Chapter 3
Variational Foundations of Modern Structural Dynamics

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Abstract The foundations of modern structural dynamic analysis are presented from a historical perspective. Underlying variational principles due to d'Alembert, Hamilton and Lagrange are reviewed, followed by subsequent key contributions to approximate analysis. Closely related procedures introduced by Ritz, Galerkin and Trefftz are described in terms of their original application to continuum boundary value problems. The role these procedures played in the development of the Finite Element Method and Matrix Structural Analysis is described. Finally, the continuing influence of variational principles in structural dynamics and mathematical physics in general is outlined.

Nomenclature

$[A_p]$	Acoustic surface area matrix
[C]	Damping matrix
$[C_p]$	Acoustic compliance matrix
E	Elastic stiffness property
F	Force
[G]	Transformation matrix
[K]	Stiffness matrix
L	Lagrangian
[M]	Mass matrix
N.B.C	Natural boundary condition
[P]	Modal participation factor matrix
P.D.E	Partial differential equation
{Q}	Modal generalized forces
S	Surface area
S_p	Acoustic susceptance matrix
T	Kinetic energy
U	Potential or strain energy
V	Volume
W	Work
m	Mass
{p}	Acoustic pressure array
q	Generalized coordinate (displacement)
t	Time
u	Displacement
x, y, z	Position
[Φ]	Modal matrix

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$[\Gamma]$	Force allocation matrix
$[I]$	Identity matrix
Ψ	Shape function
α, β	Proportional damping constants
δ	Variation
ϵ	Strain
λ	Eigenvalue
ρ	Mass density
ω_n	Natural frequency
ζ_n	Critical damping ration

3.1 Introduction

The development of mathematical physics, of which structural dynamics is a branch, owes its present state to two general viewpoints. The first viewpoint due to Newton [1] sees nature following postulates describing the dynamic equilibrium of interacting bodies. The second viewpoint, due to d'Alembert [2], Hamilton [3] and Lagrange [4], sees nature following postulates describing efficient organization of energies (variational principles). The variational viewpoint, specifically Hamilton's principle, has guided the development of partial differential equations and natural boundary conditions for technical structural theories [5, 6].

Difficulties encountered in the quest for exact solutions of partial differential equations, subjected to natural boundary conditions, led to the introduction of approximate techniques based on Hamilton's principle. Ritz [7] employed assumed shape functions and generalized coordinates to deduce mass and stiffness matrices, which are foundational to the Finite Element Method [8] and Matrix Structural Analysis [9]. Galerkin [10] similarly applied assumed functions to a variational integral associated with a system's partial differential equations; his formulation, while influential in development of the Finite Element Method, has also been applied to solve nonlinear dynamic problems [11]. Trefftz [12] introduced a novel variational technique focusing on approximate satisfaction of natural boundary conditions, which are a byproduct of Hamilton's principle; his employment of shape functions that automatically satisfy the system's partial differential equations forms a basis of the boundary element method [13].

Hamilton's principle and the Ritz method, in particular, have been instrumental in the development of both the Finite Element Method and Matrix Structural Analysis. The basic building blocks of the Finite Element Method are most often developed on the basis of assumed boundary displacement referenced and optional interior displacement shape polynomials, which are used to define element mass and stiffness matrices based on the Ritz method. Assembly and analysis of structural system models falls in the category of Matrix Structural Analysis, which for a time before general acceptance of the Finite Element Method employed a force (degree of freedom) method. The force method was based on a complementary variational principle introduced by Castigliano [14].

Free vibration analysis of a matrix structural system is typically performed on an undamped system, for which the modes or real eigenvectors satisfy a mass weighted orthogonality relationship. Free vibration analysis of a damped structural system is rarely performed due to difficulties associated with theoretical damping matrices that do not relate to empirical modal damping. An alternative to the explicit damping matrix, namely a complex structural damping constant, is often employed in aeroelastic analysis [15]. The more widely used description of damping is based on critical damping ratio factors assigned to a truncated set of undamped system modes.

Several strategies for (a) computation of system modes and natural frequencies, and (b) accurate computation of system responses are consequences of Mach's interpretation of d'Alembert's principle [16]. In particular the iterative modal analysis procedures due to Vianello [17], Stodola [18] and Bathe [19] successively update inertial forces initially described by assumed mode shapes. Accuracy of computed system dynamic response associated with a truncated set of modes was greatly enhanced by a mode acceleration method introduced by Williams [20], which automatically accounts for the quasi-static response of higher frequency modes without explicitly solving for them.

During the 1960s the complexity of finite element models outpaced the development of digital computers and several authors developed procedures to treat system models piecemeal. Hurty [21] and Craig and Bampton [22] developed equivalent techniques known as component mode synthesis that described major structural components in terms of (a) generalized modal coordinates, and (b) physical boundary (interface) degrees of freedom. The Ritz-based building blocks (sometimes called substructures or superelements) associated with component mode synthesis exhibit the same features as finite elements.

Mathematical formulations and procedures based on variational principles continue to influence developments both in structural dynamics and other branches of applied physics. The complementary viewpoint introduced by Castigliano in 1873 was generalized for dynamics by Toupin [23]. Toupin's variational principle was found to be quite useful in fluid–structure interaction studies [24]. In addition, variational formulations that are direct consequences of Galerkin's method have been introduced by Biot [25] for heat transfer and MacNeal et al. [26] for electromagnetics.

3.2 Economy in Nature and Basic Variational Formulations

Philosophers of antiquity, well before the advent of modern science initiated by Newton [1], believed that nature operated in accordance with a rule of economy as noted by Aristotle (d. 312 BC), "Nature follows the easiest path that requires the least amount of effort". In the medieval era, William of Ockham (1347) suggested an economic principle for human reasoning with his famous saying, "It is futile to employ many principles when it is possible to employ fewer" (popularly known as Ockham's razor). It is fascinating that Newton's second law may be used as a postulate to deduce variational formulations of mechanics [2–4] (as theorems). Moreover, when the variational principle due to Hamilton [3], is postulated, Newton's second law follows as a theorem.

The postulate-theorem interrelationship is demonstrated below for a mechanical system composed of a single particle restricted to motion along one direction. Newton's second law [1] states the postulate,

$$F = \frac{d}{dt}\left(m\frac{du}{dt}\right) \tag{3.1}$$

D'Alembert [2] claimed that the system follows an "economy" in variational terms that

$$\left(F - \frac{d}{dt}\left(m\frac{du}{dt}\right)\right)\cdot \delta u = 0 \tag{3.2}$$

where "δu" is defined as a variation off the true path of "u". Hamilton [3] added to D'Alembert's principle by claiming that the following time integral is true:

$$\int_{t_1}^{t_2}\left(F - \frac{d}{dt}\left(m\frac{du}{dt}\right)\right)\cdot \delta u \cdot dt = 0 \tag{3.3}$$

After rearranging terms and integrating by parts, the above integral is expressed as

$$\int_{t_1}^{t_2}\left(\delta\left(\frac{1}{2}m\left(\frac{du}{dt}\right)^2\right) + F\cdot\delta u\right)\cdot dt - \left(m\frac{du}{dt}\right)\cdot\delta u\Big|_{t_1}^{t_2} = 0 \tag{3.4}$$

This variational integral is more compactly expressed as

$$\int_{t_1}^{t_2}(\delta T + \delta W)\cdot dt = 0 \text{ subjected to the end conditions } \left(m\frac{du}{dt}\right)\cdot\delta u\Big|_{t_1}^{t_2} = 0 \tag{3.5}$$

$T = \frac{1}{2}m\left(\frac{du}{dt}\right)^2$ is the system's kinetic energy and $\delta W = F\delta u$ is the system's virtual work. In addition, in order to curtail mathematical imposition of "Calvinist" predestination doctrine, we avoid imposition of the initial and final time constraints by extending t_2 to infinity and enforce two initial (state) conditions, namely that $u(0)$ and $\dot{u}(0)$ are specified. The variational integral defined in Eq. (3.5) is known as Hamilton's principle. A more familiar form of Hamilton's principle results from dividing the total applied force into two components, namely (a) a conservative potential energy (U) based force, and (b) a non-conservative component, which are expressed as

$$F = -\frac{dU}{du} + F_e \tag{3.6}$$

Thus the variational integral becomes

$$\int_{t_1}^{t_2} (\delta T - \delta U + \delta W) \cdot dt = \int_{t_1}^{t_2} (\delta L + \delta W) \cdot dt = 0 \tag{3.7}$$

where $L = T - U$ is known as the Lagrangian function.

The postulate-theorem relationship may be reversed by starting with Hamilton's principle as a postulate. Two consequences follow from this starting point in mechanics of distributed as well as single degree of freedom systems, namely, (a) Lagrange's equations [4], and (b) Newton's second law; both may be viewed as theorems resulting from the starting postulate.

3.3 Mathematical Physics and Hamilton's Principle

Much like the above discussion on dynamics of a particle, the partial differential equations of mathematical physics (in particular mechanics) may be derived on the basis of dynamic equilibrium (Newton's laws and free body diagrams) or on the basis of Hamilton's principle (energy and virtual work). Results of both approaches produce the same description. Application of Hamilton's principle to a dynamic system described as a continuum yields a volume integral of the type

$$\int_{t_1}^{t_2} \int_V (\delta T_V - \delta U_V + \delta W_V) \cdot dV \cdot dt = 0, \tag{3.8}$$

where T_V, U_V, and δW_V are the kinetic energy, potential (or strain) energy, and virtual work functions per unit volume, respectively. Analysis of any particular dynamic system, described in terms of displacement variables, u(x, y, z, t), which may be vectors, results in the following type of functionals

$$\int_{t_1}^{t_2} \int_V (\text{P.D.E}) \cdot \delta u \cdot dV \cdot dt + \int_{t_1}^{t_2} \int_S (\text{N.B.C}) \cdot \delta u \cdot dS \cdot dt = 0 \tag{3.9}$$

"P.D.E" represents the particular partial differential equation(s) within the system's volume. "N.B.C" represents the natural boundary conditions, which are mathematically and physically admissible on the system's boundary surface(s). The general process for derivation of a system's partial differential equations and natural boundary conditions has provided a consistent basis for the development of technical structural theories for prismatic bars, beams, rings, plates and shells [5, 6].

3.4 The Contributions of Ritz, Galerkin, and Trefftz

Three outstanding contributions that led to the development of approximate analysis techniques date back to the early part of the twentieth century. The methods bearing the names of Ritz, Galerkin and Trefftz are all consequences of Hamilton's principle and the assumption of approximate solution functions.

3.4.1 The Ritz Method

A monumental contribution to approximate analysis was introduced by Ritz [7], who described the displacement field in variable separable terms

$$u(x, y, z, t) = \sum_{n=1}^{N} \Psi_n(x, y, z) \cdot q_n(t), \tag{3.10}$$

where $\Psi_n(x, y, z)$ are assumed shape functions and $q_n(t)$ are temporal generalized coordinates (displacements). In addition, the strain field,

$$\varepsilon(x, y, z, t) = \sum_{n=1}^{N} \Psi_{\varepsilon,n}(x, y, z) \cdot q_n(t), \tag{3.11}$$

is linearly related to the assumed displacement field employing the appropriate partial derivatives. It should be noted that exact closed form solutions of partial differential equations are often expressed in variable separable form, whenever such solutions are possible. By assuming a series of functions that satisfy particular boundary conditions (or generally permit solution of natural boundary conditions), substitution of Eq. (3.10) into Hamilton's principle (Eq. 3.8), the following symmetric matrix equations are deduced

$$[M]\{\ddot{q}\} + [K]\{q\} = [\Gamma]\{Q\} \tag{3.12}$$

where the positive semi-definite, symmetric mass and stiffness matrix terms are

$$M_{mn} = \int_V \text{"}\rho\text{"} \cdot \Psi_m \Psi_n dV, \; K_{mn} = \int_V \text{"E"} \cdot \Psi_{\varepsilon,m} \Psi_{\varepsilon,n} dV \tag{3.13}$$

[Note: "ρ" and "E" are representative of mass density and elastic stiffness material properties]. In addition, the generalized forcing terms are governed by volume and surface integrals associated with applied body forces and surface loads, respectively.

The Ritz method, outlined above, was initially employed to approximately solve difficult problems described by partial differential equations and associated natural boundary conditions. Ultimately, it was extensively applied in development of the Finite Element Method [8] and Matrix Structural Analysis [9].

3.4.2 Galerkin's Method

Galerkin [10] defined an approximate method using the variable separable displacement field and associated generalized coordinates (Eqs. 3.10 and 3.11) by substitution of the assumed functions into Eq. (3.9) where the boundary conditions are automatically satisfied by choice of an appropriate set of spatial functions. The general statement of Galerkin's method is

$$\int_V (\text{P.D.E}) \cdot \delta u \cdot dV = 0 \tag{3.14}$$

An appealing aspect of Galerkin's method is that it can be applied to any set of partial differential equations (even if a suitable variational formulation is unknown). The method has been successfully applied in the study of nonlinear dynamic systems [11].

3.4.3 Trefftz's Method

Trefftz [12] proposed an approximate method that employs a set of assumed functions that automatically satisfy the partial differential equations. Therefore the form of Eq. (3.9) that must be satisfied is

$$\int_S (\text{N.B.C}) \cdot \delta u \cdot dS = 0 \tag{3.15}$$

This method was generally ignored for about three decades until its advantage in the approximate solution of infinite domain problems was recognized. Trefftz's method has been instrumental in the development of the boundary element method [13].

3.5 Automated Formulations in Structural Dynamics

Among the methodologies that owe their foundations to Hamilton's principle and the contributions of Ritz, Galerkin and Trefftz, two leading techniques are of prominence, namely (a) the Finite Element Method [8], and (b) systematic Matrix Structural Analysis [9]. Discussion of these two topics will be limited to linear systems.

3.5.1 The Finite Element Method

The Finite Element Method encompasses three mathematical disciplines, namely (1) definition of building blocks (finite elements), (2) assembly of structural system models, and (3) solution of structural system equations. The third discipline is addressed by systematic Matrix Structural Analysis.

3.5.1.1 Finite Elements: The Building Blocks

The basic building block, a finite element, is generally derived as a specialized application of the Ritz Method. The displacement field associated with a single element is most generally described in terms of the following family of shape functions:

$$\begin{aligned} &u(x,y,z,t) = \sum_{n=1}^{N_h} \Psi_{h,n}(x,y,z)\cdot u_n(t) + \sum_{n=1}^{N_p} \Psi_{p,n}(x,y,z)\cdot q_n(t) \text{ or in matrix form,} \\ &\{u\} = [\Psi_h]\{u_h\} + [\Psi_p]\{q_p\} \end{aligned} \tag{3.16}$$

The "h" shape functions are referenced to physical displacements at specific grid points on the element's (boundary) surface and the "p" shapes are polynomial functions that have null value along the element's boundary surface. Substitution of the above shape family functions (and their appropriate strain partial derivatives) into Hamilton's principle (Eq. 3.8) results in positive semi-definite mass and stiffness matrices of the forms

$$[M]_{element} = \begin{bmatrix} M_{pp} & M_{ph} \\ M_{hp} & M_{hh} \end{bmatrix}_{element}, [K]_{element} = \begin{bmatrix} K_{pp} & K_{ph} \\ K_{hp} & K_{hh} \end{bmatrix}_{element} \tag{3.17}$$

It should be emphasized here that definition of well-posed and accurate finite elements depends upon expert selection of shape functions and numerical integration schemes. To be sure, development of advanced finite elements is a very important engineering/mathematics specialty.

The most commonly employed finite elements in commercial finite element codes is the "h" element which does not include "p" generalized coordinates. The more general elements are often called "h-p" elements.

3.5.1.2 Assembly of Structural System Models

Assembly of a system structural dynamic model involves the allocation and superposition (overlapping) of individual finite elements onto a system degree-of-freedom map. This process defines sparse system mass and stiffness matrices that are positive semi-definite and symmetric. A system composed of an assembly of "h" elements is defined by "grid" set mass and stiffness matrices denoted by $[M_{gg}]$ and $[K_{gg}]$, respectively. Two other matrix quantities complete the ingredients for a structural dynamic model, namely (1) allocation of excitation forces to system grid points, and (2) formation of an assumed damping matrix (which unfortunately does not resemble physical reality in most commercial finite element codes). The grid set equations for the structural dynamic model are of the form

$$\left[M_{gg}\right]\left\{\ddot{u}_g\right\}+\left[C_{gg}\right]\left\{\dot{u}_g\right\}+\left[K_{gg}\right]\left\{u_g\right\}=\left[\Gamma_g\right]\{F\} \tag{3.18}$$

Constraints describing boundary conditions, described by transformations of the form

$$\left\{u_g\right\}=\left[G_{ga}\right]\left\{u_a\right\}, \tag{3.19}$$

are applied in a symmetric manner as a consequence of the quadratic forms and integrals defined by the Ritz method, resulting in the "analysis" set equations and matrices

$$\begin{gathered}\left[M_{aa}\right]\left\{\ddot{u}_a\right\}+\left[C_{aa}\right]\left\{\dot{u}_a\right\}+\left[K_{aa}\right]\left\{u_{aa}\right\}=\left[\Gamma_a\right]\{F\} \\ \left[M_{aa}\right]=\left[G_{ga}\right]^T\left[M_{gg}\right]\left[G_{ga}\right],\left[C_{aa}\right]=\left[G_{ga}\right]^T\left[C_{gg}\right]\left[G_{ga}\right],\left[K_{aa}\right]=\left[G_{ga}\right]^T\left[K_{gg}\right]\left[G_{ga}\right],\left[\Gamma_a\right]=\left[G_{ga}\right]^T\left[\Gamma_g\right]\end{gathered} \tag{3.20}$$

3.5.2 Matrix Structural Analysis

Matrix Structural Analysis predates the Finite Element Method by several decades. Historically, matrix formulations for structural systems developed along two paths, namely (a) the force method, and (b) the displacement method. The force method owes its foundations to Castigliano [14], who introduced a complementary variational formulation for statics. The displacement method, based primarily on Hamilton's principle and the Ritz method, ultimately eclipsed the force method due to the advent of the Finite Element Method.

3.5.2.1 Free Vibration and Modal Analysis

Free vibration of a structural system generally relates to the undamped eigenvalue problem (dropping the "a" subscript in Eq. 3.20)

$$[M]\{\ddot{u}\}+[K]\{u\}=\{0\}, \tag{3.21}$$

which has solutions described by the (real) mode transformation,

$$\{u\}=[\Phi]\{q\} \tag{3.22}$$

Solutions of the resulting two matrix eigenvalue problem

$$[K]\left\{\Phi_n\right\}-[M]\left\{\Phi_n\right\}\lambda_n=\{0\}\quad\left(\lambda_n=\omega_n{}^2\right) \tag{3.23}$$

are expressed by symmetric transformations, dictated by Hamilton's principle and the Ritz method, which have the following mathematical properties (for unit mass normalized modes):

$$[\Phi]^T[M][\Phi]=[I],[\Phi]^T[K][\Phi]=[\lambda]\ \text{(where }[\lambda]\text{ is a diagonal matrix)} \tag{3.24}$$

These orthogonality properties are guaranteed by symmetries in the mass and stiffness matrices, which are consequences of the Ritz method.

It should be noted that inclusion of the damping matrix in the free vibration equations will generally result in complex modes and eigenvalues. Analysis of complex modes of structural systems, fortunately, is not often pursued. This is "fortunate" for two primary reasons, namely (a) experimental modal analysis generally indicates that "real" modes accurately fit measured data, and (b) popular damping matrix formulations generally do not relate to measured damping characteristics.

3.5.2.2 Proportional Damping Foolishness

Many texts in the field of mechanical vibration and structural dynamics (e.g., [6]) introduce the convenient notion of proportional damping, which assumes a physical damping matrix of the form

$$[C] = \alpha\,[M] + \beta\,[K] \tag{3.25}$$

This assumption is mathematically convenient in that the undamped system modes define a decoupling transformation for the damping matrix, which is

$$[\Phi]^T\,[C]\,[\Phi] = \alpha\,[I] + \beta\,[\lambda]\,. \tag{3.26}$$

However, the on-diagonal terms must correspond to the following relationship for the critical damping ratio, ζ_n

$$\{\Phi_n\}^T\,[C]\,\{\Phi_n\} = 2\zeta_n\omega_n = \alpha + \beta\omega_n^2 \quad \text{or} \quad \zeta_n = \frac{\alpha + \beta\omega_n^2}{2\omega_n}, \tag{3.27}$$

which is restrictive and does not relate to experimental reality for many structural systems displaying relatively uniform critical damping ratios.

There are two practical alternatives to proportional damping "foolishness". The first alternative, namely complex proportional damping, is widely used in aeroelasticity [15]. The second, more widely used alternative circumvents the explicit damping matrix altogether and inserts assumed (or experimentally determined) modal critical damping ratios in truncated modal response analysis of structures.

3.5.2.3 Normal Modes and Mach's Interpretation of d'Alembert's Principle

Several iterative strategies developed for the computation of normal modes have, perhaps unconsciously, benefited from Ernst Mach's interpretation of d'Alembert's principle [16], which simply views inertia (the time derivative of momentum) as an equivalent "static" force. Therefore, the undamped modal problem (Eq. 3.21) is posed as

$$[K]\,\{\Phi\} = \{F_I\} \approx [M]\,\{\Psi\} \ \text{(individual mode)}\,, \ \text{or} \ [K]\,[\Phi] = [F_I] \approx [M]\,[\Psi] \ \ \text{(mode subspace)} \tag{3.28}$$

where $\{\Psi\}$ and $[\Psi]$ are individual and grouped subspace modal guesses, respectively, and $\{\Phi\}$ and $[\Phi]$ are corresponding iterated updates of the respective guesses. The single (fundamental) mode iteration method is attributed to Vianello [17] and Stodola [18], while the grouped mode "subspace iteration" method was more recently introduced by Bathe [19]. More recently introduced methods for treatment of very large order structural systems are now widely utilized.

3.5.2.4 Response to Dynamic Loads: Modal Truncation and d'Alembert's Remedy

Employment of a truncated modal transformation (modes with natural frequency below a threshold) results in a series of ordinary differential equations of the form

$$\ddot{q}_n + 2\zeta_n\omega_n\dot{q}_n + \omega_n^2 q_n = \left[\Phi_n^T\Gamma\right]\{F(t)\} \tag{3.29}$$

Recovery of physical displacements and loads using the truncated modal transformation

$$\{u\} = [\Phi]\,\{q\}\,, \tag{3.30}$$

in many applications is inaccurate because the quasi-static response of all higher frequency modes is ignored. A clever approach, that implicitly employs Mach's interpretation of d'Alembert's principle, known as the mode acceleration method was defined by Williams [20]. The method casts the matrix equations governing system dynamics (Eq. 3.20) as follows

$$[K]\,\{u\} = [\Gamma]\,\{F(t)\} - \{F_I(t)\} \approx [\Gamma]\,\{F(t)\} - [M]\,\{\ddot{q}\} \tag{3.31}$$

where the damping forces are generally ignored. Solution of the above equation set (quasi-statically) at each time point automatically accounts for the quasi-static response of all higher frequency modes (without the need to explicitly know them!). This fact is mathematically provable.

3.5.2.5 Large Order Systems: Component Mode Synthesis

During the 1960s the complexity of finite element models outpaced the development of digital computers and several authors developed procedures to treat system models piecemeal. Hurty [21] and Craig and Bampton [22] developed equivalent techniques known as component mode synthesis that described major structural components in terms of (a) generalized modal coordinates, and (b) physical boundary (interface) degrees of freedom. The general approach is defined by partitioning of a structural component (in a state of free vibration) into "internal" or "i" degrees of freedom and "boundary" or "b" degrees of freedom, as follows:

$$\begin{bmatrix} M_{ii} & M_{ib} \\ M_{bi} & M_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{u}_i \\ \ddot{u}_b \end{Bmatrix} + \begin{bmatrix} K_{ii} & K_{ib} \\ K_{bi} & K_{bb} \end{bmatrix} \begin{Bmatrix} u_i \\ u_b \end{Bmatrix} = \begin{Bmatrix} 0_i \\ 0_b \end{Bmatrix} \tag{3.32}$$

Ritz vectors are then computed as an assembly of (a) internal modes associated with a fixed boundary, and (b) static shapes (or constraint modes) relating internal displacements to boundary displacements, as follows:

$$\begin{Bmatrix} u_i \\ u_b \end{Bmatrix} = \begin{bmatrix} \Phi_{iq} & -K_{ii}^{-1}K_{ib} \\ 0_{bq} & I_{bb} \end{bmatrix} \begin{Bmatrix} q \\ u_b \end{Bmatrix} \tag{3.33}$$

By applying the above transformation in a symmetric manner (following the Ritz method), reduced order mass and stiffness matrices for the component are realized. The form of the matrices is

$$[M] = \begin{bmatrix} I_{qq} & P_{qb} \\ P_{bq} & \overline{M}_{bb} \end{bmatrix}, [K] = \begin{bmatrix} \lambda_{qq} & 0_{qb} \\ 0_{bq} & \overline{K}_{bb} \end{bmatrix} \tag{3.34}$$

It is of significance to note that the description of the reduced order structural component is the same as that of a finite element of the h-p type described in Sect. 3.5.1.1, where (a) the internal modes serve as "p" functions, and (b) the constraint modes serve as "h" functions. Assembly of the large-order structural dynamic system then follows the overlapping procedure used for assembling an ordinary finite element model. Component mode synthesis procedures, which now include additional particular strategies beyond the Hurty–Craig–Bampton method, are widely used in the aerospace industry.

3.6 Additional Variational Principles and Applications

The complementary viewpoint introduced by Castigliano in 1873 was generalized for dynamics by Toupin [23]. The complementary form of Hamilton's principle, known as Toupin's principle, may be derived in a manner similar to the approach described in Chap. 3 of this paper.

The postulate-theorem interrelationship is demonstrated below for a mechanical system composed of a single particle restricted to motion along one direction. The time integral of Newton's second law states the postulate,

$$\widehat{F} = \int_{-\infty}^{t} F dt = m\frac{du}{dt} \text{ or } \frac{\widehat{F}}{m} = \frac{du}{dt} \tag{3.35}$$

A d'Alembert type variational statement is that the system follows an "economy" in variational terms such that

$$\left(\frac{\widehat{F}}{m} - \frac{du}{dt}\right) \cdot \delta\widehat{F} = 0 \tag{3.36}$$

where "$\delta\widehat{F}$" is defined as a variation off the true path of "$\widehat{F}$". The complementary form of Hamilton's principle is therefore

$$\int_{t_1}^{t_2} \left(\frac{\widehat{F}}{m} - \frac{du}{dt}\right) \cdot \delta\widehat{F} \cdot dt = 0 \tag{3.37}$$

This variational integral is more compactly expressed as

$$\int_{t_1}^{t_2} (\delta T_C + \delta W_C) \cdot dt = 0, \tag{3.38}$$

where $T_C = \frac{\widehat{F}^2}{m}$ is the system complementary kinetic energy and $\delta W_C = -\frac{du}{dt} \cdot \delta\widehat{F}$ is the system complementary virtual work. By introducing a complementary strain energy function (not described here), the variational principle takes the form

$$\int_{t_1}^{t_2} (\delta T_C - \delta U_C + \delta W_C) \cdot dt = 0 \tag{3.39}$$

This complementary variational principle was employed in the study and resolution of difficulties encountered with fluid–structure interaction models, which employed pressure as the physical variable for the fluid, by Coppolino [24]. One of the more interesting peculiarities of Toupin's principle is the role reversal of kinetic and strain energies in that the time derivative is associated with strain energy. As illustrated in the acoustic matrix equation

$$\left[C_p\right]\left\{\ddot{\widehat{p}}\right\} + \left[S_p\right]\{\widehat{p}\} = -\left[A_p\right]^T\{\dot{u}\} \quad \text{or} \quad \left[C_p\right]\{\ddot{p}\} + \left[S_p\right]\{p\} = -\left[A_p\right]^T\{\ddot{u}\}, \tag{3.40}$$

$[C_p]$ is the acoustic compliance (flexibility) matrix, $[S_p]$ is the susceptance (inverse mass) matrix, and $[A_p]$ is the surface area matrix. Toupin's principle, while mathematically curious, has not found many convenient applications to this date.

Additional variational formulations have been derived and employed for the study of other applications in applied physics. It appears that use of Galerkin's method as a starting point facilitates definition of novel versions of Hamilton's principle. Biot's variational principle [25] for heat conduction and Hamilton's principle for finite element solution of Maxwell's equations by MacNeal et al. [26], are just two examples of alternative expressions of Hamilton's principle.

3.7 Conclusions

This paper offers a historical thread tracing the development of modern structural dynamics, which owes much of its present state to variational principles. The foundational variational principles are presented as a logical progression stemming from (a) ancient and medieval notions of economy in nature and human reasoning, (b) Newton's laws, (c) d'Alembert's principle, and (d) Hamilton's principle. A fifth principle or consequence, (e) Lagrange's equations, is derivable from Hamilton's principle (in spite of the fact that it was introduced before Hamilton's principle). All of the variational principles were fully developed by the first half of the nineteenth century.

Three approaches to approximate analysis based on Hamilton's principle were introduced during the early part of the twentieth century by Ritz, Galerkin, and Trefftz. Each of the three approaches has led to significant progress in structural dynamics, which can be summarized as follows:

1. The Ritz method, which substitutes an assumed shape function series into Hamilton's principle, is responsible for definition of symmetric, positive semi-definite mass and stiffness matrices. Moreover, the Ritz method provides a foundation for generalized coordinates (displacements) and generalized forces. It forms the foundation for much of the Finite Element Method and Matrix Structural Analysis.
2. Galerkin's method substitutes an assumed shape series into a continuum mechanics statement of Hamilton's principle consisting of (a) a stationary integral with a partial differential equation kernel, and (b) a stationary integral with a natural boundary condition kernel. By selecting shape functions that automatically satisfy natural boundary conditions, Galerkin's method focuses on (a) and it is employed to treat a very wide range of problems in applied physics for which variational principles are not known or well understood. It has been employed in the study of nonlinear phenomena.
3. Trefftz's method takes on a complementary approach with respect to the Galerkin method in that the assumed functions automatically satisfy (a) the partial differential kernel, and it seeks a least squares solution to (b) the natural boundary condition kernel integral. This strategy is well-suited for problems related to unbounded media, and it forms much of the basis of the Boundary Element Method.

Two tasks in Matrix Structural Analysis have benefited from consequences of Mach's viewpoint of d'Alembert's principle, published near the end of the nineteenth century. They are as follows:

1. Computation of the lowest eigenvalues and eigenvectors for an undamped structural system is performed by (a) employing a single mode iteration procedure introduced by Vianello and Stodola by the first decade of the twentieth century, or (b) employing a modal block (subspace) iteration procedure introduced by Bathe in the early 1970s.
2. Accurate computation of structural dynamic response by treating a truncated set of modal accelerations as equivalent "static loads". The technique introduced by Williams in 1945 automatically accounts for the quasi-static response of all high frequency modes without explicitly solving for them.

The influence of variational principles reaches a much broader set of applications than structural dynamics based on (a) a complementary variational principle introduced by Toupin in 1952, and generalizations of Galerkin's method which resulted in (b) Biot's variational principle for heat transfer in 1957, and (c) a variational formulation for electromagnetics introduced by MacNeal et al. in 1987.

It is clear that progress in the development of variational principles, approximate methods, and practical applications has occurred in successive stages depending on prior contributions. There is a wisdom statement, generally attributed to Newton, "If I have seen further . . . ", which actually has more ancient roots. The Italian-Jewish sage, Isaiah di Trani (d. 1250) stated: "Who sees further a dwarf or a giant? Surely a giant for his eyes are situated at a higher level than those of a dwarf. But if the dwarf is placed on the shoulders of the giant, who sees further? So too we are dwarfs astride the shoulders of giants. We master their wisdom and move beyond it".

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