# Chapter 4
Embedded GOE for Spinless Fermion Systems: EGOE(2) and EGOE(k)

Matrix ensembles generated by random two-body interactions, called two-body random ensembles (TBRE), model what one may call many-body chaos or stochasticity or complexity exhibited by these systems. These ensembles are defined by representing the two-particle Hamiltonian by one of the classical ensembles (GOE or GUE or GSE) and then the $m > 2$ particle H matrix is generated by the m-particle Hilbert space geometry [1–3]. The key element here is the recognition that there is a Lie algebra that transports the information in the two-particle spaces to many-particle spaces [3–5]. Thus, in these ensembles (for many particle systems) a random matrix ensemble in two-particle spaces is embedded in the m-particle H matrix and therefore these ensembles are more generically called embedded ensembles (EE) [3, 6]. With GOE (GUE) embedding we have then EGOE(2) [EGUE(2)] with '2' denoting that in two-particles spaces the H matrix is represented by a GOE. Due to the two-body selection rules, many of the m-particle matrix elements will be zero. Figure 1.1 gives an example of a H-matrix displaying the structure due to two-body selection rules which form the basis for the EE description. Present understanding is that EE generate paradigmatic models for many-body chaos [7, 8] (one-body chaos is well understood using classical ensembles). Simplest of EE is EGOE(2) [BEGOE(2)], the embedded Gaussian orthogonal ensemble of random matrices for spinless fermion (boson) systems generated by random two-body interactions. Let us begin with EGOE for spinless fermion systems.

4.1 EGOE(2) and EGOE(k) Ensembles: Definition and Construction

The embedding algebra for EGOE(k) and EGUE(k) [also BEGOE(k) and BEGUE(k)] for a system of m spinless particles (fermions or bosons) in N single particle (sp) states with k-body interactions ($k < m$) is $SU(N)$. These ensembles are defined by the three parameters (N, m, k). The EGOE(2) ensemble for spinless fermion systems is generated by defining the two-body Hamiltonian H to be GOE

N=12, m=6
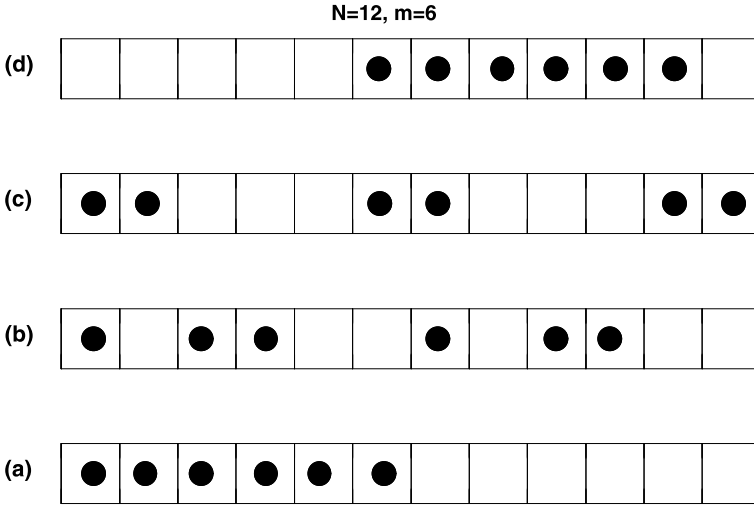
Fig. 4.1 Figure showing some configurations for the distribution of $m = 6$ spinless fermions in $N = 12$ single particle states. The m-particle configurations or basis states are similar to the distributions obtained by putting m particles in N boxes with the conditions that the occupancy of each box can be either zero or one and the total number of occupied boxes equals m. In the figure, (**a**) corresponds to the basis state $|\nu_1\nu_2\nu_3\nu_4\nu_5\nu_6\rangle$, (**b**) corresponds to the basis state $|\nu_1\nu_3\nu_4\nu_7\nu_9\nu_{10}\rangle$, (**c**) corresponds to the basis state $|\nu_1\nu_2\nu_6\nu_7\nu_{11}\nu_{12}\rangle$ and (**d**) corresponds to the basis state $|\nu_6\nu_7\nu_8\nu_9\nu_{10}\nu_{11}\rangle$

in two-particle spaces and then propagating it to many-particle spaces by using the geometry of the many-particle spaces [this is in general valid for k-body Hamiltonians, with $k < m$, generating EGOE(k)]. Let us consider a system of m spinless fermions occupying N sp states. Each possible distribution of fermions in the sp states generates a configuration or a basis state; see Fig. 4.1. Given the sp states $|\nu_i\rangle$, $i = 1, 2, \ldots, N$, EGOE(2) is defined by the Hamiltonian operator,

$$\widehat{H} = \sum_{\nu_i < \nu_j, \nu_k < \nu_\ell} \langle \nu_k \nu_\ell | \widehat{H} | \nu_i \nu_j \rangle a_{\nu_\ell}^\dagger a_{\nu_k}^\dagger a_{\nu_i} a_{\nu_j}. \tag{4.1}$$

The action of the Hamiltonian operator defined by Eq. (4.1) on the basis states $|\nu_1\nu_2\cdots\nu_m\rangle$ (Fig. 4.1 gives examples) generates the EGOE(2) ensemble in m-particle spaces. The symmetries for the antisymmetrized two-body matrix elements $\langle \nu_k \nu_\ell | \widehat{H} | \nu_i \nu_j \rangle$ are

$$\begin{aligned}
\langle \nu_k \nu_\ell | \widehat{H} | \nu_j \nu_i \rangle &= -\langle \nu_k \nu_\ell | \widehat{H} | \nu_i \nu_j \rangle, \\
\langle \nu_k \nu_\ell | \widehat{H} | \nu_i \nu_j \rangle &= \langle \nu_i \nu_j | \widehat{H} | \nu_k \nu_\ell \rangle.
\end{aligned} \tag{4.2}$$

Note that a_{ν_i} and $a_{\nu_i}^\dagger$ in Eq. (4.1) annihilate and create a fermion in the sp state $|\nu_i\rangle$ respectively. The Hamiltonian matrix $H(m)$ in m-particle spaces contains three different types of non-zero matrix elements (all other matrix elements are zero due

to two-body selection rules) and explicit formulas for these are [7],

$$
\begin{aligned}
\langle \nu_1 \nu_2 \cdots \nu_m | \widehat{H} | \nu_1 \nu_2 \cdots \nu_m \rangle &= \sum_{\nu_i < \nu_j \leq \nu_m} \langle \nu_i \nu_j | \widehat{H} | \nu_i \nu_j \rangle, \\
\langle \nu_p \nu_2 \nu_3 \cdots \nu_m | \widehat{H} | \nu_1 \nu_2 \cdots \nu_m \rangle &= \sum_{\nu_i = \nu_2}^{\nu_m} \langle \nu_p \nu_i | \widehat{H} | \nu_1 \nu_i \rangle, \\
\langle \nu_p \nu_q \nu_3 \cdots \nu_m | \widehat{H} | \nu_1 \nu_2 \nu_3 \cdots \nu_m \rangle &= \langle \nu_p \nu_q | \widehat{H} | \nu_1 \nu_2 \rangle.
\end{aligned}
\tag{4.3}
$$

Note that, in Eq. (4.3), the notation $|\nu_1 \nu_2 \cdots \nu_m\rangle$ denotes the orbits occupied by the m spinless fermions. The EGOE(2) is defined by Eqs. (4.2) and (4.3) with GOE(v^2) representation for $\widehat{H}$ in the two-particle spaces, i.e.,

$$
\langle \nu_k \, \nu_\ell | \widehat{H} | \nu_i \nu_j \rangle \text{ are independent Gaussian random variables}
$$

$$
\overline{\langle \nu_k \, \nu_\ell | \widehat{H} | \nu_i \, \nu_j \rangle} = 0,
\tag{4.4}
$$

$$
\overline{\left| \langle \nu_k \nu_\ell | \widehat{H} | \nu_i \nu_j \rangle \right|^2} = v^2 (1 + \delta_{(ij),(k\ell)}).
$$

In Eq. (4.4), 'overline' indicates ensemble average and v is a constant. Now the m-fermion EGOE(2) Hamiltonian matrix ensemble is denoted by $\{H(m)\}$ where $\{\ldots\}$ denotes ensemble, with $\{H(2)\}$ being GOE. Note that, the m-particle H-matrix dimension is $d_f(N, m) = \binom{N}{m}$ and the number of independent matrix elements is $d_f(N, 2)[d_f(N, 2) + 1]/2$; the subscript '$f$' in $d_f(N, m)$ stands for 'fermions'. Computer codes for constructing EGOE(2) ensemble have been developed by many research groups; see for example [7, 9–12]. Just as the EGOE(2) ensemble, one can define k-body ($k < m$) ensembles EGOE(k) (these are also called 2-BRE, 3-BRE, $\ldots$ in [13]) with GOE representation for the H matrix in k particle spaces (thus here we have random k-body interactions). It is possible to derive analytical results, using BCA, for some properties of the general EGOE(k). We will turn to these now.

4.2 Eigenvalue Density: Gaussian Form

4.2.1 Basic Results from Binary Correlation Approximation

Binary correlation theory for the moments of the eigenvalue density generated by spinless EGOE(k) has been developed by Mon and French [3, 14] and the moments given by BCA correspond to the moments in the dilute limit defined by $m \to \infty$, $N \to \infty$, $k \to \infty$ and $m/N \to 0$ and $k/m \to 0$. Alternatively one can use the condition that k is finite and $k/m \to 0$. We will describe the BCA for EGOE(k) in some detail here.

Let us begin with a k_H-body operator,

$$
H(k_H) = \sum_{\alpha, \beta} v_H^{\alpha\beta} \alpha^\dagger(k_H) \beta(k_H).
\tag{4.5}
$$

Here, $\alpha^\dagger(k_H)$ is the k_H particle creation operator and $\beta(k_H)$ is the k_H particle anni-hilation operator. Similarly, $v_H^{\alpha\beta}$ are matrix elements of the operator H in k_H particle space i.e., $v_H^{\alpha\beta} = \langle k_H\beta|H|k_H\alpha\rangle$ (some authors use operators with daggers to denote annihilation operators and operators without daggers to denote creation operators). Following basic traces will be used throughout,

$$\sum_\alpha \alpha^\dagger(k)\alpha(k) = \binom{\hat{n}}{k} \quad\Rightarrow\quad \left\langle\left(\sum_\alpha \alpha^\dagger(k)\alpha(k)\right)^m\right\rangle = \binom{m}{k}. \tag{4.6}$$

$$\sum_\alpha \alpha(k)\alpha^\dagger(k) = \binom{N-\hat{n}}{k} \quad\Rightarrow\quad \left\langle\left(\sum_\alpha \alpha(k)\alpha^\dagger(k)\right)^m\right\rangle = \binom{\widetilde{m}}{k}; \quad \widetilde{m} = N - m. \tag{4.7}$$

$$\sum_\alpha \alpha^\dagger(k)B(k')\alpha(k) = \binom{\hat{n}-k'}{k}B(k')$$

$$\Rightarrow\quad \left\langle\left(\sum_\alpha \alpha^\dagger(k)B(k')\alpha(k)\right)^m\right\rangle = \binom{m-k'}{k}B(k'). \tag{4.8}$$

$$\sum_\alpha \alpha(k)B(k')\alpha^\dagger(k) = \binom{N-\hat{n}-k'}{k}B(k')$$

$$\Rightarrow\quad \left\langle\left(\sum_\alpha \alpha(k)B(k')\alpha^\dagger(k)\right)^m\right\rangle = \binom{\widetilde{m}-k'}{k}B(k'). \tag{4.9}$$

Equation (4.6) follows from the fact that the average should be zero for $m < k$ and one for $m = k$ and similarly, Eq. (4.7) follows from the same argument except that the particles are replaced by holes. Equation (4.8) follows first by writing the k'-body operator $B(k')$ in operator form using Eq. (4.5),

$$B(k') = \sum_{\beta,\gamma} v_B^{\beta\gamma}\beta^\dagger(k')\gamma(k'), \tag{4.10}$$

and then applying the commutation relations for the fermion creation and anni-hilation operators. This gives $\sum_{\beta,\gamma} v_B^{\beta\gamma}\beta^\dagger(k')\sum_\alpha \alpha^\dagger(k)\alpha(k)\gamma(k')$. Now applying Eq. (4.6) to the sum involving α gives Eq. (4.8). Equation (4.9) follows from the same arguments except one has to assume that $B(k')$ is a fully irreducible k'-body operator (Chap. 5 makes clear the notion of 'irreducible' operators) and therefore, it has particle-hole symmetry. For a general $B(k')$ operator, this is valid only in the $N \to \infty$ limit. Therefore, this equation has to be applied with cau-tion.

Using the definition of the H operator in Eq. (4.5), we have

$$\overline{\langle H(k_H)H(k_H)\rangle^m} = \sum_{\alpha,\beta} \overline{\{v_H^{\alpha\beta}\}^2}\langle \alpha^\dagger(k_H)\beta(k_H)\beta^\dagger(k_H)\alpha(k_H)\rangle^m$$

$$= v_H^2 \left\langle \sum_\alpha \alpha^\dagger(k_H)\left\{\sum_\beta \beta(k_H)\beta^\dagger(k_H)\right\}\alpha(k_H)\right\rangle^m$$

$$= v_H^2 T(m,N,k_H). \tag{4.11}$$

As H is taken as EGOE(k_H) with all the k_H particle matrix elements being Gaussian variables with zero center and same variance (diagonal matrix elements variance being twice that of off-diagonal matrix elements). This gives $\overline{(v_H^{\alpha\beta})^2} = v_H^2$ to be independent of the α and β labels. It is important to note that in the dilute limit, the diagonal terms [$\alpha = \beta$ in Eq. (4.11)] in the averages are neglected as they are smaller by at least one power of $1/N$ and the individual H's are irreducible K_H-body operators. These assumptions are no longer valid for finite-N systems and hence here the evaluation of averages is more complicated. In the dilute limit, we have

$$T(m,N,k_H) = \left\langle \sum_\alpha \alpha^\dagger(k_H)\left\{\sum_\beta \beta(k_H)\beta^\dagger(k_H)\right\}\alpha(k_H)\right\rangle^m$$

$$= \binom{\widetilde{m}+k_H}{k_H}\left\langle \sum_\alpha \alpha^\dagger(k_H)\alpha(k_H)\right\rangle^m$$

$$= \binom{\widetilde{m}+k_H}{k_H}\binom{m}{k_H}. \tag{4.12}$$

Note that, we have used Eq. (4.7) to evaluate the summation over β and Eq. (4.6) to evaluate summation over α in Eq. (4.12). In the 'strict' $N \to \infty$ limit, we have

$$T(m,N,k_H) \overset{N\to\infty}{\to} \binom{m}{k_H}\binom{N}{k_H}. \tag{4.13}$$

In order to incorporate the finite-N corrections, we have to consider the contribution of the diagonal terms. Then, we have,

$$T(m,N,k_H) = \binom{m}{k_H}\left[\binom{\widetilde{m}+k_H}{k_H}+1\right]. \tag{4.14}$$

Going beyond $\langle HH\rangle$, let us consider averages involving product of four operators of the form

$$\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^m,$$

where the operators H and G being of body ranks k_H and k_G respectively and they are represented by independent EGOE(k_H) and EGOE(k_G) ensembles respectively. Now, there are two possible ways of evaluating this trace. Either (a) first contract the

H operators across the G operator using Eq. (4.9) and then contract the G operators using Eq. (4.8), or (b) first contract the G operators across the H operator using Eq. (4.9) and then contract the H operators using Eq. (4.8). However, (a) and (b) give the same result only in the 'strict' $N \to \infty$ limit and also for the result incorporating finite N corrections as discussed below. In general, the final result can be expressed as,

$$\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^m} = v_H^2\, v_G^2\, F(m, N, k_H, k_G).\qquad (4.15)$$

In the 'strict' dilute limit, we have

$$(v_H^2 v_G^2)^{-1}\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^m}$$

$$= \sum_{\alpha,\beta,\gamma,\delta} \langle \alpha^\dagger(k_H)\beta(k_H)\gamma^\dagger(k_G)\delta(k_G)\beta^\dagger(k_H)\alpha(k_H)\delta^\dagger(k_G)\gamma(k_G)\rangle^m$$

$$= \binom{\widetilde{m} - k_G + k_H}{k_H} \sum_{\alpha,\gamma,\delta} \langle \alpha^\dagger(k_H)\gamma^\dagger(k_G)\delta(k_G)\alpha(k_H)\delta^\dagger(k_G)\gamma(k_G)\rangle^m$$

$$= \binom{\widetilde{m} - k_G + k_H}{k_H}\binom{m - k_G}{k_H} \sum_{\gamma,\delta} \langle \gamma^\dagger(k_G)\delta(k_G)\delta^\dagger(k_G)\gamma(k_G)\rangle^m$$

$$= \binom{\widetilde{m} - k_G + k_H}{k_H}\binom{m - k_G}{k_H}\binom{\widetilde{m} + k_G}{k_G}\binom{m}{k_G}.\qquad (4.16)$$

Here in the first step β and $\beta^\dagger$ are contracted using Eq. (4.9) giving $\binom{\widetilde{m}-k_G}{k_H}$ and then it is taken out of the trace. In the second step $\alpha^\dagger$ and α are contracted. Then we are left with a term that is similar to Eq. (4.12) and this gives the final result. Now in the 'strict' $N \to \infty$ limit, $F(m, N, k_H, k_G)$ is

$$F(m, N, k_H, k_G) = \binom{m - k_H}{k_G}\binom{m}{k_H}\binom{N}{k_H}\binom{N}{k_G}$$

$$= \binom{m - k_G}{k_H}\binom{m}{k_G}\binom{N}{k_H}\binom{N}{k_G}.\qquad (4.17)$$

In order to obtain correct finite-N corrections to $F(\cdots)$, we have to contract over operators whose lower symmetry parts can not be ignored. The operator $H(k_H)$ decomposes into irreducible symmetry (or tensorial) parts $\mathscr{F}(s)$ denoted by $s = 0, 1, 2, \ldots, k_H$ with respect to the unitary group $SU(N)$; see Chap. 5. For a k_H-body number conserving operator [3, 15], we have (see also Chap. 5)

$$H(k_H) = \sum_{s=0}^{k_H} \binom{m - s}{k_H - s}\mathscr{F}(s).\qquad (4.18)$$

Here, the $\mathscr{F}(s)$ are orthogonal with respect to m-particle averages, i.e., $\langle \mathscr{F}(s)\mathscr{F}^{\dagger}(s')\rangle^{m} = \delta_{ss'}$. Now, $\langle H(k_H)G(k_G)H(k_H)G(k_G)\rangle^{m}$ will have four parts,

$$
\begin{aligned}
\overline{\langle H(k_H) &G(k_G)H(k_H)G(k_G)\rangle^{m}} \\
&= v_H^2 v_G^2 \sum_{\alpha,\beta,\gamma,\delta} \langle \alpha^{\dagger}(k_H)\beta(k_H)\gamma^{\dagger}(k_G)\delta(k_G)\beta^{\dagger}(k_H)\alpha(k_H)\delta^{\dagger}(k_G)\gamma(k_G)\rangle^{m} \\
&\quad + v_H^2 v_G^2 \sum_{\alpha,\gamma,\delta} \langle \alpha^{\dagger}(k_H)\alpha(k_H)\gamma^{\dagger}(k_G)\delta(k_G)\alpha^{\dagger}(k_H)\alpha(k_H)\delta^{\dagger}(k_G)\gamma(k_G)\rangle^{m} \\
&\quad + v_H^2 v_G^2 \sum_{\alpha,\beta,\gamma} \langle \alpha^{\dagger}(k_H)\beta(k_H)\gamma^{\dagger}(k_G)\gamma(k_H)\beta^{\dagger}(k_H)\alpha(k_H)\gamma^{\dagger}(k_G)\gamma(k_G)\rangle^{m} \\
&\quad + v_H^2 v_G^2 \sum_{\alpha,\delta} \langle \alpha^{\dagger}(k_H)\alpha(k_H)\delta^{\dagger}(k_G)\delta(k_G)\alpha^{\dagger}(k_H)\alpha(k_H)\delta^{\dagger}(k_G)\delta(k_G)\rangle^{m} \\
&= X + Y_1 + Y_2 + Z.
\end{aligned}
\tag{4.19}
$$

Note that we have decomposed each operator into diagonal and off-diagonal parts. We have used the condition that the variance of the diagonal matrix elements is twice that of the off-diagonal matrix elements in the defining spaces to convert the restricted summations into unrestricted summations appropriately to obtain the four terms in the RHS of Eq. (4.19). Following [14, 16, 17] and applying unitary decomposition to $\gamma\delta^{\dagger}$ (also $\delta\gamma^{\dagger}$) in the first two terms and $\alpha\beta^{\dagger}$ (also $\beta\alpha^{\dagger}$) in the third term we will get X, Y_1 and Y_2. To make things clear, we will discuss the derivation for X term in detail before proceeding further. Applying unitary decomposition to the operators $\gamma^{\dagger}(k_G)\delta(k_G)$ and $\gamma(k_G)\delta^{\dagger}(k_G)$ using Eq. (4.18), we have

$$
X = v_H^2 v_G^2 \sum_{\alpha,\beta,\gamma,\delta} \sum_{s=0}^{k_G} \binom{m-s}{k_G-s}^2 \langle \alpha^{\dagger}(k_H)\beta(k_H)\mathscr{F}_{\gamma\delta}^{\dagger}(s)\beta^{\dagger}(k_H)\alpha(k_H)\mathscr{F}_{\gamma\delta}(s)\rangle^{m}.
\tag{4.20}
$$

Contracting the operators $\beta\beta^{\dagger}$ across $\mathscr{F}$'s using Eq. (4.9) and operators $\alpha^{\dagger}\alpha$ across $\mathscr{F}$ using Eq. (4.8) gives,

$$
X = v_H^2 v_G^2 \sum_{s=0}^{k_G} \binom{m-s}{k_G-s}^2 \binom{\widetilde{m}+k_H-s}{k_H}\binom{m-s}{k_H} \sum_{\gamma,\delta} \langle \mathscr{F}_{\gamma\delta}^{\dagger}(s)\mathscr{F}_{\gamma\delta}(s)\rangle^{m}. \tag{4.21}
$$

Inversion of the equation,

$$
\sum_{\gamma,\delta} \langle \gamma^{\dagger}(k_G)\delta(k_G)\delta^{\dagger}(k_G)\gamma(k_G)\rangle^{m} = Q(m) = \sum_{s=0}^{k_G} \binom{m-s}{k_G-s}^2 \sum_{\gamma,\delta} \langle \mathscr{F}_{\gamma\delta}^{\dagger}(s)\mathscr{F}_{\gamma\delta}(s)\rangle^{m},
\tag{4.22}
$$

gives,

$$
\binom{m-s}{k_G-s}^2 \sum_{\gamma,\delta} \langle \mathscr{F}_{\gamma\delta}^{\dagger}(s)\mathscr{F}_{\gamma\delta}(s) \rangle^m
$$

$$
= \binom{m-s}{k_G-s}^2 \binom{N-m}{s}\binom{m}{s}[(k_G-s)!s!]^2
$$

$$
\times (N-2s+1) \sum_{t=0}^{s} \frac{(-1)^{t-s}[(N-t-k_G)!]^2}{(s-t)!(N-s-t+1)!t!(N-t)!} Q(N-t). \quad (4.23)
$$

For the average required in Eq. (4.22), we have

$$
Q(m) = \sum_{\gamma,\delta} \langle \gamma^{\dagger}(k_G)\delta(k_G)\delta^{\dagger}(k_G)\gamma(k_G) \rangle^m = \binom{\widetilde{m}+k_G}{k_G}\binom{m}{k_G}. \quad (4.24)
$$

Simplifying Eq. (4.23) using Eq. (4.24) and using the result in Eq. (4.21) along with the series summation [14]

$$
\sum_{t=0}^{s} \frac{(-1)^{t-s}(N-t-k_G)!(k_G+t)!}{(s-t)!(t!)^2(N-s-t+1)!} = \frac{k_G!(N-k_G-s)!}{(N+1-s)!}\binom{k_G}{s}\binom{N+1}{s},
$$
$$
\quad (4.25)
$$

the expression for X is,

$$
X = v_H^2 v_G^2 \, F(m,N,k_H,k_G);
$$

$$
F(m,N,k_H,k_G) = \sum_{s=0}^{k_G} \binom{m-s}{k_G-s}^2 \binom{\widetilde{m}+k_H-s}{k_H}\binom{m-s}{k_H}\binom{\widetilde{m}}{s}\binom{m}{s}\binom{N+1}{s}
$$

$$
\times \frac{N-2s+1}{N-s+1}\binom{N-s}{k_G}^{-1}\binom{k_G}{s}^{-1}. \quad (4.26)
$$

Although not obvious, X has $k_H \leftrightarrow k_G$ symmetry. This is easy to verify for $k_H, k_G \leq 2$. In the large N limit, Y_1, Y_2 and Z are neglected as X will make the dominant contribution; Ref. [17] gives the formulas for Y_1, Y_2 and Z. Thus, in all the applications, we use

$$
\overline{\langle H(k_H)G(k_G)H(k_H)G(k_G) \rangle^m} = X = v_H^2 v_G^2 F(m,N,k_H,k_G) \quad (4.27)
$$

with Eq. (4.17) or (4.26) for $F(m,N,k_H,k_G)$ as appropriate.

4.2.2 Dilute Limit Formulas for the Fourth and Sixth Order Moments and Cumulants

In this section throughout we will use Eqs. (4.13) and (4.17) for the functions T and F respectively, i.e. we will use the strict $N \to \infty$ limit. Also in this section, we will take H to be a k-body operator. As odd order cumulants vanish for EGOE(k), the lowest two cumulants that give information about the shape of the eigenvalue density are the fourth (k_4) and sixth (k_6) order cumulants. For these we need to consider first the fourth moment and the sixth moment.

For the fourth moments given by $\overline{\langle H^4(k) \rangle^m}$, in BCA there will be three different correlation patterns that will contribute (we must correlate in pairs the operators for all moments of order >2),

$$\overline{\langle H^4(k) \rangle^m} = \overline{\langle H(k)H(k)H(k)H(k) \rangle^m}$$
$$+ \overline{\langle H(k)H(k)H(k)H(k) \rangle^m}$$
$$+ \overline{\langle H(k)H(k)H(k)H(k) \rangle^m}. \qquad (4.28)$$

In Eq. (4.28), we denote the binary correlated pairs of operators with the symbol $H\,H$. The first two terms on the RHS of Eq. (4.28) are equal due to cyclic invariance and follow from Eq. (4.11),

$$\overline{\langle H(k)H(k)H(k)H(k) \rangle^m} = \overline{\langle H(k)H(k)H(k)H(k) \rangle^m}$$
$$= \left[\overline{\langle H^2(k) \rangle^m} \right]^2. \qquad (4.29)$$

Similarly, the third term on the RHS of Eq. (4.28) follows from Eq. (4.27),

$$\overline{\langle H(k)H(k)H(k)H(k) \rangle^m} = v_H^4 F(m, N, k, k). \qquad (4.30)$$

Combining Eqs. (4.28), (4.29) and (4.30), $\overline{\langle H^4(k) \rangle^m}$ is given by,

$$\overline{\langle H^4(k) \rangle^m} = v_H^4 \left[2\{T(m, N, k)\}^2 + F(m, N, k, k) \right]. \qquad (4.31)$$

Finally, fourth order cumulant k_4 in the dilute limit is

$$k_4 = \gamma_2 = \left[\overline{\langle H^2(k) \rangle^m} \right]^{-2} \overline{\langle H^4(k) \rangle^m} - 1 = \binom{m-k}{k} \binom{m}{k}^{-1} - 1$$
$$\to -\frac{k^2}{m} + \frac{k^2(k-1)^2}{2m^2} + O\left(1/m^3\right). \qquad (4.32)$$

In the last step we have used the expansion of binomials in powers of $1/m$ using,

$$\binom{m-r}{k} = \frac{m^k}{k!}\left[1 - \frac{1}{m}\left\{kr + \frac{k(k-1)}{2}\right\}\right.$$
$$\left. + \frac{1}{m^2}\left\{\frac{k(k-1)}{2}\left[r^2 + (k-1)r + \frac{(3k-1)(k-2)}{12}\right]\right\} + O\left(\frac{1}{m^3}\right)\right].$$
(4.33)

Therefore, for example for a two-body operator (as in nuclei and atoms) as m increases, the excess parameter γ_2 (or k_4) goes to zero indicating that the density approaches Gaussian. We will confirm this further by deriving a formula for k_6. Before turning to this, it should be added that formulas for lower order moments for EGOE(2) were also derived by Gervios [18].

For the sixth moment $\langle (H(k))^6 \rangle^m$ there are 15 binary association diagrams and they are

$$\overline{\langle H^6(k)\rangle^m}$$

$$= \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m}$$

$$\oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m} \oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m}$$

$$\oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m} \oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m}$$

$$\oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m} \oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m}$$

$$\oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m} \oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m}$$

$$\oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m} \oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m}$$

$$\oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m} \oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m}$$

$$\oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m} \oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m}.$$
(4.34)

As all the correlated H's in Eq. (4.34) are dummy operators, it is easy to see that the first five terms on RHS of Eq. (4.34) are all same. Similarly, the next six terms and also the following three terms are same. This gives,

$$\overline{\langle H^6(k)\rangle^m}$$

$$= 5\overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m} \oplus 6\overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m}$$

$$\oplus 3\overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m}$$

$$\oplus \overline{\langle H(k)H(k)H(k)H(k)H(k)H(k)\rangle^m}. \tag{4.35}$$

The first correlation diagram in Eq. (4.35) is simply $\{\langle H^2(k)\rangle^m\}^3$. With the normalization, which we will use from now onwards, $v_H^2\binom{N}{k} = 1$, this gives $\binom{m}{k}^3$. The second correlation diagram is also simple as we can take out the two directly correlated H's outside the average and then we are left with $\overline{\langle HGHG\rangle^m}$ type term. This gives $\binom{m-k}{k}\binom{m}{k}^2$. For the third correlation diagram, we can use the rule, that follows from Eqs. (4.8) and (4.9),

$$\alpha^\dagger(k)\beta(k)H(k)\beta^\dagger(k)\alpha(k) = v^2\binom{N}{k}\alpha^\dagger(k)H(k)\alpha(k)$$

$$= v^2\binom{N}{k}\binom{m-k}{k}H(k) = \binom{m-k}{k}H(k). \tag{4.36}$$

By contracting the first and third correlated H's and similarly the fourth and the sixth H's in the average gives the third term to be $\binom{m-k}{k}^2\binom{m}{k}$. In the last correlation diagram, we have to necessarily contract across two H's i.e., we have to contract two H's across an effectively $2k$-body operator. Then, first contracting the first and the fourth correlated H's, we are left with $\overline{\langle HGHG\rangle^m}$ type term. This gives $\binom{m-2k}{k}\binom{m-k}{k}\binom{m}{k}$. Substituting these results, Eq. (4.35) gives

$$\overline{\langle H^6(k)\rangle^m} = 5\binom{m}{k}^3 + 6\binom{m-k}{k}\binom{m}{k}^2$$

$$+ 3\binom{m-k}{k}^2\binom{m}{k} + \binom{m-2k}{k}\binom{m-k}{k}\binom{m}{k}. \tag{4.37}$$

First converting the sixth order moment into sixth order cumulant k_6 using Eq. (B.5) gives,

$$k_6 = 5 - 9\binom{m}{k}^{-1}\binom{m-k}{k} + 3\binom{m}{k}^{-2}\binom{m-k}{k}^2 + \binom{m}{k}^{-2}\binom{m-k}{k}\binom{m-2k}{k}. \tag{4.38}$$

Now, expanding the binomials in Eq. (4.38) in powers of $1/m$ using Eq. (4.33), we have

$$k_6 = \frac{k^3(6k-1)}{m^2} + O\left(\frac{1}{m^3}\right). \tag{4.39}$$

Similarly, for the eight order cumulant [7] we have

$$k_8 = \frac{-4k^5(23k - 9)}{m^3} + O\left(\frac{1}{m^4}\right). \tag{4.40}$$

Equations (4.32), (4.39) and (4.40) clearly show that in the dilute limit, as m increases (from $m = k$) the density approaches Gaussian as the cumulants k_r approach zero. In fact if we neglect all the cross correlated terms in the moment expressions, clearly we have $\mu_{2r} = (2r - 1)!!$ and they are the reduced central moments of a Gaussian. Although this result is derived for the dilute limit, in practice Gaussian form is seen even when the stringent dilute limit conditions are not valid (see Chap. 5 for examples). Thus the eigenvalue density tends to Gaussian form for EGOE(k).

For $m = k$, EGOE (EGUE) reduces to GOE (GUE) and the state density then is a semi-circle. For fixed k as we increase m starting from k (or vice verse) there will be semi circle to Gaussian transition in state densities. Numerically this was studied in the past [6] but the transition point was not known. Simplifying Eq. (4.32) for fixed (m, k) and $N \to \infty$, it is seen that $\gamma_2 \to -1$ for $m < 2k$. This is suggestive that $m = 2k$ is the transition point. To prove this conclusively, Benet et al. [19, 20] solved EGUE(k) [it is possible to solve EGOE(k) also] using super symmetry (SUSY) method and showed that the density is semi-circle for $m < 2k$. It is also proved that there will be non-vanishing corrections to the semi-circle shape for $m \geq 2k$. However the SUSY method fails for $m > 2k$ and therefore SUSY method could not be used to prove that for $m \gg 2k$ the eigenvalue density takes Gaussian form. In conclusion, as m increases from k, state densities exhibit semi-circle to Gaussian transition with $m = 2k$ being the transition point.

4.3 Average-Fluctuation Separation and Lower-Order Moments of the Two-Point Function

4.3.1 Level Motion in Embedded Ensembles

Given a normalized state density $\rho(E)$, it is possible to expand it in terms of its asymptotic (or smoothed) form $\overline{\rho}(E)$ and the orthonormal polynomials $P_\mu(E)$ defined by the asymptotic density. For EGOE ensembles $\overline{\rho}(E)$ is a Gaussian, i.e. $\overline{\rho}(E) = \rho_{\mathscr{G}}(E) = (\sqrt{2\pi}\sigma)^{-1}\exp[-(E - E_c)^2/2\sigma^2]$. Then the Gram-Charlier (GC) expansion [21] gives,

$$\rho(E) = \rho_{\mathscr{G}}(E)\left\{1 + \sum_{\zeta \geq 3}(\zeta!)^{-1}S_\zeta He_\zeta(\widehat{E})\right\}. \tag{4.41}$$

In Eq. (4.41), $\widehat{E} = (E - E_c)/\sigma$ is the standardized E. The centroid $E_c = \langle H \rangle^m$ and the variance $\sigma^2 = \langle H^2 \rangle^m - E_c^2$ of the Gaussian $\rho_{\mathscr{G}}$ are same as that of ρ. He_ζ are Hermite polynomials and S_ζ are, in principle, related to higher moments of the

state density $\rho(E)$. One can apply Eq. (4.41) to EGOE(2) and BEGOE(2) by noting that for fermions in the dilute limit and for bosons in the dense limit (see Chap. 9), $\overline{\rho}(E) = \rho_{\mathscr{G}}(E)$. Thus, at this stage distinction between boson and fermion systems is not important. We will not consider boson systems here and return to them in Chap. 9. Since S_ζ's change from member to member of the EGOE(2) ensemble, one can treat them as independent random variables with zero center,

$$\overline{S_\zeta} = 0, \qquad \overline{S_\zeta S_{\zeta'}} = 0 \quad \text{for } \zeta \neq \zeta'. \tag{4.42}$$

This is consistent with the result $\overline{\rho}(E) = \rho_{\mathscr{G}}(E)$ where the 'bar' denotes ensemble average. Each ζ term in Eq. (4.41) represents an excitation 'mode' and the wavelength of the modes is proportional to ζ^{-1}. Therefore small ζ terms are long wavelength modes and large ζ are short wavelength modes. The distribution function $F(E)$, the integrated version of $\rho(E)$, is $F(E) = d \int_\infty^E \rho(E')dE'$ where d is the dimensionality. Deviation of a given level with energy E from its smoothed (with respect to the ensemble) counter part $\overline{E}$ gives the level motion. In terms of $F(E)$ and the local mean spacing $\overline{D(E)}$, we have $\delta E = E - \overline{E} = [F(E) - \overline{F(E)}]\overline{D(E)}$. Then, the variance of the level motion is given by the ensemble average of $\frac{(\delta E)^2}{D(E)^2}$. Using Eq. (4.41) we have easily,

$$\overline{\frac{(\delta E)^2}{D(E)^2}} = \overline{\left[F(x) - \overline{F(x)}\right]^2}$$

$$= d^2\sigma^2[\rho_{\mathscr{G}}(E)]^2 \left\{ \sum_{\zeta \geq 3} (\zeta!)^{-2} \overline{S_\zeta^2}[He_{\zeta-1}(\widehat{E})]^2 \right\}. \tag{4.43}$$

By adding centroid and variance fluctuations, the summation in Eq. (4.43) extends to $\zeta \geq 1$. Then,

$$\overline{\frac{(\delta E)^2}{D(E)^2}} = d^2\sigma^2[\rho_{\mathscr{G}}(E)]^2 \left\{ \sum_{\zeta \geq 1} (\zeta!)^{-2} \overline{S_\zeta^2}[He_{\zeta-1}(\widehat{E})]^2 \right\}. \tag{4.44}$$

Thus we need $\overline{S_\zeta^2}$ for EGOE and BEGOE and we will address this now.

4.3.2 $\overline{S_\zeta^2}$ in Binary Correlation Approximation

Definition of the co-variances $\Sigma_{p,q}$ and an expression for them in terms of $\overline{S_\zeta^2}$ are,

$$\Sigma_{p,q} = \overline{\langle H^p \rangle \langle H^q \rangle} - \overline{\langle H^p \rangle} \, \overline{\langle H^q \rangle}$$

$$= \sum_{\zeta \geq 1} \overline{S_\zeta^2}(\sigma)^{p+q} \binom{p}{\zeta}\binom{q}{\zeta}(p - \zeta - 1)!!(q - \zeta - 1)!!. \tag{4.45}$$

The above relation follows from Eq. (4.41) as

$$\langle H^p \rangle = \overline{\langle H^p \rangle} + \sum_{\zeta \geq 3} (\zeta!)^{-1} S_\zeta \int E^p \rho_{\mathscr{G}}(E) He_\zeta(\widehat{E}) \, dE. \qquad (4.46)$$

We have used in Eq. (4.45) the fact that $\sigma^p \binom{p}{\zeta}(p - \zeta - 1)!!$ is the pth (central) moment of $\rho(E)He_\zeta(\widehat{E})$. Note that $E_c = \overline{E_c} = 0$. On the other hand, using BCA we have [3]

$$\Sigma_{p,q} = \overline{\langle H^p \rangle \langle H^q \rangle} - \overline{\langle H^p \rangle} \, \overline{\langle H^q \rangle}$$

$$= \sum_{\zeta=0}^{\infty} \binom{p}{\zeta}\binom{q}{\zeta} \overline{\{\langle H^\zeta \rangle \overline{\langle H^{p-\zeta} \rangle}\} \, \{\langle H^\zeta \rangle \overline{\langle H^{q-\zeta} \rangle}\}} - \overline{\langle H^p \rangle} \, \overline{\langle H^q \rangle}. \qquad (4.47)$$

The last term of Eq. (4.47) will cancel with the $\zeta = 0$ term of the first term. Then we have,

$$\Sigma_{p,q} = \sum_{\zeta \geq 1} \binom{p}{\zeta}\binom{q}{\zeta} \overline{\langle H^{p-\zeta} \rangle} \, \overline{\langle H^{q-\zeta} \rangle} \, \overline{\underline{\langle H^\zeta \rangle \langle H^\zeta \rangle}}. \qquad (4.48)$$

The Gaussian moments of $\overline{\langle H^{p-\zeta} \rangle}$ are $(p - \zeta - 1)!!(\sigma)^{p-\zeta}$. Therefore,

$$\Sigma_{p,q} = \sum_{\zeta \geq 1} \binom{p}{\zeta}\binom{q}{\zeta} (p - \zeta - 1)!!(q - \zeta - 1)!!(\sigma)^{p+q-2\zeta} \overline{\underline{\langle H^\zeta \rangle \langle H^\zeta \rangle}}. \qquad (4.49)$$

Comparing Eqs. (4.49) and (4.45) will give the important relation,

$$\overline{S_\zeta^2} = \overline{\underline{\langle H^\zeta \rangle \langle H^\zeta \rangle}} (\sigma)^{-2\zeta} = (\sigma)^{-2\zeta} \, \tilde{\Sigma}_{\zeta \zeta}. \qquad (4.50)$$

Thus for studying $\overline{\frac{(\delta E)^2}{D^2}}$ via (4.44), all we need to evaluate is $\overline{\underline{\langle H^\zeta \rangle \langle H^\zeta \rangle}}$.

4.3.3 Average-Fluctuations Separation in the Spectra of Dilute Fermion Systems: Results for EGOE(1) and EGOE(2)

For one body interactions as discussed by Bloch in 1969 [22], fluctuations are of Poisson type. The argument is that without interactions there are many conserved symmetries. An example is $U(N_1) \oplus U(N_2) \oplus --$, where $N_i = 2j + 1$ for a nuclear or atomic shell model j-orbit. Note that the nearest neighbor spacing S_n for the n'th level is $S_n = E_{n+1} - E_n$ where $E_{n+1} = \sum_{i=1}^{m} \varepsilon_i'$ and $E_n = \sum_{i=1}^{m} \varepsilon_i''$. Here for example ε_i' are the energies of the single particle states that are occupied by the m fermions for generating the $(n + 1)$-th state. Similarly ε_i'' generate the n-th state.

Then, obviously S_n's will be uncorrelated giving Poisson fluctuations. In [23] (see also [8])), the authors argued that there will be effects in the lowing part of the many particle spectrum that depend explicitly on the structure of the single particle spectrum. These specific effects are not yet verified in any data analysis. However, after a critical excitation strength Poisson fluctuations set in. Thus generically, for correlations and hence for the level repulsion we require k-body interactions with $k \geq 2$. Now, we will consider $k = 2$ and the results extend to any $k > 2$ [3, 6].

In the dilute limit $H = H(2)$ will be effectively an irreducible two-body operator. Chapter 5 gives details of the decomposition of $H(2)$ into irreducible zero, one and two-body operators. Then, using trace propagation results discussed in Chap. 5, we have $\sigma^2(m) = \langle H^2(2) \rangle^m \overset{\text{dilute limit}}{\Rightarrow} \binom{m}{2} \langle H^2(2) \rangle^2$. Here on-wards we will use the normalization $\langle H^2(2) \rangle^2 = 1$. Then, $\sigma^2(m) = \langle H^2(2) \rangle^m = \binom{m}{2}$. Also, in the dilute limit, as $H(2)$ is an irreducible two-body operator, the propagation equation for $\langle H^p \rangle^m$ is

$$
\begin{aligned}
\langle H^p \rangle^m \quad = \quad & \frac{m(m-1)(N-m)(N-m-1)}{N(N-1)(N-2)(N-3)} x_2^{(p)} \\
& + \frac{m(m-1)(m-2)(N-m)(N-m-1)(N-m-2)}{N(N-1)(N-2)(N-3)(N-4)(N-5)} x_3^{(p)} \\
& + \cdots \\
\xrightarrow[m/N \to 0]{m \to \infty, N \to \infty} \quad & \frac{m^2}{N^2} x_2^{(p)} + \frac{m^3}{N^3} x_3^{(p)} + \cdots \\
\rightarrow \quad & \frac{m^2}{N^2} x_2^{(p)} = \frac{m^2}{2} \left(\frac{N^2}{2} \right)^{-1} x_2^{(p)} = \binom{m}{2} \langle H^p \rangle^2.
\end{aligned}
\tag{4.51}
$$

Equation (4.51) gives the correct result for $p = 2$. Now the cross correlated trace is,

$$
\overline{\langle H^\zeta(2) \rangle^m \langle H^\zeta(2) \rangle^m} = \binom{m}{2}^2 \overline{\langle H^\zeta(2) \rangle^2 \langle H^\zeta(2) \rangle^2}
$$

$$
= 2\zeta \binom{m}{2}^2 \binom{N}{2}^{-2}.
\tag{4.52}
$$

Here, as H in 2-particle spaces is a GOE, we used the GOE result for $\overline{\langle H^\zeta(2) \rangle^2 \langle H^\zeta(2) \rangle^2}$ given by Eq. (2.60). Now, Eqs. (4.50) and (4.52) along with $\sigma^2(m) = \binom{m}{2}$ will give the important result

$$
\overline{S_\zeta^2} = 2\zeta \binom{m}{2}^{2-\zeta} \binom{N}{2}^{-2}.
\tag{4.53}
$$

Substituting Eq. (4.53) in Eq. (4.44) will give the final result for level motion in EGOE(2),

$$
\begin{aligned}
\frac{\overline{(\delta E)^2}}{\overline{D(E)}^2} &= \binom{N}{m}^2 \binom{m}{2}^2 \left[\rho_{\mathscr{G}}(E)\right]^2 \\
&\quad \times \left\{ \sum_{\zeta \geq 1} (\zeta!)^{-2} 2\zeta \binom{m}{2}^{2-\zeta} \binom{N}{2}^{-2} \left[He_{\zeta-1}(\widehat{E})\right]^2 \right\} \\
&\xrightarrow{\hat{E}=0} \frac{1}{\pi} \binom{N}{m}^2 \binom{m}{2} \binom{N}{2}^{-2} \\
&\quad \times \left\{ 1 + \frac{1}{12} \binom{m}{2}^{-2} + \frac{1}{320} \binom{m}{2}^{-4} + \cdots \right\}.
\end{aligned}
\tag{4.54}
$$

Thus, as ζ increases, deviations in $\overline{(\delta E)^2}$ from the leading term rapidly go to zero due to the $\binom{m}{2}^{-2r}$, $r = 1, 2, \ldots$ terms in Eq. (4.54). There will be no change until $\zeta \sim m/2$, thereby defining separation. Beyond this, for $\zeta \gg m/2$ the deviations grow, i.e. fluctuations set in and they will tend to that of GOE [the GOE nature of fluctuations is seen in large number of numerical calculations and therefore it is conjectured in [3, 6] that the EGOE fluctuations in energy levels and strengths will follow GOE—however there is no analytical proof]. Note that for GOE, from Eq. (2.67), we have

$$
\frac{\overline{(\delta E)^2}}{\overline{D(E)}^2} \xrightarrow{\widehat{E}=0} \frac{\gamma}{\pi^2} \ln 2d,
\tag{4.55}
$$

where γ is Euler constant and d is m-particle H matrix dimension. It is important to stress that the BCA for EGOE(2), that gave Eq. (4.54) fails for $\zeta > m/2$. However, before this limit is reached separation sets in. An important consequence of the separation is that the only a few long wavelength modes are required to define the averages. Thus we need a few lower order moments for spectral averages and they can be calculated using trace propagation equations without recourse to H matrix construction and diagonalization. The separation and the GOE nature of fluctuations (then they will be small) form the basis for statistical spectroscopy (SS) [24]. We will discuss this further in Chaps. 5 and 7.

4.3.4 Lower-Order Moments of the Two-Point Function and Cross Correlations in EGOE

Unlike GOE, for EGOE's with N the number of single particle states fixed, two-point function involves in general the two energies drawn from the spectra for two different particle numbers say m_1 and m_2. It is important to note that the GOE in

the defining space will be same for the systems with m_1 fermions and m_2 fermions as N is fixed. The two-point function $S^{\rho:m_1,m_2,N}(x_1, x_2)$ is defined by

$$S^{\rho:m_1,m_2,N}(x_1, x_2) = \overline{\langle\delta(H-x_1)\rangle_N^{m_1}\langle\delta(H-x_2)\rangle_N^{m_2}} - \overline{\langle\delta(H-x_1)\rangle_N^{m_1}}\ \overline{\langle\delta(H-x_2)\rangle_N^{m_1}}.$$
(4.56)

Here, in the densities we have also shown N explicitly to stress that N is same in all the densities. In general we have $m_1 = m_2$ or $m_1 \neq m_2$. The bivariate moments $\Sigma_{p,q}$ in Eq. (4.45) are the moments for the two-point function with $m_1 = m_2$. Similarly the level motion, discussed in the previous subsections, for a (m, N) system derives from $S^{\rho:m,m,N}(x_1, x_2)$. More importantly, Eq. (4.56) shows that EGOE generates cross correlations, that is correlations between spectra with different particle numbers, as the bivariate moments

$$\Sigma_{p,q}(m_1, m_2, N) = \overline{\langle H^p\rangle_N^{m_1}\langle H^q\rangle_N^{m_2}} - \overline{\langle H^p\rangle_N^{m_1}}\ \overline{\langle H^q\rangle_N^{m_2}}$$
(4.57)

will be in general non-zero for $m_1 \neq m_2$. It is important to stress that so far all attempts to derive the form of $S^{\rho:m_1,m_2,N}(x_1, x_2)$ for EGOE have failed; see for example [3, 25, 26]. However, it is possible to derive the formulas for the lower order bivariate moments, i.e. $\Sigma_{p,q}(m_1, m_2, N)$ with $p + q \leq 4$. These give some information about cross correlations generated by EGOE. We will discuss this important aspect in later chapters and in detail in Chap. 12.

4.4 Transition Strength Density: Bivariate Gaussian Form

The strength $R(E_i, E_f)$ generated by a transition operator $\mathcal{O}$ in the H-diagonal basis is $R(E_i, E_f) = |\langle E_f | \mathcal{O} | E_i\rangle|^2$. Correspondingly, the bivariate strength density $I_{biv;\mathcal{O}}(E_i, E_f)$ or $\rho_{biv;\mathcal{O}}(E_i, E_f)$ which is positive definite and normalized to unity is defined by

$$\begin{aligned}
I_{biv;\mathcal{O}}(E_i, E_f) &= \langle\langle\mathcal{O}^\dagger\delta(H-E_f)\mathcal{O}\delta(H-E_i)\rangle\rangle \\
&= I^f(E_f)|\langle E_f|\mathcal{O}|E_i\rangle|^2 I^i(E_i) \\
&= \langle\langle\mathcal{O}^\dagger\mathcal{O}\rangle\rangle\rho_{biv;\mathcal{O}}(E_i, E_f).
\end{aligned}$$
(4.58)

With ε_i and ε_f being the centroids and σ_i^2 and σ_f^2 being the variances of the marginal densities $\rho_{i;\mathcal{O}}(E_i)$ and $\rho_{f;\mathcal{O}}(E_f)$ respectively of the bivariate density $\rho_{biv;\mathcal{O}}$, the bivariate reduced central moments of are $\mu_{pq} = \langle\mathcal{O}^\dagger(\frac{H-\varepsilon_f}{\sigma_f})^q\mathcal{O}(\frac{H-\varepsilon_i}{\sigma_i})^p\rangle$ $/\langle\mathcal{O}^\dagger\mathcal{O}\rangle$ and $\zeta = \mu_{11}$ is the bivariate correlation coefficient. In order to obtain the asymptotic form of $\rho_{biv;\mathcal{O}}$ for EGOE, formulas for μ_{pq} with $p + q = 4$ and 6 are derived using BCA and thereby the reduced cumulants k_{pq} with $p + q = 4$ and 6.

Firstly, H is represented by EGOE(k). Given the transition operator $\mathcal{O}$ of body rank t, we can decompose it into a part that is correlated with H and represent the remaining part say R by a EGOE(t) independent of EGOE(k) representing H.

Then $\mathscr{O} = \alpha H + R$ and the αH term generates the expectation values or the diagonal matrix elements $\langle E|\mathscr{O}|E\rangle$ where E are H eigenvalues. Note that, as H and R are independent, $\alpha = \langle \mathscr{O}H\rangle/\langle H^2\rangle$. Therefore R generates the off-diagonal, in the H diagonal basis, transition matrix elements $|\langle E_f|\mathscr{O}|E_i\rangle|^2$, $E_i \neq E_f$. Thus, by removing the diagonal or expectation value producing part of $\mathscr{O}$, we can assume that H and the part R of $\mathscr{O}$ can be represented by EGOE(k) and EGOE(t) respectively and further they can be assumed to be independent. Once we remove the αH part from $\mathscr{O}$, we need not to make a distinction between $\mathscr{O}$ and R and hence from now on we use only $\mathscr{O}$. Thus, the theory for transition strengths should be applied only to the off-diagonal matrix elements. Now, we proceed to derive formulas for the bivariate moments μ_{pq} using BCA with independent EGOE(k) and EGOE(t) representations for H and $\mathscr{O}$ respectively [16, 27]. The matrix elements variances v_H^2 and $v_{\mathscr{O}}^2$ respectively in the defining space will be in general different for EGOE(k) and EGOE(t). However they will not appear in the formulas for μ_{pq} as these are reduced moments. It is useful to point out that the correlations in μ_{pq} arise due to the non-commutability of H and $\mathscr{O}$ operators. Firstly it is seen that all μ_{pq} with $p+q$ odd will vanish on ensemble average and also $\mu_{pq} = \mu_{qp}$. Moreover $\sigma_i^2 = \langle \mathscr{O}^\dagger \mathscr{O} H^2\rangle^m/\langle \mathscr{O}^\dagger \mathscr{O}\rangle^m = \langle H^2\rangle$ and $\sigma_f^2 = \sigma_i^2$. Thus the first non-trivial moment is μ_{11} and it is given by,

$$\zeta = \mu_{11} = \left\{ \overline{\langle O^\dagger(t) O(t)\rangle^m} \; \overline{\langle H(k) H(k)\rangle^m} \right\}^{-1} \overline{\langle O^\dagger(t) H(k) O(t) H(k)\rangle^m}. \tag{4.59}$$

Applying Eqs. (4.11), (4.13), (4.15) and (4.17) will then give,

$$\zeta = \binom{m}{k}^{-1} \binom{m-t}{k} = 1 - \frac{kt}{m} + \frac{k(k-1)t(t-1)}{2m^2} + O\left(\frac{1}{m^3}\right). \tag{4.60}$$

In the cases with $p+q = 4$, the moments to be evaluated are $\mu_{40} = \mu_{04}$, $\mu_{31} = \mu_{13}$ and μ_{22}. The diagrams for these follow by putting $\mathscr{O}^\dagger$ and $\mathscr{O}$ at appropriate places in the $\langle H^4\rangle$ diagrams in Eq. (4.28). Firstly, μ_{04} is given by

$$\mu_{04} = \left[\overline{\langle \mathscr{O}^\dagger \mathscr{O}\rangle^m} \left(\overline{\langle H^2\rangle^m}\right)^2 \right]^{-1} \overline{\langle \mathscr{O}^\dagger H^4(k) \mathscr{O}\rangle^m}$$

$$= \left[\overline{\langle \mathscr{O}^\dagger \mathscr{O}\rangle^m} \left(\overline{\langle H^2\rangle^m}\right)^2 \right]^{-1} \overline{\langle \mathscr{O}^\dagger \mathscr{O}\rangle^m} \; \overline{\langle H^4(k)\rangle^m}$$

$$= 2 + \binom{m}{k}^{-1} \binom{m-k}{k} = \mu_{40}. \tag{4.61}$$

Here we have used independence of $\mathscr{O}$ and H ensembles and used BCA that led to Eq. (4.32). Similarly,

$$\mu_{13} = \left[\overline{\langle \mathscr{O}^\dagger \mathscr{O}\rangle^m} \left(\overline{\langle H^2\rangle^m}\right)^2 \right]^{-1} \left[\overline{\langle \mathscr{O}^\dagger H(k) H(k) H(k) \mathscr{O} H(k)\rangle^m} \right.$$

$$\oplus \overline{\langle \mathscr{O}^\dagger H(k) H(k) H(k) \mathscr{O} H(k)\rangle^m}$$

$$\left. \oplus \overline{\langle \mathscr{O}^\dagger H(k) H(k) H(k) \mathscr{O} H(k)\rangle^m} \right]. \tag{4.62}$$

The first two terms in Eq. (4.62) are equal and the directly correlated H–H pair can be removed from the trace giving $\binom{m}{k}$. Then we are left with $\langle \mathscr{O}^\dagger H \mathscr{O} H \rangle^m$ term that gives $\binom{m-t}{k}\binom{m}{t}$. In the last term, we have to first contract the first and third H's across the second H giving $\binom{m-k}{k}$ factor. Then we are left with $\langle \mathscr{O}^\dagger H \mathscr{O} H \rangle^m$ term that gives $\binom{m-t}{k}\binom{m}{t}$ using Eq. (4.36) for contracting H's across the $\mathscr{O}$ operator. Combining all these, we have

$$\mu_{13} = \binom{m}{k}^{-2}\left[2\binom{m-t}{k}\binom{m}{k} + \binom{m-t}{k}\binom{m-k}{k}\right] = \zeta\mu_{04}. \qquad (4.63)$$

Alternatively, it is possible to consider μ_{31} and this gives immediately Eq. (4.63). Note that μ_{31} involves $\langle \mathscr{O}^\dagger H \mathscr{O} H^3 \rangle^m$ with $\mathscr{O}^\dagger$ and $\mathscr{O}$ correlated and $[\binom{m}{t}]^{-1}\binom{m-k}{t} = [\binom{m}{k}]^{-1}\binom{m-t}{k}$. This proof also gives immediately that $\mu_{15} = \mu_{51} = \zeta\mu_{06}$. Now, we will consider μ_{22} where

$$\mu_{22} = \left[\overline{\langle \mathscr{O}^\dagger \mathscr{O} \rangle^m}\left(\overline{\langle H^2 \rangle^m}\right)^2\right]^{-1}\left[\overline{\langle \mathscr{O}^\dagger H(k)H(k)\mathscr{O} H(k)H(k) \rangle^m}\right.$$

$$\oplus \overline{\langle \mathscr{O}^\dagger H(k)H(k)\mathscr{O} H(k)H(k) \rangle^m}$$

$$\oplus \left.\overline{\langle \mathscr{O}^\dagger H(k)H(k)\mathscr{O} H(k)H(k) \rangle^m}\right]$$

$$= \binom{m}{k}^{-2}\left[\binom{m}{k}^2 + \binom{m-t}{k}^2 + \binom{m-k-t}{k}\binom{m-t}{k}\right]. \qquad (4.64)$$

The first term in Eq. (4.64) is simple as we can take out the correlated pairs of H's from the trace. The second term follows by applying Eq. (4.36) twice for the contraction of H's across $\mathscr{O}$. The third term follows by first contracting two H's across $H\mathscr{O}$ operator (effective body rank $k+t$) and then we are left with the $\overline{\langle \mathscr{O}^\dagger H \mathscr{O} H \rangle^m}$ term. Using (4.60), (4.61), (4.63) and (4.64), formulas for the 4th order cumulants are obtained and they are

$$k_{04} = k_{40} = \mu_{04} - 3 = \binom{m}{k}^{-1}\binom{m-k}{k} - 1$$

$$= -\frac{k^2}{m} + \frac{k^2(k-1)^2}{2m^2} + O\left(\frac{1}{m^3}\right),$$

$$k_{13} = k_{31} = \mu_{13} - 3\mu_{11} = \zeta\, k_{04} = -\frac{k^2}{m} + \frac{k^2[(k-1)^2 + 2kt]}{2m^2} + O\left(\frac{1}{m^3}\right),$$

$$k_{22} = \mu_{22} - 2\mu_{11}^2 - 1 = \zeta^2\left\{\binom{m-k-t}{k}\binom{m-t}{k}^{-1} - 1\right\}$$

$$= -\frac{k^2}{m} + \frac{k^2[(k-1)^2 + 4kt - 2t]}{2m^2} + O\left(\frac{1}{m^3}\right).$$

$$(4.65)$$

In order to establish the structure of the bivariate cumulants, the cumulants to order $p + q = 6$ are also derived starting with the 15 diagrams in Eq. (4.34). Following the μ_{04} and μ_{13} derivations, we have simply,

$$
\begin{aligned}
\mu_{06} &= \left[\left(\overline{\langle H^2 \rangle^m} \right)^3 \right]^{-1} \overline{\langle H^6 \rangle^m} \\
&= 5 + 6 \binom{m}{k}^{-1} \binom{m-k}{k} + 3 \binom{m}{k}^{-2} \binom{m-k}{k}^2 \\
&\quad + \binom{m}{k}^{-2} \binom{m-2k}{k} \binom{m-k}{k},
\end{aligned}
\tag{4.66}
$$

$$
\mu_{15} = \zeta \mu_{06}.
$$

Here we have used Eq. (4.37) and ζ is given by Eq. (4.60). Now, we will consider μ_{24} and it is given by,

$$
\begin{aligned}
\mu_{24} = \Big[&\langle \mathcal{O}^\dagger \mathcal{O} \rangle^m \left(\langle H^2 \rangle^m \right)^3 \Big]^{-1} \\
\times \Big[\Big\langle \big(\big\{ & \mathcal{O}^\dagger H(k) H(k) \mathcal{O} H(k) H(k) H(k) H(k) \\
\oplus & \mathcal{O}^\dagger H(k) H(k) \mathcal{O} H(k) H(k) H(k) H(k) \big\} \\
\oplus \big\{ & \mathcal{O}^\dagger H(k) H(k) \mathcal{O} H(k) H(k) H(k) H(k) \\
\oplus & \mathcal{O}^\dagger H(k) H(k) \mathcal{O} H(k) H(k) H(k) H(k) \\
\oplus & \mathcal{O}^\dagger H(k) H(k) \mathcal{O} H(k) H(k) H(k) H(k) \big\} \\
\oplus \big\{ & \mathcal{O}^\dagger H(k) H(k) \mathcal{O} H(k) H(k) H(k) H(k) \\
\oplus & \mathcal{O}^\dagger H(k) H(k) \mathcal{O} H(k) H(k) H(k) H(k) \\
\oplus & \mathcal{O}^\dagger H(k) H(k) \mathcal{O} H(k) H(k) H(k) H(k) \big\} \\
\oplus \big\{ & \mathcal{O}^\dagger H(k) H(k) \mathcal{O} H(k) H(k) H(k) H(k) \\
\oplus & \mathcal{O}^\dagger H(k) H(k) \mathcal{O} H(k) H(k) H(k) H(k) \big\} \\
\oplus \big\{ & \mathcal{O}^\dagger H(k) H(k) \mathcal{O} H(k) H(k) H(k) H(k)
\end{aligned}
$$

Table 4.1 Diagrams for the bivariate reduced moment μ_{24} and the corresponding BCA formulas. In the table, $X = [\langle \mathscr{O}^\dagger \mathscr{O} \rangle^m \, (\langle H^2 \rangle^m)^3]$

Correlation diagram	Formula in BCA
$X^{-1}2\overline{\langle \mathscr{O}^\dagger AA\mathscr{O}CCBB \rangle^m}$	2
$X^{-1}3\overline{\langle \mathscr{O}^\dagger AC\mathscr{O}CABB \rangle^m}$	$3\dbinom{m}{k}^{-2}\dbinom{m-t}{k}^2$
$X^{-1}3\overline{\langle \mathscr{O}^\dagger AB\mathscr{O}ABCC \rangle^m}$	$3\dbinom{m}{k}^{-2}\dbinom{m-t}{k}\dbinom{m-t-k}{k}$
$X^{-1}2\overline{\langle \mathscr{O}^\dagger AC\mathscr{O}BCBA \rangle^m}$	$2\dbinom{m}{k}^{-3}\dbinom{m-k}{k}\dbinom{m-t}{k}^2$
$X^{-1}2\overline{\langle \mathscr{O}^\dagger AC\mathscr{O}BABC \rangle^m}$	$2\dbinom{m}{k}^{-3}\dbinom{m-k}{k}\dbinom{m-t}{k}\dbinom{m-t-k}{k}$
$X^{-1}\overline{\langle \mathscr{O}^\dagger AA\mathscr{O}CBCB \rangle^m}$	$\dbinom{m}{k}^{-1}\dbinom{m-k}{k}$
$X^{-1}\overline{\langle \mathscr{O}^\dagger AC\mathscr{O}BACB \rangle^m}$	$\dbinom{m}{k}^{-3}\dbinom{m-2k}{k}\dbinom{m-t-k}{k}\dbinom{m-t}{k}$
$X^{-1}\overline{\langle \mathscr{O}^\dagger AC\mathscr{O}BCAB \rangle^m}$	$\dbinom{m}{k}^{-3}\dbinom{m-t}{k}\sum\limits_{v=k}^{2k}\dbinom{m-v}{k}\dbinom{m-k-t}{k}\dbinom{k}{v-k}\dbinom{k}{2k-v}$

$$\oplus \mathscr{O}^\dagger H(k)H(k)\mathscr{O}H(k)H(k)H(k)H(k)\Big\}$$

$$\oplus \mathscr{O}^\dagger H(k)H(k)\mathscr{O}H(k)H(k)H(k)H(k)$$

$$\oplus \mathscr{O}^\dagger H(k)H(k)\mathscr{O}H(k)H(k)H(k)H(k)$$

$$\oplus \mathscr{O}^\dagger H(k)H(k)\mathscr{O}H(k)H(k)H(k)H(k)\Big)^m \Big]. \tag{4.67}$$

For simplicity, the 'overline' symbol is dropped in Eq. (4.67). All the terms in '{}' brackets are equal and we show in Table 4.1, using the same alphabet for correlated pairs of H's, the diagrams and the formula for them in BCA. The first seven terms in Table 4.1 are easy to recognize following the results already given before using BCA. The last term is special as we need to contract over two operators that are correlated in a different way than in all the other diagrams we have considered so far. Therefore, this needs special treatment as discussed in the context of the 8th moment of the eigenvalue density in [3]. Finally, in BCA μ_{33} can be written as follows (again here also the 'overline' symbol is dropped everywhere),

$$\mu_{33} = \left[\langle \mathscr{O}^\dagger \mathscr{O} \rangle^m \left(\langle H^2 \rangle^m \right)^3 \right]^{-1}$$

$$\times \left[\langle \left\{ \left\{ \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k) \right. \right. \right.$$

$$\oplus \, \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k)$$

$$\oplus \, \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k)$$

$$\oplus \, \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k) \}$$

$$\oplus \, \left\{ \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k) \right.$$

$$\oplus \, \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k)$$

$$\oplus \, \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k)$$

$$\oplus \, \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k) \}$$

$$\oplus \, \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k)$$

$$\oplus \, \left\{ \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k) \right.$$

$$\oplus \, \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k) \}$$

$$\oplus \, \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k)$$

$$\oplus \, \left\{ \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k) \right.$$

$$\oplus \, \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k) \}$$

$$\oplus \, \mathscr{O}^\dagger H(k)H(k)H(k)\mathscr{O}H(k)H(k)H(k) \rangle^m \right]. \qquad (4.68)$$

Just as for μ_{24}, we can write Eq. (4.68) as a sum of seven terms by recognizing that the terms in a given '{}' will give the same result. In Table 4.2 given are the BCA formulas for these terms.

From the previous discussion, it is easy to derive all the formulas given in Table 4.2. Using the formulas given in Appendix B, all the bivariate reduced moments can be converted into bivariate cumulants and then the $1/m$ expansions for the 6th

Table 4.2 Diagrams for the bivariate reduced moment μ_{33} and the corresponding BCA formulas. In the table, $X = [\langle \mathscr{O}^\dagger \mathscr{O} \rangle^m (\langle H^2 \rangle^m)^3]$

Correlation diagram	Formula in BCA
$X^{-1}4\overline{\langle \mathscr{O}^\dagger ACC\mathscr{O}ABB \rangle^m}$	$4\binom{m}{k}^{-1}\binom{m-t}{k}$
$X^{-1}4\overline{\langle \mathscr{O}^\dagger ABA\mathscr{O}BCC \rangle^m}$	$4\binom{m}{k}^{-2}\binom{m-k}{k}\binom{m-t}{k}$
$X^{-1}\overline{\langle \mathscr{O}^\dagger ABC\mathscr{O}CBA \rangle^m}$	$\binom{m}{k}^{-3}\binom{m-t}{k}^3$
$X^{-1}2\overline{\langle \mathscr{O}^\dagger ACB\mathscr{O}CAB \rangle^m}$	$2\binom{m}{k}^{-3}\binom{m-t}{k}\binom{m-t-k}{k}^2$
$X^{-1}\overline{\langle \mathscr{O}^\dagger ABA\mathscr{O}CBC \rangle^m}$	$\binom{m}{k}^{-3}\binom{m-k}{k}^2\binom{m-t}{k}$
$X^{-1}2\overline{\langle \mathscr{O}^\dagger ABC\mathscr{O}CAB \rangle^m}$	$2\binom{m}{k}^{-3}\binom{m-t}{k}^2\binom{m-k-t}{k}$
$X^{-1}\overline{\langle \mathscr{O}^\dagger ACB\mathscr{O}ACB \rangle^m}$	$\binom{m}{k}^{-3}\binom{m-2k-t}{k}\binom{m-t-k}{k}\binom{m-t}{k}$

order cumulants are,

$$k_{06} = k_{60} = \mu_{06} - 15\mu_{04} + 30$$
$$= \frac{k^3(6k-1)}{m^2} - \frac{k^3(k-1)^2(7k-1)}{m^3} + O\left(\frac{1}{m^4}\right),$$

$$k_{15} = k_{51} = \zeta k_{06} = \frac{k^3(6k-1)}{m^2} - \frac{k^3[(k-1)^2(7k-1) + kt(6k-1)]}{m^3} + O\left(\frac{1}{m^4}\right),$$

$$k_{24} = k_{42} = \mu_{24} - \mu_{04} - 8\zeta\mu_{13} - 6\mu_{22} + 24\zeta^2 + 6$$
$$= \frac{k^3(6k-1)}{m^2} - \frac{k^3[(k-1)^2(7k-1) + t(12k^2 - 6k - 1)]}{m^3} + O\left(\frac{1}{m^4}\right),$$

$$k_{33} = \mu_{33} - 6\mu_{13} - 9\zeta\mu_{22} + 12\zeta^3 + 18\zeta$$
$$= \frac{k^3(6k-1)}{m^2} - \frac{k^3[(k-1)^2(7k-1) + t(16k^2 - 13k + 2)]}{m^3} + O\left(\frac{1}{m^4}\right).$$
$$\tag{4.69}$$

As discussed in Appendix B, for a bivariate Gaussian all cumulants k_{pq} with $p + q \geq 3$ should be zero. Therefore, using Eqs. (4.65) and (4.69), it is seen that in the dilute limit (just as in the case of state densities, here also one needs $k^2/m \to 0$), the transition strength densities approach bivariate Gaussian form. Thus, we have

$$\rho_{biv;\mathscr{O}}(E_i, E_f)$$

$$\xrightarrow{EGOE} \overline{\rho_{biv;\mathscr{O}}(E_i, E_f)} = \rho_{biv-\mathscr{G};\mathscr{O}}(E_i, E_f \; ; \; \varepsilon_i, \varepsilon_f, \sigma_i, \sigma_f, \zeta)$$

$$= \frac{1}{2\pi\sigma_i\sigma_f\sqrt{1-\zeta^2}}$$

$$\times \exp\left\{-\frac{1}{2(1-\zeta^2)}\left[\left(\frac{E_i-\varepsilon_i}{\sigma_i}\right)^2 - 2\zeta\left(\frac{E_i-\varepsilon_i}{\sigma_i}\right)\left(\frac{E_f-\varepsilon_f}{\sigma_f}\right)\right.\right.$$

$$\left.\left. + \left(\frac{E_f-\varepsilon_f}{\sigma_f}\right)^2\right]\right\}. \tag{4.70}$$

However, for the strict validity of the Gaussian form, $k_{pq} = 0$ for $p + q \geq 3$ should be valid for any rotation of the (E_i, E_f) variables. To examine this, we convert the bivariate moments μ_{pq} given above in the (E_i, E_f) variables into those defined for the sum and difference variables $(E_i + E_f, E_i - E_f)$. Reduced moments and cumulants defined by these new variables will be denoted by μ'_{pq} and k'_{pq} respectively. For example, denoting E_i by x_1 and E_f by x_2, we have (without loss of generality, we assume (x_1, x_2) are standardized variables)

$$\begin{aligned}
\mu'_{20} &= \langle(x_1+x_2)^2\rangle^m = \langle 2x_1^2 + 2x_1x_2\rangle^m = 2(1+\zeta), \\
\mu'_{02} &= \langle(x_1-x_2)^2\rangle^m = \langle 2x_1^2 - 2x_1x_2\rangle^m = 2(1-\zeta), \\
\mu'_{40} &= [4(1+\zeta)^2]^{-1}\langle(x_1+x_2)^4\rangle^m \\
&= [4(1+\zeta)^2]^{-1}\langle 2x_1^4 + 6x_1^2x_2^2 + 8x_1x_2^3\rangle^m \\
&= [4(1+\zeta)^2]^{-1}(2\mu_{40} + 6\mu_{22} + 8\mu_{31}), \\
\mu'_{04} &= [4(1-\zeta)^2]^{-1}\langle(x_1-x_2)^4\rangle^m \\
&= [4(1-\zeta)^2]^{-1}\langle 2x_1^4 + 6x_1^2x_2^2 - 8x_1x_2^3\rangle^m \\
&= [4(1-\zeta)^2]^{-1}(2\mu_{40} + 6\mu_{22} - 8\mu_{31}).
\end{aligned} \tag{4.71}$$

Here, we have used the results $\langle x_1^2\rangle = \langle x_2^2\rangle$ and $\langle x_i^2 x_j\rangle = \langle x_j^2 x_i\rangle$. Converting the moments μ_{pq} into cumulants k_{pq}, we obtain (it should be noted that $\zeta' = 0$) using Eq. (4.65),

$$\begin{aligned}
k'_{40} &= [2(1+\zeta)^2]^{-1}(k_{40} + 3k_{22} + 4k_{31}) = -\frac{k^2}{m} + O\left(\frac{1}{m^2}\right), \\
k'_{04} &= [2(1-\zeta)^2]^{-1}(k_{40} + 3k_{22} - 4k_{31}) = \frac{k-3/2}{t} + O\left(\frac{1}{m}\right).
\end{aligned} \tag{4.72}$$

Similarly, it is easy to see that $\mu'_{13} = \mu'_{31} = 0$ and $k'_{13} = k'_{31} = 0$. The μ'_{22} and k'_{22} are given by

$$
\begin{aligned}
\mu'_{22} &= \left[4\left(1 - \zeta^2\right)\right]^{-1}\left\langle(x_1 + x_2)^2(x_1 - x_2)^2\right\rangle^m \\
&= \left[4\left(1 - \zeta^2\right)\right]^{-1}(2\mu_{40} - 2\mu_{22}), \\
k'_{22} &= \left[2\left(1 - \zeta^2\right)\right]^{-1}(k_{40} - k_{22}) = \frac{k(1 - 2k)}{4m} + O\left(\frac{1}{m^2}\right).
\end{aligned}
\tag{4.73}
$$

From these equations, it is clearly seen that k_{04} in the difference variable will not approach zero even if m is large although all the other cumulants approach zero as $m \to \infty$. Therefore, even in the dilute limit, EGOE will not generate a strict bivariate Gaussian. To further confirm this result, sixth order cumulants k'_{pq} with $p + q = 6$ are considered. Following the same procedure as in Eq. (4.72) for the sixth order cumulants, we get the following results using Eqs. (4.65) and (4.69),

$$
\begin{aligned}
k'_{60} &= \left[4(1 + \zeta)^3\right]^{-1}[k_{60} + 6k_{51} + 15k_{42} + 10k_{33}] \\
&= \frac{k^3(6k - 1)}{m^2} + O\left(\frac{1}{m^3}\right), \\
k'_{51} &= k'_{15} = 0, \qquad k'_{33} = 0, \\
k'_{42} &= \left[4(1 - \zeta)(1 + \zeta)^2\right]^{-1}[k_{60} + 2k_{51} - k_{42} - 2k_{33}] \\
&= \frac{k^2(32k^2 - 30k + 3)}{16m^2} + O\left(\frac{1}{m^3}\right), \\
k'_{24} &= \left[4(1 + \zeta)(1 - \zeta)^2\right]^{-1}[k_{60} - 2k_{51} - k_{42} + 2k_{33}] \\
&= \frac{k(-8k^2 + 18k - 5)}{8mt} + O\left(\frac{1}{m^2}\right), \\
k'_{06} &= \left[4(1 - \zeta)^3\right]^{-1}[k_{60} - 6k_{51} + 15k_{42} - 10k_{33}] \\
&= \frac{16k^2 - 46k + 35}{4t^2} + O\left(\frac{1}{m}\right).
\end{aligned}
\tag{4.74}
$$

It is seen from Eq. (4.74) that the cumulants k'_{24} and k'_{06} will not approach zero even if m is large. Thus, in practice one has to apply the bivariate Edgeworth corrections (given in Appendix B) to the bivariate Gaussian form of the transition strength density.

The peculiar behavior of k'_{rs} is a result of the behavior of the bivariate correlation coefficient ζ in the original (E_i, E_f) variables. It is seen from Eq. (4.60) that $\zeta \to 1$ as $m \to \infty$ (with $k/m \to 0$ and $t/m \to 0$). This implies that as m increases, the strength density will become narrower. The value $\zeta = 1$ is unphysical as this implies H and $\mathcal{O}$ commute. In practice, $\zeta = 0.6\text{--}0.8$ and it will not be very close to 1. Note that $\zeta = 0$ implies that the strengths are constant, i.e. the system reduces to a GOE

representation. An expansion for the strengths that starts from the GOE result can be obtained by expanding the delta functions in Eq. (4.58) in terms of polynomials defined by the H eigenvalue density. Given the eigenvalue density $\rho(E)$ and the corresponding orthonormal polynomials $P_\mu(E)$, we have [28]

$$\delta(H - E) = \rho(E) \sum_\mu P_\mu(H) P_\mu(E). \qquad (4.75)$$

Given the moments of $\rho(E)$, we can write the polynomials $P_\mu(E)$. As odd moments vanish for EGOE, the lowest four polynomials, in terms of standardized variables x, are

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{x^2 - 1}{\sqrt{\mu_4 - 1}}, \qquad P_3(x) = \frac{x^3 - \mu_4 x}{\sqrt{\mu_6 - \mu_4^2}}. \qquad (4.76)$$

Substituting in Eq. (4.58) the delta function expansion given by Eq. (4.75), we obtain [29]

$$\left| \langle m_f, E_f | \mathcal{O} | m_i E_i \rangle \right|^2 = \sum_{\mu,\nu} \langle \mathcal{O}^\dagger P_\mu^{m_f}(H) \mathcal{O} P_\nu^{m_i}(H) \rangle^{m_i} P_\mu^{m_f}(E_f) P_\nu^{m_i}(E_i). \quad (4.77)$$

For simplicity we assume that $m_i = m_f = m$. Now, using the results for μ_{pq} given before one can write down formulas using BCA for $g_{\mu\nu} = [\langle \mathcal{O}^\dagger \mathcal{O} \rangle^m]^{-1} \langle \mathcal{O}^\dagger P_\mu(H) \times \mathcal{O} P_\nu(H) \rangle^m$, $\mu + \nu \leq 6$. Then,

$$g_{00} = 1, \qquad g_{11} = \zeta, \qquad g_{22} = \zeta^2 \left[1 - \frac{k^2 t}{2m^2} + O\left(\frac{1}{m^3}\right) \right],$$

$$g_{33} = \zeta^3 \left[1 - 3\frac{k^2 t}{2m^2} + O\left(\frac{1}{m^3}\right) \right]. \qquad (4.78)$$

All other $g_{\mu\nu} = 0$ or at least $O(\frac{1}{m^3})$. For example,

$$g_{24} = g_{42} = \frac{k^3 t (2k - 1)}{4\sqrt{3} m^3} + O\left(\frac{1}{m^4}\right). \qquad (4.79)$$

Generalizing the results in Eq. (4.78) we have in the dilute limit

$$\left[\langle \mathcal{O}^\dagger \mathcal{O} \rangle^m \right]^{-1} \langle \mathcal{O}^\dagger P_\mu(H) \mathcal{O} P_\nu(H) \rangle^m = \delta_{\mu\nu}(\zeta)^\mu$$

$$\Rightarrow \quad \rho_{biv:\mathcal{O}}(E_i, E_f) = \rho_1(E_i) \rho_2(E_f) \sum_{\mu=0}^{\infty} (\zeta)^\mu P_\mu(E_f) P_\mu(E_i)$$

$$= \rho_{biv-\mathscr{G};\mathcal{O}}(E_i, E_f). \qquad (4.80)$$

For EGOE the eigenvalue densities are Gaussians and hence the polynomials are Hermite polynomials. Then the sum over the polynomials gives exactly

$\rho_{biv-\mathcal{G};\mathcal{O}}(E_i, E_f)$ with correlation coefficient ζ. Therefore the polynomial expansion has to be summed to very high orders to recover the bivariate Gaussian form. This implies that larger the ζ value, slower will be the convergence of the polynomial expansion for transition strengths. For EGOE, the correlation coefficient $\zeta = \binom{m}{k}^{-1}\binom{m-t}{k}$ and this will be closer to unity. Therefore, expansions for transition strength densities starting with a bivariate Gaussian form will be appropriate. In practice, it is important to employ the bivariate Edgeworth expansion given by Eq. (B.15) incorporating k_{rs}, $r + s = 3, 4$ corrections.

4.5 Strength Sums and Expectation Values: Ratio of Gaussians

Given a operator $\mathcal{O}$ acting on an eigenstate with energy E_i, the transition strength sum, sum of the strengths going to all states with energies E_f, is $\sum_{E_f} |\langle E_f|\mathcal{O}|E_i\rangle|^2$ and this is nothing but the expectation value $\langle \mathcal{O}^\dagger \mathcal{O}\rangle^{E_i} = \langle E_i|\mathcal{O}^\dagger\mathcal{O}|E_i\rangle$. However, taking degeneracies into account, one has to deal with strength sum or expectation value densities. Given a positive definite operator $K = \mathcal{O}^\dagger\mathcal{O}$, the expectation density $I_K(E) = I_{\mathcal{O}^\dagger\mathcal{O}}(E)$ and its normalized version $\rho_K(E)$ are

$$
\begin{aligned}
I_K^m(E) &= \langle m, E|K|m, E\rangle I^m(E) = \langle K\rangle^{m,E} I^m(E) \\
&= \langle\!\langle K\delta(H - E)\rangle\!\rangle^m; \\
\rho_k(E) &= \frac{\langle K\delta(H - E)\rangle^m}{\langle K\rangle^m}.
\end{aligned}
\tag{4.81}
$$

Clearly, expectation value will be the ratio of expectation value density and state density. More importantly, strength sum density [for this $K = \mathcal{O}^\dagger\mathcal{O}$ in Eq. (4.81)] will be a marginal density of the bivariate strength density. For EGOE(k), as the bivariate strength density is a bivariate Gaussian, the strength sum density will be a Gaussian and strength sum will be a ratio of Gaussians [27],

$$
\langle \mathcal{O}^\dagger \mathcal{O}\rangle^E = \frac{I_{\mathcal{O}^\dagger\mathcal{O}}^m(E)}{I^m(E)} \xrightarrow{\text{EGOE}} \frac{I_{\mathcal{O}^\dagger\mathcal{O}:\mathcal{G}}^m(E)}{I_{\mathcal{G}}^m(E)} = \langle \mathcal{O}^\dagger\mathcal{O}\rangle^m \frac{\rho_{\mathcal{O}^\dagger\mathcal{O}:\mathcal{G}}^m(E)}{\rho_{\mathcal{G}}^m(E)}.
\tag{4.82}
$$

Moments of the strength sum density are

$$
M_p(\mathcal{O}^\dagger\mathcal{O}) = \frac{\langle \mathcal{O}^\dagger\mathcal{O}H^p\rangle^m}{\langle \mathcal{O}^\dagger\mathcal{O}\rangle^m}.
\tag{4.83}
$$

Using the moments to fourth order it is possible to add Edgeworth corrections to the Gaussian densities in Eq. (4.82). With $\mathcal{O} = a_i$, Eq. (4.82) gives expectation values of the number operator $\hat{n}_i$ or the occupancies of the sp states $|i\rangle$. Similarly $\mathcal{O}$ is GT operator gives GT strength sums [30] in nuclei and dipole operator gives dipole strength sums in atoms [31].

4.6 Level Fluctuations

In this section, we will briefly discuss the various attempts made in literature to derive the two-point correlation function in energy levels for EGOE(k) and similarly for EGUE(k).

French [3, 6] has conjectured in early 70's, as already stated in Sect. 4.3, that the level and strength fluctuations for EGOE(k) follow GOE. This inference came from many numerical examples (with unfolding of each member of the ensemble with Edgeworth corrected Gaussian defined by the moments generated by individual members, i.e. using spectral unfolding rather than ensemble unfolding) both from EGOE(2) and random two-body interactions in nuclear shell model. These showed that the NNSD is close to Wigner form, $\overline{\Delta_3(\overline{n})}$ fits Dyson-Mehta formula and strength fluctuations follow P-T law. See for example Figs. 2.2, 2.4, 5.3 and [6, 7]. However, the two-point correlation function could not be derived as the BCA fails here.

In 1984, Verbaarschot and Zirnbauer [32] used the replica trick, developed in statistical mechanics for the study of spin glasses and Anderson localization, to derive the two-point function for EGOE(k). However their attempted was not successful. Later in 2000, Weidenmüller's group made another attempt [19, 20]. They have used two extreme models, one called EGE$_{min}(k)$ where all the k-particle matrix elements are assumed to be same. Thus it will have only one independent variable. The other ensemble is called EGE$_{max}(k)$ where all matrix elements, in the m particle space H matrix, allowed by symmetries are assumed to be independent Gaussian random variable and the rest are put to zero. Clearly EGE$_{min}(k)$ represents an integrable system and therefore follows Poisson. Similarly, it was shown explicitly using the SUSY method that EGE$_{max}(k)$ follows GOE. Then, using the sparsity of EGOE(k) ensemble it is argued that EGOE(k) fluctuations should be in between Poisson and GOE. However, explicit form of the two-point correlation function could not be derived [25, 26]. More recently Papenbrock et al. [33], made another attempt to establish the nature of fluctuations generated by EGOE(k). They have, motivated by the analogy to metal-insulator transition (MIT) and a special power-law random band matrix (PLRBM) that simulates the critical statistic at the MIT, constructed a random matrix ensemble called scaffolding ensemble (ScE) having properties: (i) ScE is more sparse than EGOE(k) ensemble; (ii) ScE spectral fluctuations are those of the critical ensemble. Using arguments based on a combination of analytical results, numerical examples and application of a criterion due to Levitov [34], it is argued that EGOE(k) H matrices (with $k \geq 2$) lie on the delocalized side of the MIT and is therefore chaotic or equivalently EGOE(k) fluctuations follow GOE.

It is important and also of interest to understand ergodicity and universality of embedded ensembles. Width of the fluctuations in energy centroids and spectral variances, discussed in detail in Chaps. 11 and 12, clearly indicate that in the dilute limit (for boson systems in the dense limit) EE will be ergodic. However there is not yet an explicit analytical derivation of the result that EE are ergodic. Larger variety of EE described in Chaps. 5–11 and 13 also show that EE have universality—their results apply to a variety of physical systems. Finally, there are some attempts to

study fluctuations in energy levels near the ground state in EE. For example, Bohigas and Flores [35] compared the properties of the low-lying part of the spectrum generated by random interactions in shell model (called TBRE—see Chap. 13) and showed that the widths of the positions of individual eigenvalues were much larger for the TBRE than for the GOE. Cota et al. [36, 37] analyzed NNSD and obtained for the Brody parameter the value ∼0.8. More recent results by Flores et al. [38] show that the semi-Poisson distribution gives a better fit than the Brody distribution, if spectral unfolding is used.

4.7 Summary

In summary, EGOE(k) [similarly EGUE(k) discussed in Chap. 11] generates for $m \gg k$ Gaussian form for state densities with $\gamma_2 \to -k^2/m$ and this is established using BCA. In fact, as m increases from $m = k$, state densities exhibit semi-circle to Gaussian transition with $m = 2k$ being the transition point. The semi-circle form for $m < 2k$ has been proved using SUSY method and the result beyond this follows from the BCA method. Thus, the one-point function for EGOE(k) differs from that of GOE. Secondly, using BCA it is established that the smoothed transition strength densities will take close to a bivariate Gaussian form. Then smoothed transition strength sums, being marginal densities divided by the state density, will be ratio of two Gaussians. Thirdly, EGOE(k) exhibits average-fluctuation separation (as m increases) and also non-zero cross correlations between spectra with different particle numbers (Chap. 12 gives details). Finally, it is seen (from transition strengths and level fluctuations with both being essentially two-point in nature) that there are important differences between GOE and EGOE in the two-point functions.

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