Introduction to Part III
Adaptivity and Model Reduction

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Despite all effort and progress in the numerical techniques to solve PDE-constrained optimization and control problems, the cost for their solution is still substantially higher than that for solving the associated forward problem for the PDE. In a practical situation, the associated computational work and memory requirement may still be too high to be acceptable, e.g., in an engineering design process. Therefore, further techniques are needed to reduce the computational cost.

In the section "*Adaptivity and Model Reduction*" of this book, two different techniques are discussed for reducing the complexity of solving PDE-constrained optimization problems numerically. One possibility is to use tailored discretizations, e.g., finite element (FE) Galerkin methods, that adapt the mesh size locally according to the optimization goal. This usually leads to meshes different from a possibly optimal, adapted finite element mesh used for solving the forward problem alone. High accuracy of the PDE solution may not be necessary in the same regions as needed for an accurate computation of the cost functional of the optimization problem and the associated (sub)optimal control. Also, a changing control input during an optimization algorithm leads to different PDE solutions that may require different locally refined meshes, necessitating the adaptation of the mesh during the optimization procedure. Therefore, mesh adaptivity should be based on error bounds taking this goal-orientation into account. A further reduction of the computational cost may be achieved by adaptive stopping criteria providing a proper balancing

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G. Leugering et al. (eds.), *Trends in PDE Constrained Optimization*, International
Series of Numerical Mathematics 165, DOI 10.1007/978-3-319-05083-6_16

with the discretization error for the various outer and inner algebraic iterations in solving the discretized optimization problems.

A different approach for reducing the complexity of solving PDE-constrained optimization problems consists in model order reduction, i.e., the application of mathematical methods for automatically reducing the state-space dimension of the control problem while preserving the accuracy in the map from the input function or design parameters to the optimized quantity-of-interest. Like in adaptive FE methods, reduced-order models generated to accelerate the simulation of a PDE may not be sufficient in the optimization context: in an iterative optimization algorithm, the control function changes from step to step, and this variation of input may not be covered by a snapshot-based model reduction method like proper orthogonal decomposition (POD) that is based on a pre-defined training input. On the other hand, re-computing a reduced-order model in each optimization step may become too expensive. Thus, the construction of the reduced-order model should reflect the optimization goal.

The section "*Adaptivity and Model Reduction*" consists of four papers, two dealing with local mesh adaptation for optimal control problems and two are concerned with model reduction techniques. In the survey paper "*Model reduction by adaptive discretization in optimal control*" by Rannacher, an overview is given of goal-oriented adaptive FE methods for PDE-constrained optimization problems. For problems with singularities, a quasi-uniform mesh refinement is known to result in a reduced order of convergence. In the survey "*Graded meshes in optimal control for elliptic partial differential equations*" by Apel, Pfefferer and Rösch, the strategy of local mesh grading, known to recover the full convergence order for the forward problem, is discussed for elliptic optimal control problems. Regarding model reduction, the paper "*Model order reduction for PDE-constrained optimization*" by Benner, Sachs, and Volkwein provides a survey on approaches based on reduced-order models for solving PDE-constrained optimization problems. Due to the above-mentioned shortcomings of traditional model reduction methods used in forward simulation, special model management strategies are required. These either update the reduced-order model from a previous step or determine when a new reduced-order model must be computed in an optimization loop. Two successful strategies are discussed serving these purposes, adaptive POD and trust-region POD. Moreover, the application of snapshot-free methods based on system-theoretical considerations, having a wide validity range w.r.t. input variations, to PDE-constrained optimization problems is also considered. The use of trust-region POD is also the topic of "*Adaptive trust-region POD methods in PDE-constrained optimization*" by Sachs, Schneider, and Schu. Here, this model reduction strategy is extended to solving optimization problems subject to partial integro-differential equations such as occurring in calibration problems for the pricing of financial derivatives.