))\\
&= \sum_{i=1,i\neq j}^{n-1}(w_i\cdot\rho_i(x,y)-w_i\cdot\rho_i(y,x))\\
&\quad +w_j\cdot\rho_j'(x,y)-w_j\cdot\rho_j'(y,x)\\
&= \sum_{i=1}^{n}(w_i\cdot\rho_i(x,y)-w_i\cdot\rho_i(y,x))-w_j\cdot(\rho_j(x,y)-\rho_j(y,x))\\
&\quad +w_j\cdot(\rho_j'(x,y)-\rho_j'(y,x))\\
&= -w_j\cdot(\rho_j(x,y)-\rho_j(y,x))+w_j\cdot(\rho_j'(x,y)-\rho_j'(y,x))\\
&> 0,
\end{aligned}$$

where the inequality holds if either (1) or (2) hold. Hence, the weighted mean satisfies positive responsiveness with respect to $\pi_{(3)}$. The desired result now follows from the definition of regularity and Proposition 4.36. □

The weighted mean also satsifies the Pareto Condition under specific definitions of strict preference.

Proposition 4.38. *Let $\tilde{f}$ be an FPAR as defined in Definition 4.35. Then $\tilde{f}$ satisifies the Pareto Condition with respect to $\pi = \pi_{(1)}$ and $\pi = \pi_{(3)}$.*

Proof. Let $x,y \in X$. Let $m_{x,y} = \min_{i\in N}\{\pi_i(x,y)\}$. There is no loss in generality is assuming $m_{x,y} = \pi_1(x,y)$. If $m_{x,y} = 0$, the proof is complete. Suppose otherwise. Then,

$$\begin{aligned}
1 &\leq w_1+\ldots+w_n+w_2(\frac{\pi_2(x,y)}{m_{x,y}}-1)+\ldots+w_n(\frac{\pi_n(x,y)}{m_{x,y}}-1)\\
&= w_1\cdot\frac{\pi_1(x,y)}{m_{x,y}}+\ldots+w_n\cdot\frac{\pi_n(x,y)}{m_{x,y}}.
\end{aligned}$$

Thus,

$$\pi_1(x,y) = m_{x,y} \leq w_1\cdot\pi_1(x,y)+\ldots+w_n\cdot\pi_n(x,y). \tag{4.4}$$

Because $m_{x,y} > 0$, $\pi_i(x,y) > 0$ for all $i \in N$. Thus, $\rho_i(x,y) > \rho_i(y,x)$ for all $i \in N$. Hence,

$$\sum_{i=1}^{n} w_i\cdot\rho_i(x,y) > \sum_{i=1}^{n} w_i\cdot\rho_i(y,x)\ .$$

Suppose $\pi = \pi_{(1)}$. By (3.3), $\rho_1(x,y) \leq \sum_{i=1}^{n} w_i\cdot\rho_i(x,y)$, or $m_{x,y} \leq \pi(x,y) = \rho(x,y)$. Hence $\tilde{f}$ satisfies the Pareto Condition with respect to $\pi_{(1)}$.

Suppose $\pi = \pi_{(3)}$. Similarly by (3.3),

$$\rho_1(x,y) - \rho_1(y,x) \leq \sum_{i=1}^{n} w_i \cdot [\rho_i(x,y) - \rho_i(y,x)] = \sum_{i=1}^{n} w_i \cdot \rho_i(x,y) - \sum_{i=1}^{n} w_i \cdot \rho_i(y,x),$$

or $m_{x,y} \leq \rho(x,y) - \rho(y,x) = \pi(x,y)$. Hence $\tilde{f}$ satisfies the Pareto Condition with respect to $\pi_{(3)}$. □

Unlike Positive Responsiveness, the relationship between the weighted mean and the Pareto Condition in Proposition 4.38 cannot be generalized to the case of any regular strict preference relation. The reason for this is the lack of some type of behavioral assumptions on the relationship between the strict and weak preference relationship, such as monotonicity or $\rho = \iota \cup \pi$ for a specified t-conorm $\cup$. The following example presents a case where the weighted mean aggregation rule does not satisfy the Pareto Condition with respect to a regular strict preference rule.

Example 4.39. Let $X = \{x,y\}, N = \{1,2\}$ and $\tilde{f}$ be an FPAR as defined in Definition 4.35, where $w_i = \frac{1}{2}$ for all $i \in N$. Suppose the strict preference relation is defined as follows:

$$\pi(x,y) = \begin{cases} .3 & \text{if } \rho(x,y) = .6 \text{ and } \rho(y,x) = .4, \\ 0 & \text{if } \rho(y,x) > \rho(x,y), \\ 1 & \text{otherwise.} \end{cases}$$

It is obvious that π is regular. Now consider a profle $\bar{\rho} \in \mathscr{FR}^2$ such that $\rho_1(x,y) = .5$, $\rho_2(x,y) = .7$, and $\rho_1(y,x) = \rho_2(y,x) = .4$. For this profile, the social preference relation is $\rho(x,y) = .6$ and $\rho(y,x) = .4$ because $\tilde{f}$ is the weighted mean. Then the individual and social strict preference relations are as follows:

$$\begin{aligned} \pi_1(x,y) &= 1, \\ \pi_2(x,y) &= 1, \\ \pi(x,y) &= .3, \end{aligned}$$

and the weighted mean does not satsify the Pareto Condition with respect to π although π is regular.

Currently, the only Arrowian condition unaccounted for is max-$*$ transitivity. Because the assumption max-$*$ transitivity is unusually general in the fuzzy framework, the weighted mean does not lend itself to developing one single formal argument detailing whether the FPAR satisfies the condition. Nonetheless, we can use the concept of a zero divsor to determine what type of transitivity conditions to consider.

Example 4.40. Let $\tilde{f}$ be an FPAR that is defined in 4.35. Let $N = \{1,2\}, X = \{x,y,z\}$ and $w_i = \frac{1}{2}$ for all $i \in N$. Suppose $\bar{\rho} \in \mathscr{FR}^2$ is defined as follows:

$$\begin{aligned} \rho_1(x,z) &= \rho_2(x,z) = 0, \\ \rho_1(x,y) &= \rho_2(y,z) = 1, \\ \rho_1(y,z) &= \rho_2(x,y) = 0, \\ \rho_i(z,x) &= \rho_i(y,x) = \rho_i(z,y) = 1, \end{aligned}$$

for all $i \in N$. It is easily verified that ρ_i is max-$*$ transitive under all t-norms using the boundary conditions. Then the social preference relation, $\tilde{f}(\bar{\rho})$, is, by Definition 4.35, as follows:

$$\rho(x,z) = 0,$$
$$\rho(x,y) = \rho(y,z) = .5,$$
$$\rho(z,x) = \rho(y,x) = \rho(z,y) = 1.$$

If the social preference relation is to be max-$*$, then $\rho(x,z) \geq \rho(x,y) * \rho(y,z)$ for all $x,y,z \in X$. If $*$ has no zero divisors, $\rho(x,y) * \rho(y,z) > 0$. However, $\rho(x,z) = 0 \not\geq \rho(x,y) * \rho(y,z)$, a contradiction. Thus, $\tilde{f}$ cannot be max-$*$ transitive when $*$ has no zero divisors.

Example 4.40 suggests that when considering max-$*$ transitivity conditions for the weighted mean rule, we should consider definitions in which $*$ has a zero divisor. If not, it is obvious then that the weighted mean will not satisfy the fuzzy Arrowian condition of transitivity. However, the converse of this relationship is not necessarily true as shown in the following example.

Example 4.41. Suppose $a * b = \begin{cases} \min\{a,b\} & \text{if } a+b > 1, \\ 0 & \text{otherwise.} \end{cases}$.

In this case, $*$ is the nilpotent minimum, and $*$ has a zero divisor. Let $X = \{x,y,z\}$ and $N = \{1,2\}$. Suppose $\bar{\rho} = \{\rho_1, \rho_2\} \in \mathscr{FR}^2$ and is defined as follows:

$$\rho_1(x,y) = .8$$
$$\rho_1(a,b) = .3, \forall (a,b) \in X \times X \backslash \{(x,y)\}, \text{ where } a \neq b$$
$$\rho_2(x,z) = .4;\ \rho_2(y,z) = .8$$
$$\rho_2(a,b) = .5, \forall (a,b) \in X \times X \backslash \{(x,z),(y,z)\}, \text{ where } a \neq b.$$

Suppose $\tilde{f}$ is an FPAR defined in Definition 4.35 and $w_i = \frac{1}{2}$ for all $i \in N$. Then $\tilde{f}(\bar{\rho})(x,z) = .35$, $\tilde{f}(\bar{\rho})(x,y) = .65$, and $\tilde{f}(\bar{\rho})(y,z) = .55$. However, $.35 \not\geq .65 * .55 = .55$. Hence, $\tilde{f}$ is not max-$*$ trasitive when $*$ is the nilpotent minimum.

Given this relationship, we illustrate two transitivity conditions that use a t-norm with zero divisors.

Proposition 4.42. *Let $\tilde{f}$ be a fuzzy aggregation ruled defined in Definition 4.35, and let $\bar{\rho} \in \mathscr{FR}^n$ be max-$*$ transitive. Then $\tilde{f}$ is max-$*$ transitive if, for all $a,b \in [0,1]$, $*$ is defined as follows for all $a,b \in [0,1]$:*

$$(1)\ a * b = \max\{a+b-1, 0\}$$

or

$$(2)\ a * b = \begin{cases} a & \text{if } b = 1, \\ b & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. (1) Let $x, y, z \in X$. Then by max-$*$ transitivity of ρ_i and the definition of $*$, $\rho_i(x,z) \geq \rho_i(x,y) + \rho_i(y,z) - 1$ for all $i \in N$. By definition of $\tilde{f}$,

$$\sum_{i=1}^{n} w_i \cdot \rho_i(x,z) \geq \sum_{i=1}^{n} w_i \cdot \rho_i(x,y) + \sum_{i=1}^{n} w_i \cdot \rho_i(y,z) - \sum_{i=1}^{n} w_i$$
$$\rho(x,z) \geq \rho(x,y) + \rho(y,z) - 1.$$

(2) Let $x, y, z \in X$. Suppose $\rho(x,y) * \rho(y,z) = 0$. Then the proof is complete. Suppose $\rho(x,y) * \rho(y,z) > 0$. Then there are two cases to consider.

a. First, suppose $\rho(x,y) = 1$ and $\rho(y,z) > 0$. Then, $\rho_i(x,y) = 1$ for all $i \in N$, by the definition of $\tilde{f}$. Further, by max-$*$ transitivity of ρ_i, $\rho_i(x,z) \geq \rho_i(x,y) * \rho_i(y,z)$ and $\rho_i(x,z) \geq \rho_i(y,z)$ for all $i \in N$. Hence, $\sum_{i=1}^{n} w_i \cdot \rho_i(x,z) \geq \sum_{i=1}^{n} w_i \cdot \rho_i(y,z)$. Thus, $\rho(x,y) * \rho(y,z) = \rho(y,z) \leq \rho(x,z)$.
b. Second, a similar argument can be made for the case when $\rho(y,z) = 1$ and $\rho(x,y) > 0$. Hence, $\rho(x,z) \geq \rho(x,y) * \rho(y,z)$. □

Proposition 4.42 provides two examples of t-norms under which the weighted mean is max-$*$ transitive. Proposition 4.42(1) uses the Łukasiewicz t-norm, and Proposition 4.42(2) uses the drastic t-norm. Let $H_L, H_D \subset \mathscr{FR}$ be such that H_L and H_D contain all the fuzzy preference relations that are max-$*$ transitive under the Łukasiewicz and drastic t-norm, respectively. We are now able to state two *possibility* results in the fuzzy Arrowian context.

Theorem 4.43. *Let strict preference be regular. Then there exists a nondictatorial $\tilde{f} : H_L^n \to H_L$ or $\tilde{f} : H_D^n \to H_D$ and satisfying IIA-1, Positive Responsiveness and weak Paretianism.*

Proof. Let $\tilde{f}$ be an FPAR as defined in 4.35. The result follows from Propositions 4.36, 4.37 and 4.42, and the immediacy of IIA-1 from the definition of the weighted mean.

By specifying a strict preference relation we can obtain another possibility result that includes an FPAR satsifying the Pareto Condition. □

Theorem 4.44. *Let strict preference be $\pi_{(1)}$ or $\pi_{(3)}$. Then there exists a nondictatorial $\tilde{f} : H_L^n \to H_L$ or $\tilde{f} : H_D^n \to H_D$ and satisfying IIA-1, Positive Responsiveness, weak Paretianism and the Pareto Condition.*

Proof. Let $\tilde{f}$ be an FPAR as defined in Definition 4.35. The result follows from Theorem 4.43 and Proposition 4.42. □

The transitivity conditions in Theorems 4.43 and 4.44 are quite restrictive and can be relaxed given another FPAR.

Definition 4.45. Define the fuzzy aggregation rule $\tilde{f} : \mathscr{FR}^n \to \mathscr{FR}$ as follows. For all $\bar{\rho} \in \mathscr{FR}^n$, all $x, y \in X$ and all $\tau : \mathscr{FR}^n \to (0,1)$,

$$\tilde{f}(\bar{\rho})(x,y) = \begin{cases} 1 & \text{if } x = y, \\ 1 & \text{if } \pi_i(x,y) > 0, \forall i \in N , \\ \tau(\bar{\rho}) & \text{otherwise.} \end{cases}$$

In words, Definition 4.45 is similar to a fuzzy Pareto rule, where the social strict preference for one alternative x over another y is positive if every individual strictly prefers x to y and the social strict preference is regular. The FPAR in Definition 4.45 is clearly reflexive, complete, weakly Paretian and IIA-1. It also satisfies IIA-3. To see that Definition 4.45 satisfies IIA-3 consider the following proposition.

Proposition 4.46. *Let strict preference be regular. Let $\tilde{f}$ be a fuzzy aggregation rule defined in Definition 4.45. Then $\tilde{f}$ is IIA-3.*

Proof. Let $\bar{\rho}, \bar{\rho}' \in \mathscr{FR}^n$ and $x, y \in X$. Suppose $\rho_i\rceil_{\{x,y\}} \sim \rho_i'\rceil_{\{x,y\}}$ for all $i \in N$. Then by Proposition 3.13, $\rho_i(x,y) > \rho_i(y,x)$ if and only if $\rho_i'(x,y) > \rho_i'(x,y)$ for all $i \in N$. By the definition of $\tilde{f}$, $\tilde{f}(\bar{\rho})(x,y) = 1$ if and only if $\tilde{f}(\bar{\rho}')(x,y) = 1$, and $\tilde{f}(\bar{\rho})(y,x) = \tau(\bar{\rho})$ if and only if $\tilde{f}(\bar{\rho}')(y,x) = \tau(\bar{\rho})$. Hence, $\tilde{f}(\bar{\rho})\rceil_{\{x,y\}} \sim \tilde{f}(\bar{\rho}')\rceil_{\{x,y\}}$. □

To see that see when $\tilde{f}(\bar{\rho})$ in Definition 4.45 is max-$*$ transitive, we use a series of propositions that first consider max-min transitivity and then generalize to an arbitrary t-norm.

Proposition 4.47. *Let $\pi = \pi_{(1)}$ and $\rho \in \mathscr{FR}$. If ρ is max-min transitive, then π is max-min transitive.*

Proof. Let $\rho \in \mathscr{FR}$ be such that ρ is max-min transitive, i.e.

$$\rho(x,z) \geq \min\{\rho(x,y), \rho(y,z)\}$$

for all $x, y, z \in X$. This proof will show that

$$\pi(x,z) \geq \min\{\pi(x,y), \pi(y,z)\} .$$

To do so, suppose contrary. Then there exists an $x, y, z \in Z$ such that

$$\rho(x,z) \geq \min\{\rho(x,y), \rho(y,z)\} \tag{4.5}$$

and

$$\pi(x,z) < \min\{\pi(x,y), \pi(y,z)\}. \tag{4.6}$$

Then

$$0 < \pi(x,y) = \rho(x,y) > \rho(y,x) \tag{4.7}$$

and

$$0 < \pi(y,z) = \rho(y,z) > \rho(z,y). \tag{4.8}$$

By Eqs. (4.7) and (4.8), $\rho(x,y) > 0$ and $\rho(y,z) > 0$, which implies $\rho(x,z) > 0$ by Eq. (4.5). Suppose $\pi(x,z) > 0$. Then by definition of $\pi_{(1)}$, $\pi(x,z) = \rho(x,z) \geq \min\{\rho(x,y),\rho(y,z)\} = \min\{\pi(x,y),\pi(y,z)\}$, where the latter equality holds by Eqs. (4.7) and (4.5). Since this contradicts Eq. (4.6), $\pi(x,z) = 0$. Hence,

$$\rho(z,x) \geq \rho(x,z). \tag{4.9}$$

There are now two cases to consider.

Proof. Suppose $\min\{\rho(x,y),\rho(y,z)\} = \rho(x,y)$. Then $\rho(y,z) \geq \rho(x,y)$. Hence, by transitivity,

$$\rho(x,z) \geq \rho(x,y). \tag{4.10}$$

By transitivity, $\rho(y,x) \geq \min\{\rho(y,z),\rho(z,x)\}$. Because $\rho(y,z) \geq \rho(x,y)$ and Eq. (3.8), $\min\{\rho(y,z),\rho(z,x)\} \geq \min\{\rho(x,y),\rho(x,z)\} = \rho(x,y)$. Then $\rho(y,x) \geq \rho(x,y)$; however, this contradicts Eq. (4.7).

Suppose $\min\{\rho(x,y),\rho(y,z)\} = \rho(y,z)$, which implies $\rho(x,y) \geq \rho(y,z)$. Then by transitivity,

$$\rho(x,z) \geq \rho(y,z) \tag{4.11}$$

By transitivity, $\rho(z,y) \geq \min\{\rho(z,x),\rho(x,y)\}$; and $\rho(z,y) \geq \rho(z,x)$, or $\rho(z,y) \geq \rho(x,y)$. If $\rho(z,y) \geq \rho(z,x)$, then $\rho(z,y) \geq \rho(z,x) \geq \rho(x,z) \geq \rho(y,z)$ by Eqs. (4.9) and (4.11). However, this contradicts Eq. (4.8). If $\rho(z,y) \geq \rho(x,y)$, then $\rho(z,y) \geq \rho(x,y) \geq \rho(y,z)$ by the assumption of $\rho(x,y) \geq \rho(y,z)$. However, this also contradicts Eq. (4.8). Thus, $\pi(x,z) \geq \min\{\pi(x,y),\pi(y,z)\}$, and π is also max-min transitive. □

□

Proposition 4.47 demonstrates that when strict preference is of type one, max-min transitivity of an FWPR ρ implies max-min transitivity of the strict preference relation derived from ρ. Like Proposition 4.38 and Example 4.39, this relationship between the max-min transitivity of ρ and $\pi_{(1)}$ cannot be generalized to the case of all regular strict relations because the ordinal concept of strict preference is insufficient for the cardinal concept of max-min transitivity. Nonetheless, assuming that strict preference is of type one allows us to show the max-min transitivity of individual preference relations and obtain the following result.

Proposition 4.48. *Let $\pi = \pi_{(1)}$, $\bar{\rho} \in \mathscr{FR}^n$ and $\tilde{f}$ be an FPAR defined in Definition* (4.45). *Suppose ρ_i is max-min transitive for all $i \in N$. Then $\tilde{f}$ is max-min transitive.*

Proof. Let $x,y,z \in X$. Suppose $\bar{\rho} \in \mathscr{FR}$ be such that ρ_i is max-min transitive for all $i \in N$. If $\min\{\tilde{f}(\bar{\rho})(x,y),\tilde{f}(\bar{\rho})(y,z)\} = \tau(\bar{\rho})$, then the proof is complete. Suppose the contrary. Then $\min\{\tilde{f}(\bar{\rho})(x,y),\tilde{f}(\bar{\rho})(y,z)\} = 1$. By the definition of $\tilde{f}$, $\pi_i(x,y) > 0$ and $\pi_i(y,z) > 0$ for all $i \in N$. By Proposition 4.47, we have $\pi_i(x,z) \geq \min\{\pi_i(x,y),\pi_i(y,z)\}$ for all $i \in N$. Thus, $\pi_i(x,z) > 0$ for all $i \in N$ and $\tilde{f}(\bar{\rho})(x,z) = 1$. Hence, $\tilde{f}$ is max-min transitive. □

To see when $\tilde{f}(\bar{\rho})$ in Definition 4.45 is max-$*$ transitive under any specified t-norm, consider the following proposition, which uses the boundary condition to prove the result.

Proposition 4.49. *Let $\rho \in \mathscr{FR}$ be such that ρ is max-min transitive. Let $*$ be an arbitrary t-norm. Then ρ is max-$*$ transitive.*

Proof. For any $a,b \in [0,1]$, $a * b \leq a * 1 = a$ and $a * b \leq 1 * b = b$ by the boundary condition of $*$. Because $a * b \leq a$ and $a * b \leq b$, $a * b \leq \min\{a,b\}$. Let $x,y,z \in X$. By transitivity of ρ, $\rho(x,z) \geq \min\{\rho(x,y),\rho(y,z)\} \geq \rho(x,y) * \rho(y,z)$. Hence, ρ is max-$*$ transitive. □

We can now state another possibility result with less restrictive transitivity conditions.

Theorem 4.50. *Let $\pi = \pi_{(1)}$. Then there exists a nondictatorial $\tilde{f} : \mathscr{FR}^{*n} \to \mathscr{FR}^{*}$ satisfying IIA-1, IIA-3, weak Paretianism and the Pareto Condition.*

Proof. Let $\tilde{f}$ be defined by Definition 4.45. Clearly, $\tilde{f}$ is reflexive and complete, and it satisfies IIA-1, weak Paretianism and the Pareto Condition. By Proposition 4.46, $\tilde{f}$ is IIA-3, and Proposition 4.49 generalizes Propositions 4.47 and 4.48. Thus, $\tilde{f}$ is max-$*$ transitive. □

Theorem 4.50 achieves a more general possibility result, but using Definition (4.45) has two important consequences. First, individual and social preferences must be max-$*$ transitive under the same t-norm definition. For example, given some $\bar{\rho} \in \mathscr{FR}^n$, it is impossible to guarantee the max-min transitivity of $\tilde{f}(\bar{\rho})$ when $\bar{\rho}$ is only max-$*$ transitive under the drastic t-norm. Second, as illustrated by Dutta (1987), adding the requirement of positive responsiveness to Theorem 4.50 will void the possibility results. This occurs because, when $\tilde{f}(\bar{\rho})(x,y) = \tilde{f}(\bar{\rho})(y,x)$ for some $x,y \in X$, an individual $i \in N$ switching from complete indifference between x and y ($\rho_i(x,y) = \rho_i(y,x)$) to some strict preference between the two ($\rho_i(x,y) \neq \rho_i(y,x)$) does not necessarily imply that the social preference will exhibit strict preference as well ($\tilde{f}(\bar{\rho})(x,y) \neq \tilde{f}(\bar{\rho})(y,x)$).

Even with these two considerations, the importance of Theorems 4.43, 4.44, and 4.50 remains: the fuzzy Arrowian framework allows for the nondictatorial aggregation of fuzzy preferences in a manner that satisfies normative democratic criteria. Further, as Theorem 4.31 demonstrates, the concept of a neutral FPAR can be used to derive an aggregation rule that is unique and not necessarily dictatorial when, in the exact case, neutrality implies dictatorship. Not only do the results in the fuzzy preference framework reveal substantive conclusions that are distinct from previous approaches using exact preferences, but also they suggest that the traditional, negative results of social choice theory are unsubstantiated when groups possess fuzzy preferences.

4.3 Empirical Application II: The Spatial Model and Fuzzy Aggregation

Section (4.2) discussed the difficulty that arises when using FWPRs in empirical analyses. Most often, researchers will not have the necessary data to create individual FWPRs for every member in a group of political actors. However, fuzzy numbers can be used to represent the degree to which an actor views an alternative as ideal i.e., the σ function, and an FWPR can be estimated using such a function. This section further extends the analysis in Section (4.2) by illustrating how a fuzzy preference aggregation rule can be used to predict policy decisons of a group of actors.

In the spatial model, alternatives can be represented by k-dimensional Euclidean space or $\mathbb{R}^k$. When $k = 1$, σ is identical to the fuzzy numbers presented in the previous empirical example, where, for some $x \in X$, $\sigma(x)$ denotes the degree to which x is ideal. In this case, $\sigma_i : \mathbb{R}^1 \to [0,1]$ for all $i \in N$. It is often assumed that σ is *normal*, which requires there exists $x \in X$ such that $\sigma(x) = 1$. In words, normality ensures that every actor views at least one alternative as ideal. Let $\mathscr{F}\mathscr{N}(X)$ denote all the fuzy subsets of X such that the fuzzy subset is normal. When N is the set of actors, it is assumed each actor prossesses a preference function, *preference function profile* can be written as $\bar{\sigma} = (\sigma_1, \ldots, \sigma_n)$.

Table 4.1 Sigma Values of Four Alternatives

	$\sigma_1(\cdot)$	$\sigma_2(\cdot)$	$\sigma_3(\cdot)$
$w = .1$	0	0	0
$x = .5$	0.33	1.0	0
$y = .57$	0.1	1.0	0.1
$z = .68$	0	0.2	0.65

Let $N = \{1,2,3\}$ and let $X = [0,1] \subset \mathbb{R}^1$. Figure 4.1 presents a fairly traditional profile of preference functions over the set of alternatives where no actor possesses more than three areas of discrete indifference. For example, player 1 is indifferent between all alternatives in the intervals $[0,.1]$ and $[.6,1]$ ($\sigma_1(x) = 0$) and between all alternatives in the interval $[.35,.5]$ ($\sigma_1(x) = .33$). The fuzzy numbers presented in Figure 4.1 are sufficient to characterize the degree to which any alternative in X is ideal for all three actors. Table 4.1 provides the sigma values of four alternatives in X. Here, $w \in X$ is outside the support of ideal alternatives for all three players, and $y \in X$ is in the support of ideal alternatives for all three players. In addition, y is in the core of player two's set of ideal alternatives.

Given the fuzzy preference functions in Figure 4.1, we can create preference relations based on the degree to which each alternative is ideal. Section (4.2) gave two examples of such procedures. First, for all $x, y \in X$ and all $\sigma \in \mathscr{F}\mathscr{N}(X)$,

$$\rho_{(G)}(x,y) = \vee\{t \in [0,1] \mid \sigma(y) * t \leq \sigma(x)\},$$

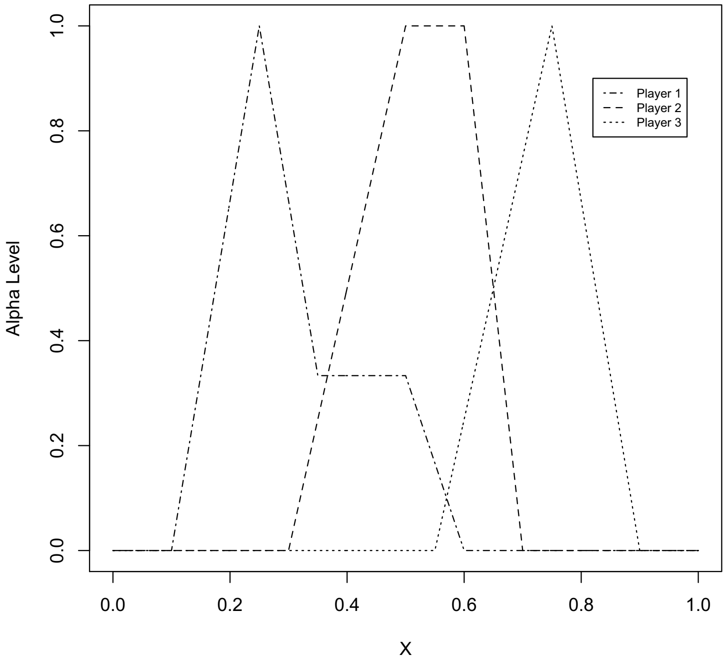


Fig. 4.1 Example of a Three Player Fuzzy Spatial Model

which can be simplified to the following if $* = \min$:

$$\rho_{(G)}(x,y) = \begin{cases} 1 & \text{if } \sigma(x) \geq \sigma(y), \\ \sigma(x) & \text{otherwise.} \end{cases}$$

We have already shown that $\rho_{(G)}$ is reflexive, strongly connected and max-min transitive. Table 4.2 presents the preference profile $\bar{\rho}_{(G)}$ over the four alternatives selected in Table 4.1.

Table 4.2 Inferred FWPRs Using $\rho_{(G)}$

$i=1$	w	x	y	z
w	1	0	0	1
x	1	1	1	1
y	1	.1	1	1
z	1	0	0	1

$i=2$	w	x	y	z
w	1	0	0	0
x	1	1	1	1
y	1	1	1	1
z	1	.2	.2	1

$i=3$	w	x	y	z
w	1	1	0	0
x	1	1	0	0
y	1	1	1	.1
z	1	1	1	1

A second procudure used for inferring an FWPR from a preference function is

$$\rho_{(M)}(x,y) = \begin{cases} 1 & \text{if } x = y, \\ (\sigma(x) - \sigma(y) + c) \wedge 1 & \text{if } \sigma(x) \geq \sigma(y), \\ 1 - [\sigma(y) - \sigma(x) + 1 - c) \wedge 1] & \text{otherwise.} \end{cases}$$

where $c \in [0,1]$ for all $x,y \in X$ and $\sigma \in \mathscr{F}\mathscr{N}(X)$. It is obvious that $\rho_{(M)}$ is reflexive and complete when $c > 0$.We also know $\rho_{(M)}$ is weakly transitive (see Proposition (4.20)). If we set c to a specific value, we can infer another preference profile as well. Let $c = .5$; Table 4.3 illustrates the preference profile $\bar{\rho}_{(M)}$ in this case. In contrast to $\bar{\rho}_{(G)}$, the image of $\bar{\rho}_{(M)}$ contains more elements for each actor than image of $\bar{\rho}_{(G)}$.

Table 4.3 Inferred FWPRs Using $\rho_{(M)}$ when $c = .5$

$i=1$	w	x	y	z
w	1	.17	.4	.5
x	.83	1	.73	.83
y	.6	.27	1	.6
z	.5	.17	.4	1

$i=2$	w	x	y	z
w	1	0	0	.3
x	1	1	.5	1
y	1	.5	1	1
z	.7	0	0	1

$i=3$	w	x	y	z
w	1	.5	.4	0
x	.5	1	.4	0
y	.6	.6	1	0
z	1	1	1	1

We can now apply an FPAR $\tilde{f}$ to the preference profiles $\bar{\rho}_{(G)}$ and $\bar{\rho}_{(M)}$. When $\tilde{f}$ is the weighted mean rule from Definition (4.33), assume $w_i = \frac{1}{3}$ for all $i \in N$. Then Table 4.4 illustrates $\tilde{f}(\bar{\rho}_{(G)})$ and $\tilde{f}(\bar{\rho}_{(M)})$ over the four alternatives $\{w,x,y,z\}$. When $\tilde{f}$ is the fuzzy Pareto rule from Definition (4.45), assume $\tau(\bar{\rho}_{(G)}) = (\bar{\rho}_{(M)}) = .5$, and Table 4.5 reports the results of the fuzzy Pareto rule over the same four alternatives.

Table 4.4 The Weighted Mean Rule Using $\bar{\rho}_{(G)}$ and $\bar{\rho}_{(M)}$ when $w_i = \frac{1}{3}$

$\tilde{f}(\bar{\rho}_{(G)})$	w	x	y	z
w	1	.33	0	.33
x	1	1	.67	.67
y	1	.7	1	.7
z	1	.4	.4	1

$\tilde{f}(\bar{\rho}_{(M)})$	w	x	y	z
w	1	.22	.27	.27
x	.78	1	.54	.61
y	.73	.46	1	.53
z	.73	.39	.47	1

Tables 4.4 and 4.5 reveal an important distinction between the weighted mean and fuzzy Pareto rule. The weighted mean is more susceptible to the specific procedure chosen to infer fuzzy preference relations than the fuzzy Pareto rule. While the fuzzy Pareto rule returns two identical social preference relations regardless of how the inidividual preference relations were created, the weighted mean exhibits significant differences between the social preference relation from $\bar{\rho}_{(G)}$ and the one from $\bar{\rho}_{(M)}$.

We can also calculate the maximal sets from the four newly aggregated social preference relations. In Section (4.2), the fuzzy maximal set is defined as follows: for all $x \in X$,

Table 4.5 The Fuzzy Pareto Rule Using $\bar{\rho}_{(G)}$ and $\bar{\rho}_{(M)}$ when $\tau(\bar{\rho}) = .5$

$\tilde{f}(\bar{\rho}_{(G)})$	w	x	y	z
w	1	.5	.5	.5
x	.5	1	.5	.5
y	1	.5	1	.5
z	.5	.5	.5	1

$\tilde{f}(\bar{\rho}_{(M)})$	w	x	y	z
w	1	.5	.5	.5
x	.5	1	.5	.5
y	1	.5	1	.5
z	.5	.5	.5	1

$$M(\rho,\mu)(x) = \mu(x) * (\circledast(\vee\{t \in [0,1] \mid \mu(w) * \rho(w,x) * t \leq \rho(x,w),\ \forall w \in \text{Supp}(\mu)\})),$$

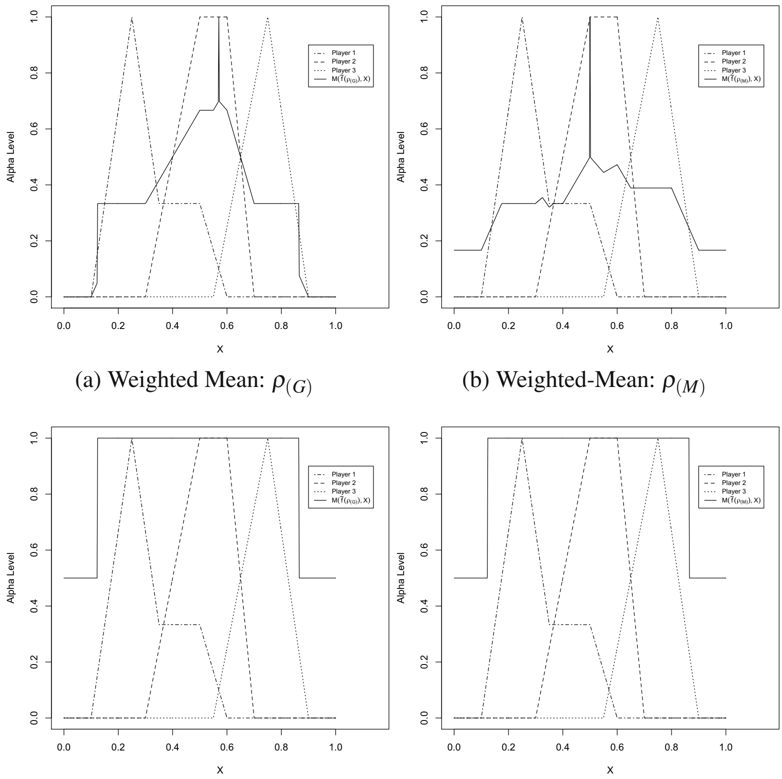
where $\mu \in \mathscr{F}(X)$. $M(\rho,\mu)$ can be simplified by assuming that $\mu(x) = 1$ for all $x \in X$ and $* = \circledast = \min$. The first assumption acknowleges that all alternatives are fully possible. The second merely specifies a t-nrom. With these two assumptions, the maximal set can be written as

$$M(\rho,X)(x) = (\wedge(\vee\{t \in [0,1] \mid \rho(w,x) \wedge t \leq \rho(x,w),\ \forall w \in X\})).$$

As before, $M(\rho,X)(x)$ signifies the degree to which $x \in X$ is a maximal alternative given the FWPR ρ. Let $S = \{w,x,y,z\} \subseteq X$. Then Table 4.6 shows the final calculations for $M(\tilde{f}(\bar{\rho}_{(G)},S)$ and $M(\tilde{f}(\bar{\rho}_{(M)},S)$ where $\tilde{f}$ is either the weighted mean rule or the fuzzy Pareto rule. Furthermore, Figure 4.2 plots the four maximal sets over the entire set of alternatives. As before, the fuzzy Pareto rule returns identical results regardless of the specific profile, and the core of the fuzzy Pareto's maximal set is the support all three players' preference functions. In these cases (Figures 4.2(c) and 4.2(d)), the researcher could predict almost any alternative to be selected by the group of players. In contrast, the core of the weighted mean rule differs from $\bar{\rho}_{(G)}$ and $\bar{\rho}_{(M)}$, which lead to different predictions about what alternative would be selected. In Figure 4.2(a), the core of $\tilde{f}(\bar{\rho}_{(G)})$ is the alternative where all three players' fuzzy preference functions intersect at the maximum degree. In Figure 4.2(b), however, the core of $\tilde{f}(\bar{\rho}_{(M)})$ is the maximum intersection between players 2 and 3, which is the maximum intersection for any two players in the example. Hence, $\tilde{f}(\bar{\rho}_{(G)})$ appears to be more collegial and consensus-driven than $\tilde{f}(\bar{\rho}_{(M)})$ when $\tilde{f}$ is the wieghted mean rule.

Table 4.6 Results for $M(\tilde{f}(\bar{\rho}_{(G)},S)$ and $M(\tilde{f}(\bar{\rho}_{(M)},S)$

	Weighted Mean		Fuzzy Pareto	
	$M(\tilde{f}(\bar{\rho}_{(G)},S)$	$M(\tilde{f}(\bar{\rho}_{(M)},S)$	$M(\tilde{f}(\bar{\rho}_{(G)},S)$	$M(\tilde{f}(\bar{\rho}_{(M)},S)$
w	0	.22	.5	.5
x	0.67	1.0	1	1
y	1.0	.46	1	1
z	0.4	.39	1	1



(a) Weighted Mean: $\rho_{(G)}$ (b) Weighted-Mean: $\rho_{(M)}$

(c) Fuzzy Pareto: $\rho_{(G)}$ (d) Fuzzy Pareto: $\rho_{(M)}$

Fig. 4.2 Maximal Set

Using either FPAR, the procedures described in the definitions of $\rho_{(G)}$ and $\rho_{(M)}$ allow for easy estimation of individual FWPRs without requiring researchers to gather data concerning the degree to which an individual prefers every alternative over every other alternative. When aggregating the individual preference relations, the researcher can choose any number of FPARs, and the maximal set can clearly relate the social preference relation back to individual preference functions. In the example presented in this section, the weighted mean rule generates a maximal set with one alternative in its core while the fuzzy Pareto rule results in a maximal set whose core spans the support of the individual preference function.

References

Arrow, K.: Social Choice and Individual Values. Wiley, New York (1951)

Austen-Smith, D., Banks, J.S.: Positive Political Theory I: Collective Preference. University of Michigan Press, Ann Arbor (1999)

Banerjee, A.: Fuzzy preferences and Arrow-type problems. Social Choice and Welfare 11, 121–130 (1994)

Barrett, C.R., Pattanaik, P.K., Salles, M.: Rationality and aggregation of preferences in an ordinally fuzzy framework. Fuzzy Sets and Systems 49, 9–13 (1992)

Billot, A.: Economic theory of fuzzy equilibria: an axiomatic analysis. Lecture notes in economics and mathematical systems. Springer (1992), http://books.google.com/books?id=ml-7AAAAIAAJ

Black, D.: On Arrow's impossibility theorem. Journal of Law and Economics 12(2), 227–248 (1969), http://EconPapers.repec.org/RePEc:ucp:jlawec:v:12:y:1969:i:2:p:227-48

Blau, J.H.: A direct proof of Arrow's theorem. Econometrica 40(1), 61–67 (1972), http://EconPapers.repec.org/RePEc:ecm:emetrp:v:40:y:1972:i:1:p:61-67

Dasgupta, M., Deb, R.: An impossibility theorem with fuzzy preferences. In: Logic, Game Theory and Social Choice: Proceedings of the International Conference, LGS, vol. 99, pp. 13–16 (1999)

Duddy, C., Perote-Peña, J., Piggins, A.: Arrow's theorem and max-star transitivity. Social Choice and Welfare 36(1), 25–34 (2011), http://dx.doi.org/10.1007/s00355-010-0461-x

Dutta, B.: Fuzzy preferences and social choice. Mathematical Social Sciences 13(3), 215–229 (1987)

Fishburn, P.C.: On collective rationality and a generalized impossibility theorem. Review of Economic Studies 41(4), 445–457 (1974), http://ideas.repec.org/a/bla/restud/v41y1974i4p445-57.html

Fono, L.A., Andjiga, N.G.: Fuzzy strict preference and social choice. Fuzzy Sets Syst. 155, 372–389 (2005), http://dx.doi.org/10.1016/j.fss.2005.05.001

Fono, L.A., Donfack-Kommogne, V., Andjiga, N.G.: Fuzzy Arrow-type results without the pareto principle based on fuzzy pre-orders. Fuzzy Sets and Systems 160(18), 2658–2672 (2009)

Fung, L.W., Fu, K.S.: An axiomatic approach to rational decision making in a fuzzy environment. In: Zadah, L.A., Fu, K.S., Tanaka, K., Shimura, M. (eds.) Fuzzy Sets and Their Applications to Cognitive and Decision Processes, ch. 10, pp. 227–256. Academic Publishers, New York (1975)

García-Lapresta, J.L., Llamazares, B.: Aggregation of fuzzy preferences: Some rules of the mean. Social Choice and Welfare 17(4), 673–690 (2000), http://dx.doi.org/10.1007/s003550000048

Inada, K.I.: Alternative incompatible conditions for a social welfare function. Econometrica 23(4), 396–399 (1955)

Little, I.M.D.: Social choice and individual values. Journal of Political Economy 60, 422 (1952)

May, K.O.: A set of independent necessary and sufficient conditions for simple majority decision. Econometrica 20(4), 680–684 (1952), http://dx.doi.org/10.2307/1907651

Mordeson, J.N., Clark, T.D.: Fuzzy Arrow's theorem. New Mathematics and Natural Computation 05(02), 371–383 (2009), http://www.worldscientific.com/doi/abs/10.1142/S1793005709001362

Ovchinnikov, S.: Social choice and łukasiewicz logic. Fuzzy Sets and Systems 43(3), 275–289 (1991), Aggregation and Best Choices of Imprecise Opinions

Perote-Peña, J., Piggins, A.: Strategy-proof fuzzy aggregation rules. Journal of Mathematical Economics 43(5), 564–580 (2007)
Richardson, G.: The structure of fuzzy preferences: Social choice implications. Social Choice and Welfare 15, 359–369 (1998)
Salles, M.: Fuzzy utility. In: Handbook of Utility Theory: vol. 1: Principles, p. 321. Springer (1998)
Sen, A.K.: Liberty, unanimity and rights. Economica 43(171), 217–245 (1976), http://dx.doi.org/10.2307/2553122
Skala, H.: Arrow's impossibility theorem: Some new aspects. In: Gottinger, H., Leinfellner, W. (eds.) Decision Theory and Social Ethics, Theory and Decision Library, vol. 17, pp. 215–225. Springer, Netherlands (1978), http://dx.doi.org/10.1007/978-94-009-9838-4_11
Ubeda, L.: Neutrality in arrow and other impossibility theorems. Economic Theory 23(1), 195–204 (2004), http://dx.doi.org/10.1007/s00199-002-0353-0