Chapter 2
Lambert Summability and the Prime Number Theorem

2.1 Introduction

The *prime number theorem* (PNT) was stated as conjecture by German mathematician Carl Friedrich Gauss (1777–1855) in the year 1792 and proved independently for the first time by Jacques Hadamard and Charles Jean de la Vallée-Poussin in the same year 1896. The first elementary proof of this theorem (without using integral calculus) was given by Atle Selberg of Syracuse University in October 1948. Another elementary proof of this theorem was given by Erdös in 1949.

The PNT describes the asymptotic distribution of the prime numbers. The PNT gives a general description of how the primes are distributed among the positive integers.

Informally speaking, the PNT states that if a random integer is selected in the range of zero to some large integer N, the probability that the selected integer is prime is about $1/\ln(N)$, where $\ln(N)$ is the natural logarithm of N. For example, among the positive integers up to and including $N = 10^3$, about one in seven numbers is prime, whereas up to and including $N = 10^{10}$, about one in 23 numbers is prime (where $\ln(103) = 6.90775528$ and $\ln(1010) = 23.0258509$). In other words, the average gap between consecutive prime numbers among the first N integers is roughly $\ln(N)$.

Here we give the proof of this theorem by the application of Lambert summability and Wiener's Tauberian theorem. The Lambert summability is due to German mathematician Johann Heinrich Lambert (1728–1777) (see Hardy [41, p. 372]; Peyerimhoff [80, p. 82]; Saifi [86]).

2.2 Definitions and Notations

(i) **Möbius Function.** The classical *Möbius function* $\mu(n)$ is an important multiplicative function in number theory and combinatorics.This formula is due to German mathematician August Ferdinand Möbius (1790–1868) who

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introduced it in 1832. $\mu(n)$ is defined for all positive integers n and has its values in $\{-1, 0, 1\}$ depending on the factorization of n into prime factors. It is defined as follows (see Peyerimhoff [80, p. 85]):

$$\mu(n) = \begin{cases} 1 & , n \text{ is a square-free positive integer with an even number of prime factors,} \\ -1 & , n \text{ is a square-free positive integer with an odd number of prime factors,} \\ 0 & , n \text{ is not square-free,} \end{cases}$$

that is,

$$\mu(n) = \begin{cases} 1 & , n = 1, \\ (-1)^k & , n = p_1 p_2 \cdots p_k,\ p_i \text{ prime, } p_i \neq p_j, \\ 0 & , \text{otherwise.} \end{cases} \tag{2.2.1}$$

Thus

(a) $\mu(2) = -1$, since $2 = 2$;
(b) $\mu(10) = 1$, since $10 = 2 \times 5$;
(c) $\mu(4) = 0$, since $4 = 2 \times 2$.

We conclude that $\mu(p) = -1$, if p is a prime number.

(ii) **The Function $\pi(x)$.** The *prime-counting function* $\pi(x)$ is defined as the number of primes not greater than x, for any real number x, that is, $\pi(x) = \sum_{p<x} 1$ (Peyerimhoff [80, p. 87]). For example, $\pi(10) = 4$ because there are four prime numbers (2, 3, 5, and 7) less than or equal to 10. Similarly, $\pi(1) = 0, \pi(2) = 0, \pi(3) = 1, \pi(4) = 2, \pi(1000) = 168, \pi(10^6) = 78498$, and $\pi(10^9) = 50847478$ (Hardy [43, p. 9]).

(iii) **The von Mangoldt Function Λ_n.** The function Λ_n is defined as follows (Peryerimhoff [80, p. 84]):

$$\Lambda_n = \begin{cases} \log p & , n = p^\alpha \text{ for some prime } p \text{ and } \alpha \geq 1, \\ 0 & , \text{otherwise.} \end{cases}$$

(iv) **Lambert Summability.** A series $\sum_{n=1}^{\infty} a_n$ is said to be *Lambert summable* (or summable $\mathcal{L}$) to s, if

$$\lim_{x \to 1^-} (1 - x) \sum_{k=1}^{\infty} \frac{k a_k x^k}{1 - x^k} = s. \tag{2.2.2}$$

In this case, we write $\sum a_n = s(\mathcal{L})$. Note that if a series is convergent to s, then it is Lambert summable to s.

This series is convergent for $|x| < 1$, which is true if and only if $a_n = O((1 + \varepsilon)^n)$, for every $\varepsilon > 0$ (see [6, 52, 99]).

If we write $x = e^{-\frac{1}{y}}$ $(y > 0)$, $s(t) = \sum_{k \le t} a_k$ $(a_0 = 0)$, $g(t) = \frac{te^{-t}}{1-e^{-t}}$, then $\sum a_k$ is summable $\mathcal{L}$ to s if and only if (note that $1 - x \approx \frac{1}{y}$)

$$\lim_{y\to\infty} \frac{1}{y} \int_0^\infty \frac{te^{-\frac{t}{y}}}{1-e^{-\frac{t}{y}}} ds(t) = \lim_{y\to\infty} -\int_0^\infty s(t) dg\left(\frac{t}{y}\right)$$
$$= \lim_{y\to\infty} -\frac{1}{y} \int_0^\infty g'\left(\frac{t}{y}\right) s(t) dt = s.$$

The method $\mathcal{L}$ is regular.

2.3 Lemmas

We need the following lemmas for the proof of the PNT which is stated and proved in the next section. In some cases, Tauberian condition(s) will be used to prove the required claim. The general character of a Tauberian theorem is as follows. The ordinary questions on summability consider two related sequences (or other functions) and ask whether it will be true that one sequence possesses a limit whenever the other possesses a limit, the limits being the same; a Tauberian theorem appears, on the other hand, only if this is untrue, and then asserts that the one sequence possesses a limit provided the other sequence both possesses a limit and satisfies some additional condition restricting its rate of increase. The interest of a Tauberian theorem lies particularly in the character of this additional condition, which takes different forms in different cases.

Lemma 2.3.1 (Hardy [41, p. 296]; Peyerimhoff [80, p. 80]). *If* $g(t), h(t) \in L(0, \infty)$, *and if*

$$\int_0^\infty g(t) t^{ix} dt \ne 0 \; (-\infty < x < \infty), \tag{2.3.1}$$

then $s(t) = O(1)$ ($s(t)$ *real and measurable) and*

$$\lim_{x\to\infty} \frac{1}{x} \int_0^\infty g\left(\frac{t}{x}\right) s(t) dt = 0 \;\; \textit{implies} \;\; \lim_{x\to\infty} \frac{1}{x} \int_0^\infty h\left(\frac{t}{x}\right) s(t) dt = 0.$$

Lemma 2.3.2 (Peyerimhoff [80, p. 84]). *If* $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ $(\alpha_i = 1, 2, \ldots,$ p_i *prime*), *then* $\sum_{d/n} \Lambda_d = \log n$.

Proof. Since d runs through divisors of n and we have to consider only $d = p_1, p_1^2, \ldots, p_1^{\alpha_1}, \ldots, p_k^{\alpha_k}$, therefore $\sum_{d/n} \Lambda_d = \alpha_1 \log p_1 + \alpha_2 \log p_2 + \cdots + \alpha_k \log p_k = \log n$.

This completes the proof of Lemma 2.3.2. □

Lemma 2.3.3 (Peyerimhoff [80, p. 84]).

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)(s > 1), \tag{2.3.2}$$

where ζ is a Riemann's Zeta function.

Proof. We have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} &= 1 - \frac{1}{2^s} + \frac{1}{3^s} - \frac{1}{4^s} + \cdots \\
&= \left(1 + \frac{1}{2^s} + \frac{1}{3^s}\right) - 2\left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \cdots\right) \\
&= \zeta(s) - \frac{2}{2^s}\left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\right) \\
&= \zeta(s) - 2^{1-s}\zeta(s) \\
&= (1 - 2^{1-s})\zeta(s).
\end{aligned}$$

This completes the proof of Lemma 2.3.3. □

Lemma 2.3.4 (Hardy [41, p. 246]). *If $s > 1$, then*

$$\zeta(s) = \prod_p \frac{p^s}{p^s - 1}. \tag{2.3.3}$$

Lemma 2.3.5 (Hardy [43, p. 253]).

$$-\zeta'(s) = \zeta(s) \sum_{n=1}^{\infty} \frac{\Lambda_n}{n^s} \tag{2.3.4}$$

Proof. From 2.3.3, we have

$$\log \zeta(s) = \sum_p \log \frac{p^s}{p^s - 1}.$$

Differentiating with respect to s and observing that

$$\frac{d}{ds}\left(\log \frac{p^s}{p^s - 1}\right) = -\frac{\log p}{p^s - 1},$$

we obtain

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \frac{\log p}{p^s - 1}. \tag{2.3.5}$$

The differentiation is legitimate because the derived series is uniformly convergent for $s \geq 1 + \delta > 1$, $\delta > 0$.

We can write (2.3.5) in the form

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_p \log p \sum_{m=1}^{\infty} p^{-ms}$$

and the double series $\sum \sum p^{-ms} \log p$ is absolutely convergent when $s > 1$. Hence it may be written as

$$\sum_{p,m} p^{-ms} \log p = \sum_{n=0}^{\infty} \Lambda_n n^{-s}.$$

This completes the proof of Lemma 2.3.5. □

Lemma 2.3.6 (Peyerimhoff [80, p. 84]). *$s_n \to s(\mathcal{L})$, as $n \to \infty$ and $a_n = O\left(\frac{1}{n}\right)$ imply $s_n \to s$, as $n \to \infty$.*

Proof. We wish to show that $a_n = O(1/n)$ is a Tauberian condition. In order to apply Wiener's theory we must show that (2.3.1) holds. But for $\varepsilon > 0$

$$\begin{aligned}
-\int_0^\infty t^{ix+\varepsilon} g'(t)dt &= (ix+\varepsilon)\int_0^\infty t^{ix+\varepsilon-1} g(t)dt \\
&= (ix+\varepsilon)\sum_{k=0}^{\infty}\int_0^\infty t^{ix+\varepsilon} e^{-(k+1)t} dt \\
&= (ix+\varepsilon)\Gamma(1+\varepsilon+ix)\sum_{k=0}^{\infty}\frac{1}{(k+1)^{1+\varepsilon+ix}}
\end{aligned}$$

i.e.,

$$-\int_0^\infty t^{ix} g'(t)dt = \Gamma(1+ix) \lim_{\varepsilon\to 0} (ix+\varepsilon)\zeta(1+\varepsilon+ix).$$

This has a simple pole at 1 and is $\neq 0$ on the line $\mathrm{Re} z = 1$. A stronger theorem is true, namely, $\mathcal{L} \subseteq Abel$, i.e., every Lambert summable series is also Abel summable (see [42]), which implies this theorem. For the sake of completeness we give a proof that $\zeta(1+ix) \neq 0$ for real x. The formula (2.3.4) implies $\zeta(1+ix) \neq 0$.

This completes the proof of Lemma 2.3.6. □

Lemma 2.3.7 (Peyerimhoff [80, p. 86]).

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0$$

Proof. This follows from O-Tauberian theorem for Lambert summability, if $\sum_{n=1}^{\infty} \frac{\mu(n)}{n} = O(L)$. But

$$\begin{aligned}(1-x)\sum_{n=1}^{\infty} \frac{\mu(n)x^n}{1-x^n} &= (1-x)\sum_{n=1}^{\infty} \mu(n) \sum_{k=0}^{\infty} x^{n(k+1)} \\ &= (1-x)\sum_{m=1}^{\infty} \sum_{n/m} \mu(n) = x(1-x).\end{aligned}$$

A consequence is (by partial summation)

$$\sum_{k \le n} \mu(k) = O(n) \tag{2.3.6}$$

which follows with the notation

$$m(t) = \sum_{1 \le k \le t} \frac{\mu(k)}{k} \text{ from } \sum_{k \le n} \mu(k) = \int_{1-0}^{n} t\,dm(t) = nm(n) - \int_{1}^{n} m(t)dt.$$

This completes the proof of Lemma 2.3.7. □

Lemma 2.3.8 (Hardy [43, p. 346]). *Suppose that* $c_1, c_2, \ldots,$ *is a sequence of numbers such that*

$$C(t) = \sum_{n \le t} c_n$$

and that $f(t)$ *is any function of* t. *Then*

$$\sum_{n \le x} c_n f(n) = \sum_{n \le x-1} C(n)\{f(n) - f(n+1)\} + C(x)f([x]). \tag{2.3.7}$$

If, in addition, $c_j = 0$ *for* $j < n_1$ *and* $f(t)$ *has a continuous derivative for* $t \ge n_1$, *then*

$$\sum_{n \le x} c_n f(n) = C(x)f(x) - \int_{n_1}^{x} C(t)f'(t)dt. \tag{2.3.8}$$

Proof. If we write $N = [x]$, the sum on the left of (2.3.7) is

$$C(1)f(1) + \{C(2) - C(1)\}f(2) + \cdots + \{C(N) - C(N-1)\}f(N)$$
$$= C(1)\{f(1) - f(2)\} + \cdots + C(N-1)\{f(N-1) - f(N)\} + C(N)f(N).$$

Since $C(N) = C(x)$, this proves (2.3.7). To deduce (2.3.8), we observe that $C(t) = C(n)$ when $n \le t < n+1$ and so

$$C(n)[f(n) - f(n+1)] = -\int_n^{n+1} C(t)\ f'(t)dt.$$

Also $C(t) = 0$ when $t < n_1$.

This completes the proof of Lemma 2.3.8. □

Lemma 2.3.9 (Hardy [43, p. 347]).

$$\sum_{n \le x} \frac{1}{n} = \log x + C + O\left(\frac{1}{x}\right),$$

where C is Euler's constant.

Proof. Put $c_n = 1$ and $f(t) = 1/t$. We have $C(x) = [x]$ and (2.3.8) becomes

$$\sum_{n \le x} \frac{1}{n} = \frac{[x]}{x} + \int_1^x \frac{[t]}{t^2} dt$$
$$= \log x + C + E,$$

where

$$C = 1 - \int_1^\infty \frac{t - [t]}{t^2} dt$$

is independent of x and

$$E = \int_x^\infty \frac{t - [t]}{t^2} dt - \frac{x - [x]}{x}$$
$$= \int_x^\infty \frac{O(1)}{t^2} dt + O\left(\frac{1}{x}\right)$$
$$= O\left(\frac{1}{x}\right)$$

This completes the proof of Lemma 2.3.9. □

Lemma 2.3.10 (Peyerimhoff [80, p. 86]). *If*

$$\chi(x) = \sum_{k \le x} \left[\psi\left(\frac{x}{k}\right) - \frac{x}{k} + \log \frac{x}{k} + C \right] \text{ and } \psi(x) = \sum_{n \le x} \Lambda_n,$$

then $\chi(x) = O(\log(x+1))$.

Proof. Möbius formula (2.2.1) yields that

$$\psi(x) - x + \log x + C = \sum_{d \le x} \chi\left(\frac{x}{d}\right) \mu(d). \tag{2.3.9}$$

From $\log n = \sum_{d/n} \Lambda_d$ Lemma 2.3.2, it follows that

$$\sum_{n \le x} \log n = \sum_{n \le x} \sum_{kd=n} \Lambda_d = \sum_{k \le x} \sum_{d \le x/k} \Lambda_d = \sum_{k \le x} \psi\left(\frac{x}{k}\right).$$

Therefore, we obtain

$$\chi(x) = \sum_{n \le x} \log n - x \left[\log x + C + O\left(\frac{1}{x}\right) \right] + [x] \log x - \sum_{k \le x} \log k + [x]C,$$

i.e.,

$$\chi(x) = O(\log(x+1)). \tag{2.3.10}$$

This completes the proof of Lemma 2.3.10. □

Lemma 2.3.11 ([Axer's Theorem] (Peyerimhoff [80, p. 87])). *If*

(a) $\chi(x)$ *is of bounded variation in every finite interval* $[1, T]$,
(b) $\sum_{1 \le k \le x} a_k = O(x)$,
(c) $a_n = O(1)$,
(d) $\chi(x) = O(x^\alpha)$ *for some* $0 < \alpha < 1$,

then

$$\sum_{1 \le k \le x} \chi\left(\frac{x}{k}\right) a_k = O(x).$$

Proof. Let $0 < \delta < 1$. Then

$$\sum_{1 \le k \le \delta x} \chi\left(\frac{x}{k}\right) a_k = O(x^\alpha)\, \delta^{1-\alpha}\, x^{1-\alpha} = O\left(x \delta^{1-\alpha}\right).$$

Assuming that $m-1<\delta x\le m$, $N\le x<N+1$ (m and N integers), we have

$$\begin{aligned}\sum_{\delta x\le k\le x}\chi\left(\frac{x}{k}\right)a_k &= \sum_{k=m}^{N-1}\left[\chi\left(\frac{x}{k}\right)-\chi\left(\frac{x}{k+1}\right)\right]s_k+\chi\left(\frac{x}{N}\right)s_N-\chi\left(\frac{x}{m}\right)s_{m-1}\\ &= O(x)\int_{\delta x}^{x}\left|d\chi\left(\frac{x}{t}\right)\right|+O(x)\\ &= O(x)\int_{1}^{1/\delta}|d\chi(t)|+O(x).\end{aligned}$$

This completes the proof of Lemma 2.3.11. □

Lemma 2.3.12 (Peyerimhoff [80, p. 87]). $\psi(x)-x=O(x)$.

Proof. It follows from (2.3.6), (2.3.9), (2.3.10), and Axer's theorem, that $\psi(x)-x=O(x)$.

This completes the proof of Lemma 2.3.12. □

Lemma 2.3.13 (Peyerimhoff [80, p. 87]). *Let* $\vartheta(x)=\sum_{p\le x}\log p$ *(p prime), then*

(a) $\vartheta(x)\le\psi(x)=O(x)$;
(b) $\psi(x)=\vartheta(x)+\vartheta(\sqrt{x})+\cdots+\vartheta(\sqrt{x})$, *for every* $k>\frac{\log x}{\log 2}$.

Lemma 2.3.14 (Peyerimhoff [80, p. 87]).

$$\psi(x)=\vartheta(x)+O(1)\frac{\log x}{\log 2}\sqrt{x}.$$

Proof. It follows from part (b) of Lemma 2.3.13 that

$$\psi(x)=\vartheta(x)+O(1)\frac{\log x}{\log 2}\sqrt{x}.$$

This completes the proof of Lemma 2.3.14. □

Lemma 2.3.15 (Peyerimhoff [80, p. 87]).

$$\vartheta(x)=x+O(x). \qquad (2.3.11)$$

Proof. Lemma 2.3.14 implies that $\vartheta(x)=x+O(x)$. □

2.4 The Prime Number Theorem

Theorem 2.4.1. *The PNT states that* $\pi(x)$ *is asymptotic to* $x/\log x$ *(see Hardy [41, p. 9]), that is, the limit of the quotient of the two functions* $\pi(x)$ *and* $x/\ln x$ *approaches 1, as* x *becomes indefinitely large, which is the same thing as* $[\pi(x)\log x]/x \to 1$*, as* $x \to \infty$ (Peyerimhoff [80, p. 88]).

Proof. By definition and by

$$\begin{aligned}\pi(x) &= \int_{3/2}^{x} \frac{1}{\log t}\, d\vartheta(t)\\ &= \frac{\vartheta(x)}{\log x} + \int_{3/2}^{x} \frac{\vartheta(t)}{t(\log t)^2}\, dt\\ &= \frac{\vartheta(x)}{\log x} + O\left(\frac{x}{(\log x)^2}\right)\end{aligned}$$

[note that $\vartheta(x) = O(x)$].

Using (2.3.11) we obtain the PNT, i.e.,

$$\lim_{x\to\infty} \pi(x) = \frac{x}{\log x},$$

or

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1.$$

This completes the proof of Theorem 2.4.1. □