# Chapter 3
# Governing Equations

The governing equations are presented in the PDE form solved numerically by finite-difference methods as well as the integral form solved numerically by finite-volume methods. In addition, the quasi-one-dimensional Euler equations and the shock-tube problem are given, along with a means for obtaining their exact solutions. These form the basis of the programming assignments in this and subsequent chapters.

## 3.1 The Euler and Navier-Stokes Equations

### *3.1.1 Partial Differential Equation Form*

Flow of a continuum fluid is governed by a set of partial differential equations collectively known as the Navier-Stokes equations.[1] They can be written in various different forms. We present the following form, known as conservative form, because it is advantageous for numerical solution, as we shall see later, and restrict our interest to two-dimensional Cartesian coordinates for simplicity of exposition. Extension to three dimensions is straightforward. In two dimensions, there are four equations, representing the conservation of mass, two components of momentum, and energy. For an unsteady compressible flow, these can be written as follows:

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = \frac{\partial E_{\mathrm{v}}}{\partial x} + \frac{\partial F_{\mathrm{v}}}{\partial y}, \tag{3.1}$$

[1] Formally, the Navier-Stokes equations are the equations arising from the conservation of momentum; they do not include the equations describing conservation of mass and energy. We follow the prevailing usage and term the whole set the Navier-Stokes equations.

T. H. Pulliam and D. W. Zingg, *Fundamental Algorithms in Computational Fluid Dynamics*, Scientific Computation, DOI: 10.1007/978-3-319-05053-9_3,

where

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ e \end{bmatrix}, \quad E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ u(e+p) \end{bmatrix}, \quad F = \begin{bmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ v(e+p) \end{bmatrix}, \tag{3.2}$$

$$E_{\mathrm{v}} = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ f_4 \end{bmatrix}, \quad F_{\mathrm{v}} = \begin{bmatrix} 0 \\ \tau_{xy} \\ \tau_{yy} \\ g_4 \end{bmatrix}. \tag{3.3}$$

The variable $Q$ represents the conservative dependent variables per unit volume, including the density, $\rho$, the components of momentum per unit volume, $\rho u$ and $\rho v$, where $u$ and $v$ are the Cartesian velocity components, and the total energy per unit volume, $e$. The total energy includes internal and kinetic energy and can be written as

$$e = \rho \left( \epsilon + \frac{u^2 + v^2}{2} \right), \tag{3.4}$$

where $\epsilon$ is the internal energy per unit mass. The vectors $E$ and $F$ are known as the inviscid flux vectors. They contain convective fluxes plus terms associated with pressure. For some flow problems, other terms, such as gravitational forces, can be important and should be included. In the momentum equations, the pressure terms represent forces; in the energy equation they are associated with the work done by the pressure forces. Although it is important to understand these equations as conservation laws for mass, momentum, and energy, it is also instructive to recognize that the momentum equations are an expression of the fact that in an inertial reference frame, the time rate of change of momentum of a particle or collection of particles is equal to the net force acting on the particle or collection of particles. In other words, the momentum equations are a statement of Newton's second law, force equals mass times acceleration.

We will restrict our attention here to thermally and calorically perfect gases, giving the relations

$$p = \rho RT \tag{3.5}$$

and

$$\epsilon = c_{\mathrm{v}} T, \tag{3.6}$$

where $p$ is the pressure, $R$ is the specific gas constant, $T$ is the temperature, and $c_{\mathrm{v}}$ is the specific heat capacity at constant volume. The equation of state enables the pressure to be expressed in terms of the conservative flow variables as follows:

$$p = \rho RT \tag{3.7}$$

$$= \rho R \left( \frac{\epsilon}{c_\mathrm{v}} \right) \tag{3.8}$$

$$= (\gamma - 1)\rho\epsilon \tag{3.9}$$

$$= (\gamma - 1)\left( e - \frac{\rho}{2}(u^2 + v^2) \right) \tag{3.10}$$

$$= (\gamma - 1)\left[ e - \frac{1}{2\rho}\left( (\rho u)^2 + (\rho v)^2 \right) \right], \tag{3.11}$$

where $\gamma$ is the ratio of specific heats, $c_\mathrm{p}/c_\mathrm{v}$, $c_\mathrm{p}$ is the specific heat capacity at constant pressure, and we have used the relation

$$c_\mathrm{v} = \frac{R}{\gamma - 1}. \tag{3.12}$$

For a perfect gas, the speed of sound, $a$, satisfies the relations

$$a^2 = \frac{\gamma p}{\rho} = \gamma R T. \tag{3.13}$$

Alternative equations of state must be used under conditions when the perfect gas law does not apply, such as flows at very high temperatures.

The vectors $E_\mathrm{v}$ and $F_\mathrm{v}$ include terms associated with viscosity and heat conduction. We will consider Newtonian fluids here, but the reader is reminded that this assumption is not universally applicable. For a Newtonian fluid, the viscous stresses are given in two dimensions by

$$\begin{aligned}
\tau_{xx} &= \mu \left( \frac{4}{3}\frac{\partial u}{\partial x} - \frac{2}{3}\frac{\partial v}{\partial y} \right), \\
\tau_{xy} &= \mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right), \\
\tau_{yy} &= \mu \left( -\frac{2}{3}\frac{\partial u}{\partial x} + \frac{4}{3}\frac{\partial v}{\partial y} \right),
\end{aligned} \tag{3.14}$$

where $\mu$ is the dynamic viscosity, which is typically a function of temperature, and for air can often be determined using Sutherland's law. The viscous terms appearing in the momentum equations are forces. The terms $f_4$ and $g_4$ in the energy equation represent the work done by the viscous forces as well as heat conduction.

Heat conduction is governed by Fourier's law, which states that the local heat flux, which is the rate of flow of heat per unit area per unit time, is directly proportional to the local gradient of the temperature. The constant of proportionality, $k$, is known as the thermal conductivity. Based on Fourier's law, the heat conduction terms can be written in two-dimensional Cartesian coordinates as

$$\frac{\partial}{\partial x}\left( k \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y}\left( k \frac{\partial T}{\partial y} \right). \tag{3.15}$$

It is convenient to introduce the Prandtl number, $Pr$, which is the ratio of kinematic viscosity to thermal diffusivity. It is given by

$$Pr = \frac{\mu c_{\mathrm{p}}}{k}. \tag{3.16}$$

This dimensionless number depends on the properties of the fluid. For air, the Prandtl number is close to 0.71 for a wide range of temperatures. For a perfect gas, the heat conduction terms can thus be written as

$$\frac{\partial}{\partial x}\left(\frac{\mu}{Pr(\gamma-1)}\frac{\partial a^2}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\mu}{Pr(\gamma-1)}\frac{\partial a^2}{\partial y}\right), \tag{3.17}$$

where we have used the relation

$$c_{\mathrm{p}} = \frac{\gamma R}{\gamma - 1}. \tag{3.18}$$

Hence we obtain the following expressions for the terms $f_4$ and $g_4$ in the energy equation:

$$\begin{aligned} f_4 &= u\tau_{xx} + v\tau_{xy} + \frac{\mu}{Pr(\gamma-1)}\frac{\partial a^2}{\partial x}, \\ g_4 &= u\tau_{xy} + v\tau_{yy} + \frac{\mu}{Pr(\gamma-1)}\frac{\partial a^2}{\partial y}. \end{aligned} \tag{3.19}$$

It is often convenient to non-dimensionalize the equations. In order to do so, we require a reference length, $l$, normally chosen as some characteristic physical length scale in the problem, a reference density, $\rho_\infty$, often chosen for an external flow as the density of the undisturbed fluid far from the body, and a reference velocity scale. It is traditional in fluid dynamics to choose a velocity scale such as $u_\infty$, the velocity of the body moving through the fluid. For our purpose here, it is more convenient to use $a_\infty$, the speed of sound in the undisturbed air far from the body, since $u_\infty$ could be zero for some flow problems, such as a helicopter in hover. The conditions far from the body are often called free stream conditions. With these reference quantities, we obtain the following non-dimensional quantities (indicated by the tilde):

$$\begin{aligned} &\widetilde{x} = \frac{x}{l}, \quad \widetilde{y} = \frac{y}{l}, \quad \widetilde{t} = \frac{ta_\infty}{l}, \\ &\widetilde{\rho} = \frac{\rho}{\rho_\infty}, \quad \widetilde{u} = \frac{u}{a_\infty}, \quad \widetilde{v} = \frac{v}{a_\infty}, \\ &\widetilde{e} = \frac{e}{\rho_\infty a_\infty^2}, \quad \widetilde{\mu} = \frac{\mu}{\mu_\infty}. \end{aligned} \tag{3.20}$$

Substituting these non-dimensional quantities into the Navier-Stokes equations, dropping the tildes, and defining the Reynolds number as

$$Re = \frac{\rho_\infty l a_\infty}{\mu_\infty}, \tag{3.21}$$

we obtain the following non-dimensional form of the equations:

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = Re^{-1}\left(\frac{\partial E_{\mathrm{v}}}{\partial x} + \frac{\partial F_{\mathrm{v}}}{\partial y}\right), \tag{3.22}$$

where all terms are as previously defined except in terms of non-dimensional quantities. It is important to note that this definition of the Reynolds number based on $a_\infty$ differs from the conventional definition based on $u_\infty$. The two are related by the free stream Mach number, $M_\infty = u_\infty / a_\infty$.

The Euler equations are obtained from the Navier-Stokes equations by neglecting the terms associated with viscosity and heat conduction, i.e. setting $E_{\mathrm{v}}$ and $F_{\mathrm{v}}$ to zero. Numerical solutions of the Euler equations can be useful if the effect of viscosity and heat conduction on the quantities of interest is small. There are many other simplified forms of the Navier-Stokes equations that can be useful for specific classes of problems. It is important that their limitations be well understood.

We stated earlier that the above equations are in conservative form. There are two aspects to this. The first is that we choose the conserved quantities, mass, momentum, and energy, per unit volume as the dependent variables. It is also possible to write a system of equations in terms of other variables, such as the *primitive* variables, density, velocity, and pressure, that is analytically equivalent but can lead to different solutions when solved numerically. For example, for a perfect gas the one-dimensional Euler equations can be written in terms of the *primitive* variables $R = [\rho, u, p]^{\mathrm{T}}$ as follows:

$$\frac{\partial R}{\partial t} + \tilde{A}\frac{\partial R}{\partial x} = 0\,, \tag{3.23}$$

where

$$\tilde{A} = \begin{bmatrix} u & \rho & 0 \\ 0 & u & \rho^{-1} \\ 0 & \gamma p & u \end{bmatrix}.$$

The second aspect is related to the products appearing in the fluxes. In the conservative form, the product rule of differentiation is not applied. A term such as

$$\frac{\partial}{\partial x}(\rho u)$$

appearing in the mass conservation equation is not expanded as

$$\rho \frac{\partial u}{\partial x} + u \frac{\partial \rho}{\partial x},$$

which is in non-conservative form. Again the two forms are analytically equivalent, but under some circumstances, such as flows with nonstationary shock waves, an algorithm that is not conservative can produce substantially inaccurate solutions.

Although we do not normally solve non-conservative forms of the equations, they can be useful for analysis. For example, consider the one-dimensional Euler equations in conservative form:

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} = 0, \tag{3.24}$$

where

$$Q = \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} = \begin{bmatrix} \rho \\ \rho u \\ e \end{bmatrix}, \quad E = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ u(e+p) \end{bmatrix}. \tag{3.25}$$

If the solution is smooth, (3.24) can be rewritten in the following form:

$$\frac{\partial Q}{\partial t} + A \frac{\partial Q}{\partial x} = 0\ , \tag{3.26}$$

where

$$A = \frac{\partial E}{\partial Q} \tag{3.27}$$

is known as the flux Jacobian. The flux Jacobian is derived by first writing the flux vector in terms of the conservative variables

$$E = \begin{bmatrix} Q_2 \\ \\ (\gamma - 1)Q_3 + \frac{3-\gamma}{2}\frac{Q_2^2}{Q_1} \\ \\ \gamma \frac{Q_3 Q_2}{Q_1} - \frac{\gamma-1}{2}\frac{Q_2^3}{Q_1^2} \end{bmatrix}, \tag{3.28}$$

which gives, for a perfect gas,

$$A = \frac{\partial E_i}{\partial Q_j} = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}\left(\frac{Q_2}{Q1}\right)^2 & (3-\gamma)\frac{Q_2}{Q_1} & \gamma - 1 \\ A_{31} & A_{32} & \gamma\left(\frac{Q_2}{Q_1}\right) \end{bmatrix}, \tag{3.29}$$

where

$$A_{31} = (\gamma - 1)\left(\frac{Q_2}{Q_1}\right)^3 - \gamma\left(\frac{Q_3}{Q_1}\right)\left(\frac{Q_2}{Q_1}\right)$$

$$A_{32} = \gamma\left(\frac{Q_3}{Q_1}\right) - \frac{3(\gamma - 1)}{2}\left(\frac{Q_2}{Q_1}\right)^2. \tag{3.30}$$

This can be rewritten in terms of $\rho$, $u$, and $e$ as

$$A = \begin{bmatrix} 0 & 1 & 0 \\ \frac{\gamma-3}{2}u^2 & (3-\gamma)u & \gamma - 1 \\ A_{31} & A_{32} & \gamma u \end{bmatrix}, \tag{3.31}$$

where

$$A_{31} = (\gamma - 1)u^3 - \gamma\frac{ue}{\rho}$$

$$A_{32} = \gamma\frac{e}{\rho} - \frac{3(\gamma - 1)}{2}u^2. \tag{3.32}$$

The eigenvalues of the flux Jacobian $A$ are $u$, $u + a$, $u - a$. Since these are all real, and the eigenvectors of $A$ are linearly independent, the system (3.26) is *hyperbolic*. Hence some important properties of these equations can be obtained from characteristic theory. First, the eigenvalues represent the characteristic speeds at which information is propagated. The convection of the fluid propagates information at speed $u$, while sound waves propagate information at speeds $u + a$ and $u - a$. If the flow is supersonic, i.e. $|u| > a$, then all of the eigenvalues have the same sign, and information is propagated in one direction only. If the flow is subsonic, i.e. $|u| < a$, then the eigenvalues are of mixed sign, and information is propagated in both directions. This is critical in the design of numerical methods and in the development of boundary conditions. Riemann invariants can be found that are propagated at the characteristic speeds, as long as the solution remains smooth. The entropy $\ln(p/\rho^\gamma)$ propagates at speed $u$, while the quantities $u \pm 2a/(\gamma - 1)$ propagate at speeds $u \pm a$.

The flux Jacobian $A$ in (3.26) is related to the matrix $\tilde{A}$ in (3.23) by the following similarity transform:

$$A = S\tilde{A}S^{-1}, \tag{3.33}$$

where $S = \partial Q/\partial R$. Hence the eigenvalues of the two matrices are identical, consistent with the fact that (3.26) and (3.23) are different representations of the same physical processes.

### 3.1.2 Integral Form

The Navier-Stokes equations governing an unsteady compressible flow can also be written in the following *integral* form in two-dimensional Cartesian coordinates:

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{V(t)} Q\mathrm{d}x\mathrm{d}y + \oint_{S(t)}(E\mathrm{d}y - F\mathrm{d}x) = Re^{-1}\oint_{S(t)}(E_{\mathrm{v}}\mathrm{d}y - F_{\mathrm{v}}\mathrm{d}x) \quad (3.34)$$

for an arbitrary control volume $V(t)$ bounded by the surface $S(t)$, with all variables as defined and non-dimensionalized previously. This form is obtained from the more general coordinate-free form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} Q\mathrm{d}V + \oint_{S(t)} \hat{n}\cdot\mathcal{F}\mathrm{d}S = 0, \quad (3.35)$$

where $\hat{n}$ is the unit vector normal to the surface pointing outwards, and $\mathcal{F}$ is the flux tensor, including inviscid, viscous, and heat conduction terms. In two-dimensional Cartesian coordinates, the flux tensor is given by

$$\mathcal{F} = (E - Re^{-1}E_{\mathrm{v}})\hat{i} + (F - Re^{-1}F_{\mathrm{v}})\hat{j}, \quad (3.36)$$

where $\hat{i}$ and $\hat{j}$ are unit vectors in the $x$ and $y$ directions, respectively. The contour in (3.34) is traversed in a counter-clockwise direction; hence the area-weighted outward normal can be written as

$$\hat{n}\mathrm{d}S = \hat{i}\mathrm{d}y - \hat{j}\mathrm{d}x. \quad (3.37)$$

### 3.1.3 Physical Boundary Conditions

The physical boundary conditions that must be satisfied at a rigid body surface are as follows. For an inviscid flow governed by the Euler equations, the flow must be tangent to the surface; in other words, the velocity component normal to the surface must be zero:

$$(u\hat{i} + v\hat{j})\cdot\hat{n} = 0\,. \quad (3.38)$$

For viscous flows governed by the Navier-Stokes equations, the no-slip condition must be satisfied at the surface: all components of velocity must be zero. In addition, for viscous flows, it is normally assumed that the surface is either held at a fixed temperature or is adiabatic. In the latter case, the gradient of the temperature in a direction normal to the surface is zero at the surface:

$$\nabla T\cdot\hat{n} = 0. \quad (3.39)$$

Other physical boundary conditions can vary from problem to problem. For external flow problems, there is often a requirement that as the distance from the body approaches infinity, the flow must approach its undisturbed state. This condition is usually applied at a boundary some finite distance from the body. Other problems may involve specified incoming flows.

## 3.2 The Reynolds-Averaged Navier-Stokes Equations

When the Navier-Stokes equations are time-averaged over a time interval that is long in comparison with the turbulent time scales but short in comparison to other physical time scales, apparent stresses known as Reynolds stresses as well as additional heat flux terms appear. It is the function of a turbulence model, which typically involves the solution of one or more partial differential equations, to furnish these additional terms and thereby to provide closure to the system. For the remainder of this book, all algorithms will be presented in the context of the Euler and Navier-Stokes equations rather than the Reynolds-averaged Navier-Stokes (RANS) equations, although these algorithms are routinely used for the RANS equations. In order to apply these algorithms to the RANS equations, the Reynolds stresses must be added to the Navier-Stokes equations in the form given by the particular turbulence model selected, and the solution algorithm must be applied to any partial differential equations associated with the turbulence model.

## 3.3 The Quasi-One-Dimensional Euler Equations and the Shock-Tube Problem

The quasi-one-dimensional Euler equations and the shock-tube problem are used throughout this book as examples and in the programming assignments. The quasi-one-dimensional Euler equations govern the inviscid flow in a quasi-one-dimensional channel with varying cross-sectional area per unit depth $S(x)$ and can be written as follows [1]:

$$\frac{\partial(\rho S)}{\partial t} + \frac{\partial(\rho u S)}{\partial x} = 0, \tag{3.40}$$

$$\frac{\partial(\rho u S)}{\partial t} + \frac{\partial[(\rho u^2 + p)S]}{\partial x} = p\frac{\mathrm{d}S}{\mathrm{d}x}, \tag{3.41}$$

$$\frac{\partial(eS)}{\partial t} + \frac{\partial[u(e+p)S]}{\partial x} = 0, \tag{3.42}$$

where the variables $t, x, \rho, u, p$, and $e$ have the same definitions as in Sect. 3.1. These are typically solved for a steady flow in a channel with prescribed boundary conditions.

The shock-tube problem is an initial-value problem. Viscosity is again neglected, and the above equations are solved with $S(x) = 1$. The initial conditions are such that there are two initial fluid states separated by a diaphragm at $t = 0$. These are typically quiescent with different pressures and densities. Using $x_0$ to represent the location of the diaphragm and subscripts $L$ and $R$ to indicate the fluid states to the left and right of the diaphragm, the initial conditions can be written as

$$u = 0,\ p = p_L,\ \rho = \rho_L,\ x < x_0 \tag{3.43}$$

$$u = 0,\ p = p_R,\ \rho = \rho_R,\ x \geq x_0\,. \tag{3.44}$$

When the diaphragm is removed instantaneously, a flow is initiated in the direction from high pressure to low. For the example given later in the section, where $p_R < p_L$, a contact discontinuity separating the original two states propagates to the right, an expansion wave propagates to the left, and a shock wave propagates to the right at a speed higher than that of the contact surface. We assume that the process is terminated before any of these waves reach the ends of the shock tube. Hence boundary conditions are not required.

### *3.3.1 Exact Solution: Quasi-One-Dimensional Channel Flow*

We present the equations needed to write a computer program to determine the exact solution for a quasi-one-dimensional channel flow as a reference solution for comparison with numerical solutions. The relevant theory and explanation can be found in most good gasdynamics textbooks (see Shapiro [1] for example). A problem is defined by specifying the channel area variation, $S(x)$, the total pressure and temperature at the inlet, $p_{01}$ and $T_{01}$, the critical area, $S^*$, an indication of whether the initial Mach number is subsonic or supersonic, and a shock location, $x_{\text{shock}}$, if applicable. The solution is calculated by marching from inlet to outlet. At a given $x$ location, both $S$ and $S^*$ are known, so the local Mach number, $M = u/a$, can be calculated from the following nonlinear equation using an iterative technique:

$$\frac{S}{S^*} = \frac{1}{M}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2}M^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}\,. \tag{3.45}$$

A subsonic or supersonic initial Mach number guess should be used, depending on the problem specification. The temperature and pressure can then be determined from the isentropic relations:

$$T = \frac{T_{01}}{1 + \frac{\gamma-1}{2}M^2} \tag{3.46}$$

$$p = p_{01}\left(1 + \frac{\gamma - 1}{2}M^2\right)^{-\left(\frac{\gamma}{\gamma-1}\right)}. \tag{3.47}$$

Other variables, such as density, velocity, and sound speed, can be calculated using the perfect gas relations and the definition of the Mach number. Once the specified location of the shock is reached, if applicable, the Rankine-Hugoniot relations are used to find the conditions downstream of the shock:

$$T_{0R} = T_{0L} \tag{3.48}$$

$$M_R^2 = \frac{2 + (\gamma - 1)M_L^2}{2\gamma M_L^2 - (\gamma - 1)} \tag{3.49}$$

$$\frac{p_R}{p_L} = \frac{2\gamma M_L^2 - (\gamma - 1)}{\gamma + 1} \tag{3.50}$$

$$\frac{p_{0R}}{p_{0L}} = \frac{\left([(\gamma + 1)/2]M_L^2/\{1 + [(\gamma - 1)/2]M_L^2\}\right)^{\frac{\gamma}{\gamma-1}}}{\{[2\gamma/(\gamma + 1)]M_L^2 - (\gamma - 1)/(\gamma + 1)\}^{\frac{1}{\gamma-1}}}. \tag{3.51}$$

The density and sound speed downstream of the shock can then be found using the perfect gas relations. The value of $S^*$ must also be recalculated to correspond to conditions downstream of the shock from:

$$S_R^* = S_L^* \frac{\rho_L^* a_L^*}{\rho_R^* a_R^*}, \tag{3.52}$$

where

$$\rho_{01} = \frac{p_{01}}{RT_{01}}$$

$$\rho_0^R = \frac{p_0^R}{RT_{01}}$$

$$a_{01} = \sqrt{\frac{\gamma p_{01}}{\rho_{01}}}$$

$$a_0^R = \sqrt{\frac{\gamma p_0^R}{\rho_0^R}}$$

$$\rho_L^* a_L^* = \rho_{01} a_{01}\left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

$$\rho_R^* a_R^* = \rho_0^R a_0^R \left(\frac{2}{\gamma + 1}\right)^{\frac{\gamma+1}{2(\gamma-1)}}.$$

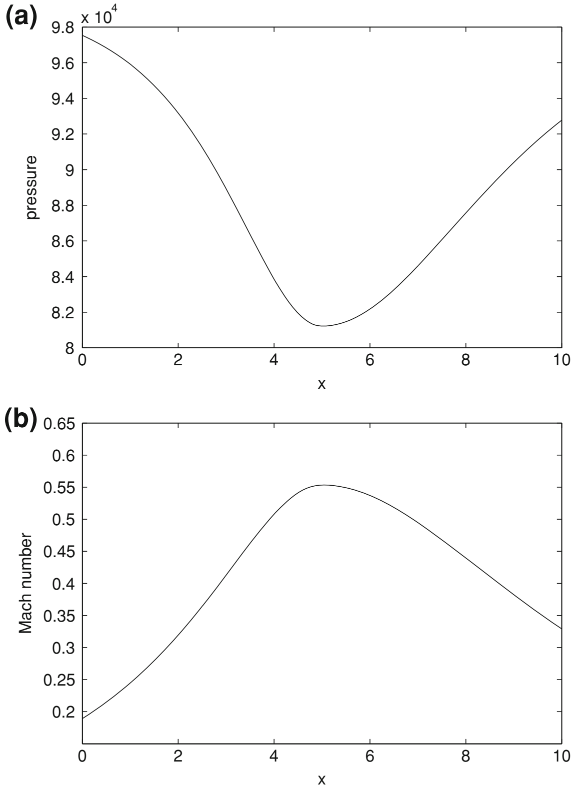


**Fig. 3.1** Exact solution for the subsonic channel flow problem. **a** Pressure (in Pa). **b** Mach number

The solution downstream of the shock can then be calculated using (3.45) with these new values of $S^*$ and $p_0$.

We will consider two examples from Hirsch [2]. In both cases, $S(x)$ is given by

$$S(x) = \begin{cases} 1 + 1.5\left(1 - \frac{x}{5}\right)^2 & 0 \leq x \leq 5 \\ 1 + 0.5\left(1 - \frac{x}{5}\right)^2 & 5 \leq x \leq 10 \end{cases} \tag{3.53}$$

where $S(x)$ and $x$ are in meters. In both cases, the fluid is air, which is considered to be a perfect gas with $R = 287\ \mathrm{N \cdot m \cdot kg^{-1} \cdot K^{-1}}$, and $\gamma = 1.4$, the total temperature is $T_0 = 300$ K, and the total pressure at the inlet is $p_{01} = 100$ kPa. For the first case, the flow is subsonic throughout the channel, with $S^* = 0.8$. The pressure and Mach number for this case are plotted in Fig. 3.1. For the second case, the flow is transonic, with subsonic flow at the inlet, a shock at $x = 7$, and $S^* = 1$. The pressure and Mach number for this case are plotted in Fig. 3.2.

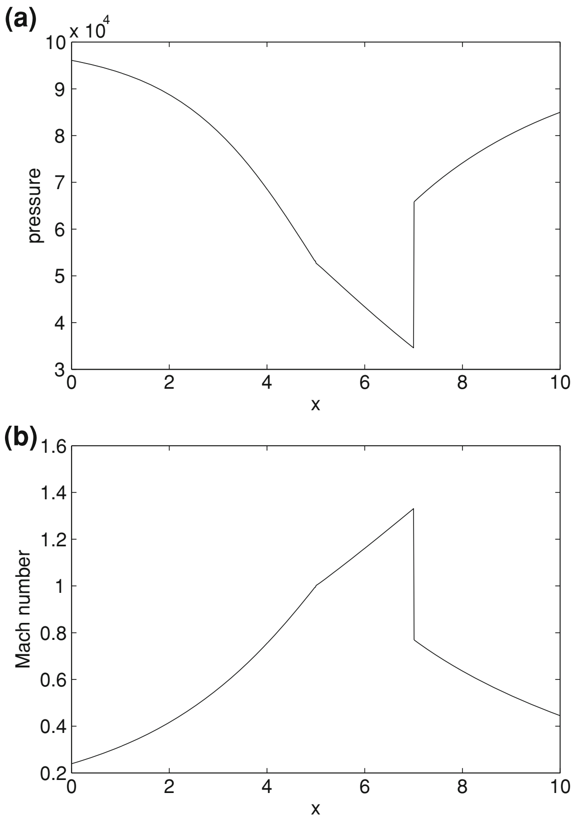


**Fig. 3.2** Exact solution for the transonic channel flow problem. **a** Pressure (in Pa). **b** Mach number

## *3.3.2 Exact Solution: Shock-Tube Problem*

As in the previous section, we present without explanation the equations needed to solve a shock-tube problem. See Hirsch [2] for more details. We assume initial conditions as described earlier in Sect. 3.3, which lead to a solution with an expansion wave traveling to the left, a contact surface moving to the right at speed $V$, and a shock wave moving to the right at a speed $C$, where $C > V$. We thus define the following states: The state to the left of the head of the expansion fan is denoted by the subscript $L$; it is the original quiescent state to the left of the diaphragm. The state within the expansion wave, where the variables vary continuously, is denoted by the subscript 5. The constant state between the tail of the expansion fan and the contact surface is denoted by the subscript 3. The constant state between the contact surface and the shock wave is denoted by the subscript 2. Finally, the quiescent state to the right of the shock, which is the original state to the right of the diaphragm, is denoted by the subscript $R$.

The normal shock relations must hold across the shock. Following Hirsch [2], we define the pressure ratio across the shock as $P = p_2/p_R$. Across the contact surface, pressure and velocity are continuous. The flow in the expansion wave is isentropic, and characteristic theory can be applied. After some algebra, the following implicit equation is found which must be solved for P:

$$\sqrt{\frac{2}{\gamma(\gamma-1)}}\frac{P-1}{\sqrt{1+\alpha P}} = \frac{2}{\gamma-1}\frac{a_L}{a_R}\left[1-\left(\frac{p_R}{p_L}P\right)^{\frac{\gamma-1}{2\gamma}}\right], \tag{3.54}$$

where

$$\alpha = \frac{\gamma+1}{\gamma-1},$$

and $p_L$, $p_R$, $a_L$, and $a_R$ are the pressures and sound speeds associated with the initial states. Recall that the sound speeds can be determined from the specified pressures and densities using (3.13). Once the above equation has been solved by an iterative method for nonlinear algebraic equations, such as Newton's method, the pressure to the left of the shock, $p_2$, is known. The density to the left of the shock can be found from

$$\frac{\rho_2}{\rho_R} = \frac{1+\alpha P}{\alpha + P}. \tag{3.55}$$

Since the pressure is continuous across the contact surface, we know that $p_3 = p_2$. The propagation speed of the contact surface can then be found from

$$V = \frac{2}{\gamma-1}a_L\left[1-\left(\frac{p_3}{p_L}\right)^{\frac{\gamma-1}{2\gamma}}\right]. \tag{3.56}$$

The fluid velocity on either side of the contact surface must be equal to V, which gives $u_3 = u_2 = V$. To complete the state to the left of the contact surface, the density can be found by exploiting the fact that the flow in the expansion wave is isentropic, and hence the entropy to the left of the contact surface is equal to that of the original quiescent left state, giving

$$\rho_3 = \rho_L\left(\frac{p_3}{p_L}\right)^{\frac{1}{\gamma}}. \tag{3.57}$$

The speed at which the shock wave propagates is given by

$$C = \frac{(P-1)a_R^2}{\gamma u_2}. \tag{3.58}$$

The head of the expansion wave travels to the left at speed $a_L$. Therefore, for $x \leq x_0 - a_L t$, the fluid state is defined by the original state to the left of the diaphragm. The tail of the expansion wave moves to the left at a speed given by $a_L - V(\gamma+1)/2$. Thus the state between the tail of the expansion wave and the contact surface (state 3) is the solution for $x_0 + [V(\gamma+1)/2 - a_L]t < x \leq x_0 + Vt$. State 2 is the solution for $x_0 + Vt < x \leq x_0 + Ct$, and finally, for $x > x_0 + Ct$, the solution is the original state to the right of the diaphragm. To complete the solution, we require the state within the expansion fan, that is for $x_0 - a_L t < x \leq x_0 + [V(\gamma+1)/2 - a_L]t$. It is given by

$$\begin{aligned}
u_5 &= \frac{2}{\gamma+1}\left(\frac{x-x_0}{t} + a_L\right) \\
a_5 &= u_5 - \frac{x-x_0}{t} \\
p_5 &= p_L\left(\frac{a_5}{a_L}\right)^{\frac{2\gamma}{\gamma-1}} \\
\rho_5 &= \frac{\gamma p_5}{a_5^2}.
\end{aligned}$$

As an example, we consider the following shock-tube problem from Hirsch [2]: $p_L = 10^5$, $\rho_L = 1$, $p_R = 10^4$, and $\rho_R = 0.125$, where the pressures are in Pa and the densities in Kg/m$^3$. The fluid is a perfect gas with $\gamma = 1.4$. Figure 3.3 displays the density and Mach number at $t = 6.1$ ms. Along with the steady channel flow solutions shown in Figs. 3.1 and 3.2, this exact solution provides an excellent reference for use in verifying numerical solutions.

## 3.4 Exercises

**3.1** Write a computer program to determine the exact solution of the quasi-one-dimensional Euler equations for the following subsonic problem. $S(x)$ is given by

$$S(x) = \begin{cases} 1 + 1.5\left(1 - \frac{x}{5}\right)^2 & 0 \leq x \leq 5 \\ 1 + 0.5\left(1 - \frac{x}{5}\right)^2 & 5 \leq x \leq 10 \end{cases} \tag{3.59}$$

where $S(x)$ and $x$ are in meters. The fluid is air, which is considered to be a perfect gas with $R = 287\ \text{N} \cdot \text{m} \cdot \text{kg}^{-1} \cdot \text{K}^{-1}$, and $\gamma = 1.4$, the total temperature is $T_0 = 300$ K, and the total pressure at the inlet is $p_{01} = 100$ kPa. The flow is subsonic throughout the channel, with $S^* = 0.8$. Compare your solution with that plotted in Fig. 3.1.

**3.2** Repeat Exercise 3.1 for a transonic flow in the same channel. The flow is subsonic at the inlet, there is a shock at $x = 7$, and $S^* = 1$. Compare your solution with that plotted in Fig. 3.2.

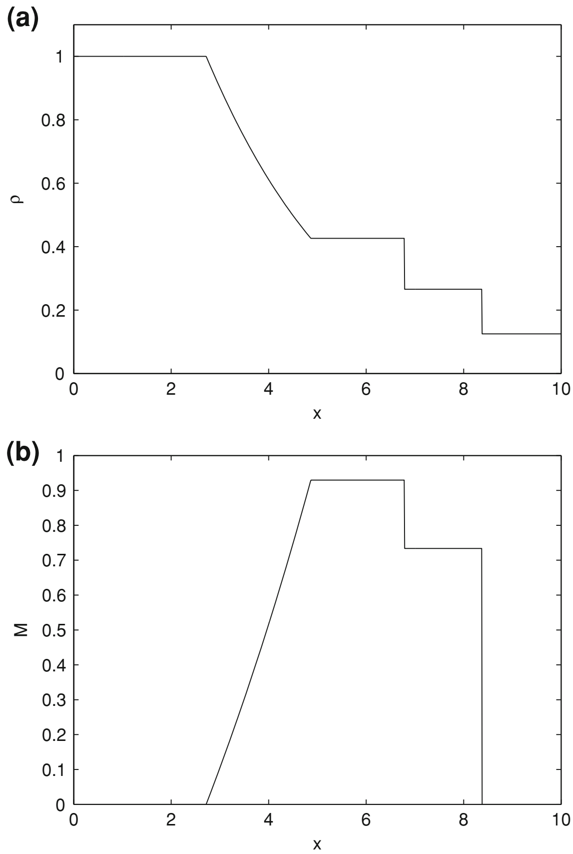


**Fig. 3.3** Exact solution for the shock-tube problem at $t = 6.1$ ms. **a** Density (in Kg/m$^3$ ). **b** Mach number

**3.3** Write a computer program to determine the exact solution for the following shock-tube problem: $p_L = 10^5$, $\rho_L = 1$, $p_R = 10^4$, and $\rho_R = 0.125$, where the pressures are in Pa and the densities in Kg/m$^3$. The fluid is a perfect gas with $\gamma = 1.4$. Compare your solution at $t = 6.1$ ms with that plotted in Fig. 3.3.

## References


1. Shapiro, A.H.: The Dynamics and Thermodynamics of Compressible Fluid Flow. Ronald Press, New York (1953)
2. Hirsch, C.: Numerical Computation of Internal and External Flows, vol. 2. Wiley, Chichester (1990)