Chapter 8
Matrix Summability of Fourier and Walsh-Fourier Series

8.1 Introduction

In this chapter we apply regular and almost regular matrices to find the sum of derived Fourier series, conjugate Fourier series, and Walsh-Fourier series (see [4] and [69]). Recently, Móricz [67] has studied statistical convergence of sequences and series of complex numbers with applications in Fourier analysis and summability.

8.2 Summability of Fourier Series

Let f be L-integrable and periodic with period 2π, and let the *Fourier series* of f be

$$\frac{1}{a_0} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)\,. \qquad (8.2.1)$$

Then, the *series conjugate* to it is

$$\sum_{k=1}^{\infty} (b_k \cos kx - a_k \sin kx)\,, \qquad (8.2.2)$$

and the *derived series* is

$$\sum_{k=1}^{\infty} k\,(b_k \cos kx - a_k \sin kx)\,. \qquad (8.2.3)$$

M. Mursaleen, *Applied Summability Methods*, SpringerBriefs in Mathematics,
DOI 10.1007/978-3-319-04609-9_8,

Let $S_n(x)$, $\tilde{S}_n(x)$, and $S_n'(x)$ denote the partial sums of series (8.2.1), (8.2.2), and (8.2.3), respectively. We write

$$\psi_x(t) = \psi(f,t) = \begin{cases} f(x+t) - f(x-t)\ , & 0 < t \leq \pi; \\ g(x), & t = 0 \end{cases}$$

and

$$\beta_x(t) = \frac{\psi_x(t)}{4 \sin \frac{1}{2} t},$$

where $g(x) = f(x+0) - f(x-0)$. These formulae are correct a.e..

Theorem 8.2.1. *Let $f(x)$ be a function integrable in the sense of Lebesgue in $[0, 2\pi]$ and periodic with period 2π. Let $A = (a_{nk})$ be a regular matrix of real numbers. Then for every $x \in [-\pi, \pi]$ for which $\beta_x(t) \in BV[0, \pi]$,*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} S_k'(x) = \beta_x(0+) \tag{8.2.4}$$

if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} \sin\left(k + \frac{1}{2}\right) t = 0 \tag{8.2.5}$$

for every $t \in [0, \pi]$, where $BV[0, \pi]$ denotes the set of all functions of bounded variations on $[0, \pi]$.

We shall need the following well-known *Dirichlet-Jordan Criterion for Fourier series* [101].

Lemma 8.2.2 (Dirichlet-Jordan Criterion for Fourier Series). *The trigonometric Fourier series of a 2π-periodic function f having bounded variation converges to $[f(x+0) - f(x-0)]/2$ for every x and this convergence is uniform on every closed interval on which f is continuous.*

We shall also need the following result on the weak convergence of sequences in the Banach space of all continuous functions defined on a finite closed interval [11].

Lemma 8.2.3. *Let $C[0, \pi]$ be the space of all continuous functions on $[0, \pi]$ equipped with the sup-norm $\|.\|$. Let $g_n \in C[0, \pi]$ and $\int_0^\pi g_n dh_x \to 0$, as $n \to \infty$, for all $h_x \in BV[0, \pi]$ if and only if $\|g_n\| < \infty$ for all n and $g_n \to 0$, as $n \to \infty$.*

Proof. We have

$$
\begin{aligned}
S_k'(x) &= \frac{1}{\pi}\int_0^{\pi} \psi_x(t)\left(\sum_{m=1}^{k} m \sin mt\right) dt \\
&= -\frac{1}{\pi}\int_0^{\pi} \psi_x(t)\frac{d}{dt}\left[\frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{t}{2}}\right] dt \\
&= I_k + \frac{2}{\pi}\int_0^{\pi} \sin\left(k+\frac{1}{2}\right) t\, d\beta_x(t),
\end{aligned}
$$

where

$$
I_k = \frac{1}{\pi}\int_0^{\pi} \beta_x(t)\cos\frac{t}{2}\left[\frac{\sin\left(k+\frac{1}{2}\right)t}{\sin\frac{t}{2}}\right] dt.
$$

Then,

$$
\sum_{k=1}^{\infty} a_{nk} S_k'(x) = \sum_{k=1}^{\infty} a_{nk} I_k + \frac{2}{\pi}\int_0^{\pi} L_n(t)\, d\beta_x(t),
$$

where

$$
L_n(t) = \sum_{k=1}^{\infty} a_{nk} \sin\left(k+\frac{1}{2}\right) t.
$$

Since $\beta_x(t)$ is of bounded variation on $[0,\pi]$ and $\beta_x(t) \to \beta_x(0+)$ as $t \to 0$, $\beta_x(t)\cos\frac{t}{2}$ has also the same properties. Hence, by Lemma 8.2.2, $I_k \to \beta_x(0+)$ as $k \to \infty$.

Since the matrix $A = (a_{nk})$ is regular, we have

$$
\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk} I_k = \beta_x(0+). \tag{8.2.6}
$$

Now, it is enough to show that (8.2.5) holds if and only if

$$
\lim_{n\to\infty}\int_0^{\pi} L_n(t)\, d\beta_x(t) = 0. \tag{8.2.7}
$$

Hence, by Lemma 8.2.3, it follows that (8.2.7) holds if and only if

$$
\| L_n(t) \| \leq M \text{ for all } n \text{ and for all } t \in [0,\pi], \tag{8.2.8}
$$

and (8.2.5) holds, where M is a constant. Since (8.2.8) is satisfied by the regularity of A, it follows that (8.2.7) holds if and only if (8.2.5) holds. Hence the result follows immediately.

This completes the proof. □

Similarly we can prove the following result for almost regularity.

Theorem 8.2.4. *Let f be a function integrable in the sense of Lebesgue in $[0, 2\pi]$ and periodic with period 2π. Let $A = (a_{nk})$ be an almost regular matrix of real numbers. Then for every $x \in [-\pi, \pi]$ for which $\beta_x(t) \in BV[0, \pi]$,*

$$\lim_{p\to\infty} \frac{1}{p+1} \sum_{j=n}^{n+p} \sum_{k=1}^{\infty} a_{jk} S'_k(x) = \beta_x(0+) \text{ uniformly in } n$$

if and only if

$$\lim_{p\to\infty} \frac{1}{p+1} \sum_{j=n}^{n+p} \sum_{k=1}^{\infty} a_{jk} \sin\left(k + \frac{1}{2}\right) t = 0 \text{ uniformly in } n$$

for every $t \in [0, \pi]$.

Theorem 8.2.5. *Let $f(x)$ be a function integrable in the sense of Lebesgue in $[0, 2\pi]$ and periodic with period 2π. Let $A = (a_{nk})$ be a regular matrix of real numbers. Then A-transform of the sequence $\{k\tilde{S}_k(x)\}$ converges to $g(x)/\pi$, i.e.,*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} k a_{nk} \tilde{S}_k(x) = \frac{1}{\pi} g(x) \tag{8.2.9}$$

if and only if

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} \cos kt = 0 \tag{8.2.10}$$

for every $t \in (0, \pi]$, where each $a_k, b_k \in BV[0, 2\pi]$.

Proof. We have

$$\begin{aligned}\tilde{S}_n(x) &= \frac{1}{\pi} \int_0^{\pi} \psi_x(t) \sin nt \, dt, \\ &= \frac{g(x)}{n\pi} + \frac{1}{n\pi} \int_0^{\pi} \cos nt \, d\psi_x(t).\end{aligned}$$

Therefore

$$\sum_{k=1}^{\infty} k a_{nk} \tilde{S}_k(x) = \frac{g(x)}{\pi} \sum_{k=1}^{\infty} a_{nk} + \frac{1}{\pi} \int_0^{\pi} K_n(t)\, d\psi_x(t), \tag{8.2.11}$$

where

$$K_n(t) = \sum_{k=1}^{\infty} a_{nk} \cos kt.$$

Now, taking limit as $n \to \infty$ on both sides of (8.2.10) and using Lemma 8.2.3 and regularity conditions of A as in the proof of Theorem 8.2.1, we get the required result. □

Remark 8.2.6. Analogously, we can state and prove Theorem 8.2.4 for almost regular matrix A.

8.3 Summability of Walsh-Fourier Series

Let us define a sequence of functions $h_0(x), h_1(x), \ldots, h_n(x)$ which satisfy the following conditions:

$$h_0(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{2}, \\ -1, & \frac{1}{2} \le x < 1, \end{cases}$$

$h_0(x+1) = h_0(x)$ and $h_n(x) = h_0(2^n x)$, $n = 1, 2, \ldots$. The functions $h_n(x)$ are called the *Rademacher's functions*.

The *Walsh functions* are defined by

$$\phi_n(x) = \begin{cases} 1, & n = 0, \\ h_{n_1}(x) h_{n_2}(x) \cdots h_{n_r}(x), & n > 1,\ 0 \le x \le 1 \end{cases}$$

for $n = 2^{n_1} + 2^{n_2} + \cdots + 2^{n_r}$, where the integers n_i are uniquely determined by $n_{i+1} < n_i$.

Let us recall some basic properties of Walsh functions (see [34]). For each fixed $x \in [0, 1)$ and for all $t \in [0, 1)$

(i) $\phi_n(x \dot{+} t) = \phi_n(x)\phi_n(t)$,
(ii) $\int_0^1 f(x \dot{+} t)dt = \int_0^1 f(t)dt$, and
(iii) $\int_0^1 f(t)\phi_n(x \dot{+} t)dt = \int_0^1 f(x \dot{+} t)\phi_n(t)dt$,

where $\dot{+}$ denotes the operation in the dyadic group, the set of all sequences $s = (s_n)$, $s_n = 0, 1$ for $n = 1, 2, \ldots$ is addition modulo 2 in each coordinate.

Let for $x \in [0, 1)$,

$$J_k(x) = \int_0^x \phi_k(t)dt, \ k = 0, 1, 2, \ldots$$

It is easy to see that $J_k(x) = 0$ for $x = 0, 1$.

Let f be L-integrable and periodic with period 1, and let the *Walsh-Fourier series of* f be

$$\sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_n = \int_0^1 f(x)\phi_n(x)dx$$

are called the *Walsh-Fourier coefficients of* f.

The following result is due to Siddiqi [91].

Theorem 8.3.1. *Let $A = (a_{nk})$ be a regular matrix of real numbers. Let $z_k(x) = c_k\phi_k(x)$ for an L-integrable function $f \in BV[0, 1)$. Then for every $x \in [0, 1)$*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} z_k(x) = 0$$

if and only if

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} J_k(x) = 0,$$

where x is a point at which $f(x)$ is of bounded variation.

This can be proved similarly as our next result which is due to Mursaleen [69] in which we use the notion of F_A-summability. Recently, Alghamdi and Mursaleen [4] have applied Hankel matrices for this purpose.

Theorem 8.3.2. *Let $A = (a_{nk})$ be a regular matrix of real numbers. Let $z_k(x) = c_k\phi_k(x)$ for an L-integrable function $f \in BV[0, 1)$. Then for every $x \in [0, 1)$, the sequence $\{z_k(x)\}_k$ is F_A-summable to 0 if and only if the sequence $\{J_k(x)\}_k$ is F_A-summable to 0, that is,*

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} z_{k+p}(x) = 0, \ \textit{uniformly in } p$$

if and only if

$$\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{nk}J_k(x) = 0 \text{ uniformly in } p,$$

where x is a point at which $f(x)$ is of bounded variation.

Proof. We have

$$\begin{aligned} z_k(x) = c_k\phi_k(x) &= \int_0^1 f(t)\phi_k(t)\phi_k(x)dt, \\ &= \int_0^1 f(t)\phi_k(x\dot{+}t)dt = \int_0^1 f(x\dot{+}t)\phi_k(t)dt, \end{aligned}$$

where $x\dot{+}t$ belongs to the set Ω of dyadic rationals in $[0, 1)$; in particular each element of Ω has the form $p/2^n$ for some nonnegative integers p and $n, 0 \le p < 2^n$. Now, on integration by parts, we obtain

$$\begin{aligned} z_k(x) &= [f(x\dot{+}t)J_k(t)]_0^1 - \int_0^1 J_k(t)df(x\dot{+}t), \\ &= -\int_0^1 J_k(t)df(x\dot{+}t), \text{ since } J_k(x) = 0 \text{ for } x \in \{0, 1\}. \end{aligned}$$

Hence, for a regular matrix $A = (a_{nk})$ and $p \ge 0$, we have

$$\sum_{k=1}^{\infty} a_{nk}z_{k+p}(x) = -\int_0^1 D_{np}(t)\, dh_x(t), \tag{8.3.1}$$

where

$$D_{np}(t) = \sum_{k=1}^{\infty} a_{nk}J_{k+p}(t), \tag{8.3.2}$$

and $h_x(t) = f(x\dot{+}t)$. Write, for any $t \in \mathbb{R}$, $g_{np} = (D_{np}(t))$.

Since A is regular (and hence almost regular), it follows that $\|g_{np}\| < \infty$ for all n and p, and $g_{np} \to 0$, as $n \to \infty$ pointwise, uniformly in p. Hence by Lemma 8.2.3,

$$\int_0^1 D_{np}(t)dh_x(t) \to 0$$

as $n \to \infty$ uniformly in p. Now, letting $n \to \infty$ in (8.3.1) and (8.3.2) and using Lemma 8.2.3, we get the desired result.

This completes the proof. □

Remark 8.3.3. If we take the matrix A as the Cesàro matrix $(C, 1)$, then we get the following result for almost summability.

Theorem 8.3.4. *Let* $A = (a_{nk})$ *be almost regular matrix of real numbers. Let* $z_k(x) = c_k \phi_k(x)$ *for an* L *-integrable function* $f \in BV[0,1)$*. Then for every* $x \in [0, 1)$

$$F - \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} z_k(x) = 0$$

if and only if

$$F - \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} J_k(x) = 0,$$

where x *is a point at which* f *is of bounded variation.*