Chapter 3
Model Reformulations and Tightening

The general MIP model, discussed in Chap. 2, is reconsidered hereinafter, investigating some possible reformulations, from different points of view (Sect. 3.1). The objective of enucleating implicit implications and introducing *valid* inequalities, to *tighten* the model, is examined next (Sect. 3.2).

3.1 Alternative Models

The issue discussed in this section focuses mainly on the case occurring when the packing problem is expressed in terms of *feasibility*, i.e. when all the given items have to be placed and no *objective* function is stated a priori. This situation can arise, for instance, when the items are the elements of a device and, as such, they all have to be installed inside an appropriate container, as essential parts of the same equipment. The thus defined *feasibility* subproblem is also of interest, as it represents one of the basic concepts of the heuristic procedures put forward in Chap. 4. As far as this specific subproblem is concerned, since no *objective* function is specified a priori, an arbitrary one can be introduced, in order to simplify the task of finding an *integer-feasible* solution.

The general model of Sect. 2.1 (including the additional conditions of Sect. 2.3) is reconsidered hereinafter in terms of *feasibility*, providing three different reformulations of it (Sects. 3.1.1, 3.1.2 and 3.1.4). In all of them, it is understood that either all the given items can be loaded or the instance to solve is infeasible. In each of these reformulations, an ad hoc *objective* function is defined, with the scope of minimizing (even if indirectly) the overall overlap of items. In the first (Sect. 3.1.1) and second (Sect. 3.1.2, except the variation outlined at the end), no sooner does the solver obtain the first *integer-feasible* solution than the optimization is stopped (even if just a suboptimal solution of the ad hoc *objective* function has been found). In all reformulations, both the *orthogonality* and *domain* conditions are maintained, as defined in Sect. 2.1. (i.e. consisting of constraints (2.1), (2.2), (2.3) and (2.4)). The second reformulation (Sect. 3.1.2) is subject to

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straightforward variations. One in particular (Sect. 3.1.3) is an actual alternative to the general MIP model, no longer restricted to the *feasibility* subproblem. It could also be utilized (at least partially) in the heuristics of Chap. 4. This aspect would definitely represent an interesting objective for future research.

3.1.1 General MIP Model First Linear Reformulation

The rationale of the general MIP model reformulation presented hereinafter stresses the introduction of an ad hoc *objective* function. This aims at reducing the solution search region, as much as possible, in order to obtain any *integer-feasible* solution.

The approach adopted draws on the work achieved by Suhl (1984), dealing with (large-scale) *fixed-charge* models. Suhl's work provides an efficient preprocessing technique aimed at reducing the *big-M* terms, associated to the *fixed-charge* constraints, i.e. at 'minimizing' (a priori) the related region, in the LP *relaxation*.

As far as the model reformulation in question is concerned, an approach, intended to 'minimize' the search region R_S, relative to the *non-intersection* (*big-M*) constraints (2.5a) and (2.5b), is investigated, to tackle efficiently the relative *feasibility* subproblem. These constraints are then reformulated in an *LP-relaxed* form and an ad hoc *objective* function, substituting (2.7), is introduced. The reformulated model is described as follows.

All variables χ are set to one, as all the given items must be inside the domain and the *non-intersection* constraints (2.5a) and (2.5b) are rewritten as

$$\forall \beta \in B, \forall i,j \in I / i < j, \forall h \in C_i, \forall k \in C_j$$
$$w_{\beta 0hi} - w_{\beta 0kj} \geq \frac{1}{2} \sum_{\omega \in \Omega} \left(L_{\omega\beta hi} \vartheta_{\omega i} + L_{\omega\beta kj} \vartheta_{\omega j} \right) + d^{+}_{\beta hkij} - D_{\beta}, \tag{3.1a}$$

$$\forall \beta \in B, \forall i,j \in I / i < j, \forall h \in C_i, \forall k \in C_j$$
$$w_{\beta 0kj} - w_{\beta 0hi} \geq \frac{1}{2} \sum_{\omega \in \Omega} \left(L_{\omega\beta hi} \vartheta_{\omega i} + L_{\omega\beta kj} \vartheta_{\omega j} \right) + d^{-}_{\beta hkij} - D_{\beta}. \tag{3.1b}$$

Constraints (2.6) are substituted with the following:

$$\forall \beta \in B, \forall i,j \in I / i < j, \forall h \in C_i, \forall k \in C_j \quad d^{+}_{\beta hkij} \geq \sigma^{+}_{\beta hkij} D_{\beta}, \tag{3.2a}$$

$$\forall \beta \in B, \forall i,j \in I / i < j, \forall h \in C_i, \forall k \in C_j \quad d^{-}_{\beta hkij} \geq \sigma^{-}_{\beta hkij} D_{\beta}, \tag{3.2b}$$

$$\forall i,j \in I / i < j, \forall h \in C_i, \forall k \in C_j$$
$$\sum_{\beta \in B} \left(\sigma^{+}_{\beta hkij} + \sigma^{-}_{\beta hkij} \right) = 1, \tag{3.3}$$

where $d^{+}_{\beta hkij}, d^{-}_{\beta hkij} \in [0, D_{\beta}]$.

The adopted ad hoc *objective* function is

$$\max \sum_{\substack{\beta\in B,\\ i,j\in I/i<j,\\ h\in C_i,k\in C_j}} \frac{d^+_{\beta hkij}+d^-_{\beta hkij}}{D_\beta}. \tag{3.4}$$

Any optimal solution of the reformulated model identifies a minimal subset of the *feasibility* region, relative to the general MIP model (Sect. 2.1).

Proposition 3.1 *For any given set of items, the feasibility regions, associated to the general MIP model and its first linear reformulation respectively (neglecting the subspace associated to the variables d^+ and d^-), are coincident.*

Proof Dealing with the feasibility subproblem, all variables χ are set to one. Constraints (2.1), (2.2), (2.3) and (2.4) are obviously coincident in both models, and it is thus sufficient to demonstrate that constraints (2.5a), (2.5b) and (2.6) of the general MIP model are equivalent to constraints (3.1a), (3.1b), (3.2a), (3.2b) and (3.3) of the reformulated one. It is immediately seen that given that all variables χ are set to one, constraints (2.6) can be substituted with (3.3). To show that constraints (2.5a) and (2.5b) are equivalent to (3.1a), (3.1b), (3.2a) and (3.2b), we shall distinguish the cases where the variables σ are zero from those where they are equal to one.

Consider, for instance, $\sigma^+_{\beta hkij}=0$. This implies that constraints (2.5a) become $w_{\beta 0hi}-w_{\beta 0kj}\geq \frac{1}{2}\sum_{\omega\in\Omega}\left(L_{\omega\beta hi}\vartheta_{\omega i}+L_{\omega\beta kj}\vartheta_{\omega j}\right)-D_\beta$.

These are equivalent to constraints (3.1a), with $d^+_{\beta hkij}=0$, in compliance with constraints (3.2a). Considering, instead, $\sigma^+_{\beta hkij}=1$, this implies that constraints (2.5a) become $w_{\beta 0hi}-w_{\beta 0kj}\geq \frac{1}{2}\sum_{\omega\in\Omega}\left(L_{\omega\beta hi}\vartheta_{\omega i}+L_{\omega\beta kj}\vartheta_{\omega j}\right)$.

These are equivalent to constraints (3.1a), with $d^+_{\beta hkij}=D_\beta$, in compliance with constraints (3.2a). As the same reasoning can be carried out, taking into account the cases relative to the variables $\sigma^-_{\beta hkij}$, the two models are equivalent. □

Remark 3.1 To better understand the meaning of the general MIP model first linear reformulation, we shall make some intuitive considerations. Let us define, for each β, for every pair of components h and k of item i and j, respectively, the squared subspace $S_\beta=[0, D_\beta]\times[0, D_\beta]\subset \boldsymbol{R}^2$, associated to variables $d^-_{\beta hkij}$ and $d^+_{\beta hkij}$. The bound $d^+_{\beta hkij}+d^-_{\beta hkij}\leq 2D_\beta-\sum_{\omega\in\Omega}\left(L_{\omega\beta hi}\vartheta_{\omega i}+L_{\omega\beta kj}\vartheta_{\omega j}\right)$ is implicitly determined by inequalities (3.1a) and (3.1b). The objective function induces the solution projection on S_β to stay along the straight line $d^+_{\beta hkij}+d^-_{\beta hkij}=2D_\beta-\sum_{\omega\in\Omega}\left(L_{\omega\beta hi}\vartheta_{\omega i}+L_{\omega\beta kj}\vartheta_{\omega j}\right)$. If $D_\beta-\sum_{\omega\in\Omega}\left(L_{\omega\beta hi}\vartheta_{\omega i}+L_{\omega\beta kj}\vartheta_{\omega j}\right)\geq 0$, this intersects S_β in the points

$\left(D_\beta, D_\beta - \sum_{\omega \in \Omega} \left(L_{\omega\beta hi}\vartheta_{\omega i} + L_{\omega\beta kj}\vartheta_{\omega j}\right)\right)$ and $\left(D_\beta - \sum_{\omega \in \Omega} \left(L_{\omega\beta hi}\vartheta_{\omega i} + L_{\omega\beta kj}\vartheta_{\omega j}\right), D_\beta\right)$, respectively, determining an internal segment. In this occurrence, if the linear solver (utilized by the MIP optimizer) looks for vertex solutions (as in the case of a simplex-based one), the above extreme points are more likely to be selected than the ones internal to the segment (although this expectation is not based on rigorous reasoning). One has to bear in mind, moreover, that either $d^+_{\beta hkij} = D_\beta$ or $d^-_{\beta hkij} = D_\beta$ (for any β) guarantees that no intersection occurs between the two corresponding items.

As a partially alternative version of this model reformulation, the constraints $\forall \beta \in B, \forall i, j \in I/i < j, \forall h \in C_i, \forall k \in C_j, d^+_{\beta hkij} + d^-_{\beta hkij} \leq D_\beta$ could also be added to *tighten* the *feasibility* region (creating in the subspace S_β the two extreme points $(D_\beta, 0)$ and $(0, D_\beta)$, without excluding any solution. These inequalities are obviously *tighter* than the bounds $d^+_{\beta hkij} + d^-_{\beta hkij} \leq 2D_\beta - \sum_{\omega \in \Omega} \left(L_{\omega\beta hi}\vartheta_{\omega i} + L_{\omega\beta kj}\vartheta_{\omega j}\right)$, when $D_\beta - \sum_{\omega \in \Omega} \left(L_{\omega\beta hi}\vartheta_{\omega i} + L_{\omega\beta kj}\vartheta_{\omega j}\right) \geq 0$. The conditions $d^-_{\beta hkij}, d^+_{\beta hkij} \in [0, D_\beta]$, moreover, if explicitly introduced in the model, can be of computational advantage, when the linear solver adopted treats the variable bounds independently (as in the case of *simplex-based* ones).

3.1.2 General MIP Model Second Linear Reformulation

To discuss this alternative model, we shall consider, for each item *component*, the set of all concentric parallelepipeds containing it. The reformulation examined hereinafter is also based on an ad hoc *objective* function. It is aimed at finding, for each *component*, the enclosing parallelepiped (included in D) of maximum volume that does not intersect any other enclosing parallelepipeds, associated to *components* of different items.

To this purpose, the *non-intersection* conditions of Sect. 2.1 are properly changed. Whilst (2.6) is kept, inequalities (2.5a) and (2.5b) are substituted with the constraints below. For each *component* h of i, the non-negative variables $l_{\beta hi}$ are introduced, assuming that all variables χ are set to one:

$$\forall \beta \in B, \forall i, j \in I/i < j, \forall h \in C_i, \forall k \in C_j$$
$$w_{\beta 0hi} - w_{\beta 0kj} \geq \frac{1}{2}\left(l_{\beta hi} + l_{\beta kj}\right) - D_\beta\left(1 - \sigma^+_{\beta hkij}\right), \tag{3.5a}$$

$$\forall \beta \in B, \forall i, j \in I/i < j, \forall h \in C_i, \forall k \in C_j$$
$$w_{\beta 0kj} - w_{\beta 0hi} \geq \frac{1}{2}\left(l_{\beta hi} + l_{\beta kj}\right) - D_\beta\left(1 - \sigma^-_{\beta hkij}\right), \tag{3.5b}$$

$$\begin{aligned}&\forall \omega \in \Omega, \forall \beta \in B, \forall i \in I, \forall h \in C_i\\&l_{\beta hi} \geq L_{\omega\beta hi}\vartheta_{\omega i}.\end{aligned} \tag{3.6}$$

The following (*surrogate*) *objective* function is defined:

$$\max \sum_{\substack{\beta \in B,\\ i \in I, h \in C_i}} l_{\beta hi}. \tag{3.7}$$

For each *component h* of each item *i*, the terms $l_{\beta hi}$ represent (for the orientation ω assumed by *i*) the projections, on the axes w_β, of an enclosing parallelepiped, containing *component h* and centred with it. Inequalities (3.5a), (3.5b) and (3.6) (together with (2.6)) guarantee that the enclosing parallelepipeds, belonging to different items, do not intersect.

Remark 3.2 Rigorously speaking, as the objective function (3.7) refers to the total sum of the component sides, it should be considered as a surrogate expression of $\max \sum_{i \in I, h \in C_i} \prod_{\beta \in B} l_{\beta hi}$.

As previously mentioned, possible variations of the approach discussed above could be considered. One is obtained simply by inverting inequalities (3.6) as follows and keeping all remaining constraints, as well as the *objective* function, unaltered:

$$\begin{aligned}&\forall \omega \in \Omega, \forall \beta \in B, \forall i \in I, \forall h \in C_i\\&l_{\beta hi} \leq L_{\omega\beta hi}\vartheta_{\omega i}.\end{aligned} \tag{3.8}$$

In this case, an *integer-optimal* solution (and not just any *integer-feasible* one) has necessarily to be found, in order to guarantee that no intersections occur among the given items. It should be noticed that, at each step, the optimization process is induced to minimize the overall overlap, without assigning items a volume that exceeds their own. Moreover, since, in this case, the value of the global optimal solution is known a priori, it can be advantageously utilized as *cutoff* parameter (to get rid of suboptimal solutions).

3.1.3 A Non-restrictive Reformulation of the General MIP Model

A possible reformulation of the general MIP model, without renouncing its original objective of maximizing either the overall loaded volume or mass, is also quite straightforward. The problem is no longer expressed in terms of *feasibility* (i.e. without the possibility of rejecting items), so that all variables χ are set free again, as in Sect. 2.1.

As a first step, inequalities (3.6) are transformed into the equations:

$$\begin{aligned}&\forall \omega \in \Omega, \forall \beta \in B, \forall i \in I, \forall h \in C_i\\&l_{\beta hi} = L_{\omega\beta hi}\vartheta_{\omega i}.\end{aligned} \tag{3.9}$$

In order to define the new *objective* function (substituting (2.7)), the terms K_{hi} are introduced (with obvious meaning) for each *component* h of each item i, where $\forall i \in I \quad \sum_{h \in C_i} K_{hi} = K_i$, cf. (2.7). The dimensions of *component* h of i are indicated with $L_{\alpha hi}$, $\alpha \in \{1, 2, 3\} = A$, assuming, from now on, that $L_{1hi} \leq L_{2hi} \leq L_{3hi}$. The new *objective* function is then expressed by the following:

$$\max \sum_{\substack{\beta \in B,\\ i \in I, h \in C_i}} \frac{K_{hi}}{\sum_{\alpha \in A} L_{\alpha hi}} l_{\beta hi}. \tag{3.10}$$

It is easily seen that the two *objective* function*s* (2.7) and (3.10) are equivalent for any *integer-feasible* solution (by (3.9)). The expression (3.10), differently from (2.7), provides the significant computational advantage of minimizing the item overall overlap at each step of the optimization process. Just to summarize the reformulation in question, we could point out that it consists of constraints (2.1), (2.2) (*orthogonality*), (2.3), (2.4) (*domain*), (2.6), (3.5a), (3.5b) and (3.9) (*non-intersection*), in addition to *objective* function (3.10). It is also understood that in all the relevant expressions above, the variables $l_{\beta hi}$ could be eliminated. They may, indeed, be substituted by their corresponding terms, on the basis of (3.9) (that could also be eliminated).

3.1.4 General MIP Model Nonlinear Reformulation

The general packing problem presented in Sect. 2.1 is notoriously subject to nonlinear (MINLP) formulations (e.g. Birgin and Lobato 2010; Birgin et al. 2006; Cassioli and Locatelli 2011). We shall introduce, hereinafter, a nonlinear reformulation of the general MIP model *non-intersection* constraints, assuming, as previously, that all variables χ are set to one (as the *feasibility* subproblem is in question). It is straightforward to prove that the nonlinear constraints below are equivalent to (2.5a), (2.5b) and (2.6):

$$\forall \beta \in B, \forall i, j \in I / i < j, \forall h \in C_i, \forall k \in C_j$$

$$\left(w_{\beta 0hi} - w_{\beta 0kj}\right)^2 - \left[\frac{1}{2}\sum_{\omega \in \Omega}\left(L_{\omega\beta hi}\vartheta_{\omega i} + L_{\omega\beta kj}\vartheta_{\omega j}\right)\right]^2 = s_{\beta hkij} - r_{\beta hkij}, \tag{3.11}$$

$$\forall \beta \in B, \forall i, j \in I / i < j, \forall h \in C_i, \forall k \in C_j$$

$$\prod_{\beta \in B} r_{\beta hkij} = 0, \tag{3.12}$$

where $s_{\beta hkij} \in [0, D_\beta^2]$ and $r_{\beta hkij} \in [0, D_\beta^2]$ (actually, smaller upper bounds could be chosen for both sets of variables).

Indeed, for each pair of *components* h and k, of items i and j, respectively, equations (3.12) guarantee that for at least one β, the corresponding term $r_{\beta hkij}$ is zero, and equations (3.11) that the *non-intersection* conditions hold for such a β, i.e. $\left|w_{\beta 0hi} - w_{\beta 0kj}\right| \geq \frac{1}{2}\sum_{\omega \in \Omega}\left(L_{\omega\beta hi}\vartheta_{\omega i} + L_{\omega\beta kj}\vartheta_{\omega j}\right)$. More precisely, constraints (2.5a) and (2.5b) correspond to equations (3.11), whilst equations (2.6) correspond to (3.12).

As the *non-intersection* constraints (3.11) and (3.12) are most likely hard to tackle, they are therefore considered in terms of *fixed penalization* in the ad hoc *objective* function we are going to introduce. All remaining linear (MIP), constraints are kept as such. A formulation aimed at satisfying as much *non-intersection* conditions as possible is the following:

$$\min\left\{\sum_{\substack{\beta \in B,\\ i,j \in I/i<j,\\ h \in C_i, k \in C_j}}\left\{\left(w_{\beta 0hi} - w_{\beta 0kj}\right)^2 - \left[\frac{1}{2}\sum_{\omega \in \Omega}\left(L_{\omega\beta hi}\vartheta_{\omega i} + L_{\omega\beta kj}\vartheta_{\omega j}\right)\right]^2 - s_{\beta hkij} + r_{\beta hkij}\right\}^2 + K_P \sum_{\substack{i,j \in I/i<j,\\ h \in C_i, k \in C_j}}\prod_{\beta \in B} r_{\beta hkij}\right\} \tag{3.13}$$

where K_P is a positive coefficient (that represents an appropriate 'weight' associated to the product terms).

It is immediately seen that the *objective* function (3.13) is non-negative. A zero-global-optimal solution exists if and only if the constraints ((2.1), (2.2), (2.3), (2.4), (2.5a), (2.5b) and (2.6) of the general MIP model of Sect. 2.1 (with all variables χ set to one) delimit a feasible region. This *objective* function thus 'minimizes' the intersection between items. Its global optima, moreover, guarantee an ultimate (non-approximate) solution to the *feasibility* subproblem under discussion. It could be observed that for each set of variables ϑ, (3.13) is a polynomial function (providing, as such, potential algorithmic advantages; on global polynomial optimization, see, for instance, De Loera et al. 2012; Hanzon and Jibetean 2003; Schweighofer 2006).

Alternative *fixed penalization* can be considered (e.g. Cassioli and Locatelli 2011). We shall introduce here one *objective function* with *fixed penalization* correlated to the *non-intersection* constraints only:

$$\min\left\{\sum_{\substack{\beta\in B,\\ i,j\in I/i<j,\\ h\in C_i,k\in C_j}} \max\left\{-\left(w_{\beta 0hi}-w_{\beta 0kj}\right)^2+\left[\frac{1}{2}\sum_{\omega\in\Omega}\left(L_{\omega\beta hi}\vartheta_{\omega i}+L_{\omega\beta kj}\vartheta_{\omega j}\right)\right]^2,0\right\} + K_P\sum_{\substack{i,j\in I/i<j,\\ h\in C_i,k\in C_j,}}\prod_{\beta\in B} r_{\beta hkij}\right\} \tag{3.14}$$

As the previous one, this *objective* function is also non-negative and each zero-global-optimum corresponds to a solution of the *feasibility* problem.

Remark 3.3 Both the MINLP formulations discussed above contain only linear (MIP) constraints. This aspect could be advantageous, when the MINLP solvers utilized treat the model linear sub-structure independently (e.g. The MathWorks 2012). It is moreover worth noticing that all functions involved in both MINLP formulations are Lipschitz-continuous and, consequently, Lipschitzian solvers can be profitably adopted (e.g. Pintér 1997, 2009). Indeed, all constraints are of the MIP type and (3.13) is smooth. As far as (3.14) is concerned, it is sufficient to observe that the terms $\max\left\{-\left(w_{\beta 0hi}-w_{\beta 0kj}\right)^2+\left[\frac{1}{2}\sum_{\omega\in\Omega}\left(L_{\omega\beta hi}\vartheta_{\omega i}+L_{\omega\beta kj}\vartheta_{\omega j}\right)\right]^2,0\right\}$ keep their Lipshitz-continuous characteristic (e.g. Pintér 1996).

3.2 Implications and Valid Inequalities

As is well known, in the MIP context, remarkable research effort has been devoted to looking into general approaches to *tighten* the model. This means to make its *linear relaxation* an as precise as possible approximation of the *convex hull* relative to the *mixed-integer* solutions (e.g. Andersen et al. 2005; Ceria et al. 1998; De Farias et al. 1998; Jünger et al. 2009; Marchand et al 1999; Nemhauser and Wolsey 1990; Van Roy and Wolsey 1987; Weismantel 1996; Wolsey 1989).

Polyhedral analysis (e.g. Atamtürk 2005; Constantino 1998; Dash et al. 2010; Hamacher et al. 2004; Padberg 1995; Pochet and Wolsey 1994; Yaman 2009) is adopted to this purpose, in order to find *valid* inequalities (e.g. Aardal et al. 1995; Cornuéjols 2008; Padberg et al. 1985; Wolsey 1990, 2003). These are aimed at *tightening* the MIP model under consideration. The introduction of such auxiliary conditions is particularly suitable when a *branch-and-cut* approach (e.g. Andreello et al. 2007; Balas et al. 1996; Cordier et al. 2001; Padberg 2001; Padberg and Rinaldi 1991) is followed.

Differently from more traditional MIP algorithms, such as *branch-and-bound* (where all model constraints have to be set a priori) with a *branch-and-cut* process, the *valid* inequalities are activated just when needed and dropped when not required.

With reference to the general MIP model (Sect. 2.1), for items consisting of single parallelepipeds to load into a parallelepiped (see Sect. 2.1, special case), some *valid* inequalities, holding under specific assumptions, have been put forward by Padberg (1999). This has been done to tackle the problem by means of a dedicated *branch-and-cut* approach. Some quite simple conditions, not restricted to the case of single parallelepipeds, are considered hereinafter (limited subsets of them can be advantageously taken into account also when a *branch-and-bound* approach is adopted). A first group of inequalities is hence introduced:

$$\forall i, j \in I / i < j, \forall h \in C_i, \forall k \in C_j$$
$$\sum_{\beta \in B} \left(\sigma^{+}_{\beta hkij} + \sigma^{-}_{\beta hkij} \right) \le \chi_i, \tag{3.15a}$$

$$\forall i, j \in I / i < j, \forall h \in C_i, \forall k \in C_j$$
$$\sum_{\beta \in B} \left(\sigma^{+}_{\beta hkij} + \sigma^{-}_{\beta hkij} \right) \le \chi_j. \tag{3.15b}$$

These, together with (2.6), for each pair of *components* h and k of items i and j, respectively, imply that one, and only one, of the relative variables $\sigma^{+}_{\beta hkij}$ and $\sigma^{-}_{\beta hkij}$ has to be equal to one if both items are loaded; all of them are equal to zero otherwise. It is immediate to notice that in the general MIP model of Sect. 2.1, in case both items are picked, more than one of the variables $\sigma^{+}_{\beta hkij}$ and $\sigma^{-}_{\beta hkij}$ could be non-zero. The above extended version is hence *tighter* than the previous, without any loss of generality, as no *integer-feasible* solutions are excluded.

Some straightforward examples of necessary conditions, concerning pairs of items, in particular situations, can be considered. Firstly, let us consider the very simple case when item i and j cannot be aligned with respect to the axis w_β (because they would exceed the dimension D_β, for all possible orientations of both). In such an occurrence, the conditions below can be explicitly posed:

$$\forall h \in C_i, \forall k \in C_j \quad \sigma^{+}_{\beta hkij} = \sigma^{-}_{\beta hkij} = 0.$$

In addition to these, a set of more complicated implications, correlating alignment and orientation, could be introduced. An example, dealing with the special case of Sect. 2.1, relative to single parallelepipeds, is reported here.[1] Considering items i and j, *if* $L_{1i} + L_{2j} > D_\beta$, *they cannot be aligned along the axis* w_β, *with either* L_{2j} *or* L_{3j} *parallel to it*. And analogously, this holds if $L_{1j} + L_{2i} > D_\beta$. The following inequalities can hence be set:

$$\forall \beta, \forall i,j \in I / i<j, L_{1i}+L_{2j}>D_\beta \quad \delta_{2\beta j}+\delta_{3\beta j} \le 1-\sigma^+_{\beta ij}-\sigma^-_{\beta ij},$$
$$\forall \beta, \forall i,j \in I / i<j, L_{1j}+L_{2i}>D_\beta \quad \delta_{2\beta i}+\delta_{3\beta i} \le 1-\sigma^+_{\beta ij}-\sigma^-_{\beta ij}.$$

These conditions can easily be extended when tetris-like items are involved, i.e. when the general MIP model of Sect. 2.1 is considered. This gives rise to inequalities of the type $\sum_{\omega \in \Omega'_{\beta hkij}} \vartheta_{\omega j} \le 1-\sigma^+_{\beta hkij}-\sigma^-_{\beta hkij}$, where $\Omega'_{\beta hkij}$ $(i<j)$ is the set of orientations (of j), incompatible with the alignment conditions of the components h (of i) and k (of j). Similar expressions hold for i, with $\Omega'_{\beta hkji}$ $(i<j)$.

Straightforward *transitivity* conditions (e.g. Padberg 1999; Fasano 2008) can, moreover, be looked upon, when triplets of single parallelepipeds are taken into account. They can easily be extended when actual tetris-like items are involved. Focusing on the triplet of *components* h, h', h'' of items i, i', i'', respectively, *if, along the axis* w_β, *h precedes* h' *and* h' *precedes* h'', *then h precedes* h'', *along the same axis*. This implication is expressed by

$$\forall \beta \in B, \forall i,i',i'' \in I / i<i'<i'', \forall h \in C_i, \forall h' \in C_{i'}, \forall h'' \in C_{i''}$$
$$\sigma^-_{\beta hh''ii''} \ge \sigma^-_{\beta hh'ii'} + \sigma^-_{\beta h'h''i'i''} - 1.$$

Still referring to the same triplet of *components*, the further implication holds: *if* $L_{1hi}+L_{1h'i'}+L_{1h''i''}>D_\beta$, *then the whole triplet cannot be aligned along the axis* w_β. This is expressed by the following constraints:

$$\forall \beta \in B, \forall i,i',i'' \in I / i<i'<i'', \forall h \in C_i, \forall h' \in C_{i'}, \forall h'' \in C_{i''} / L_{1hi}+L_{1h'i'}+L_{1h''i''}>D_\beta$$
$$\sigma^+_{\beta hh'ii'}+\sigma^-_{\beta hh'ii'}+\sigma^+_{\beta h'h''i'i''}+\sigma^-_{\beta h'h''i'i''}+\sigma^+_{\beta hh''ii''}+\sigma^-_{\beta hh''ii'} \le 2.$$

The proof is straightforward. It is sufficient to notice that being the hypothesis stated, at the most, two *components* may be aligned along the axis w_β and that for each pair of *components*, either the corresponding variable σ^+_β or σ^-_β must be zero.

[1] Note These conditions have been introduced by S. Gliozzi, senior managing consultant at IBM GBS Advanced Analytics and Optimization.

As a further observation, note that the implications correlating alignment and orientation, as presented in this section, would be susceptible to extensions involving chains of more than three *components*. Their introduction could provide practical advantages in the perspective of a dedicated *branch-and-cut* approach.

Remark 3.4 When the layer constraints reported in Sect. 2.3.5 are introduced in the model, inequalities (3.15a) and (3.15b) can properly be extended. Moreover, the necessary conditions $\forall i \in I \; w_{3i} \geq \min_{i' \neq i} \{L_{1i'}\} \widehat{\chi}_i$ can explicitly be added, following the perspective presented in this section.