# Chapter 3
# Summability Tests for Singular Points

## 3.1 Introduction

A point at which the function $f(z)$ ceases to be analytic, but in every neighborhood of which there are points of analyticity is called singular point of $f(z)$.

Consider a function $f(z)$ defined by the power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{3.1.1}$$

having a positive radius of convergence. Every power series has a circle of convergence within which it converges and outside of which it diverges. The radius of this circle may be infinite, including the whole plane, or finite. For the purposes here, only a finite radius of convergence will be considered. Since the circle of convergence of the series passes through the singular point of the function which is nearest to the origin, the modulus of that singular point can be determined from the sequence $a_n$ in a simple manner. The problem of determining the exact position of the singular point on the circle of convergence is considered; tests can be devised to determine whether or not that point is a singular point of the function defined by the series. It may be supposed, without loss of generality, that the radius of convergence of the series is 1. In this chapter we apply Karamata/Euler summability method to determine or test if a particular point on the circle of convergence is a singular point of the function defined by the series (3.1.1).

## 3.2 Definitions and Notations

*Karamata's summability method* $K[\alpha, \beta]$ was introduced by Karamata (see [8]) and the summability method associated with this matrix is called Karamata method or $K[\alpha, \beta]$-method (c.f. [86]).

M. Mursaleen, *Applied Summability Methods*, SpringerBriefs in Mathematics,
DOI 10.1007/978-3-319-04609-9_3, 

The *Karamata matrix* $K[\alpha, \beta] = (c_{nk})$ is defined by

$$c_{nk} = \begin{cases} 1 \ , \ n = k = 0, \\ 0 \ , \ n = 0, k = 1, 2, 3, \dots, \end{cases}$$

$$\left[\frac{\alpha + (1-\alpha-\beta)z}{1-\beta z}\right]^n = \sum_{k=0}^{\infty} c_{nk} z^k, \quad n = 1, 2, \dots .$$

$K[\alpha, \beta]$ is the Euler matrix for $K[1-r, 0] = E(r)$ (see [2]); the Laurent matrix for $K[1-r, r] = S(r)$ (see [95]), and with a slight change, the Taylor matrix for $K[0, r] = T(r)$ (see [28]). If $T(r) = (c_{nk}),$ then

$$\left[\frac{(1-r)z}{1-rz}\right]^{n+1} = \sum_{k=0}^{\infty} c_{nk} z^{k+1}, \quad n = 0, 1, 2, \dots$$

## 3.3 Tests for Singular Points

King [49] devised two tests in the form of following theorems, each of which provides necessary and sufficient condition that $z = 1$ be a singular point of the function defined by the series (3.1.1).

**Theorem 3.3.1.** *A necessary and sufficient condition that $z = 1$ be a singular point of the function defined by the series (3.1.1) is that*

$$\limsup_{n\to\infty} \left| \sum_{m=0}^{n} \binom{n}{m} r^m (1-r)^{n-m} \, a_m \right|^{1/n} = 1,$$

*for some* $0 < r < 1$.

*Proof.* Consider the function

$$F(t) = \frac{1}{1-(1-r)t} f\left(\frac{rt}{1-(1-r)t}\right).$$

$F(t)$ is regular in the region

$$D_r = \left\{ t : \left| \frac{rt}{1-(1-r)t} \right| < 1 \right\}.$$

Furthermore, $z = 1$ is a singular point of $f(z)$ if and only if $t = 1$ is a singular point of $F(t)$. A simple calculation gives

$$D_r = \{t : \mathrm{Re}(t) < 1\},$$
$$D_r = \left\{ t : \left| t - \frac{1-r}{1-2r} \right| > \frac{r}{1-2r} \right\},$$
$$D_r = \left\{ t : \left| t - \frac{1-r}{1-2r} \right| < \frac{r}{2r-1} \right\},$$

for $r = 1/2,\ 0 < r < 1/2,$ and $1/2 < r < 1$, respectively. In each case $t = 1$ is on the boundary of $D_r$ and $D_r$ contains all points of the closed unit disk except $t = 1$. If we write $F(t) = \sum_{n=0}^{\infty} b_n t^n$, it follows that $t = 1$ is a singular point of $F(t)$ if and only if the radius of convergence of the series is exactly 1. That is, if and only if

$$\limsup_{n\to\infty} |b_n|^{1/n} = 1.$$

The function $F(t)$ is given by

$$\begin{aligned} F(t) &= \frac{1}{1-(1-r)t} \sum_{m=0}^{\infty} a_m \left[ \frac{rt}{1-(1-r)t} \right]^m \\ &= \sum_{m=0}^{\infty} a_m r^m t^m \sum_{n=m}^{\infty} \binom{n}{m} (1-r)^{n-m} t^{n-m} \end{aligned}$$

provided that $(1-r)|t| < 1$. It is easy to verify the interchange of summation in the last expression. Hence, $F(t) = \sum_{n=0}^{\infty} t^n \sum_{m=0}^{n} \binom{n}{m} r^m (1-r)^{n-m} a_m$. Therefore,

$$b_n = \sum_{m=0}^{n} \binom{n}{m} r^m (1-r)^{n-m} a_m. \tag{3.3.1}$$

This completes the proof. □

**Theorem 3.3.2.** *A necessary and sufficient condition that $z = 1$ be a singular point of the function defined by the series (3.1.1) is that*

$$\limsup_{m\to\infty} \left| \sum_{n=m}^{\infty} \binom{n}{m} r^{n-m} (1-r)^{m+1}\, a_n \right|^{1/n} = 1,$$

*for some* $0 < r < 1$.

*Proof.* Consider the function

$$G(t) = (1-r)\ f(r + (1-r)t).$$

$G(t)$ is regular in the region $R_r = \{t : |r + (1-r)t| < 1\}$. A simple calculation gives

$$R_r = \left\{ t : \left| t - \frac{r}{r-1} \right| < \frac{1}{1-r} \right\}.$$

The point $t = 1$ is on the boundary of $R_r$ and $R_r$ contains all points of the closed unit disk except $t = 1$. If we write

$$G(t) = \sum_{n=0}^{\infty} c_n t^n,$$

it follows that $z = 1$ is a singular point of $f(z)$ if and only if

$$\limsup_{n\to\infty} |c_n|^{1/n} = 1.$$

The function $G(t)$ is given by

$$\begin{aligned} G(t) &= (1-r)\sum_{n=0}^{\infty} a_n (r + (1-r)t)^n \\ &= (1-r)\sum_{n=0}^{\infty} a_n \sum_{m=0}^{\infty} \binom{n}{m} r^{n-m}(1-r)^m t^m \\ &= \sum_{m=0}^{\infty} t^m \sum_{n=m}^{\infty} \binom{n}{m} r^{n-m}(1-r)^{m+1} a_n. \end{aligned}$$

Hence,

$$c_m = \sum_{n=m}^{\infty} \binom{n}{m} r^{n-m}(1-r)^{m+1} a_n. \tag{3.3.2}$$

This completes the proof. □

These theorems yield the following corollaries.

**Corollary 3.3.3.** *If the sequence* $(a_n)$ *is* $E(r)$*-summable,* $0 < r < 1$*, to a nonzero constant, then* $z = 1$ *is a singular point of the function defined by the series (3.1.1).*

**Corollary 3.3.4.** *If the sequence* $(a_n)$ *is* $T(r)$*-summable,* $0 < r < 1$*, to a nonzero constant, then* $z = 1$ *is a singular point of the function defined by the series (3.1.1).*

Extending the above results, Hartmann [44] proved Theorem 3.3.6. The following lemma is needed for the proof of Theorem 3.3.6.

**Lemma 3.3.5.** *If $K[\alpha,\beta] = (c_{nk})$ for $|\alpha| < 1, |\beta| < 1$, then there exists $\rho > 0$, independent of $k$, such that for $|t| < \rho$ and $k = 0, 1, 2, \ldots$,*

$$\sum_{n=0}^{\infty} c_{n,k+1}t^n = \frac{(1-\alpha)(1-\beta)t}{(1-\alpha t)^2}\left[\frac{\beta+(1-\alpha-\beta)t}{1-\alpha t}\right]^k.$$

*Proof.* Let $f(z) = [\alpha + (1-\alpha-\beta)z]/(1-\beta z)$. If $0 < R < 1 < 1/|\beta|$, then there exists $\rho_1 > 0$ such that if $|t| \le \rho_1$ and let

$$\phi_t(z) = \frac{1}{1-tf(z)} = \sum_{n=0}^{\infty} t^n[f(z)].$$

Since this convergence is uniform in $|z| \le R$, one can apply Weierstrass theorem on uniformly convergent series of analytic functions (see [53]) to write

$$\sum_{n=0}^{\infty} t^n[f(z)]^n = \sum_{n=0}^{\infty} t^n\left(\sum_{k=0}^{\infty} c_{nk}z^k\right) = \sum_{k=0}^{\infty} z^k\left(\sum_{n=0}^{\infty} c_{nk}t^n\right). \tag{3.3.3}$$

But

$$\frac{1}{1-tf(z)} = \frac{1-\beta z}{(1-\alpha t)\left[1-\frac{\beta+(1-\alpha-\beta)t}{1-\alpha t}z\right]}. \tag{3.3.4}$$

There exits $\rho_2 > 0$ such that $|t| \le \rho_2$ and $|z| \le R$ imply $|[\beta + (1-\alpha-\beta)t]z/[1-\alpha t]| < 1$. Thus (3.3.4) may be expanded in a power series,

$$\frac{1}{1-tf(z)} = \sum_{k=0}^{\infty} \frac{1-\beta z}{1-\alpha t}\left[\frac{\beta+(1-\alpha-\beta)t}{1-\alpha t}\right]^k z^k. \tag{3.3.5}$$

Then, for $|t| \le \min(\rho_1, \rho_2)$, one has, by equating coefficients in (3.3.3) and (3.3.5), the results of the lemma. □

**Theorem 3.3.6.** *A necessary and sufficient condition that $z = 1$ be a singular point of the function defined by the series (3.1.1) is that*

$$\limsup_{n\to\infty}\left|\sum_{k=0}^{\infty} c_{n,k+1}a_k\right|^{1/n} = 1 \tag{3.3.6}$$

*for some $\alpha < 1, \beta < 1$ and $\alpha + \beta > 0$.*

*Proof.* Consider the function

$$F(t) = \frac{(1-\alpha)(1-\beta)t}{(1-\alpha t)^2} f\left(\frac{\beta + (1-\alpha-\beta)t}{1-\alpha t}\right).$$

$F(t)$ is regular in the region $D$, where

$$D = \left\{t : \left|\frac{\beta + (1-\alpha-\beta)t}{1-\alpha t}\right| < 1\right\}.$$

Furthermore, $z = 1$ is a singular point of $f(z)$ if and only if $t = 1$ is a singular point of $F(t)$. A simple calculation gives

$$D = \begin{cases} t : |t + \frac{\alpha+\beta}{1-\beta-2\alpha}| < |\frac{1-\alpha}{1-\beta-2\alpha}| & , 1-\beta-2\alpha > 0; \\ t : \text{ Re } (t) < 1 & , 1-\beta-2\alpha = 0; \\ t : |t + \frac{\alpha+\beta}{1-\beta-2\alpha}| > |\frac{1-\alpha}{1-\beta-2\alpha}| & , 1-\beta-2\alpha < 0. \end{cases}$$

In each case $t = 1$ is on the boundary of $D$ and $D$ contains all points of the closed unit disk except $t = 1$. Writing $F(t)$ in series form yields

$$F(t) = \frac{(1-\alpha)(1-\beta)t}{(1-\alpha t)^2} \sum_{k=0}^{\infty} a_k \left[\frac{\beta + (1-\alpha-\beta)t}{1-\alpha t}\right]^k,$$

provided $t \varepsilon D$. By Lemma 3.3.5, there exists $\rho > 0$ such that for $|t| \le \rho_1 < \rho$ and $k = 0, 1, 2, \ldots$

$$\sum_{n=0}^{\infty} c_{n,k+1} t^n = \frac{(1-\alpha)(1-\beta)t}{(1-\alpha t)^2} \left[\frac{\beta + (1-\alpha-\beta)t}{1-\alpha t}\right]^k. \tag{3.3.7}$$

Since $(1-\alpha)(1-\beta)t/(1-\alpha t)^2$ vanishes for $t = 0$ and $[\beta + (1-\alpha-\beta)t]/[1-\alpha t]$ is equal to $\beta$ for $t = 0$, with $|\beta| < 1$, there exists $\rho_2(\alpha, \beta) < \rho_1$ such that $|t| \le \rho_2$ implies $|\sum_{n=0}^{\infty} c_{n,k+1} t^n| \le M r^k$ for some $r = r(\alpha, \beta) < 1$. Thus

$$\begin{aligned} \left|\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} c_{n,k+1} a_k t^n\right| &\le \sum_{k=0}^{\infty} |a_k| \left|\sum_{n=0}^{\infty} c_{n,k+1} t^n\right| \\ &\le M \sum_{k=0}^{\infty} |a_k| r^k, \end{aligned}$$

which converges since (3.3.7) has radius of convergence one. Weierstrass theorem now implies

$$F(t) = \sum_{k=0}^{\infty}\sum_{n=0}^{\infty} c_{n,k+1} a_k t^n, \tag{3.3.8}$$

for $|t| \leq \rho_2$. By analytic continuation (3.3.8) holds in a disk whose boundary contains the singularity of $F(t)$ nearest the origin and $t = 1$ is a singular point of $F(t)$ if and only if the radius of convergence of series (3.3.8) is exactly 1, i.e.,

$$\limsup_{n\to\infty} \left| \sum_{k=0}^{\infty} c_{n,k+1} a_k \right|^{1/n} = 1. \tag{3.3.9}$$

This completes the proof of the theorem. □

From this, following result may be deduced.

**Corollary 3.3.7.** *If the sequence* $(0, a_0, a_1, \ldots)$ *is* $K[\alpha, \beta]$ *summable* $\alpha < 1, \beta < 1, \alpha + \beta > 0$*, to a nonzero constant, then* $z = 1$ *is a singular point of the function given by (3.1.1).*

*Remark 3.3.8.* Notice $K[\alpha, \beta]$ is regular for $\alpha < 1, \beta < 1$ and $\alpha+\beta > 0$ (see [8]). If $(b_n)$ is the $K[\alpha, \beta]$ transform of $(0, a_0, a_1, \ldots)$, then $b_0 = 0$, $b_n = \sum_{k=0}^{\infty} c_{n,k+1} a_k$, $n = 1, 2, \ldots$. Now, if $(0, a_0, a_1, \ldots)$ is $K[\alpha, \beta]$ summable to a nonzero constant, then (3.3.6) holds. If the $T(r)$ transform of $(a_n)$ is $(c_n)$ and the $K[0, r]$ transform of $(0, a_0, a_1, \ldots)$ is $(\gamma_n)$, then $\gamma_0 = 0$, $\gamma_n = c_{n-1} (n \geq 1)$ and thus one has immediately Corollary 3.3.4. In [2] it is proved that $E(r)$ is translative to the right when $E(r)$ is regular, so Corollary 3.3.7 implies Corollary 3.3.3.