Risk Processes with Normal Inverse Gaussian Claims and Premiums

Dean Teneng and Kalev Pärna

Abstract We study risk processes where claims and premiums are modeled by independent normal inverse Gaussian (NIG) Lévy processes; claims by a spectrally positive NIG Lévy process. Using martingale technique, the Lundberg inequality for ruin probability is proved.

Keywords NIG · Risk process · Cramer-Lundberg

1 Introduction

Boykov [1] studies risk processes where claims and premiums are modeled by independent compound Poisson processes. In [4], the difference of premiums and claims are modeled by different Lévy processes; capitalizing on the NIG. Of recent, Stanojevic and Levajkovic [6] proposed modeling premiums with a time changed subordinated Lévy process. We implement this proposal using NIG-Lévy process since it can be represented as an inverse Gaussian time changed Brownian motion with drift [5]. Further, we model aggregate claims by a spectrally positive NIG Lévy process. Using martingale technique, we prove the Lundberg inequality for ruin probability.

2 Model Considerations

2.1 Modified Premium Process

Generally, premiums are determinate, discrete, independent, non-negative stationary increments [2] and we consider an infinite number collected within the time period.

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Modified premium incorporates the effects of inflation, dividend payouts, tax, interest rate fluctuations, claims processing costs and other administrative expenses by the insurance company but claims. It can take on negative or positive values and can be represented by a function with support on the entire real line. Our proposal is to use a finite mean and finite variance NIG Lévy process; with the mean representing constant loaded premium and variance the variation in this modified premium process i.e. stochastic premiums. This is because NIG Lévy process has paths composed of an infinite number of small jumps and exhibit diffusion-like feature with a jump driven structure [3]. NIG[1] Lévy process[2] has its Laplace exponent through Lévy-Khintchine theorem [4] as follows:

$$\Psi_1(\lambda) = a\lambda + \int_{-\infty}^{+\infty} \left[e^{\lambda x} - 1 - \lambda x I_{\{|x|\leq 1\}}\right] \nu(dx) \tag{1}$$

where a, λ are real constants and $\nu(dx)$ is a measure on $R \setminus \{0\}$ such that $\int_{-\infty}^{+\infty}(1 \wedge X^2)\nu(dx) < \infty$.

2.2 Claims

Claims also are generally independent,[3] indeterminate, stationary, non-negative increments with an infinite number collected within considered finite time interval [2]. We model these with a spectrally positive NIG Lévy process i.e. an NIG Lévy process with no negative jumps and chosen to have finite mean, finite variance and support on the positive real line (see Fig. 1). It is not the negative of a subordinator.[4] Generally, if X is spectrally positive, then $-X$ is spectrally negative. Spectrally negative Lévy processes are well studied in the literature. NIG spectrally negative has its Laplace exponent through Lévy-Khintchine theorem as follows:

$$\Psi_2(\lambda) = -a_1\lambda + \int_{-\infty}^{0} \left[e^{\lambda y} - 1 - \lambda y I_{\{y>-1\}}\right] \nu_1(dy) \tag{2}$$

[1]A random variable X is NIG distributed, (denoted NIG$(\alpha, \beta, \delta, \mu)$) if its probability density function is given by $f_{\text{NIG}}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha\delta}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)} \frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2-(x-\mu)^2}}$ where $K_\lambda(x) = \int_0^\infty (u^{\lambda-1} e^{\frac{-x}{2}(u+u^{-1})})du$ with $\delta > 0$ scaling parameter, $\alpha > 0$ shape parameter, β with $0 \leq |\beta| \leq \alpha$ skewness parameter and $\mu \in \Re$ location parameter. The mean and variance are given by $\mu + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}$ and $\delta\frac{\alpha^2}{[\sqrt{\alpha^2-\beta^2}]^3}$ respectively. It has a simple Laplace exponent $\Psi(\lambda) = -\mu\lambda + \delta(\sqrt{\alpha^2-\beta^2} - \sqrt{\alpha^2-(\beta-\lambda)^2})$, $|\beta - \lambda| < \alpha$.

[2]Let $(\Omega, F, (F(t))_{t\geq 0}, P)$ be a filtered probability space. An adapted *cádlág* $\Re$-valued process $X = \{X(t)\}_{t\geq 0}$ with $X(0) = 0$ is NIG Lévy process if $X(t)$ has independent stationary [3] increments distributed as NIG$(\alpha, \beta, \delta, \mu)$.

[3]We leave out cases of disasters and serial accidents where claims can be correlated.

[4]A subordinator is a strictly non-decreasing Lévy process.

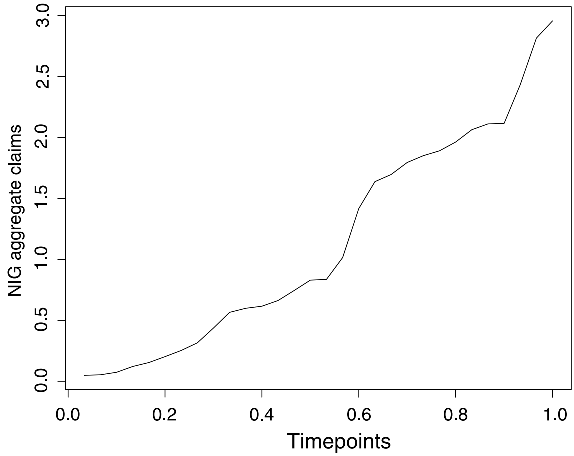

Fig. 1 Spectrally positive NIG(50, −10, 1, 0) process depicting aggregate claims

where a_1 is a real constant representing drift, λ also real and $\nu_1(dy)$ is a measure on $R \setminus \{0\}$ such that $\int_{-\infty}^{\infty}(1 \wedge Y^2)\nu_1(dy) < \infty$.

3 Proposed Model

We propose a Cramer-Lundberg risk model with modified stochastic premiums and stochastic claims both modeled by different NIG-Lévy processes. The risk process $U(t), t \geq 0$ is defined as

$$U(t) = u + X_1(t) - X_2(t) \tag{3}$$

where u is the initial capital, $X_1(t) \sim \text{NIG}(\alpha, \beta, \delta_1 t, \mu_1 t)$ the modified premium process and $X_2(t) \sim \text{NIG}(\alpha, \beta, \delta_2 t, \mu_2 t)$ the aggregate claims process. The risk process $U(t) - u$ is distributed as $\text{NIG}(\alpha, \beta, (\delta_1 - \delta_2)t, (\mu_1 - \mu_2)t)$ with its Laplace exponent through Lévy-Khintchine theorem:

$$\Psi_T(\lambda) = \Psi_1(\lambda) + \Psi_2(\lambda) = (a - a_1)\lambda + \int_0^{+\infty} \left[e^{\lambda x} - 1 - \lambda x I_{\{|x|\leq 1\}}\right]\nu(dx) \tag{4}$$

taking in to account the chosen approximation $\int_{-\infty}^{0}[e^{\lambda x} - 1 - \lambda x I_{\{|x|\leq 1\}}]\nu(dx) \approx \int_{-\infty}^{0}[e^{\lambda y} - 1 - \lambda y I_{\{y>-1\}}]\nu_1(dy)$. This approximation basically means the company has controls in such a way that most claims can be settled, but ultimate ruin or profitability is determined by (4). $\Psi_T(\lambda)$ of (4) has both a constant part $(a - a_1)$ and a stochastic part $\int_0^{+\infty}[e^{\lambda x} - 1 - \lambda x I_{\{|x|\leq 1\}}]\nu(dx)$ representing in a sense the modified risk process. Therefore, net profitability condition simply translates to $E(X(1)) > E(Y(1))$, where $X(1)$ represents modified premium and $Y(1)$ aggregate claims.[5]

[5]Making use of infinite divisibility property of Lévy processes.

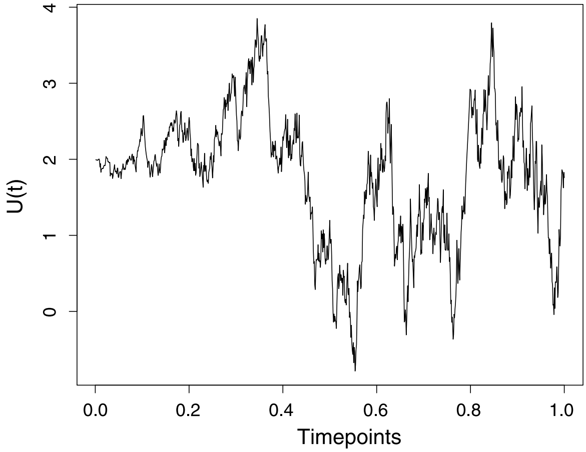

Fig. 2 Risk process which is the difference of NIG(50, −10, 1, 0) and spectrally positive NIG(50, −10, 2, 0)

Considering the Martingale[6] approach to ruin probability, if we can find a value $r = R$ in the domain of the definition of $\Psi(\lambda)$ such that $\Psi(\lambda) = 0$ and $\tau < \infty$, then we could simply write

$$\psi(u) = E_Q\big[e^{RU(\tau)}\big]e^{-Ru}, \quad u \geq 0. \tag{7}$$

We calculate such an $R = \frac{2(d\sqrt{\alpha^2-\beta^2}+\beta)}{d^2+1}$ where $d = \frac{\mu_1-\mu_2}{\delta_2-\delta_1}$, similar to [4] where their $c = \mu_1 - \mu_2$ and $\delta = \delta_2 - \delta_1$; employing similar analysis. Simulated graph (Fig. 2) demonstrates how the proposed model looks like.

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[6]If X is a Levy process with Laplace exponent $\Psi(\lambda)$, then

$$M_t^r = \frac{e^{-rX(t)}}{e^{t\Psi(r)}} \tag{5}$$

with r a real number is a local martingale. Ruin probability is defined as $\psi(u) = P\{\tau < \infty\}, u \geq 0$ and $\tau = inf\{t > 0 : U(t) < 0\}$. Using Esscher transform [4, 5] to change the martingale M_t^r from P to a locally equivalent measure Q for some r in the domain of the Laplace exponent, we can write

$$\psi(u) = E_P[I_{\{\tau<\infty\}}] = E_Q\left[\frac{I_{\{\tau<\infty\}}}{M_\tau^r}\right] = E_Q\big[e^{r(U(\tau)-u)+\tau\Psi(r)}I_{\{\tau<\infty\}}\big]. \tag{6}$$

Hence, if we can find a value $r = R$ in the domain of the definition of $\Psi(r)$ such that $\Psi(r) = 0$ and $\tau < \infty$, then we could simply write $\psi(u) = E_Q[e^{RU(\tau)}]e^{-Ru}$, $u \geq 0$.

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