Gaussian Mean Curvature Flow for Submanifolds in Space Forms

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Abstract In this chapter we investigate the convergence of the mean curvature flow of submanifolds in Euclidean and hyperbolic spaces with Gaussian density. For Euclidean case, we prove that the flow deforms a closed submanifold with pinching condition to a "round point" in finite time.

Keywords Riemannian metric • Mean curvature flow • Density • Conformal transformation

Mathematics Subject Classifications (2010): Primary 53C44, Secondary 35K93, 35R01

1 Introduction

The *mean curvature flow* (MCF) was proposed by W. Mullins (1956) to describe the formation of grain boundaries in annealing metals. Brakke [5] introduced the motion of a submanifold by its MCF in arbitrary codimension and constructed a generalized varifold solution for all time. There are many works for the classical solution of MCF on hypersurfaces. Huisken [7] showed that if the initial hypersurface in the Euclidean space is compact and uniformly convex, then MCF converges to a "round point" in a finite time. He also studied MCF of hypersurfaces in a Riemannian manifold satisfying a pinching condition in a sphere, see [1].

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V. Rovenski and P. Walczak (eds.), *Geometry and its Applications*, Springer Proceedings in Mathematics & Statistics 72, DOI 10.1007/978-3-319-04675-4_2,

For MCF of submanifolds with higher codimension, fruitful results were obtained for submanifolds with low dimension or admitting some special structures, see survey [11, 12]. Andrews and Baker [2] proved a convergence theorem for MCF of closed submanifolds satisfying a suitable pinching condition in the Euclidean space. Baker [3] and Liu–Xu–Ye–Zhao [8, 9] generalized Andrews-Baker's convergence theorem [2] for MCF of submanifolds in the Euclidean space to the case of MCF of arbitrary codimension in spherical and hyperbolic space forms and Riemannian manifolds.

Morgan [10] introduced manifolds with density, which provides a new concept of curvature. A. Borisenko and V. Miquel considered MCF with density for hypersurfaces in Euclidean space.

In this chapter we study the convergence of the MCF of submanifolds in Euclidean and hyperbolic spaces with Gaussian density. For Euclidean case, we prove that the flow deforms a closed submanifold satisfying pinching condition to a "round point" in finite time. For hyperbolic case, we find maximal radius (or minimal normal curvature) of central hypersphere in a hyperbolic space that shrinks to the origin under the MCF with Gaussian density; moreover, for central spheres of smaller radius we estimate the collapsing time.

2 The MCF in Riemannian Manifolds and Space Forms

Consider immersions of a closed manifold M^n into a space form:

$$F_t : M^n \to \bar{M}^{n+p}(c), \quad F_t(q) = F(q,t), \quad q \in M^n,\ t \in [0,T).$$

Denote by h_t the second fundamental tensor, and by $H_t = \operatorname{Tr}_g h_t$ the mean curvature vector field of the immersions (g is the induced metric on M). The *MCF* is the evolution equation (see [2, 11])

$$\partial_t F = H, \tag{1}$$

where $F_0 : M^n \to \bar{M}^{n+p}(c)$ provides initial data.

Remark 1. The *general form of the MCF* is

$$(\partial_t F)^\perp = H, \tag{2}$$

where $^\perp$ denotes the projection onto the normal space of $F_t(M)$. This equation is equivalent to (1) up to diffeomorphisms of M (see [12]; the proof is the same as for $p = 1$ in [6]).

Let $\bar{M}^{n+p}(c)$ be endowed with a continuous *density function* $f = e^\psi$, where $\psi \in C^2(\bar{M}^{n+p}(c))$. The generalization of the mean curvature of submanifolds in such spaces, obtained by the first variation of the volume, is given in [10] as

$$H_\psi = H - (\nabla\psi)^\perp.$$

It is natural to study flows governed by H_ψ instead of H:

$$\partial_t F = H - (\nabla\psi)^{\perp}. \tag{3}$$

Any n-dimensional submanifold satisfies $|h|^2 \geq \frac{1}{n}|H|^2$ (where $|H|$ and $|h|$ are norms), and totally umbilical submanifolds give the equality.

Lemma 1 ([13]). *Let M^n be an n-dimensional submanifold in an $(n+p)$-dimensional Riemannian manifold $\bar{M}^{n+p}$ and π a tangent two-plane on $T_q(M)$ at a point $q \in M$. Choose an orthonormal two-frame $\{e_1, e_2\}$ at q such that $\pi = \operatorname{span}\{e_1, e_2\}$. Then*

$$K(\pi) \geq \frac{1}{2}\left(2\bar{K}_{\min} + \frac{H^2}{n-1} - |h|^2\right) + \sum\nolimits_{a=n+1,\, j>i}^{n+p} \sum\nolimits_{(i,j)\neq(1,2)} (h_{ij}^a)^2 .$$

Recently, Andrews–Baker [2] proved convergence theorem for the MCF of closed submanifolds satisfying a pinching condition in the Euclidean space.

Theorem A ([2]). *Let $n \geq 2$, and suppose that $F_0(M^n)$ is a closed submanifold smoothly immersed in $\mathbb{R}^{n+p}$. If $F_0(M^n)$ has $H \neq 0$ everywhere and satisfies*

$$|h|^2 \leq \begin{cases} \frac{4}{3n}|H|^2, & \text{if } n = 2, 3, \\ \frac{1}{n-1}|H|^2, & \text{if } n \geq 4, \end{cases} \tag{4}$$

then MCF (1) has a unique smooth solution $F_t : M^n \times [0, T) \to \mathbb{R}^{n+p}$ on a finite maximal time interval, and F_t converges uniformly to a point $q \in \mathbb{R}^{n+p}$ as $t \to T$. The rescaled maps $\tilde{F}_t = \frac{F_t - q}{\sqrt{2n(T-t)}}$ converge in C^∞ as $t \to T$ to an embedding $\tilde{F}_T$ with image equal to a regular unit n-sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+p}$. If $n \geq 4$, pinching ratio (4) is optimal.

Liu–Wei–Zghao [8] extended Theorem A to submanifolds in hyperbolic spaces.

Theorem A′ ([8]). *Let $F_0(M^n)$ $(n \geq 2)$ be a closed submanifold smoothly immersed in hyperbolic space $\mathbb{H}^{n+p}(c)$ of constant curvature $c < 0$. If $F_0(M^n)$ satisfies*

$$|h|^2 \leq \begin{cases} \frac{4}{3n}|H|^2 + \frac{n}{2}c, & \text{if } n = 2, 3, \\ \frac{1}{n-1}|H|^2 + 2c, & \text{if } n \geq 4, \end{cases} \tag{5}$$

then MCF (1) with F_0 as initial value has a unique smooth solution $F_t : M^n \times [0, T) \to \mathbb{H}^{n+p}(c)$ on a finite maximal time interval, and $F_t(M^n)$ converges uniformly to a "round point" as $t \to T$.

3 Gaussian MCF in Euclidean Space

The *Gaussian density* $e^{-\frac{n}{2}\mu^2|x|^2}$ (for some $\mu > 0$) in $\mathbb{R}^{n+p}$ is rotational invariant and corresponds to the radial function

$$\psi(x) = -\frac{n}{2}\,\mu^2|x|^2. \tag{6}$$

In this case, $\nabla\psi(x) = -n\mu^2 x$ for all $x \in \mathbb{R}^{n+p}$. Along the submanifold $F(M)$ we have $(\nabla\psi)^\perp = -n\mu^2 F^\perp$. Since $H_\psi = H - (\nabla\psi)^\perp$, see [10], the *MCF in* $\mathbb{R}^{n+p}$ *with Gaussian density* is defined by

$$\partial_t F = H + n\mu^2 F^\perp. \tag{7}$$

Lemma 2 (see [4]). *Let ψ be a radial function on $\mathbb{R}^{n+p}$. The vector field $\nabla\psi$ is conformal if and only if*

$$\psi(x) = \pm\frac{n}{2}\,\mu^2|x|^2 \qquad (\text{hence, } \nabla\psi = \pm\, n\mu^2\, x) \qquad \text{for some } \mu > 0.$$

Borisenko–Miquel [4] proved convergence theorem for the MCF with Gaussian density on a hypersurface in $\mathbb{R}^{n+1}$.

Theorem B ([4]). *Let $F_0 : M \to \mathbb{R}^{n+1}$ be a convex hypersurface with a chosen unit normal vector N, which evolves under MCF with Gaussian density (see (7) with $p = 1$)*

$$\partial_t F = (H + n\mu^2\langle F, N\rangle)N. \tag{8}$$

Then its evolution F_t remains convex for all time $t \in [0, T)$ where it is defined.

If $h \ge \mu g$ and $h(v, v) > \mu g(v, v)$ in some vector at some point v, then there is a point q_0 inside the convex domain $F_0(M)$ such that $F_0(M)$ lies in the ball B with center q_0 of radius $1/\mu$. Moreover,

1. *$T < \infty$ and $h > \mu g$ for $t \in (0, T)$,*
2. *$F_t(M)$ belongs to a ball of radius $1/\mu$ all time and shrinks to a "round point" when $t \to T$.*

Lemma 3 ([2]). *If a solution $F_t : M^n \to \mathbb{R}^{n+p}$ $(0 \le t < T)$ of MCF (1) satisfies $|h|^2 + a < C|H|^2$ for some constants $C \le \frac{1}{n} + \frac{1}{3n}$ and $a > 0$ at $t = 0$, then this remains true for all $0 \le t < T$.*

Using Theorem A, we extend Theorem B for submanifolds in Euclidean space.

Theorem 1. *Let $F_0 : M^n \to \mathbb{R}^{n+p}$ be a complete smoothly immersed submanifold with the condition*

$$|h|^2 + \beta^2 \le C\,|H|^2 := \begin{cases} \frac{4}{3n}\,|H|^2, & \text{if } n = 2, 3, \\ \frac{1}{n-1}\,|H|^2, & \text{if } n \ge 4, \end{cases} \tag{9}$$

where

$$\beta^2 \geq (\pi\mu)^2 \frac{n+p}{n+p+1} - \Big(\frac{1}{n-1} - C\Big)|H|^2. \tag{10}$$

Then the MCF with the Gaussian density in $\mathbb{R}^{n+p}$, (7)*, has a unique smooth solution* $F_t : M^n \times [0, T) \to \mathbb{R}^{n+p}$ *on a finite maximal time interval, and* F_t *converges uniformly to a "round point" when* $t \to T$.

Proof. Its main steps coincide with ones in the proof of Theorem B.

By Lemma 1, at each point $q \in M^n$ the smallest sectional curvature $K_{\min}$ satisfies

$$K_{\min}(q) \geq \frac{1}{2}\Big(\frac{1}{n-1}\,|H(q)|^2 - |h(q)|^2\Big). \tag{11}$$

Substituting $|h|^2$ from our assumption (9) into inequality (11), for $q \in M$ we obtain

$$K_{\min}(q) \geq \frac{1}{2}\left(\left(\frac{1}{n-1} - C\right)|H|^2 + \beta^2\right).$$

Note that $\frac{1}{n-1} - C \geq 0$. By Theorem of Bonnet, Hopf-Rinow and Myers for $t = 0$, we have

$$\operatorname{diam} M \leq \pi\sqrt{2}\,\tilde{d}, \qquad \tilde{d} = \left[\left(\frac{1}{n-1} - C\right)|H|^2 + \beta^2\right]^{-1/2}.$$

Note that the inner diameter of M is greater than or equal to diameter d of $F_0(M)$. The Yung's Theorem (1901) tells us that every set $K \subset \mathbb{R}^{n+p}$ of diameter d is contained in a ball in $\mathbb{R}^{n+p}$ of radius $r_0(K) = \sqrt{\frac{n+p}{2(n+p+1)}}\,d$. Thus, $F_0(M)$ is contained in a ball in $\mathbb{R}^{n+p}$ of radius

$$r_0 \leq \pi\sqrt{\frac{n+p}{n+p+1}}\,\tilde{d}. \tag{12}$$

Recall that if $F_0(M)$ is contained in a ball $B(r_0)$ of radius $r_0 > 0$, then flow (1) must develop singularity (collapsing to a point) before the time $T = r_0^2/(2n)$, see [2].

Condition (10) for β yields the inequality $r_0^2/(2n) < 1/(2n\mu^2)$.

By Proposition 1, the MCF $\hat{F}_{\hat{t}}$ of (14) is equivalent to the flow F_t of (7) for all $\hat{t} \in [0, \hat{T}]$.

The submanifold $\hat{F}_0(M) = F_0(M)$ satisfies the conditions of Theorem A. Then (1) has a unique smooth solution $\hat{F}_{\hat{t}} : M^n \times [0, \hat{T}) \to \mathbb{R}^{n+p}$ on a finite maximal time interval, and it converges uniformly to a point $\hat{q} \in \mathbb{R}^{n+p}$ as $\hat{t} \to \hat{T}$. The rescaled maps converge in C^∞ as $\hat{t} \to \hat{T}$ to an embedding with image equal to a regular n-sphere in some $(n+1)$-dimensional subspace of $\mathbb{R}^{n+p}$.

From equivalence of flows (14) and (7) we conclude that F_t converges in a finite time uniformly to a point $q \in \mathbb{R}^{n+p}$. Since submanifolds $\hat{F}_{\hat{t}}(M)$ and $F_t(M)$ are homothetic, we obtain that F_t converges to a "round point" $q \in \mathbb{R}^{n+p}$. □

By the next proposition, one may transfer any result on MCF (1) to a result on flow (3) with ψ given in (6).

Proposition 1 (For $p = 1$, see [4]). *MCF (7) in $\mathbb{R}^{n+p}$ with Gaussian density is equivalent, up to tangential diffeomorphisms, with the parameter change*

$$\hat{t} = -\frac{1}{2n\mu^2}\big(e^{-2n\mu^2 t} - 1\big) \tag{13}$$

to the MCF in $\mathbb{R}^{n+p}$

$$\frac{\partial \hat{F}}{\partial \hat{t}} = \hat{H} \qquad \Big(\text{for } \hat{t} < \frac{1}{2n\mu^2}\Big). \tag{14}$$

Proof. The one-parameter family of diffeomorphisms $\phi_t(x) = e^{-n\mu^2 t}x$ is the solution of the ODE

$$\frac{d}{dt}\phi_t(x) = -n\mu^2\phi_t(x)$$

with the initial condition $\phi_0(x) = x$ and is associated with the vector field $X(x) = -n\mu^2 x$ on $\mathbb{R}^{n+p}$. If F flows by the mean curvature with density $f = e^{-\frac{1}{2}n\mu^2|x|^2}$, then the flow $\hat{F}_t = \phi_t \circ F_t$ has the form

$$\hat{F} = e^{-n\mu^2 t}F. \tag{15}$$

To check this and to find the corresponding reparametrization of time, we compute

$$\begin{aligned}\partial_t \hat{F} &= -n\mu^2 e^{-n\mu^2 t}F + e^{-n\mu^2 t}(H + n\mu^2 F^{\perp})\\ &= -n\mu^2 e^{-n\mu^2 t}F^{\top} + e^{-n\mu^2 t}H = -n\mu^2\hat{F}^{\top} + e^{-n\mu^2 t}H.\end{aligned}$$

By (15), the second fundamental tensors of $\hat{F}$ and F are related by $\hat{h} = e^{-n\mu^2 t}h$; hence, $\hat{H} = e^{n\mu^2 t}H$. Therefore, the evolution for $\hat{F}$ is

$$\partial_t\hat{F} = -n\mu^2\hat{F}^{\top} + e^{-2n\mu^2 t}\hat{H}. \tag{16}$$

If we define $\hat{t}$ by (13), we get $dt/d\hat{t} = (d\hat{t}/dt)^{-1} = e^{2n\mu^2 t}$, and

$$\frac{\partial\hat{F}}{\partial\hat{t}} = \frac{\partial\hat{F}}{\partial t}\cdot\frac{dt}{d\hat{t}} = -n\mu^2 e^{2n\mu^2 t}\hat{F}^{\top} + \hat{H} = \frac{1}{2}\Big(\hat{t} - \frac{1}{2n\mu^2}\Big)^{-1}\hat{F}^{\top} + \hat{H}. \tag{17}$$

Flow (17) is, up to a tangential diffeomorphism (see Remark 1), equivalent to the MCF equation $\partial_t \hat{F} = \hat{H}$ for $\hat{t} < \hat{T} = \frac{1}{2} n\mu^{-2}$ (because at $\hat{T}$ the tangential diffeomorphism giving the equivalence is not well defined: the time $\hat{t} = \hat{T}$ corresponds in (13) to $t = \infty$). □

Remark 2. For Euclidean case, we find $t = -\frac{1}{2n\mu^2}\log(1 - 2n\mu^2\hat{t})$, and the converse of (15) is

$$F = e^{n\mu^2 t}\hat{F} = (1 - 2n\mu^2 t)^{-1/2}\hat{F}.$$

In [3] Baker proved a convergence result for the MCF of submanifolds in a sphere $S^{n+p}(c)$ of constant curvature $c > 0$. Using this, one may deduce the convergence theorem for the MCF for closed submanifolds satisfying a pinching condition in the sphere with Gaussian density.

4 Gaussian MCF in Hyperbolic Space

Let r be the *distance function* from a fixed point q (the origin) on $\mathbb{H}^{n+p} := \mathbb{H}^{n+p}(-1)$.

The *Gaussian density* $e^{n\mu^2(1-\cosh r)}$ (for some $\mu > 0$) in a hyperbolic space $\mathbb{H}^{n+p}$ is rotational invariant and corresponds to the radial function

$$\psi(x) = -n\mu^2(\cosh r(x) - 1). \tag{18}$$

In this case, $\nabla\psi(x) = -n\mu^2(\sinh r(x))\partial_r$ for all $x \in \mathbb{H}^{n+p}$.

The *MCF with Gaussian density* for a submanifold $F_0 : M^n \to \mathbb{H}^{n+p}$ is

$$\partial_t F = H + n\mu^2(\sinh r(F))\partial_r^{\perp}. \tag{19}$$

For a hypersurface $F_0 : M^n \to \mathbb{H}^{n+1}$ with a chosen unit normal vector N this reads

$$\partial_t F = \big(H + n\mu^2 \sinh r(F)\langle\partial_r, N\rangle\big)\, N. \tag{20}$$

Lemma 4. *(i) Let $\psi = \varphi \circ r$ be a radial function on $\mathbb{H}^{n+p}$ (for a function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ of class C^1). Then the vector field $\nabla\psi$ is conformal if and only if $\varphi(r) = \pm n\mu^2(\cosh r - 1)$ for some $\mu \in \mathbb{R}_+$.*

(ii) In spherical coordinates $(r, \tilde{x})$ in $\mathbb{H}^{n+p}$ the conformal diffeomorphisms belonging to $X(x) = -n\mu^2(\sinh r(x))\partial_r$ have a form $\tilde{\phi}_t(r, \tilde{x}) = (\phi_t(r), \tilde{x})$, where

$$\phi_t(r) = 2\,\mathrm{arctanh}\big(\tanh(r/2)\, e^{-n\mu^2 t}\big). \tag{21}$$

Proof. (i) We have $\nabla\psi = \varphi'\,\nabla r$. The condition for the vector field $\nabla\psi$ being conformal, that is, $\mathrm{Hess}_\psi = \lambda\, g$, translates into

$$\varphi''\nabla r\otimes\nabla r+\varphi'\,\mathrm{Hess}_r=\lambda\, g. \tag{22}$$

The hessian is defined as a symmetric $(0,2)$-tensor such that $\mathrm{Hess}_\psi(X,Y) = g(S(X),Y)$, where $S(X)=\nabla_X\nabla\psi$ is a self-adjoint $(1,1)$-tensor.

The normal curvature of a sphere of radius r in $\mathbb{H}^{n+p}$ is $\coth r$. Hence,

$$\mathrm{Hess}_r=(\coth r)(g-\nabla r\otimes\nabla r).$$

Collecting terms with g and $\nabla r\otimes\nabla r$ in (22), we obtain the system

$$\varphi''=(\coth r)\,\varphi',\qquad \lambda=(\coth r)\,\varphi'.$$

The solution of the first ODE with the initial condition $\varphi(0)=0$ has the required form. Notice that $\varphi\approx\mp\frac{1}{2}\,n\mu^2 r^2$ for $r\approx 0$, see Lemma 2.

(ii) The one-parameter family $\phi_t(r)$ of conformal radial diffeomorphisms belonging to $\tilde X(r)=-n\mu^2(\sinh r)\partial_r$ is the solution of the Cauchy's problem

$$\frac{d}{dt}\,\phi_t(r)=-n\mu^2\sin h\phi_t(r),\qquad \phi_0(r)=r.$$

The unique solution has form (21). □

Remark 3. One may represent $\mathbb{H}^{n+p}$ as a unit ball $B(0,1)\subset\mathbb{R}^{n+p}$ with the metric

$$\mathrm{ds}^2=\frac{4\,\mathrm{dx}^2}{(1-x^2)^2},\quad \text{where}\quad x=(x_1,\dots,x_{n+p}),\quad x^2=\sum\nolimits_i x_i^2.$$

For the hyperbolic radial distance r we have $\mathrm{dr}=\frac{2\,\mathrm{d}|x|}{1-x^2}$ and

$$r=2\,\mathrm{arctanh}(|x|)\quad\Longleftrightarrow\quad |x|=\tanh(r/2).$$

Hence, $\sinh r=\frac{2\,|x|}{1-x^2}$, and the unit radial vector is $\partial_r=\frac{1-x^2}{2\,|x|}F$.

If F flows by the mean curvature with density $f=e^{n\mu^2(1-\cosh r)}$, for the density we obtain

$$\nabla\psi=-n\mu^2(\sinh r(F))\,\partial_r=-n\mu^2 F\,.$$

Then the flow $\hat F_t=\phi_t(F_t)$, where $\phi_t(x)=e^{-n\mu^2 t}x$, has the form, see (15),

$$\hat F=e^{-n\mu^2 t}F\,. \tag{23}$$

The derivation in t yields

$$\partial_t \hat F = \partial_t(e^{-n\mu^2 t}F) = e^{-n\mu^2 t}\big((H+n\mu^2 F^{\perp})-n\mu^2 F\big) = e^{-n\mu^2 t}H - n\mu^2 e^{-n\mu^2 t}F^{\top}.$$

Note that $\coth r = \frac{1+x^2}{2\,|x|}$ and $\coth\hat r = \frac{1+x^2 e^{-2n\mu^2 t}}{2\,|x|\,e^{-n\mu^2 t}}$, where $\hat r = 2\,\mathrm{arctanh}(e^{-n\mu^2 t}|x|)$ due to (23). Since the mapping of $\mathbb{H}^{n+p}$ into itself given in (23) is conformal, for the mean curvature vectors H and $\hat H$ of submanifolds F and $\hat F$ we have

$$\hat H = \lambda H, \quad \text{where} \quad \lambda = \frac{\coth\hat r}{\coth r} = \frac{1+x^2 e^{-2n\mu^2 t}}{(1+x^2)\,e^{-n\mu^2 t}}.$$

Thus, $e^{-n\mu^2 t}H = \frac{(1+x^2)\,e^{-2n\mu^2 t}}{1+x^2 e^{-2n\mu^2 t}}\hat H$ and the PDE above reduces to

$$\partial_t \hat F = \frac{(1+x^2)\,e^{-2n\mu^2 t}}{1+x^2 e^{-2n\mu^2 t}}\hat H - n\mu^2 \hat F^{\top}.$$

After suitable tangential transformation of M^n, we obtain the PDE that generalizes (1):

$$\partial_t \hat F = \frac{1+x^2}{e^{2n\mu^2 t}+x^2}\,\hat H\,. \tag{24}$$

Note that (24) reduces to MCF (14) when $\mu\to 0$.

In the next proposition we find maximal radius (or minimal normal curvature) of central hypersphere in a hyperbolic space that shrinks to the origin under the MCF with Gaussian density; for central spheres of smaller radius we estimate the collapsing time.

Proposition 2. *Let either the radius r_0 of the central hypersphere $S^n(r_0)\subset\mathbb{H}^{n+1}$ or its normal curvature k satisfy the certain of inequalities*

$$\cosh r_0 < \sigma_1 := \frac{1+\sqrt{1+4\mu^4}}{2\,\mu^2}, \qquad k > \mu\sqrt{\sigma_1}. \tag{25}$$

Then $S^n(r_0)$ shrinks to the origin under MCF (20) with Gaussian density by the time

$$\begin{aligned} T &= \frac{1}{n\sqrt{1+4\mu^4}}\ln\frac{(1-2\mu^2+\sqrt{1+4\mu^4})(2\mu^2\cosh r_0-1+\sqrt{1+4\mu^4})}{(2\mu^2-1+\sqrt{1+4\mu^4})(1-2\mu^2\cosh r_0+\sqrt{1+4\mu^4})}\\ &< \frac{\sigma_1}{n\mu^2(\sigma_1+1)}\cdot\frac{\cosh r_0-1}{\sigma_1-\cosh r_0}\,. \end{aligned} \tag{26}$$

The central sphere of radius $r_1 = \mathrm{arccosh}(\sigma_1)$ is a fixed point of the flow. The central sphere of radius $r > r_1$ expands without limit.

Proof. The mean curvature of the central hypersphere $S^n(r)$ of radius r is $H = -n\coth r$; hence, $N = \partial_r$ and (20) reads as the ODE for the radius $r(t) > 0$,

$$\frac{d}{dt}\,r = -n\coth r + n\mu^2 \sinh r, \qquad r(0) = r_0. \tag{27}$$

The sphere shrinks to a point when

$$\coth r - \mu^2 \sinh r > 0 \quad \Leftrightarrow \quad \mu^2\cosh^2 r - \cosh r - \mu^2 < 0.$$

The roots of quadratic equation $\mu^2\sigma^2 - \sigma - \mu^2 = 0$ are $\sigma_{1,2} = \frac{1\pm\sqrt{1+4\mu^4}}{2\mu^2}$. The positive root $\sigma_1 \geq 1$, and the negative root $\sigma_2 \in (-1, 0)$. Hence, the central sphere of radius $r_1 = \operatorname{arccosh}(\sigma_1)$ is a fixed point of the flow, the central sphere of radius $r > r_1$ expands without limit, and the central sphere of radius $r < r_1$ shrinks to the origin. The normal curvature of the r_1-sphere is $k_1 = \coth r_1 = \cosh r_1/\sqrt{\cosh^2 r_1 - 1} = \sqrt{\frac{1+\sqrt{1+4\mu^4}}{2}} > \mu$.

Assuming $\sigma(t) = \cosh r(t) > 1$ and $\sigma_o = \cosh r_0$, we reduce (27) to

$$d\sigma/dt = n(\mu^2\sigma^2 - \sigma - \mu^2) = n\mu^2(\sigma - \sigma_1)(\sigma - \sigma_2), \qquad \sigma(0) = \sigma_o.$$

We have

$$\frac{1}{(\sigma-\sigma_1)(\sigma-\sigma_2)} = -\frac{1}{\sigma_1-\sigma_2}\Big(\frac{1}{\sigma_1 - y} + \frac{1}{y-\sigma_2}\Big),$$

$$\int_{\sigma_o}^{\sigma} \frac{\mathrm{dy}}{(y-\sigma_1)(y-\sigma_2)} = n\mu^2 t.$$

If the initial value satisfies $\sigma_o \in (1, \sigma_1)$, then the integral above is $\log\frac{y-\sigma_2}{\sigma_1-y}\,|_{\sigma_o}^{\sigma} = n\mu^2(\sigma_1-\sigma_2)\,t$; hence, the solution $\sigma(t)$ is a decreasing function

$$\sigma(t) = \frac{\sigma_2\,\alpha + \sigma_1}{\alpha+1}\,, \qquad \text{where} \quad \alpha = \frac{\sigma_1-\sigma_o}{\sigma_o-\sigma_2}\,e^{\,n\mu^2(\sigma_1-\sigma_2)\,t}\,.$$

Note that $\lim\limits_{t\to\infty}\sigma(t) = \sigma_2 < 0 < 1 < \sigma_1 = \lim\limits_{t\to-\infty}\sigma(t)$. The collapse $r(T) = 0$ at $t = T$ (i.e., $\sigma(T) = 1$) appears at

$$T = \frac{1}{n\mu^2(\sigma_1-\sigma_2)}\log\frac{(\sigma_o-\sigma_2)(\sigma_1-1)}{(\sigma_1-\sigma_o)(1-\sigma_2)} > 0,$$

that is, (26). Using the inequality $\log(1+y) < y$ for $y > 0$ and relation $\sigma_2 = -1/\sigma_1$, we obtain

$$T < \frac{1}{n\mu^2(\sigma_1 - \sigma_2)}\Big(\frac{(\sigma_o - \sigma_2)(\sigma_1 - 1)}{(\sigma_1 - \sigma_o)(1 - \sigma_2)} - 1\Big)$$
$$= \frac{\sigma_o - 1}{n\mu^2(\sigma_1 - \sigma_o)(1 - \sigma_2)} = \frac{\sigma_1}{n\mu^2(\sigma_1 + 1)} \cdot \frac{\sigma_o - 1}{\sigma_1 - \sigma_o}\,.$$

Certainly, for initial value $\sigma_o > \sigma_1$, the solution $\sigma(t)$ is a monotone increasing function. □

Remark 4. For the MCF of a hypersphere in $\mathbb{H}^{n+1}$, the radius obeys the PDE $\frac{d}{dt}r = -n\coth r$; hence, $\cosh r(t) = e^{-nt}\cosh r_0$ and the existence time is $\tilde{T} = \frac{1}{n}\log(\cosh r_0)$, i.e., $r(\tilde{T}) = 0$. For $\mu = 0$, flow (20) reduces to the MCF, and in this case we have $\lim_{\mu\to 0} T = \tilde{T}$. For the MCF of a submanifold M^n in $\mathbb{H}^{n+p}$ $(n, p > 1)$, we have the course estimate $\tilde{T} < \frac{1}{n-1}\, r_0$, see [8]. We conjecture that Theorem A′ can be extended to the convergence theorem (like Theorem 1) for the MCF of closed submanifolds satisfying a pinching condition in the hyperbolic space with Gaussian density.

Acknowledgements Supported by the Marie-Curie actions grant EU-FP7-P-2010-RG, No. 276919.

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