Chapter 9
Almost Convergence in Approximation Process

9.1 Introduction

Several mathematicians have worked on extending or generalizing the Korovkin's theorems in many ways and to several settings, including function spaces, abstract Banach lattices, Banach algebras, Banach spaces, and so on. This theory is very useful in real analysis, functional analysis, harmonic analysis, measure theory, probability theory, summability theory, and partial differential equations. But the foremost applications are concerned with constructive approximation theory which uses it as a valuable tool. Even today, the development of Korovkin-type approximation theory is far from complete. Note that the first and the second theorems of Korovkin are actually equivalent to the algebraic and the trigonometric version, respectively, of the classical Weierstrass approximation theorem [1]. In this chapter we prove Korovkin type approximation theorems by applying the notion of almost convergence and show that these results are stronger than original ones.

9.2 Korovkin Approximation Theorems

Let $F(\mathbb{R})$ denote the linear space of all real-valued functions defined on $\mathbb{R}$. Let $C(\mathbb{R})$ be the space of all functions f continuous on $\mathbb{R}$. We know that $C(\mathbb{R})$ is a normed space with norm

$$\|f\|_{\infty} := \sup_{x\in\mathbb{R}} |f(x)|, \ f \in C(\mathbb{R}).$$

We denote by $C_{2\pi}(\mathbb{R})$ the space of all 2π-periodic functions $f \in C(\mathbb{R})$ which is a normed space with

$$\|f\|_{2\pi} = \sup_{t\in\mathbb{R}} |f(t)|.$$

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We write $L_n(f;x)$ for $L_n(f(s);x)$ and we say that L is a positive operator if $L(f;x) \geq 0$ for all $f(x) \geq 0$.

Korovkin type approximation theorems are useful tools to check whether a given sequence $(L_n)_{n\geq 1}$ of positive linear operators on $C[0,1]$ of all continuous functions on the real interval [0, 1] is an approximation process. That is, these theorems exhibit a variety of test functions which assure that the approximation property holds on the whole space if it holds for them. Such a property was discovered by Korovkin in 1953 for the functions 1, x, and x^2 in the space $C[0,1]$ as well as for the functions 1, cos, and sin in the space of all continuous 2π-periodic functions on the real line.

The classical *Korovkin first and second theorems* state as follows (see [1,55]):

Theorem 9.2.1. *Let (T_n) be a sequence of positive linear operators from $C[0,1]$ into $F[0,1]$. Then $\lim_{n\to\infty} \|T_n(f,x) - f(x)\|_\infty = 0$, for all $f \in C[0,1]$ if and only if $\lim_{n\to\infty} \|T_n(f_i,x) - e_i(x)\|_\infty = 0$, for $i = 0,1,2$, where $e_0(x) = 1$, $e_1(x) = x$, and $e_2(x) = x^2$.*

Theorem 9.2.2. *Let (T_n) be a sequence of positive linear operators from $C_{2\pi}(\mathbb{R})$ into $F(\mathbb{R})$. Then $\lim_{n\to\infty} \|T_n(f,x) - f(x)\|_{2\pi} = 0$, for all $f \in C_{2\pi}(\mathbb{R})$ if and only if $\lim_{n\to\infty} \|T_n(f_i,x) - f_i(x)\|_{2\pi} = 0$, for $i = 0,1,2$, where $f_0(x) = 1$, $f_1(x) = \cos x$, and $f_2(x) = \sin x$.*

9.3 Korovkin Approximation Theorems for Almost Convergence

The following result is due to Mohiuddine [60]. In [7], such type of result is proved for almost convergence of double sequences.

Theorem 9.3.1. *Let (T_k) be a sequence of positive linear operators from $C[a,b]$ into $C[a,b]$ satisfying the following conditions:*

$$F - \lim_{p\to\infty} \|T_k(1,x) - 1\|_\infty = 0, \tag{9.3.1}$$

$$F - \lim_{p\to\infty} \|T_k(t,x) - x\|_\infty = 0, \tag{9.3.2}$$

$$F - \lim_{p\to\infty} \left\|T_k(t^2,x) - x^2\right\|_\infty = 0. \tag{9.3.3}$$

Then for any function $f \in C[a,b]$ bounded on the whole real line, we have

$$F - \lim_{k\to\infty} \|T_k(f,x) - f(x)\|_\infty = 0.$$

Proof. Since $f \in C[a,b]$ and f is bounded on the real line, we have

$$|f(x)| \leq M, \ -\infty < x < \infty.$$

Therefore,

$$|f(t) - f(x)| \leq 2M, \ -\infty < t, x < \infty. \tag{9.3.4}$$

Also, we have that f is continuous on $[a, b]$, i.e.,

$$|f(t) - f(x)| < \epsilon, \ \forall |t - x| < \delta. \tag{9.3.5}$$

Using (9.3.4) and (9.3.5) and putting $\psi(t) = (t - x)^2$, we get

$$|f(t) - f(x)| < \epsilon + \frac{2M}{\delta^2}\psi, \ \forall |t - x| < \delta.$$

This means

$$-\epsilon - \frac{2M}{\delta^2}\psi < f(t) - f(x) < \epsilon + \frac{2M}{\delta^2}\psi.$$

Now, we operating $T_k(1, x)$ to this inequality since $T_k(f, x)$ is monotone and linear. Hence,

$$T_k(1, x)\left(-\epsilon - \frac{2M}{\delta^2}\psi\right) < T_k(1, x)(f(t) - f(x)) < T_k(1, x)\left(\epsilon + \frac{2M}{\delta^2}\psi\right).$$

Note that x is fixed and so $f(x)$ is a constant number. Therefore,

$$\begin{aligned} -\epsilon T_k(1, x) - \frac{2M}{\delta^2}T_k(\psi, x) &< T_k(f, x) - f(x)T_k(1, x) \\ &< \epsilon T_k(1, x) + \frac{2M}{\delta^2}T_k(\psi, x). \end{aligned} \tag{9.3.6}$$

But

$$\begin{aligned} T_k(f, x) - f(x) &= T_k(f, x) - f(x)T_k(1, x) + f(x)T_k(1, x) - f(x) \\ &= [T_k(f, x) - f(x)T_k(1, x)] + f(x)[T_k(1, x) - 1]. \end{aligned} \tag{9.3.7}$$

Using (9.3.6) and (9.3.7), we have

$$T_k(f, x) - f(x) < \epsilon T_k(1, x) + \frac{2M}{\delta^2}T_k(\psi, x) + f(x)\left[T_k(1, x) - 1\right]. \tag{9.3.8}$$

Let us estimate $T_k(\psi, x)$

$$\begin{aligned} T_k(\psi, x) &= T_k\left[(t - x)^2, x\right] \\ &= T_k(t^2 - 2tx + x^2, x) \\ &= T_k(t^2, x) + 2xT_k(t, x) + x^2T_k(1, x) \\ &= [T_k(t^2, x) - x^2] - 2x[T_k(t, x) - x] + x^2[T_k(1, x) - 1]. \end{aligned}$$

Using (9.3.8), we obtain

$$\begin{aligned}T_k(f,x)-f(x) &< \epsilon T_k(1,x)+\frac{2M}{\delta^2}\left\{\left[T_k(t^2,x)-x^2\right]+2x[T_k(t,x)-x]\right.\\&\quad\left.+x^2[T_k(1,x)-1]\right\}+f(x)\left[T_k(1,x)-1\right]\\&=\epsilon[T_k(1,x)-1]+\epsilon+\frac{2M}{\delta^2}\left\{\left[T_k(t^2,x)-x^2\right]+2x[T_k(t,x)-x]\right.\\&\quad\left.+x^2[T_k(1,x)-1]\right\}+f(x)\left[T_k(1,x)-1\right].\end{aligned}$$

Since ϵ is arbitrary, we can write

$$\begin{aligned}T_k(f,x)-f(x) &\leq \epsilon[T_k(1,x)-1]+\frac{2M}{\delta^2}\{[T_k(t^2,x)-x^2]+2x[T_k(t,x)-x]\\&\quad+x^2[T_k(1,x)-1]\}+f(x)\left[T_k(1,x)-1\right].\end{aligned}$$

Now replacing $T_k(\cdot,x)$ by $D_{n,p}(f,x)=\frac{1}{p+1}\sum_{k=n}^{n+p}T_k(\cdot,x)$, we get

$$\begin{aligned}D_{n,p}(f,x)-f(x) &\leq \epsilon[D_{n,p}(1,x)-1]+\frac{2M}{\delta^2}\{[D_{n,p}(t^2,x)-x^2]\\&\quad+2x[D_{n,p}(t,x)-x]+x^2[D_{n,p}(1,x)-1]\}\\&\quad+f(x)\left[D_{n,p}(1,x)-1\right],\end{aligned}$$

and therefore

$$\begin{aligned}\left\|D_{n,p}(f,x)-f(x)\right\|_\infty &\leq \left(\epsilon+\frac{2Mb^2}{\delta^2}+M\right)\left\|D_{n,p}(1,x)-1\right\|_\infty\\&\quad+\frac{4Mb}{\delta^2}\left\|D_{n,p}(t,x)-x\right\|_\infty+\frac{2M}{\delta^2}\left\|D_{n,p}(t^2,x)-x^2\right\|_\infty.\end{aligned}$$

Letting $p\to\infty$ and using (9.3.1)–(9.3.3), we get

$$\lim_{p\to\infty}\left\|D_{n,p}(f,x)-f(x)\right\|_\infty=0 \text{ uniformly in } n.$$

This completes the proof of the theorem. □

In the following example we construct a sequence of positive linear operators satisfying the conditions of Theorem 9.3.1, but it does not satisfy the conditions of Theorem 9.2.1.

Example 9.3.2. Consider the sequence of classical Bernstein polynomials

$$B_n(f,x):=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k};\quad 0\leq x\leq 1.$$

Let the sequence (P_n) be defined by $P_n : C[0,1] \to C[0,1]$ with $P_n(f,x) = (1+z_n)B_n(f,x)$, where z_n is defined by

$$z_n = \begin{cases} 1 \ , \ n \text{ is odd,} \\ 0 \ , \ n \text{ is even} \end{cases}$$

Then,

$$B_n(1,x) = 1, \ \ B_n(t,x) = x, \ \ B_n(t^2,x) = x^2 + \frac{x - x^2}{n},$$

and the sequence (P_n) satisfies the conditions (9.3.1)–(9.3.3). Hence, we have

$$F - \lim \|P_n(f,x) - f(x)\|_\infty = 0.$$

On the other hand, we get $P_n(f,0) = (1+z_n)f(0)$, since $B_n(f,0) = f(0)$, and hence

$$\|P_n(f,x) - f(x)\|_\infty \geq |P_n(f,0) - f(0)| = z_n|f(0)|.$$

We see that (P_n) does not satisfy the classical Korovkin theorem, since $\limsup_{n\to\infty} z_n$ does not exist.

Our next result is an analogue of Theorem 9.2.2.

Theorem 9.3.3. *Let (T_k) be a sequence of positive linear operators from $C_{2\pi}(R)$ into $C_{2\pi}(R)$. Then, for all $f \in C_{2\pi}(R)$*

$$F - \lim_{k\to\infty} \|T_k(f;x) - f(x)\|_{2\pi} = 0 \tag{9.3.9}$$

if and only if

$$F - \lim_{k\to\infty} \|T_k(1;x) - 1\|_{2\pi} = 0, \tag{9.3.10}$$

$$F - \lim_{k\to\infty} \|T_k(\cos t;x) - \cos x\|_{2\pi} = 0, \tag{9.3.11}$$

$$F - \lim_{k\to\infty} \|T_k(\sin t;x) - \sin x\|_{2\pi} = 0. \tag{9.3.12}$$

Proof. Since each f_0, f_1, and f_2 belongs to $C_{2\pi}(\mathbb{R})$, the conditions (9.3.10)–(9.3.12) follow immediately from (9.3.9). Let the conditions (9.3.10)–(9.3.12) hold and $f \in C_{2\pi}(\mathbb{R})$.

Let I be a closed subinterval of length 2π of $\mathbb{R}$. Fix $x \in I$. By the continuity of f at x, it follows that for given $\varepsilon > 0$ there is a number $\delta > 0$ such that for all t

$$|f(t) - f(x)| < \varepsilon, \tag{9.3.13}$$

whenever $|t - x| < \delta$. Since f is bounded, it follows that

$$|f(t) - f(x)| \leq 2\|f\|_{2\pi}, \tag{9.3.14}$$

for all $t \in \mathbb{R}$. For all $t \in (x - \delta, 2\pi + x - \delta]$. Using (9.3.13) and (9.3.14), we obtain

$$|f(t) - f(x)| < \varepsilon + \frac{2\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}} \psi(t), \tag{9.3.15}$$

where $\psi(t) = \sin^2[(t - x)/2]$. Since the function $f \in C_{2\pi}(\mathbb{R})$ is 2π-periodic, the inequality (9.3.15) holds for $t \in \mathbb{R}$.

Now, operating $T_k(1; x)$ to this inequality, we obtain

$$\begin{aligned}|T_k(f; x) - f(x)| &\leq [\varepsilon + |f(x)|]|T_k(1; x) - 1| + \varepsilon + \frac{\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}}[|T_k(1; x) - 1| \\ &+ |\cos x||T_k(\cos t; x) - \cos x| + |\sin x||T_k(\sin t; x) - \sin x|] \leq \varepsilon \\ &+ \left[\varepsilon + |f(x)| + \frac{\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}}\right]\{|T_k(1; x) - 1| \\ &+ |T_k(\cos t; x) - \cos x| + |T_k(\sin t; x) - \sin x|\}\end{aligned}$$

Now, taking $\sup_{x \in I}$, we get

$$\begin{aligned}\|T_k(f; x) - f(x)\|_{2\pi} &\leq \varepsilon + K\left(\|T_k(1; x) - 1\|_{2\pi}\right. \\ &\left.+ \|T_k(\cos t; x) - \cos x\|_{2\pi} + \|T_k(\sin t; x) - \sin x\|_{2\pi}\right),\end{aligned} \tag{9.3.16}$$

$$\text{where } K := \left\{\varepsilon + \|f\|_{2\pi} + \frac{\|f\|_{2\pi}}{\sin^2 \frac{\delta}{2}}\right\}.$$

Now replacing $T_k(\cdot, x)$ by $\frac{1}{m+1}\sum_{k=n}^{n+m} T_k(\cdot, x)$ in (9.3.17) on both sides and then taking the limit as $m \to \infty$ uniformly in n. Therefore, using conditions (9.3.10)–(9.3.12), we get

$$\lim_{m\to\infty}\left\|\frac{1}{m+1}\sum_{k=n}^{n+m} T_k(f, x) - f(x)\right\|_{2\pi} = 0 \text{ uniformly in } n,$$

i.e., the condition (9.3.9) is proved.

This completes the proof of the theorem. □

In the following example we see that Theorem 9.3.3 is stronger than Theorem 9.2.2.

Theorem 9.3.4. *For any $n \in \mathbb{N}$, denote by $S_n(f)$ the n-th partial sum of the Fourier series of f, i.e.,*

$$S_n(f)(x) = \frac{1}{2}a_0(f) + \sum_{k=1}^{n} a_k(f)\cos kx + b_k(f)\sin kx.$$

For any $n \in \mathbb{N}$, *write*

$$F_n(f) := \frac{1}{n+1}\sum_{k=0}^{n} S_k(f).$$

A standard calculation gives that for every $t \in \mathbb{R}$

$$\begin{aligned} F_n(f;x) &:= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1}{n+1}\sum_{k=0}^{n}\frac{\sin\frac{(2k+1)(x-t)}{2}}{\sin\frac{x-t}{2}}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\frac{1}{n+1}\sum_{k=0}^{n}\frac{\sin^2\frac{(n+1)(x-t)}{2}}{\sin^2\frac{x-t}{2}}dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\varphi_n(x-t)dt, \end{aligned}$$

where

$$\varphi_n(x) := \begin{cases} \frac{\sin^2\frac{(n+1)(x-t)}{2}}{(n+1)\sin^2\frac{x-t}{2}} & , x \text{ is not a multiple of } 2\pi, \\ n+1 & , x \text{ is a multiple of } 2\pi. \end{cases}$$

The sequence $(\varphi_n)_{n\in\mathbb{N}}$ is a positive kernel which is called the *Fejér kernel,* and the corresponding operators F_n, $n \geq 1$ are called the *Fejér convolution operators.*

Note that the Theorem 9.2.2 is satisfied for the sequence (F_n). In fact, we have for every $f \in C_{2\pi}(\mathbb{R})$, $F_n(f) \to f$, as $n \to \infty$.

Let $L_n : C_{2\pi}(\mathbb{R}) \to C_{2\pi}(\mathbb{R})$ be defined by

$$L_n(f;x) = (1+z_n)F_n(f;x), \tag{9.3.17}$$

where the sequence $z = (z_n)$ is defined as above. Now,

$$\begin{aligned} L_n(1;x) &= 1, \\ L_n(\cos t;x) &= \frac{n}{n+1}\cos x, \\ L_n(\sin t;x) &= \frac{n}{n+1}\sin x \end{aligned}$$

so that we have

$$F - \lim_{n\to\infty} \|L_n(1;x) - 1\|_{2\pi} = 0,$$

$$F - \lim_{n\to\infty} \|L_n(\cos t; x) - \cos x\|_{2\pi} = 0,$$
$$F - \lim_{n\to\infty} \|L_n(\sin t; x) - \sin x\|_{2\pi} = 0,$$

that is, the sequence (L_n) satisfies the conditions (9.3.9)–(9.3.12). Hence by Theorem 9.3.3, we have

$$F\text{-}\lim_{n\to\infty} \|L_n(f) - f\|_{2\pi} = 0,$$

i.e., our theorem holds. But on the other hand, Theorem 9.2.2 does not hold for our operator defined by (9.3.17), since the sequence (L_n) is not convergent.

Hence Theorem 9.3.3 is stronger than Theorem 9.2.2.