Chapter 3
Interpolating and Other Extended Classical Ensembles

Changes in the nature of level fluctuations in the situations such as (i) a symmetry is gradually broken, (ii) two good symmetry subspaces are gradually admixed, (iii) ordered (integrable) spectra gradually become chaotic and so on are studied by using interpolating and/or partitioned random matrix ensembles [1–7]. A simple yet useful approach for deriving the NNSD's for interpolating ensembles is to extend, as pointed out in [8–12], the simple Wigner's 2×2 matrix formalism. The appropriate 2×2 random matrix ensemble for Poisson to GOE and GUE and GOE to GUE transitions is [8, 12],

$$H = \begin{bmatrix} \alpha(X_1 + X_2) + pv\lambda & \alpha X_3 + i\alpha' X_4 \\ \alpha X_3 - i\alpha' X_4 & \alpha(X_1 - X_2) - pv\lambda \end{bmatrix}. \tag{3.1}$$

In Eq. (3.1) X_1, X_2, X_3 and X_4 are $G(0, v^2)$ variables and the usefulness of p and λ will later become clear. The H matrix in Eq. (3.1) for $\lambda = 0$, $\alpha' = 0$ is GOE, $\lambda = 0$, $\alpha' = \alpha$ is GUE, and $X_i = 0$ and λ a Poisson gives a Poisson spectrum. Thus the matrix in Eq. (3.1) interpolates Poisson, GOE and GUE (in fact also a uniform spectrum). Given λ_1 and λ_2, the two eigenvalues of the H matrix, we have

$$(\lambda_1 - \lambda_2)^2 = S^2 = 4\big[(\alpha X_2 + pv\lambda)^2 + \big(\alpha^2 X_3^2 + \alpha'^2 X_4^2\big)\big]. \tag{3.2}$$

Let us define

$$\begin{aligned} x_2 &= 2\alpha X_2 + 2pv\lambda \to G\big(2pv\lambda, (2\alpha v)^2\big), \\ x_3 &= 2\alpha X_3 \to G\big(0, (2\alpha v)^2\big), \\ x_4 &= 2\alpha' X_4 \to G\big(0, \big(2\alpha' v\big)^2\big). \end{aligned} \tag{3.3}$$

Therefore,

$$\begin{aligned} P(x_2, x_3, x_4)\, dx_2\, dx_3\, dx_4 = {} & \frac{dx_2\, dx_3\, dx_4}{(2\pi)^{3/2}(2\alpha v)^2(2\alpha' v)} \\ & \times \exp -\left(\frac{(x_2 - 2pv\lambda)^2 + x_3^2}{2(2\alpha v)^2} + \frac{x_4^2}{2(2\alpha' v)^2} \right). \end{aligned} \tag{3.4}$$

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Changing the variables (x_2, x_3, x_4) to (S, θ, ϕ) such that $x_2 = S\sin\theta\cos\phi$, $x_3 = S\sin\theta\sin\phi$, $x_4 = S\cos\theta$, we get

$$P(S)\,dS = \frac{S^2 dS}{(2\pi)^{3/2}(2\alpha v)^2(2\alpha' v)} \exp\left(-\frac{p^2\lambda^2}{2\alpha^2}\right) \int_0^{2\pi} \exp\left\{\frac{p\lambda}{2v\alpha^2} S\sin\theta\cos\phi\right\} d\phi$$
$$\times \int_0^{\pi} \exp -\frac{1}{2}\left[\frac{S^2\sin^2\theta}{4\alpha^2 v^2} + \frac{S^2\cos^2\theta}{4\alpha' v^2}\right] \sin\theta\, d\theta. \tag{3.5}$$

The integral over ϕ is $2\pi I_0(\frac{p\lambda}{2v\alpha^2} S\sin\theta)$ where I_0 is Bessel function. With $S\cos\theta = z$, the final result is,

$$P(S:\lambda)\,dS = \frac{S dS}{4v^3\alpha^2\alpha'\sqrt{2\pi}} \exp\left(-\frac{p^2\lambda^2}{2\alpha^2} - \frac{S^2}{8v^2\alpha^2}\right)$$
$$\times \int_0^S dz I_0\left(\frac{p\lambda}{2v\alpha^2}\sqrt{S^2 - z^2}\right) \exp\left[\frac{(\alpha')^2 - \alpha^2}{8v^2\alpha^2(\alpha')^2} z^2\right]. \tag{3.6}$$

With $\lambda = 0, \alpha = 1$ and $\alpha' \to \alpha$ we have GOE to GUE transition. Similarly, assuming a distribution $f(\lambda)d\lambda$ for λ (with λ independent of X_i, $i = 1, 2, 3, 4$), Eq. (3.6) defines for example the Poisson to GOE and GUE interpolations. Combining Eq. (3.6) with

$$P(S)dS = \left[\int_{-\infty}^{+\infty} P(S:\lambda) f(\lambda) d\lambda\right] dS$$
$$\text{for Poisson } f(\lambda)d\lambda = e^{-\lambda}d\lambda \quad \text{for } 0 \le \lambda \le \infty \quad \text{and} \quad 0 \quad \text{for } \lambda < 0 \tag{3.7}$$

gives the spacing distributions for Poisson to GOE and GUE. With $f(\lambda) = 1$ for $0 \le \lambda \le 1$ and 0 for $\lambda < 0$ and also for $\lambda > 1$ will give uniform to GOE and GUE transitions; Ref. [13] gives a numerical study of uniform to GOE and GUE transitions. It is also possible to consider $f(\lambda) = \delta(\lambda - \lambda_c)$. Note that we have always $\int_{-\infty}^{\infty} f(\lambda)\,d\lambda = 1$. Before going further, it is important to mention that an extension of the matrix in Eq. (3.1) including GSE with 4×4 matrices was given in [14].

Before going further, it is important to point out that the results in Refs. [15–18] are used in the simplifications of various integrals we need ahead. A list of some useful integrals are,

$$I_n(a,c) = \int_0^{\infty} x^n \left[\exp -ax^2\right] \Phi(cx)\,dx,$$
$$I_1 = \frac{c}{2a(a+c^2)^{1/2}},$$
$$I_2 = \frac{1}{2\sqrt{\pi}}\left[\frac{1}{a^{3/2}}\tan^{-1}\frac{c}{a^{1/2}} + \frac{c}{a(a+c^2)}\right], \tag{3.8}$$
$$I_3 = \frac{1}{2a^2}\frac{c}{\sqrt{a+c^2}} + \frac{c}{4a(a+c^2)^{3/2}}.$$

In Eq. (3.8), $\Phi(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp -t^2 dt$ is the Error function. An integral with the Bessel function I_0 is,

$$\int_0^\infty \left[\exp -a^2 t^2\right] t^{\mu-1} I_0(bt)\, dt = \frac{\Gamma(\mu/2)}{2a^\mu} {}_1F_1\left(\mu/2, 1, b^2/4a^2\right). \tag{3.9}$$

Finally,

$$\int_0^\infty [\exp -at] \Phi(bt) dt = \frac{1}{a} \exp\left(\frac{a^2}{4b^2}\right)\left[1 - \Phi\left(\frac{a}{2b}\right)\right]. \tag{3.10}$$

The hyper-geometric function ${}_1F_1$ in Eq. (3.9) is also denoted as $M(\mu/2,\ 1,\ b^2/4a^2)$.

3.1 GOE-GUE Transition

3.1.1 2 × 2 Matrix Results

Substituting $\lambda = 0$, $\alpha = 1$ and $\alpha' \to \alpha$ in Eq. (3.6), spacing distribution interpolating GOE to GUE is obtained [1, 8],

$$P_{\text{GOE-GUE}}(S)\, dS = dS \frac{S}{4v^2(1-\alpha^2)^{1/2}} \exp -\frac{S^2}{8v^2}\, \Phi\left[\sqrt{\frac{1-\alpha^2}{8\alpha^2 v^2}}\, S\right]. \tag{3.11}$$

Using Eq. (3.8) it is seen that $P(S)$ is normalized to unity and the average spacing $D_\alpha = \int_0^\infty SP(S) dS$ is,

$$D_\alpha = \frac{1}{\sqrt{\pi(1-\alpha^2)}}\left[\sqrt{8v^2} \tan^{-1} \frac{\sqrt{1-\alpha^2}}{\alpha} + \sqrt{8\alpha^2 v^2 \left(1-\alpha^2\right)}\right]. \tag{3.12}$$

Note that D_0 is the average spacing between the unperturbed levels,

$$D_0 = \sqrt{2\pi}\, v. \tag{3.13}$$

To proceed further it is useful to introduce the transition parameter Λ which is the r.m.s. admixing GUE matrix element $\alpha^2 v^2$ divided by D_0^2,

$$\Lambda = \frac{\alpha^2 v^2}{D_0^2} = \frac{\alpha^2}{2\pi}. \tag{3.14}$$

Note that $\Lambda = 0$ gives GOE and $\Lambda = 1/2\pi$ gives GUE. The importance of the Λ parameter is that it will allow us to extend the 2×2 matrix results to $N \times N$ matrices; see [1, 8, 12] and the results ahead. Although this was pointed out first in [8], it was

rediscovered in [9, 10]. It is easy to see that $P(x)dx$ with $x = S/D_0$ will depend only on Λ. For example, $\overline{x} = \overline{S}/D_0$ is

$$\begin{aligned}\overline{x} &= \frac{2}{\pi\sqrt{1-2\pi\Lambda}}\left[\tan^{-1}\sqrt{\frac{1-2\pi\Lambda}{2\pi\Lambda}} + \sqrt{2\pi\Lambda(1-2\pi\Lambda)}\right] \\ &\overset{\Lambda\ll 1}{\longrightarrow} (1+\pi\Lambda) + O\left(\Lambda^{3/2}\right). \end{aligned} \tag{3.15}$$

Now we can write down the expression for the variance $\sigma^2(0:\Lambda)$ of the NNSD. Note that from Eq. (3.2) we have easily $\overline{S^2} = 8v^2 + 4\alpha^2 v^2$ and then,

$$\begin{aligned}\sigma^2(0:\Lambda) &= \frac{\overline{S^2}}{(\overline{S})^2} - 1 \\ &= \frac{\overline{S^2}}{(D_0)^2\overline{x}^2} - 1 = \frac{4(1+\pi\Lambda)}{\pi\overline{x}^2} - 1 \\ &\overset{\Lambda\ll 1}{\longrightarrow} \left(\frac{4}{\pi} - 1\right) - 4\Lambda \\ &= \sigma^2(0:0) - 4\Lambda. \end{aligned} \tag{3.16}$$

Equation (3.16) extends to any $N \times N$ matrix and for most purposes this small Λ result is adequate for data analysis.

A different parametrization that gives GOE for $\Lambda = 0$ and GUE for $\Lambda = \infty$ is to put in Eqs. (3.1), (3.6), $\lambda = 0$, $\alpha \to \alpha + \sqrt{1-\alpha^2}$, $\alpha' \to \alpha$ and finally divide all the matrix elements by $\sqrt{1-\alpha^2}$. This gives GOE + $[\alpha/\sqrt{1-\alpha^2}]$ GUE ensemble. Then

$$\Lambda = \frac{1}{D_0^2}\left[\frac{\alpha}{\sqrt{1-\alpha^2}}\right]^2 v^2, \quad D_0^2 = \sqrt{2\pi}\, v. \tag{3.17}$$

Now, with $\hat{S} = S/D_0$, the NNSD is

$$P(\hat{S})\, d\hat{S} = d\hat{S}\, \frac{\pi}{2}\sqrt{1+2\pi\Lambda}\, \hat{S} \exp -\frac{\pi}{4}\hat{S}^2\, \Phi(\hat{S}/\sqrt{8\Lambda}). \tag{3.18}$$

Equation (3.18) gives correctly the GOE and GUE NNSD for $\Lambda = 0$ and $\Lambda = \infty$; note that $\Phi(ax) \to 2(ax)/\sqrt{\pi}$ as $a \to 0$. The variance of the NNSD, with Λ defined by Eq. (3.17), is

$$\begin{aligned}&\sigma^2_{\text{GOE-GUE}}(0:\Lambda) \\ &= \frac{\pi(1+3\pi\Lambda)}{[(1+2\pi\Lambda)\tan^{-1}\{(2\pi\Lambda)^{-1/2}\} + \sqrt{2\pi\Lambda}]^2} - 1 \\ &\overset{\Lambda\ll 1}{\longrightarrow} \left(\frac{4}{\pi} - 1\right) - 4\Lambda = \sigma^2(0:0) - 4\Lambda. \end{aligned} \tag{3.19}$$

It is important to mention that all the results given here reproduce exactly the results discussed in [9, 10]. Finally, it should be mentioned that 2×2 GOE-GUE transition results were first given in [19] although the transition parameter was not identified by the authors.

3.1.2 $N \times N$ Ensemble Results for $\Sigma^2(r)$ and $\overline{\Delta_3}(r)$

Let us consider $H = H^R + i\alpha H^I$ and then $\alpha = 0$ gives GOE and $\alpha = 1$ gives GUE. The matrix elements of H satisfy the following properties (with $a = H_{ij}^R$ and $b = H_{ij}^I$),

$$\begin{aligned} \overline{H_{ij}H_{ij}} &= \overline{(a + i\alpha\, b)\,(a + i\alpha\, b)} = \overline{a^2} - \overline{\alpha^2 b^2}, \\ \overline{H_{ij}H_{ji}} &= \overline{(a + i\alpha\, b)\,(a - i\alpha\, b)} = \overline{a^2} + \overline{\alpha^2 b^2}. \end{aligned} \tag{3.20}$$

Using the normalization $v^2 d(1 + \alpha^2) = 1$ ($\overline{a^2} = \overline{b^2} = v^2$), we have

$$\begin{aligned} \overline{H_{ij}H_{ij}} &= \frac{1 - \alpha^2}{d(1 + \alpha^2)} = \frac{\eta}{d}, \\ \overline{H_{ij}H_{ji}} &= \frac{1}{d}, \\ \eta &= \frac{1 - \alpha^2}{1 + \alpha^2}. \end{aligned} \tag{3.21}$$

In the product $\overline{\langle \underbrace{H^\zeta}\rangle\langle \underbrace{H^\zeta}\rangle}$ there are d number of H_{ij} terms and each with its partner comes ζ times. Hence,

$$\begin{aligned} \overline{\langle \underbrace{H^\zeta}\rangle\langle \underbrace{H^\zeta}\rangle} &= \frac{\zeta}{d^2}\, d^\zeta \left[\frac{1}{d^\zeta} + \frac{\eta^\zeta}{d^\zeta}\right] = \frac{\zeta}{d^2}\left(1 + \eta^\zeta\right) = A_\zeta \frac{\zeta}{d^2}; \\ A_\zeta &= 1 + \eta^\zeta. \end{aligned} \tag{3.22}$$

Equation (3.22) correctly reproduces the values for $\alpha = 0$ and $\alpha = 1$ given in Eq. (2.60). Now, extending Eq. (2.62) we have

$$\begin{aligned} S^F(x, y) &= \frac{1}{\pi^2 d^2} \sum_{\zeta=1}^{d} A_\zeta\, \zeta^{-1} \sin\zeta\psi(x)\, \sin\zeta\psi(y) \\ &\simeq S^F_{\mathrm{GUE}}(x, y) + \frac{1}{\pi^2 d^2} \sum_{\zeta=1}^{\infty} (\eta')^\zeta\, \zeta^{-1} \sin\zeta\psi(x)\, \sin\zeta\psi(y) \\ &= S^F_{\mathrm{GUE}}(x, y) + \frac{1}{4\pi^2 d^2} \ln \frac{1 + (\eta')^2 - 2\eta' \cos(\psi(x) + \psi(y))}{1 + (\eta')^2 - 2\eta' \cos(\psi(x) - \psi(y))} \\ &\overset{r \ll d}{\Rightarrow} S^F_{\mathrm{GUE}}(x, y) + \frac{1}{4\pi^2 d^2} \ln \frac{(1 - \eta')^2 + 4\pi^2 \overline{\rho}^2 \eta'}{(1 - \eta')^2 + r^2 \eta' / 4\pi^2 d^2 \overline{\rho}^4}; \end{aligned} \tag{3.23}$$

with

$$\begin{aligned} |x-y| &\overset{r\ll d}{\Rightarrow} r\overline{D}, \\ \eta' &\overset{r\ll d}{\Rightarrow} \eta'(x) = \eta \exp\left(-\tau/d\sin^2\psi\right) = \eta\exp\left(-\tau/d\pi^2\overline{\rho}^2\right). \end{aligned} \tag{3.24}$$

In Eq. (3.23) we have introduced an exponential cut-off in ζ in order to extend the ζ summation to ∞. The details are as follows: With a cut-off $e^{-\alpha_0\zeta}$ the ζ sum is extended to ∞ as in the case of GOE. Then $\eta^\zeta e^{-\alpha_0\zeta} = (\eta e^{-\alpha_0})^\zeta = (\eta')^\zeta$. The choice for α_0 is $\alpha_0 = \tau/d\sin^2\psi = \tau/\pi^2 d\overline{\rho}^2$. Using the simplification as it is done in the case of GOE we get step no. 3 in Eq. (3.23). Now $\cos(\psi_1+\psi_2)$ is $\cos 2\psi$ for $x\sim y$ and $\cos(\psi_1-\psi_2)$ is $1-(\delta\psi)^2/2$. Therefore,

$$\begin{aligned} \cos 2\psi &= 1-2\sin^2\psi = 1-2\pi^2\overline{\rho}^2 \\ 1-\frac{(\delta\psi)^2}{2} &= 1-\left[\frac{r}{2d\pi\overline{\rho}^2}\right]^2. \end{aligned} \tag{3.25}$$

Equation (3.25) will give the fourth equality in Eq. (3.23). Then

$$\begin{aligned} \Sigma^2_\alpha(r) &= \Sigma^2_{\text{GUE}} + \frac{1}{4\pi^2}\left\{2\ln\frac{(1-\eta')^2+4\pi^2\overline{\rho}^2\eta'}{(1-\eta')^2} - 2\ln\frac{(1-\eta')^2+4\pi^2\overline{\rho}^2\eta'}{(1-\eta')^2+\frac{r^2\eta'}{4\pi^2d^2\overline{\rho}^4}}\right\} \\ &= \Sigma^2_{\text{GUE}} + \frac{1}{2\pi^2}\ln\left[1+\frac{r^2\eta'}{4\pi^2d^2\overline{\rho}^4(1-\eta')^2}\right]. \end{aligned} \tag{3.26}$$

At this stage it is convenient to introduce the transition parameter

$$\Lambda(\alpha) = \alpha^2 d\overline{\rho}^2 \tag{3.27}$$

$$\begin{aligned} \eta &\overset{\alpha\ll 1}{\Rightarrow} \exp -2\alpha^2 \\ \eta' &= \exp\left(-2\alpha^2 - \frac{\tau}{d\pi^2\overline{\rho}^2}\right) \simeq 1 \end{aligned} \tag{3.28}$$

$$\Rightarrow \quad 1-\eta' \simeq 2\alpha^2 + \frac{\tau}{d\pi^2\overline{\rho}^2} = \frac{\tau+2\pi^2\Lambda(\alpha)}{d\pi^2\overline{\rho}^2}.$$

Using Eq. (3.28) in Eq. (3.26) we obtain

$$\Sigma^2(r:\Lambda) \overset{\alpha^2\ll 1,\ \Lambda\ll 1}{\Longrightarrow} \Sigma^2_{\text{GUE}}(r) + \frac{1}{2\pi^2}\ln\left[1+\frac{\pi^2r^2}{4[\tau+2\pi^2\Lambda(\alpha)]^2}\right] \tag{3.29}$$

and the small Λ expansion is,

$$\Sigma^2(r:\Lambda) = \Sigma^2_{\text{GUE}}(r) + \frac{1}{2\pi^2}\ln\left(1+\frac{\pi^2r^2}{4\tau^2}\right) - \frac{2\Lambda(\alpha)}{\tau[1+\frac{4\tau^2}{\pi^2r^2}]} + \cdots \tag{3.30}$$

Also note that $\Sigma^2(r:\Lambda) \to \Sigma^2_{\rm GUE}(r)$ for $\Lambda \to \infty$. The parameter τ is fixed from GOE-GUE difference for $r = 1$; $\Delta_{\Sigma^2(1)} = 0.446 - 0.344 = 0.102$ and this gives $\tau = 0.615$. Equation (3.29) was reported first in [1, 8] and later Dupuis and Montambaux [20] derived the same formula in the study of statistical behavior of the spectrum for a metallic ring pierced by a magnetic field. Here the parameter τ has a clear physical meaning. Finally we mention that an exact solution for GOE to GUE transition for $N \times N$ matrices was given by Pandey and Mehta [21].

The Δ_3 statistic for GOE to GUE transition follows by combining Eqs. (3.29) and (C.8),

$$\overline{\Delta}_3(\overline{n},\alpha) = \overline{\Delta}_3^{\rm GUE}(\overline{n}) + \frac{1}{\pi^2\overline{n}^4}\int_0^{\overline{n}} \left(\overline{n}^3 - 2\overline{n}^2 r + r^3\right) \ln\left[1 + B(\Lambda) r^2\right] dr. \quad (3.31)$$

Here we have used Eq. (3.29) with the substitution $B(\Lambda) = \pi^2/4[\tau + 2\pi^2\Lambda(\alpha)]^2$. Solving the integral in Eq. (3.31) using MATHEMATICA gives,

$$\begin{aligned}\overline{\Delta}_3(\overline{n}:\Lambda) = \overline{\Delta}_3^{\rm GUE}(\overline{n}) + \frac{1}{\overline{n}^4\pi^2}\Bigg[& \frac{2\,\overline{n}^3}{\sqrt{B(\Lambda)}} \tan^{-1}\left(\overline{n}\sqrt{B(\Lambda)}\right) \\ & + \left[\frac{B^2(\Lambda)\overline{n}^4 - 1 - 4B(\Lambda)\overline{n}^2}{4B^2(\Lambda)} \ln\left(1 + B(\Lambda)\overline{n}^2\right)\right] - \overline{n}^4 \\ & + \frac{(1 + B(\Lambda)\overline{n}^2)}{16B^4(\Lambda)}\left(6B^2(\Lambda) - 2B^3(\Lambda)\overline{n}^2 - 3\right)\Bigg]. \end{aligned} \quad (3.32)$$

3.1.3 Application to TRNI in Nucleon-Nucleon Interaction

Following the fact that GOE generates stronger level repulsion compared to GOE, as seen from 2×2 $P(S)dS$, Wigner [22] suggested that this could be used to detect time reversal breaking in nuclear force. This and the close agreement between neutron resonance data (i.e. NDE) and GOE coupled with the GOE to GUE transition theory, i.e. the transition curve defined by Eq. (3.29) for $r = 1$, allows us to derive a bound on time reversal non invariant (TRNI) part of the nucleon nucleon interaction. Firstly the NDE data with 1336 levels gives $\Sigma^2(1)$ value to be 0.445 and the GOE value is 0.446. Adding the sample size error on the theory value, within 3σ (99.7 % confidence), the upper bound on $\pi^2\Lambda$ is 0.145 [1]. As $\Lambda = \alpha^2 v^2/D^2$, the bound on αv is $\alpha v \simeq 0.1D$. Note that v is r.m.s. many particle nuclear matrix element for the TRI part of H. To convert this to a bound on α, i.e. TRNI in the effective nucleon-nucleon interaction, v has been determined using statistical spectroscopy methods (see Chap. 7). The deduced bound is $\alpha \leq 10^{-3}$ [23]. Recently, Morrison et al. [24] suggested that a similar analysis for T-odd, P-even interactions in atoms should be possible.

3.2 Poisson to GOE and GUE Transitions

3.2.1 2 × 2 *Matrix Results for Poisson to GOE Transition*

Substituting $\alpha' = 0$ in Eq. (3.6) and applying Eq. (3.7) will give the NNSD for Poisson (P) to GOE transition. To this end we use the result

$$\lim_{\alpha' \to 0} \left[\sqrt{2\pi}\left(2v\alpha'\right)\right]^{-1} \exp\left[-z^2/8\left(v\alpha'\right)^2\right] = (1/2)\delta_{z,0}$$

and the factor $1/2$ comes as we have $z \geq 0$. Then, in terms of the transition parameter

$$\Lambda = \frac{\alpha^2 v^2}{D_0^2}, \tag{3.33}$$

where the mean spacing D_0 of the unperturbed Poisson spectrum is $D_0 = 2pv$ and the mean square admixing GOE matrix element is $\alpha^2 v^2$, the NNSD for P to GOE transition, with $\hat{S} = S/D_0$, is [9, 12],

$$P_{\text{P-GOE}}(\hat{S})\, d\hat{S} = d\hat{S} \frac{\hat{S}}{4\Lambda} \exp\left\{-\hat{S}^2/8\Lambda\right\} \int_0^\infty \exp\left\{-\lambda - \frac{\lambda^2}{8\Lambda}\right\} I_0\left(\frac{\lambda \hat{S}}{4\Lambda}\right) d\lambda. \tag{3.34}$$

For $\Lambda = 0$, Eq. (3.34) gives Poisson and for $\Lambda \to \infty$ the Wigner (GOE) form. Using Eq. (3.9) with $a^2 = 1/8\Lambda$, $\mu = 2$ and $b = \lambda/4\Lambda$ (for the integral over $\hat{S}$), it is easily proved that $P_{\text{P-GOE}}(\hat{S})$ is normalized to unity.

Although we can compare $P_{\text{P-GOE}}(S)$ with $P(S)$ for various Λ values, it is more instructive to examine the $\Lambda \to 0$ and $\hat{S}$ small limit. As $\Lambda \to 0$, we can approximate $\exp -(\lambda + \lambda^2/8\Lambda)$ by $\exp -\lambda^2/8\Lambda$. Now applying Eq. (3.9) and the results given in p. 509 of [15], i.e. ${}_1F_1(\frac{1}{2}, 1, z) = [\exp z/2] I_0(z/2)$, will give

$$P_{\text{P-GOE}}(\hat{S})\, d\hat{S} = d\hat{S} \sqrt{\frac{\pi}{8}} \frac{\hat{S}}{\Lambda^{1/2}} \exp\left\{-\frac{\hat{S}^2}{16\Lambda}\right\} I_0\left(\frac{\hat{S}^2}{16\Lambda}\right). \tag{3.35}$$

Let us mention that perturbation theory also gives Eq. (3.35) for a general $N \times N$ matrix [25]. One important result that follows from Eq. (3.35) is that $P(S)$ goes to zero as S goes to zero for non-zero values of Λ (i.e. there is level repulsion as soon as GOE is switched on).

In the data analysis and applications, more useful is the variance of the NNSD, $\sigma^2(0:\Lambda) = (\overline{S^2}/\overline{S}^2) - 1$ for P to GOE transition, which defines a transition curve. Using Eq. (3.9) with $a^2 = 1/8\Lambda$, $\mu = 3$ and $b = \lambda/4\Lambda$ for the integral over $\hat{S}$ and then applying Eq. (7.628) on p. 871 in [16] will give $\overline{\hat{S}} = \sqrt{\pi}\Psi(-\frac{1}{2}, 0, 2\Lambda)$ where Ψ is Kummer's function [15]. As $\overline{\lambda^2} = 2$ for Poisson, Eq. (3.2) gives $\overline{S^2} = 8\alpha^2 v^2 + 8p^2 v^2$. Then, with $D_0 = 2pv$,

$$\sigma^2_{\text{P-GOE}}(0:\Lambda) = \frac{\overline{S^2}}{\left[\overline{\hat{S}}\right]^2 D_0^2} = \frac{8\Lambda + 2}{\pi[\Psi(-1/2, 0, 2\Lambda)]^2} - 1. \tag{3.36}$$

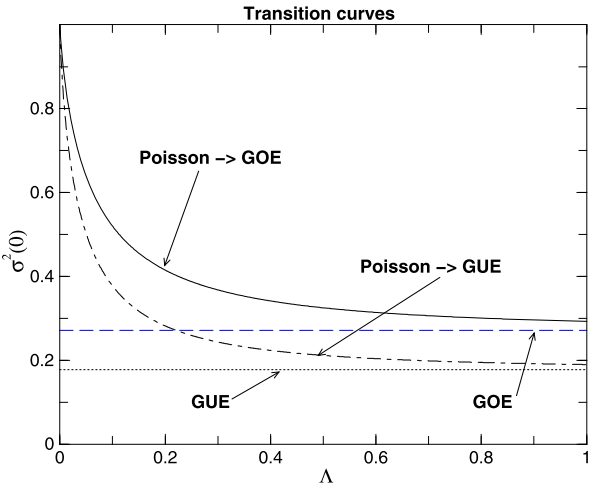

Fig. 3.1 Variance $\sigma^2(0)$ of NNSD vs transition parameter Λ for Poisson to GOE and GUE transitions. Figure is constructed using the results given in [12]. See Sect. 3.2 for details

The complete transition curve, i.e. plot of $\sigma^2(0:\Lambda)$ vs Λ is given in Fig. 3.1. It is instructive to consider small Λ expansion of $\sigma^2(0:\Lambda)$. To this end we start with the identity $\Psi(-1/2,0,2\Lambda) = (2\Lambda)\Psi(1/2,2,2\Lambda)$ and carry out small Λ expansion for $\Psi(1/2,2,2\Lambda)$. Using Eq. (13.1.6) on p. 504 of [15] we have,

$$\begin{aligned}
&\Psi(1/2,2,z)\\
&= \frac{1}{\Gamma(-\frac{1}{2})}\left[\left\{1+O(z)\right\}\ln z + \left\{\psi\left(\frac{1}{2}\right) - \psi(1) - \psi(2)\right\} + O(z)\right] + \frac{1}{\Gamma(\frac{1}{2})z}\\
&= \frac{1}{\sqrt{\pi}\,z}\left\{1 - \frac{z\ln z}{2} - \frac{z}{2}\left[\psi\left(\frac{1}{2}\right) - \psi(1) - \psi(2)\right]\right\} + O\left(z^2\right). \qquad (3.37)
\end{aligned}$$

Here we used $\Gamma(-\frac{1}{2}) = -2\sqrt{\pi}$ and $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. With $z = 2\Lambda$, $\psi(\frac{1}{2}) = -\gamma - 2\ln 2$, $\psi(1) = -\gamma$ and $\psi(2) = -\gamma + 1$ in Eq. (3.37), $\overline{\hat{S}}$ is

$$\overline{\hat{S}} = 2\sqrt{\pi}\,\Lambda\,\Psi(1/2,2,2\Lambda) = 1 - \Lambda\ln(2\Lambda) + \Lambda[2\ln 2 + 1 - \gamma] + O\left(\Lambda^2\right). \qquad (3.38)$$

Therefore the small Λ expansion for $\sigma^2(0:\Lambda)$ is

$$\sigma^2_{\text{P-GOE}}(0:\Lambda) \overset{\Lambda\ll 1}{\longrightarrow} 1 + 4\Lambda\left\{\ln(2\Lambda) + 1 + \gamma - 2\ln 2)\right\} \qquad (3.39)$$

where γ is Euler's constant. Note the $\Lambda\ln\Lambda$ term also appears in the small Λ expansion for the number variance $\Sigma^2(1)$; see Eq. (3.51) ahead.

3.2.2 2 × 2 Results for Poisson to GUE Transition

Let us now consider the Poisson to GUE transition. Simplifying Eq. (3.4) after putting $\alpha = \alpha'$ will give,

$$P(S)dS = \frac{S^2 dS}{\sqrt{2\pi}(2\alpha v)^3} \exp - \frac{S^2 + 4p^2 v^2 \lambda^2}{8\alpha^2 v^2} \times \int_0^\pi d\theta \sin\theta \exp \frac{pvS\lambda \cos\theta}{2\alpha^2 v^2}. \tag{3.40}$$

Now carrying the θ integration and applying Eq. (3.7), we obtain the NNSD for P to GUE and the final result is, with Λ defined in Eq. (3.33),

$$P_{\text{P-GUE}}(\hat{S})\, d\hat{S} = d\hat{S} \frac{\hat{S}}{\sqrt{2\pi}\,\Lambda^{1/2}} \exp\{-\hat{S}^2/8\Lambda\} \times \int_0^\infty \lambda^{-1} \exp\left\{-\lambda - \frac{\lambda^2}{8\Lambda}\right\} \sinh\left(\frac{\hat{S}\lambda}{4\Lambda}\right) d\lambda. \tag{3.41}$$

It should be noted that the mean squared GUE admixing matrix element is $2\alpha^2 v^2$ and hence in this case the transition parameter Λ, used in Eq. (3.41), is mean squared admixing GUE matrix element divided by two times the square of the mean spacing of the Poisson spectrum. Using the integrals given in p. 365 of [16], it is easy to prove that $P_{\text{P-GUE}}(\hat{S})$ is normalized to unity. Similarly, using Eq. (3.10) we have,

$$\overline{\hat{S}} = 4\Lambda \int_0^\infty \frac{1}{y}[\exp -\sqrt{8\Lambda}\, y]\Phi(y)\, dy + \left[\sqrt{\frac{8\Lambda}{\pi}} + [\exp 2\Lambda]\big(1 - \Phi(\sqrt{2\Lambda})\big)\right]. \tag{3.42}$$

Carrying out further simplifications using MATHEMATICA (the functions used here are ExpIntegralEi(–), HypergeometricU(–) and HypergeometricPFQ[{a, b}, {c, d}, z]), the final result is

$$\overline{\hat{S}} = X(\Lambda) = 2\Lambda \left[-Ei(2\Lambda) + 4\sqrt{2\Lambda/\pi}\, {}_2F_2\left(\begin{matrix}1/2, 1\\ 3/2, 3/2\end{matrix}; \quad 2\Lambda\right)\right] + \sqrt{8\Lambda/\pi} + [\exp 2\Lambda]\big[1 - \Phi(\sqrt{2\Lambda})\big]. \tag{3.43}$$

Then, the exact expression for $\sigma^2(0:\Lambda)$ is

$$\sigma^2_{\text{P-GUE}}(0:\Lambda) = \frac{12\Lambda + 2}{[X(\Lambda)]^2} - 1. \tag{3.44}$$

In Eq. (3.43) *Ei* is exponential integral (see p. 228 in [15]),

$$Ei(x) = \gamma + \ln x + \sum_{n=1}^{\infty} \frac{x^n}{n(n!)}.$$

Similarly ${}_2F_2$ is generalized hyper-geometric function,

$${}_2F_2\left(\begin{matrix} a,\ b \\ c,\ d \end{matrix};\quad x\right) = 1 + \frac{ab}{cd}x + \frac{a(a+1)b(b+1)}{c(c+1)d(d+1)}\frac{x^2}{2!} + \cdots.$$

The complete Poisson to GUE transition curve for $\sigma^2(0:\Lambda)$ vs Λ, from Eq. (3.44) is given in the Fig. 3.1. Once again it is instructive to consider the small Λ expansion for $\sigma^2_{\text{P-GUE}}(0:\Lambda)$. Note that,

$$\begin{aligned} X(\Lambda) &= 2\Lambda\left[\left\{-\gamma - \ln(2\Lambda) + O(\Lambda)\right\} + 4\sqrt{\frac{2\Lambda}{\pi}} + O\left(\Lambda^{3/2}\right)\right] \\ &\quad + \sqrt{\frac{8\Lambda}{\pi}} + \left[1 + 2\Lambda + O\left(\Lambda^2\right)\right]\left[1 - \frac{2}{\sqrt{\pi}}\sqrt{2\Lambda} + O\left(\Lambda^{3/2}\right)\right] \\ &= 1 + 2\Lambda[1 - \gamma - \ln 2 - \ln\Lambda] + O\left(\Lambda^{3/2}\right). \end{aligned} \tag{3.45}$$

Now Eq. (3.44) gives,

$$\sigma^2_{\text{P-GUE}}(0:\Lambda) \xrightarrow{\Lambda \ll 1} 1 + 8\Lambda\left(\ln(\Lambda) + \frac{1}{2} + \gamma + \ln 2\right). \tag{3.46}$$

Just as in the case of Poisson to GOE, here also there is the $\Lambda \ln \Lambda$ term. The approximation in Eq. (3.46) is good for $\Lambda \lesssim 0.05$.

3.2.3 *Relationship Between Λ Parameter for Poisson to GOE and the Berry-Robnik Chaos Parameter*

There are several different formulas, given by Brody [26], Berry and Robnik [27, 28], Izrailev [29, 30], Blocki [31] and many others for the NNSD $P_{\text{P-GOE}}(S)\,dS$ and $P_{\text{P-GUE}}(S)\,dS$. For example, the well known Brody (Br) distribution for Poisson to GOE transition, with the Brody parameter ω is [26]

$$P^{\text{B}}_{\text{P-GOE}}(S)\,dS = aS^{\omega}\exp\left\{-bS^{\omega+1}\right\};\quad a = (\omega+1)b,\ b = \left\{\Gamma\left(\frac{\omega+2}{\omega+1}\right)\right\}^{\omega+1} \tag{3.47}$$

and it reduces to Poisson for $\omega = 0$ and Wigner (GOE) form for $\omega = 1$. For $0 < \omega < 1$, the distribution given by Eq. (3.47) vanishes as $S \to 0$ but has an infinite

derivative at that point, an unrealistic feature. Recently a physical process that generates the Brody distribution has been identified [32] and here the Brody parameter corresponds to an appropriate fractal dimension.

The one parameter (ρ) Berry-Robnik (BR) formulas for Poisson to GOE and GUE are,

$$\begin{aligned} P^{\text{BR}}_{\text{P-GOE}}(S)\,dS &= (1-\rho)^2 \exp\{-(1-\rho)S\}\, erfc(\sqrt{\pi}\rho S/2) \\ &\quad + \big(2(1-\rho)\rho + \pi\rho^3 S/2\big)\exp\{-(1-\rho)S - \pi\rho^2 S^2/4\}, \\ P^{\text{BR}}_{\text{P-GUE}}(S)\,dS &= \big(2\rho(1-\rho) - (1-\rho)^2\,\rho\,S\big)\exp\{-(1-\rho)S\}\, erfc\left(\frac{2}{\sqrt{\pi}}\,\rho\,S\right) \\ &\quad + \left(\frac{32}{\pi^2}\rho^4 S^2 + \frac{8}{\pi}(1-\rho)\rho^2 S + (1-\rho)^2\right) \\ &\quad \times \exp\left\{-(1-\rho)S - \frac{4}{\pi}\rho^2 S^2\right\} \end{aligned} \tag{3.48}$$

where ρ is fractional volume, in phase space, of the chaotic region and $1-\rho$ is fractional volume of all regular regions put together. The BR forms are good when there is only one dominant chaotic region coexisting with regular regions. Note that $\rho = 0$ gives Poisson and $\rho = 1$ Wigner (GOE or GUE). Modification of BR distribution (flooding- and tunneling-improved BR) has been discussed recently in [33]. Now we will consider the relationship between the 2×2 results and the BR distribution given by Eq. (3.48) in order to give a physical meaning to the Λ parameter.

The transition curves given in Fig. 3.1 show that the Poisson to GOE and Poisson to GUE transitions are nearly complete for $\Lambda \sim \Lambda_c = 0.3$. The results in Eqs. (3.36) and (3.44) are in fact applicable to general $N \times N$ matrices (or for any interacting many particle system) through the transition parameter Λ by giving appropriate interpretations to $\alpha^2 v^2$ and D_0 in the expression for Λ; this is indeed verified by the results in Fig. 4 of [10]. With this, the results in Eqs. (3.36) and (3.44) can be applied to realistic systems. An important question is: what is the significance of the numerical value 0.3 of Λ_c for Poisson to GOE (similarly for Poisson to GUE)? Toward this end, in [12] relationship between Λ and the BR parameter ρ (ρ representing fractional volume, in phase space, of the chaotic region of a complex dynamical system) for P-GOE transition was explored. Equation (30) of [27] gives $\sigma^2(0:\rho)$ for the BR $P(S)dS$ as a function of the ρ parameter (ρ changing from 0 to 1),

$$\sigma^2_{\text{P-GOE:BR}}(0:\rho) = \frac{2}{1-\rho}\left[1 - \exp\frac{1-\rho^2}{\pi\rho^2}\,\Phi\left(\frac{1-\rho}{\sqrt{\pi}\,\rho}\right)\right]. \tag{3.49}$$

Say $\Lambda_{\text{BR}} = \rho/(1-\rho)$ so that Λ_{BR} changes from 0 to ∞ just as Λ. Fitting Eq. (3.49) to the curves in the Fig. 3.1, it is seen that [12],

$$\Lambda \simeq \frac{\Lambda_{\text{BR}}}{20} = \frac{\rho}{20(1-\rho)} \quad \text{for } \Lambda \gtrsim 0.05. \tag{3.50}$$

However for $\Lambda \lesssim 0.01$ results of Eq. (3.36) and the corresponding BR formula differ significantly. Equation (3.50) gives $\rho = 0.85$ for $\Lambda = 0.3$. Thus 85 % chaoticity can be used as a guide for deciding the marker for order (Poisson) to chaos (GOE) transition. For example, using sufficient number of energy levels near ground states or near the yrast line at high spins as the case may be in atomic nuclei (similarly in other interacting many particle systems such as atoms, molecules etc.), it is possible to deduce the corresponding $\sigma^2(0)$ values. Then from Fig. 3.1 one can read-off the value of Λ (or, depending on the sample size errors, determine a bound on Λ) for Poisson to GOE transitions in these systems. Converting this to ρ gives information about the amount of chaoticity in the system. If it is 85 % (i.e. $\Lambda \geq 0.3$), then one can argue that chaos has set in. This approach was used in deriving the order-chaos border in interacting fermion [34] and boson systems [35].

3.2.4 Poisson to GOE, GUE Transitions: $N \times N$ Ensemble Results for $\Sigma^2(r)$

Without going in details here we give the formulas, valid for $N \times N$ matrices, for $\Sigma^2(\overline{n}, \Lambda)$ for Poisson to GOE and GUE transitions. They, valid for $\overline{n} \gg \Lambda^{1/2}$, are [1]

$$\begin{aligned} \Sigma^2_{\text{P-GOE}}(\overline{n}, \Lambda) &\xrightarrow{\Lambda \ll 1} \overline{n} - 2\Lambda\left(\ln\frac{\overline{n}^2}{2\Lambda} + \gamma - 1 + \ln 4\right), \\ \Sigma^2_{\text{P-GUE}}(\overline{n}, \Lambda) &\xrightarrow{\Lambda \ll 1} \overline{n} - 4\Lambda\left(\ln\frac{\overline{n}^2}{2\Lambda} + \gamma\right). \end{aligned} \tag{3.51}$$

More general discussion of Poisson to GUE (and GOE) transitions for $N \times N$ matrices is given in [3, 5–7].

3.2.5 Onset of Chaos at High Spins via Poisson to GOE Transition

Stephens et al. [36, 37] developed a novel technique to measure the chaoticity parameter ($\Lambda^{1/2}$) for order-chaos transition in rotational nuclei. With $\overline{D}$ giving the average spacing of the levels that are mixed and v giving the r.m.s. admixing matrix element, $\Lambda^{1/2} = v/\overline{D}$. Extending Wigner's 2×2 matrix formalism, the variance of the NNSD for Poisson to GOE transition is given by Eqs. (3.36) and (3.39). As discussed before, the Poisson to GOE transition is nearly complete for $\Lambda \sim 0.3$. and $\Lambda \simeq \frac{\rho}{20(1-\rho)}$ where ρ represents fractional volume, in phase space, of the chaotic region of a complex dynamical system. From the experiments for Yb isotopes, Stephens et al. deduced that $\Lambda^{1/2} \sim 0.15$ to 1.5. Thus at present it is not possible to make a definite statement about onset of chaoticity in the Yb isotopes.

3.3 2 × 2 Partitioned GOE

Let us consider the H matrix $H = H_0 + \alpha V$ where H_0 is a 2×2 block matrix with dimension $d = d_1 + d_2$ (d_1 is dimension of the upper block and d_2 of the lower block) and V is a d dimensional GOE(v^2). Note that GOE(v^2) stands for GOE random matrix ensemble with diagonal matrix elements of the matrices in the ensemble being $G(0, 2v^2)$ and off-diagonal matrix elements $G(0, v^2)$. We put the off-diagonal blocks of H_0 to zero and represent the upper block $\{H_{0;11}\}$ with dimension d_1 by a GOE(v_1^2) where $v_1^2 = v^2(d_1 + d_2)/d_1$ and similarly the lower block $\{H_{0;22}\}$ with dimension d_2 by a GOE(v_2^2) with $v_2^2 = v^2(d_1 + d_2)/d_2$. Thus, $\alpha = 0$ corresponds to a superposition of two GOE's and $\alpha \to \infty$ gives a single GOE. For this 2×2 partitioned GOE, binary correlation approximation gives $A_\zeta = 2(1 + [1 + \alpha^2]^{-\zeta}) \sim 2(2 - \zeta\alpha^2)$. Recall that for GOE to GUE transition we have $A_\zeta = 1 + (\frac{1-\alpha^2}{1+\alpha^2})^\zeta \sim 2(1 - \zeta\alpha^2)$. Therefore it is easy to modify the GOE-GUE derivation and derive the following result, with the transition parameter $\Lambda = \alpha^2 v^2/\overline{D}^2$, for the number variance [1],

$$\Sigma^2_{2\times 2}(r, \Lambda) = \Sigma^2(r, \infty) + \frac{1}{\pi^2} \ln\left\{1 + \frac{\pi^2 r^2}{4(\tau + \pi^2\Lambda)^2}\right\}. \tag{3.52}$$

The cut-off parameter τ is determined using the result

$$\Sigma^2_{2\times 2}(r, 0) = \Sigma^2_{\rm GOE}\big([d_1/d]r\big) + \Sigma^2_{\rm GOE}\big([d_2/d]r\big)$$

and Eq. (3.52) is good for $r > 2$. Note that $\Sigma^2(r, \infty)$ is the GOE value. As discussed before, $\Sigma^2(r)$ formula gives the expression for $\overline{\Delta}_3(r)$,

$$\overline{\Delta}_{3:2\times 2}(r, \Lambda) = \overline{\Delta}_3(r, \infty) + \frac{1}{\pi^2}\left\{\left[\frac{1}{2} - \frac{2}{X^2r^2} - \frac{1}{2X^4r^4}\right]\ln\big(1 + X^2r^2\big) + \frac{4}{Xr}\tan^{-1}(Xr) + \frac{1}{2X^2r^2} - \frac{9}{4}\right\}; \quad X = \frac{\pi}{2(\tau + \pi^2\Lambda)}. \tag{3.53}$$

A direct and good test of Eq. (3.53) came recently from experiments with two coupled flat superconducting microwave billiards [38].

3.3.1 Isospin Breaking in ^{26}Al and ^{30}P Nuclear Levels

Shriner and Mitchell [39, 40] considered complete spectroscopy for levels up to ∼8 MeV excitation in ^{26}Al and ^{30}P. For ^{26}Al, there are 75 $T = 0$ and 25 $T = 1$ levels with $J^\pi = 1^+$ to 5^+. Similarly in ^{30}P there are 69 $T = 0$ and 33 $T = 1$ levels with $J^\pi = 0^\pm$ to $5^\pm$. With Coulomb interaction breaking isospin, the appropriate random matrix model here is 2×2 GOE giving 2GOE to 1GOE transition. Analysis of data for ^{26}Al and ^{30}P is in good agreement with 2GOE to 1GOE transition. In

particular, using ^{26}Al the data was analyzed using a slightly different 2×2 random matrix ensemble [41],

$$\begin{bmatrix} \text{GOE}((d/d_0)v^2) & \alpha V_c(2v^2) \\ \widetilde{\alpha V_c(2v^2)} & \text{GOE}((d/d_1)v^2) \end{bmatrix} \tag{3.54}$$

in $|T=0\rangle$ and $|T=1\rangle$ basis with the dimension d_0 of the $T=0$ space being $d_0 = 75$ and the dimension d_1 of the $T=1$ space being $d_1 = 25$. Then the total dimension $d = 100$. Analysis of data gave $\alpha = 0.056$ and $\overline{H_{ij}^2(c)} = \overline{\alpha^2 V_{c;ij}^2} \sim (20 \text{ keV})^2$. The corresponding spreading width $\Gamma = 2\pi \overline{H_{ij}^2(c)}/\overline{D}$ is $\Gamma \sim 32$ keV. There is also an analysis of reduced transition probabilities (with about 1500 transitions) in ^{26}Al showing deviations from P-T [42]. Let us consider this in some detail.

For GOE, given the strengths $R(E_i, E_f) = |\langle E_i|\mathscr{O}|E_f\rangle|^2$, the locally renormalized transition strengths $x = R(E_i, E_f)/\overline{R(E_i, E_f)}$ are distributed according to the Porter-Thomas (P-T) law. Deviations from P-T law could be ascribed to symmetry breaking and then the questions are: (i) where to look for good data; (ii) what is the appropriate random matrix ensemble and what are its predictions. Adams et al. [42] collected data for reduced electromagnetic transition matrix elements in ^{26}Al from ground state to 8 MeV excitation. The data divides into 120 different transition sequences with each of them having about 10 matrix elements; a transition sequence is defined by initial $J_i^{\pi_i} T_i$ going to all $J_f^{\pi_f} T_f$ (with no missing transitions in between) for a given $B_T^L(E \text{ or } M)$ where L is multipole rank and $T = 0$ for isoscalar (IS) and $T = 1$ for isovector (IV) transitions. In the data set there are 211 $E1$ IS, 172 $E1$ IV, 358 $M1$ IV and 132 $E2$ IS transition matrix elements. Instead of the locally renormalized strengths x, distribution of $z = \log(x)$ is plotted by combining all the data with a proper prescription. The P-T form gives maximum at $z = 0$ while data shows the peak at ≈ -0.5. It is conjectured that this deviation is a consequence of isospin breaking. The random matrix model now consists of the 2×2 partitioned GOE for the Hamiltonian as given by Eq. (3.54) and in the same basis an independent 2×2 partitioned GOE for the transition operator $\mathscr{O}$ [43],

$$\mathscr{O} = \beta_{\text{IS}} \begin{bmatrix} \mathscr{O}(0) & 0 \\ 0 & \mathscr{O}(1) \end{bmatrix} + \beta_{\text{IV}} \begin{bmatrix} 0 & \mathscr{O}_c \\ \widetilde{\mathscr{O}_c} & 0 \end{bmatrix}. \tag{3.55}$$

Here, $\beta_{\text{IS}} = 1$ and $\beta_{\text{IV}} = 0$ for IS and $\beta_{\text{IS}} = 0$ and $\beta_{\text{IV}} = 1$ for IV transitions. Determining appropriately the scale parameters of the various GOE's in the H and $\mathscr{O}$ ensembles, Barbosa et al. [43] recently constructed $P(z)\,dz$ via numerical calculations by transforming the ensemble in Eq. (3.55) into $\{H\}$ basis via the unitary matrices that diagonalize H's. The random matrix model correctly predicts the shift in the peak with respect to P-T. However the data are more strongly peaked and at present there is no quantitative understanding of this feature.

3.4 Rosenzweig-Porter Model: Analysis of Atomic Levels and Nuclear 2^+ and 4^+ Levels

Rosenzweig-Porter model [44] is the appropriate random matrix model when we consider a set of levels in a spectrum $\mathscr{S}$ and the levels in $\mathscr{S}$ differ containing conserved quantum numbers which are either unknown or ignored. Then, the spectrum can be broken into r sub-spectra $\mathscr{S}_j$ of independent sequences of levels with $j = 1, 2, \ldots, r$. In the set of levels considered for the analysis of $P(S)$ (i.e. NNSD), say the fraction of levels from $\mathscr{S}_j$ is f_j. Then, $0 < f_j \leq 1$ and $\sum_{j=1}^{r} f_j = 1$. Now an appropriate random matrix model is to represent each subspace $\mathscr{S}_j$ by independent GOE's of dimension $d_j = df_j$ where d is the size of $\mathscr{S}$. NNSD for such an ensemble was first considered by Rosenzweig-Porter (RP) [44] and they showed that, with $f_j = 1/r; r \to \infty$, $P(S)$ goes to Poisson. This model has been employed in discussing LS to JJ coupling change in atomic spectra. Exact solutions for the RP model are given very recently [45, 46] but they are not useful in data analysis. Abul-Magd derived a simplified formula for $P(S)$ in terms of the chaoticity parameter $f = \sum_{j=1}^{r} f_j^2$ and it is [47],

$$P(S)dS = \left[1 - f + Q(f)\pi S/2\right] \exp\left[-(1-f)S - Q(f)\pi S^2/4\right] \tag{3.56}$$

where $Q(f) = f(0.7 + 0.3f)$. With $f = 1/r$, $r \to \infty$, $P(S)$ goes to Poisson and $f = 1$ gives GOE. Abul-Magd, Harney, Simbel and Weidenmüller [48, 49] analyzed the NNSD of low-lying 2^+ levels (up to ~4 MeV excitation) for Poisson to GOE transition using Eq. (3.56). They considered 1306 levels belonging to 169 nuclei (with a minimum of 5 consecutive 2^+ levels in a given nucleus). The nuclei are grouped into classes defined by the collectivity parameter $E(4_1^+)/E(2_1^+)$. In the system considered, departures from GOE arise due to the neglect of possibly good quantum numbers. Using Bayesian inference method, values of f are deduced and it is found to be small for nuclei with IBM symmetries while for the intermediate nuclei $f \sim 0.6$.

Equation (3.56) was also applied in the analysis NNSD for 2^+ levels of prolate and oblate deformed nuclei by Al-Sayed and Abul-Magd [50]. They considered 30 nuclei of oblate deformation having 246 levels and 83 nuclides of prolate deformation having 590 energy levels ranging from ^{28}Si to ^{228}Ra. Analysis showed that the chaoticity parameter f is ~0.73 for prolate nuclei and ~0.59 for oblate nuclei suggesting that oblate nuclei are more regular compared to prolate nuclei.

It may be useful to note that a formula for the number variance $\Sigma^2(r)$ for the RP model was given in [1] and this is not yet used in any data analysis. All the nuclear data that is analyzed so far using RMT is shown in Fig. 3.2 in the angular momentum and energy plane.

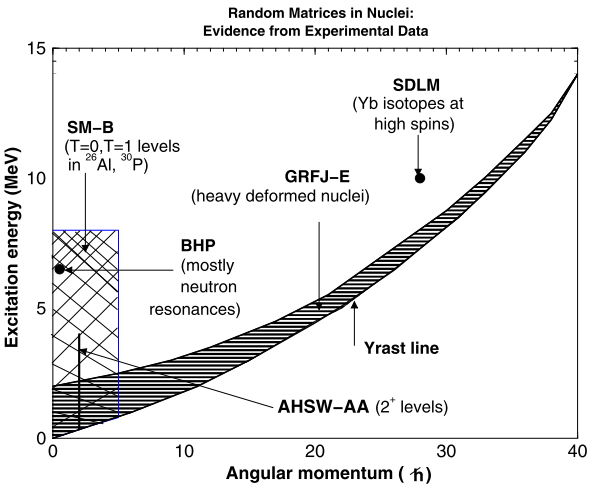

Fig. 3.2 Schematic diagram giving the regions, in the excitation energy vs angular momentum plane for nuclei, where data was analyzed for evidence for random matrices (GOE and its extensions). Details of BHP [51], SM-B [39, 40, 43], AHSW-AA [48–50], GRFJ-E [52, 53], and SDLM [36, 37] are given in the text (Color figure online)

3.5 Covariance Random Matrix Ensemble XX^T: Eigenvalue Density

Let us consider a $N \times M$ matrix X with matrix elements real and chosen to be independent $G(0, v^2)$ variables. Then the $N \times N$ random matrix ensemble $C = XX^T$, where X^T is the transpose of X, represents a GOE related covariance random matrix ensemble (GOE-CRME). It is possible to consider many other types of CRME's as discussed for example in [54–56]. The CRME's have wide ranging applications. For example: (i) they are important in multivariate statistical analysis [57]; (ii) they are used in the study of cross-correlations in financial data [58–60]; (iii) they appear in a model for mixing between distant configurations in nuclear shell model [61]; (iv) they are relevant for statistical analysis of correlations in atmospheric data [62]; (v) they determine statistical bounds on entanglement in bipartite quantum systems due to quantum chaos [63].

In this section we consider the ensemble averaged density $\rho^C(E)$ of the eigenvalues of GOE-CRME $C = XX^T$. Dyson [64, 65] first derived the result for $\rho^C(E)$ for the matrices X with $N = M$. In most applications the eigenvalue density for $N \neq M$ is needed. The result for this situation was derived using many different techniques; see [54, 55, 57, 66] and references therein. Equation (3.73) ahead gives the final result. It is indeed possible to obtain $\rho^C(E)$ using the 2×2 partitioned GOE (p-GOE:2) employed in nuclear structure studies as a statistical model for mixing between distant nuclear shell model configurations [61, 67]. This gives an easy to understand derivations of the final result [68].

3.5.1 A Simple 2 × 2 Partitioned GOE: p-GOE:2(Δ)

Let us consider two spaces #1 and #2 with dimensions d_1 and d_2 respectively . For a simple statistical model for the mixing between the spaces #1 and #2, one can assume, as a first step, that all the eigenvalues in #1 are degenerate and say their value is 0. Similarly one may assume that the eigenvalues in #2 are also degenerate with their value say Δ. More importantly, these two spaces will mix and the mixing Hamiltonian X will be a $d_1 \times d_2$ matrix. A plausible model for X is to replace it by a GOE, i.e. assume that the matrix elements of X are independent $G(0, v^2)$ variables. Then we have a 2×2 block structured random matrix ensemble,

$$H_\Delta = \begin{bmatrix} 0I_1 & X \\ X^T & \Delta\, I_2 \end{bmatrix}. \tag{3.57}$$

This ensemble is called p-GOE:2(Δ). Note that the matrices I_1 and I_2 are unit matrices with dimensions d_1 and d_2 respectively and the H matrix dimension is $d = d_1 + d_2$. Now let us consider the eigenvalue density $\rho^\Delta(E)$ for the matrix H_Δ. The $\rho^\Delta(E)$ is simply,

$$\rho^\Delta(E) = \left\langle \delta(H_\Delta - E) \right\rangle^{1+2}, \tag{3.58}$$

and its decomposition into sum of the partial densities $\rho^{\Delta;1}$ and $\rho^{\Delta;2}$ defined over the spaces #1 and #2 respectively is given by,

$$\rho^{\Delta;i}(E) = \left\langle \delta(H_\Delta - E) \right\rangle^i; \qquad \rho^\Delta(E) = d^{-1}\left[d_1\rho^{\Delta;1}(E) + d_2\rho^{\Delta;2}(E)\right]. \tag{3.59}$$

As we will see ahead, the densities $\rho^{\Delta;1}(E)$ and $\rho^{\Delta;2}(E)$ differ only in a delta function. Therefore from now on we will consider only $\rho^{\Delta;1}(E)$ and also assume that $d_1 < d_2$. For mathematical simplicity, as an intermediate step, we will consider the matrix ensemble $H_{\pm\Delta'}$,

$$H_{\pm\Delta'} = \begin{bmatrix} -\Delta' I_1 & X \\ X^T & \Delta' I_2 \end{bmatrix} \tag{3.60}$$

and the corresponding $\rho^{\pm\Delta';1}(E)$. Denoting the p-th moment of this density by $M_p^{\pm\Delta';1}$, we have, with $[\frac{p}{2}]$ being the integer part of $\frac{p}{2}$,

$$M_p^{\pm\Delta';1} = (-1)^p \sum_{\nu=0}^{[\frac{p}{2}]} \binom{[\frac{p}{2}]}{\nu} (\Delta')^{p-2\nu} \left\langle \left(XX^T\right)^\nu \right\rangle^1. \tag{3.61}$$

Equation (3.61) is derived by multiplying $H_{\pm\Delta'}$ p-times and then using the first diagonal block of the resulting 2×2 block matrix. Similarly $M_{2\nu}^{\pm\Delta';2} = (d_1/d_2) M_{2\nu}^{\pm\Delta';1}$ and $M_{2\nu+1}^{\pm\Delta';2} = -(d_1/d_2) M_{2\nu+1}^{\pm\Delta';1}$ with $M_0^{\pm\Delta';1} = M_0^{\pm\Delta';2} = 1$. Equation (3.61) shows that there should be a generalized convolution form for

$\rho^{\pm\Delta';1}(E)$ with one of the factors being $\rho^{\Delta'=0;1}$ as the moments for a density written as a convolution of two functions follow the law $M_p(\rho_A \otimes \rho_B) = \sum \binom{p}{s} M_s(A) M_{p-s}(B)$; here $\otimes$ denotes convolution. From Eq. (3.61) we have $M_{2\nu+1}^{\pm\Delta';1} = -\Delta' M_{2\nu}^{\pm\Delta';1}$. Then $\int_{-\infty}^{\infty} E^{2\nu}(E+\Delta')\rho^{\pm\Delta';1}(E)dE = 0$ and also $\int_{-\infty}^{\infty} E^{2\nu}(E-\Delta')\rho^{\pm\Delta';1}(-E)dE = 0$. They imply that $\rho^{\pm\Delta';1}(E)$ is of the form $|\frac{E-\Delta'}{E+\Delta'}|^{1/2} f(E)$ where $f(E)$ is an even function of E. This and the fact that $\langle (XX^T)^\nu \rangle^1$ is the 2ν-th moment of $\rho^{0:1}(E)$, allow us to identify the following important result,

$$\rho^{\pm\Delta';1}(E) = \left|\frac{E-\Delta'}{E+\Delta'}\right|^{\frac{1}{2}} \rho^{\Delta'=0;1}\left(\sqrt{E^2-(\Delta')^2}\right), \quad |E| \geq \Delta'. \tag{3.62}$$

Now, putting $\Delta' = \frac{\Delta}{2}$ and shifting all the eigenvalues E by $\Delta/2$ so that $E \to (E - \frac{\Delta}{2})$, the final result for $\rho^{\Delta;1}$ is obtained,

$$\rho^{\Delta;1}(E) = \left|\frac{E-\Delta}{E}\right|^{\frac{1}{2}} \rho^{\Delta=0;1}\left(\sqrt{E(E-\Delta)}\right), \quad E \geq \Delta,\ E \leq 0. \tag{3.63}$$

Equation (3.63) was reported first in [61]. Now we will consider $\rho^{\Delta=0;1}(E)$ for p-GOE:2($\Delta = 0$).

3.5.2 Moments and the Eigenvalue Density for p-GOE:2($\Delta = 0$)

Given $H_0 = \left[\begin{smallmatrix} 0I_1 & X \\ X^T & 0I_2 \end{smallmatrix}\right]$, mathematical induction gives,

$$(H_0)^{2\nu} = \begin{bmatrix} (XX^T)^\nu & 0 \\ 0 & (X^TX)^\nu \end{bmatrix}, \qquad (H_0)^{2\nu+1} = \begin{bmatrix} 0 & (XX^T)^\nu X \\ (X^TX)^\nu X^T & 0 \end{bmatrix}. \tag{3.64}$$

Then, immediately we have $\langle\langle (H_0)^p \rangle\rangle^1 = \langle\langle (H_0)^p \rangle\rangle^2$ for $p \neq 0$ and for $p = 0$, they are d_1 and d_2 respectively. Secondly, all the odd moments of $\rho^{\Delta=0;1}(E)$ are zero. Also, for $d_1 < d_2$, $\rho^{\Delta=0;2}(E) = \frac{d_1}{d_2}\rho^{\Delta=0;1}(E) + (1 - \frac{d_1}{d_2})\delta(E)$. One way of constructing $\rho^{\Delta=0;1}(E)$ is via its moments. From Eq. (3.64) we have $M_{2\nu}(\rho^{\Delta=0;1}) = \langle (XX^T)^\nu \rangle^1$ and they can be evaluated for p-GOE:2 using BCA discussed earlier. Firstly, the ensemble averaged second moment simply is,

$$M_2\left(\rho^{\Delta=0;1}\right) = (d_1)^{-1} \sum_{i,j} \overline{X_{ij}\left(X^T\right)_{ji}} = (d_1)^{-1} \sum_{i,j} \overline{X_{ij}^2} = v^2 d_2. \tag{3.65}$$

Similarly, defining $\widetilde{M}_p = d_1 M_p = \langle\langle (H_0)^p \rangle\rangle^1$, we have

$$\widetilde{M}_4\left(\rho^{\Delta=0;1}\right) = \sum_{i,j,k,l} \overline{X_{ij}\left(X^T\right)_{jk} X_{kl}\left(X^T\right)_{li}} = \sum_{i,j,k,l} \overline{X_{ij}X_{kj}X_{kl}X_{il}}.$$

In the sum here, applying BCA, we need to consider only terms that contain pairwise correlations. Then, with $k = i$ or $l = j$,

$$\begin{aligned}\widetilde{M}_4\big(\rho^{\Delta=0;1}\big) &= \sum_{i,j,l}[\overline{X_{ij}X_{ij}X_{il}X_{il}}] + \sum_{i,j,k}[\overline{X_{ij}X_{kj}X_{kj}X_{ij}}] \\ &= \sum_{i,j,l}[\overline{X_{ij}X_{ij}}\,\overline{X_{il}X_{il}}] + \sum_{i,j,k}[\overline{X_{ij}X_{ij}}\,\overline{X_{kj}X_{kj}}] \\ &= v^4\big[d_1d_2^2 + d_1^2d_2\big]. \end{aligned} \tag{3.66}$$

The two terms in Eq. (3.66) can be written as $\langle \underbrace{X}\underbrace{X^T}\,\underbrace{X}\underbrace{X^T}\rangle$ and $\langle \underbrace{X\,\underbrace{X^T\,\underbrace{X}\underbrace{X^T}}}\rangle$. The terms that are dropped in Eq. (3.66) involve cross correlations, i.e. terms with odd number of matrix elements in between those that are correlated. They will be smaller by a factor of d_2 (or d_1). Thus BCA here is good if d_1 and d_2 both are large. Proceeding further we have for $\widetilde{M}_6$,

$$\begin{aligned}\widetilde{M}_6\big(\rho^{\Delta=0;1}\big) &= \sum_{i,j,k,l,m,n} \overline{X_{ij}X_{kj}X_{kl}X_{ml}X_{mn}X_{in}} \\ &= \sum_{i,j,l,n} \overline{X_{ij}X_{ij}X_{il}X_{il}X_{in}X_{in}} + \sum_{i,j,l,m} \overline{X_{ij}X_{ij}X_{il}X_{ml}X_{ml}X_{il}} \\ &\quad + \sum_{i,j,k,n} \overline{X_{ij}X_{kj}X_{kj}X_{ij}X_{in}X_{in}} + \sum_{i,j,k,m} \overline{X_{ij}X_{kj}X_{kj}X_{mj}X_{mj}X_{ij}} \\ &\quad + \sum_{i,j,k,l} \overline{X_{ij}X_{kj}X_{kl}X_{kl}X_{kj}X_{ij}} \\ &= v^6\big(d_1d_2^3 + 3d_1^2d_2^2 + d_1^3d_2\big). \end{aligned} \tag{3.67}$$

The binary correlation structure in Eq. (3.67) is clear and let us apply it to $\widetilde{M}_8$. Writing X_{ij} as X_a, symbolically $\widetilde{M}_8 = \sum \overline{X_aX_bX_cX_dX_eX_fX_gX_h}$. Now: (i) with X_a and X_b correlated, the correlations in the remaining $X_cX_dX_eX_fX_gX_h$ are same as those in $\widetilde{M}_6$; (ii) with X_a and X_d correlated, necessarily X_b and X_c must be correlated and the remaining $X_eX_fX_gX_h$ correlations are same as those in $\widetilde{M}_4$; (iii) with X_a and X_f correlated, necessarily X_g and X_h must be correlated and the remaining $X_bX_cX_dX_e$ correlations are same as those in $\widetilde{M}_4$; (iv) with X_a and X_h correlated, the correlations in the remaining $X_bX_cX_dX_eX_fX_g$ are same as those in $\widetilde{M}_6$. Then the expression for $\widetilde{M}_8$ is,

$$\widetilde{M}_8\big(\rho^{\Delta=0;1}\big) = v^8\big[d_1^4d_2 + 6d_1^3d_2^2 + 6d_1^2d_2^3 + d_1d_2^4\big]. \tag{3.68}$$

Continuing this will lead to a recursion formula for the moments,

$$\widetilde{M}_{2\nu}\big(\rho^{\Delta=0;1}\big) = v^2 \sum_{r=0,2,\ldots,2\nu-2} \widetilde{M}_{2\nu-2-r}\big(\rho^{\Delta=0;1}\big)\widetilde{M}_r\big(\rho^{\Delta=0;2}\big); \quad \nu \geq 1,$$
$$\widetilde{M}_{2\nu}\big(\rho^{\Delta=0;1}\big) = \widetilde{M}_{2\nu}\big(\rho^{\Delta=0;2}\big) \quad \text{for } \nu \neq 0, \qquad \widetilde{M}_0\big(\rho^{\Delta=0;1}\big) = d_1, \tag{3.69}$$
$$\widetilde{M}_0\big(\rho^{\Delta=0;2}\big) = d_2.$$

For example using Eq. (3.69) we have $\widetilde{M}_{10} = v^{10}[d_1^5 d_2 + 10d_1^4 d_2^2 + 20d_1^3 d_2^3 + 10d_1^2 d_2^4 + d_1 d_2^5]$. With all the moments determined, it is possible to identify the density $\rho^{\Delta=0;1}$. Integral tables in [69], the expression for M_4 given by (3.66) and $\widetilde{M}_{2\nu}$, $\nu = 1, 2, 3, 4$ for $d_1 = d_2$ allow us to write the final solution,

$$\rho^{\Delta=0;1}(E)dE = \frac{1}{2\pi v^2 d_1}\frac{\sqrt{(R_+^2 - E^2)(E^2 - R_-^2)}}{|E|}\,dE, \quad R_- \leq |E| \leq R_+;$$
$$R_\pm = v(\sqrt{d_2} \pm \sqrt{d_1}). \tag{3.70}$$

Note that $\rho^{\Delta=0;1}(E) = 0$ for $|E| < R_-$ or $|E| > R_+$ and also it is a semicircle for $d_1 = d_2$. The reduced moments $\widetilde{M}_{2\nu} = M_{2\nu}/(M_2)^\nu$ of $\rho^{\Delta=0;1}(E)$ are,

$$\widetilde{M}_{2\nu}\big(\rho^{\Delta=0;1}\big) = \frac{(1+\sqrt{R_0})^{2\nu+2}}{\pi R_0}\int_{\overline{R_0}}^1 x^{2\nu-1}\sqrt{\big(1-x^2\big)\big(x^2 - \overline{R_0}^2\big)}\,dx;$$
$$R_0 = \frac{d_1}{d_2}, \ \overline{R_0} = \frac{1-\sqrt{R_0}}{1+\sqrt{R_0}}. \tag{3.71}$$

3.5.3 Eigenvalue Density for GOE-CRME

Our primary interest is to determine the eigenvalue density $\rho^C(E)$ for the GOE-CRME $C = XX^T$ where X is a $d_1 \times d_2$ matrix with its matrix elements being independent $G(0, v^2)$ variables; we assume $d_1 \leq d_2$. From Eq. (3.64) it is seen easily that the ν-th moment of ρ^C and the 2ν-th moment of $\rho^{\Delta=0;1}$ are simply related, $M_\nu(\rho^C) = M_{2\nu}(\rho^{\Delta=0;1})$. As $\rho^{\Delta=0;1}$ is an even function, we have

$$M_{2\nu}\big(\rho^{\Delta=0;1}\big) = 2\int_0^\infty E^{2\nu}\rho^{\Delta=0;1}(E)\,dE = \int_0^\infty y^\nu\left[\frac{\rho^{\Delta=0;1}(y^{1/2})}{y^{1/2}}\right]dy$$
$$\Longrightarrow \quad \rho^C(y) = \frac{\rho^{\Delta=0;1}(y^{1/2})}{y^{1/2}}. \tag{3.72}$$

Now, the formula for ρ^C follows simply from Eq. (3.70),

$$\rho^C(\lambda)\,d\lambda = \frac{1}{2\pi v^2 d_1}\frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{\lambda}\,d\lambda, \quad \lambda_- \leq \lambda \leq \lambda_+;$$

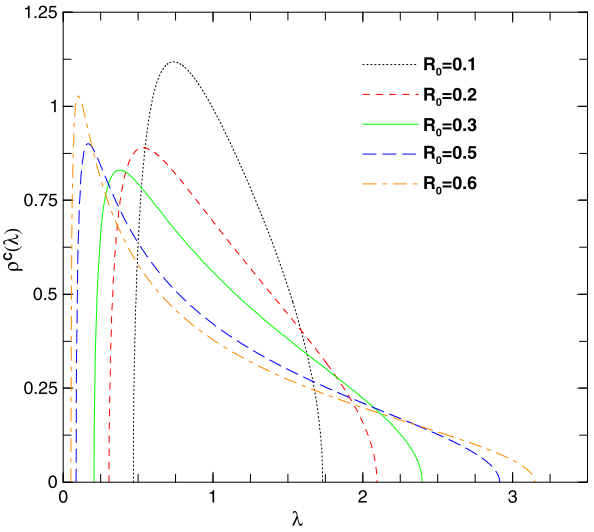

Fig. 3.3 Eigenvalue density for GOE-CRME ensemble for different values of $R_0 = d_1/d_2$. Equation (3.74) gives the formula for the eigenvalue density

$$\lambda_\pm = v^2(\sqrt{d_2} \pm \sqrt{d_1})^2 = \left(v^2 d_1 d_2\right)\frac{1}{d_1}\left[1 + R_0 \pm 2\sqrt{R_0}\right]. \tag{3.73}$$

With the normalization $v^2 d_2 = 1$, we have

$$\rho^C(\lambda)\, d\lambda = \frac{1}{2\pi R_0}\frac{\sqrt{(\lambda_+ - \lambda)(\lambda - \lambda_-)}}{\lambda}\, d\lambda, \quad \lambda_- \le \lambda \le \lambda_+;$$
$$\lambda_\pm = [1 + R_0 \pm 2\sqrt{R_0}], \;\; R_0 = \frac{d_1}{d_2}, \;\; d_1 < d_2. \tag{3.74}$$

The final solution given by Eq. (3.74) is same as the result reported for example in [59] with $Q = 1/R_0$ and $\sigma^2 = 1$. Thus, $\rho^C(E)$ follows from p-GOE:2(Δ) and it is simple to deal with this ensemble. Figure 3.3 gives a plot of $\rho^C(\lambda)$ for various values of R_0 and used here is Eq. (3.74). Before going further, some comments on generalization of p-GOE will be useful.

Given $\rho(x)$, its Stieltjes transform $f(z)$ is

$$f(z) = \int_{-\infty}^{+\infty} \frac{\rho(x)}{z - x}\, dx \tag{3.75}$$

where z is a complex variable. Since $-\pi\delta(x) = \Im\langle\frac{1}{x+i0}\rangle$, we have

$$\rho(x) = -\frac{1}{\pi}\Im\left\{\left[\lim_{\varepsilon\to 0} f(x + i\varepsilon)\right]\right\}. \tag{3.76}$$

Given a general 2×2 block matrix $\left[\begin{smallmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{smallmatrix}\right]$ that is real symmetric with dimensions for the diagonal blocks being d_1 and d_2 respectively, following Eqs. (3.58)

and (3.59), we have $d\rho(x) = \sum_{i=1}^{2} d_i \rho^i(x)$. Let us denote the Stieltjes transforms of ρ, ρ^1 and ρ^2 by f, f_1 and f_2 respectively. We assume that the matrix elements of H_{ij} are independent Gaussian variables with variances v_{ij}^2. Moreover, we can assume that all matrix elements are zero centered except that the diagonal matrix elements of H_{22} have centroid Δ. Then, using the moments recursion, one can prove that [4]

$$\begin{aligned} df_1 &= \frac{1}{z - v_{11}^2 d_1 f_1 - v_{12}^2 d_2 f_2} \\ df_2 &= \frac{1}{z - \Delta - v_{22}^2 d_2 f_2 - v_{12}^2 d_1 f_1}. \end{aligned} \tag{3.77}$$

Solving these equations for f_1 with $v_{11} = v_{22} = 0$ and $\Delta = 0$ and applying Eq. (3.76) will give $\rho^{\Delta=0;1}(x)$. This will be an alternative derivation of Eq. (3.70) given earlier. However, Eq. (3.77) allows one to solve the most general 2×2 block matrix problem with $v_{11} \neq 0$, $v_{22} \neq 0$ and $\Delta \neq 0$, i.e. most general p-GOE:2 random matrix problem. Deriving an analytical form for ρ^1 (similarly for ρ^2) for the general p-GOE:2 is of considerable interest in nuclear physics [67]. Its further generalization to p-GOE:N was analyzed in [70] and the partial densities ρ^i are reduced to multiple integrals involving commuting and anticommuting variables.

3.6 Further Extensions and Applications of RMT

Here below will give a list of various extensions and applications of RMT. This list is only partial as the subject of "Random Matrices: Theory and Applications" is too vast to be covered in completeness at one place.

1. There are many new class of random matrix ensembles that are not covered in this book and some of them are: (i) β ensembles and more general random matrix ensembles related to orthogonal polynomials [71, 72]; (ii) critical random matrix ensembles [73, 74]; (iii) ensembles with non-extensive q entropy [75–77]; (iv) ensemble with super statistics [78]; (v) special constrained Gaussian ensembles [79]; (vi) Cyclic random matrix ensembles [80]; (vii) Hussein and Pato's deformed ensembles based on maximum entropy principle [81–84]; (viii) Transition ensemble for harmonic oscillators to GUE transition [85]; (ix) New versions and new applications of circular ensembles [86–90]. (x) Non-Hermitian random matrix ensembles; see [91–98] and references therein. (x) Random density matrices for entanglement related studies [99, 100].
2. There is a nice relationship between ensembles of 2×2 Hermitian matrices and Gaussian point process [101] and similarly between Poisson point process and 2×2 complex non-Hermitian random matrices [102]. Construction and applications of many other 2×2 random matrix ensembles are discussed in [76, 103, 104]. For example, introduced in [103] are 2×2 pseudo-Hermitian random matrix ensembles and in [76] introduced are ensembles based on Tsallis entropy. Thus, 2×2 ensembles have much wider relevance. As an additional

example, briefly discussed in Appendix E are 2×2 matrix results and their extensions for open quantum systems [105–112].

3. Going beyond 2×2 ensembles, recently 3×3 random matrix ensembles (first discussion on 3×3 random matrix ensembles was given in [113]) are found to be useful in deriving some new results. Using 3×3 GE, derived in [114] is the probability distribution for the ratio of consecutive level spacings (see Chap. 16). This distribution and its relatives are suggested [115–117] to be useful in understanding localization in interacting many particle systems.
4. RMT for missing levels and incomplete spectra has been discussed for example in [65, 118–120] and this is of considerable interest in data analysis and for predictions of missing levels.
5. Using the analogy between energy levels and time series, methods of time series analysis are applied to RMT spectra showing for example $1/f^2$ noise for Poisson systems and $1/f$ noise for GOE/GUE/GSE. There are several investigations in this direction as given for example in [118, 121–126].
6. There are applications of RMT for biological networks [127], neural networks [128], small world networks [129], terrace-width distributions on vicinal surfaces of vicinal crystals [130], finding words in literary texts [131] and so on.
7. There is extensive literature on results for the rate of convergence of probability distributions in RMT and on asymptotic properties of a variety of random matrices; see for example [54, 96, 132–135].
8. Extreme statistics in RMT is another important topics that is not discussed in this Section. An example is the probability distribution for the largest or the lowest eigenvalue in GOE. Tracy-Widom distribution is the starting point for all these investigations. See [134–144] and references therein.
9. New classification of random matrices, extending Dyson's 3-fold way to 10 classes, based on group theory is given in [145–147]. These new classes have applications in condensed matter physics. They also include chiral ensembles for QCD related applications [148–152].
10. Random matrix theory for random phase approximation (RPA), a widely used quantum many-body approximate method, has been introduced in [153].
11. Random matrix theory for scattering and Ericson fluctuations is an important topic that is not discussed in this section. Good references for these are [154–158].
12. Numerical methods and algorithms for constructing and analyzing random matrices of large dimensions are available for example in [159, 160].

References

1. J.B. French, V.K.B. Kota, A. Pandey, S. Tomsovic, Statistical properties of many-particle spectra V. Fluctuations and symmetries. Ann. Phys. (N.Y.) **181**, 198–234 (1988)
2. D.M. Leitner, Real symmetric random matrix ensembles of Hamiltonians with partial symmetry breaking. Phys. Rev. E **48**, 2536–2546 (1993)
3. A. Pandey, Brownian-motion model of discrete spectra. Chaos Solitons Fractals **5**, 1275–1285 (1995)

4. A. Pandey, Statistical properties of many-particle spectra IV. New ensembles by Stieltjes transform methods. Ann. Phys. (N.Y.) **134**, 110–127 (1981)
5. T. Guhr, Transitions toward quantum chaos: with supersymmetry from Poisson to Gauss. Ann. Phys. (N.Y.) **250**, 145–192 (1996)
6. K.M. Frahm, T. Guhr, A. Muller-Groeling, Between Poisson and GUE statistics: role of the Breit-Wigner width. Ann. Phys. (N.Y.) **270**, 292–327 (1998)
7. H. Kunz, B. Shapiro, Transition from Poisson to Gaussian unitary statistics: the two-point correlation function. Phys. Rev. E **58**, 400–406 (1998)
8. J.B. French, V.K.B. Kota, Statistical spectroscopy. Annu. Rev. Nucl. Part. Sci. **32**, 35–64 (1982)
9. G. Lenz, F. Haake, Reliability of small matrices for large spectra with nonuniversal fluctuations. Phys. Rev. Lett. **67**, 1–4 (1991)
10. G. Lenz, K. Zyczkowski, D. Saher, Scaling laws of the additive random-matrix model. Phys. Rev. A **44**, 8043–8050 (1991)
11. E. Caurirer, B. Grammaticos, A. Ramani, Level repulsion near integrability: a random matrix analogy. J. Phys. A **23**, 4903–4910 (1990)
12. V.K.B. Kota, S. Sumedha, Transition curves for the variance of the nearest neighbor spacing distribution for Poisson to Gaussian orthogonal and unitary ensemble transitions. Phys. Rev. E **60**, 3405–3408 (1999)
13. T. Guhr, H.A. Weidenmüller, Coexistence of collectivity and chaos in nuclei. Ann. Phys. (N.Y.) **193**, 472–489 (1989)
14. S. Schierenberg, F. Bruckmann, T. Wettig, Wigner surmise for mixed symmetry classes in random matrix theory. Phys. Rev. E **85**, 061130 (2012)
15. M. Abramowtiz, I.A. Stegun (eds.), *Handbook of Mathematical Functions*, NBS Applied Mathematics Series, vol. 55 (U.S. Govt. Printing Office, Washington, D.C., 1972)
16. I.S. Gradshteyn, I.M. Ryzhik, *Tables of Integrals, Series and Products*, 5th edn. (Academic Press, New York, 1980)
17. A. Erdelyi, W. Magnus, F. Oberhettinger, F.G. Triconi (eds.), *Higher Transcendental Functions—The Bateman Manuscript Project*, vol. I (McGraw-Hill, New York, 1953)
18. P.B. Khan, C.E. Porter, Statistical fluctuations of energy levels: the unitary ensemble. Nucl. Phys. **48**, 385–407 (1963)
19. L.D. Favro, J.F. McDonald, Possibility of detecting a small time-reversal-noninvariant term in the Hamiltonian of a complex system by measurements of energy-level spacings. Phys. Rev. Lett. **19**, 1254–1256 (1967)
20. N. Dupuis, G. Montambaux, Aharonov-Bohm flux and statistics of energy levels in metals. Phys. Rev. B **43**, 14390–14395 (1991)
21. A. Pandey, M.L. Mehta, Gaussian ensembles of random Hermitian matrices intermediate between orthogonal and unitary ones. Commun. Math. Phys. **87**, 449–468 (1983)
22. E.P. Wigner, Random matrices in physics. SIAM Rev. **9**, 1–23 (1967)
23. J.B. French, V.K.B. Kota, A. Pandey, S. Tomsovic, Statistical properties of many-particle spectra VI. Fluctuation bounds on N-N T-noninvariance. Ann. Phys. (N.Y.) **181**, 235–260 (1988)
24. M.J. Morrison, A. Derevianko, Proposed search for T-odd, P-even interactions in spectra of chaotic atoms (2012). arXiv:1206.3607 [physics.atom.ph]
25. S. Tomsovic, Bounds on the time-reversal non-invariant nucleon-nucleon interaction derived from transition-strength fluctuations, Ph.D. Thesis, University of Rochester, Rochester, New York (1986)
26. T.A. Brody, A statistical measure for the repulsion of energy levels. Lett. Nuovo Cimento **7**, 482–484 (1973)
27. M.V. Berry, M. Robnik, Semiclassical level spacings when regular and chaotic orbits coexist. J. Phys. A **17**, 2413–2422 (1984)
28. M.V. Berry, M. Robnik, Statistics of energy levels without time-reversal symmetry: Aharonov-Bohm chaotic billiards. J. Phys. A **19**, 649–668 (1986)

29. F.M. Izrailev, Intermediate statistics of the quasi-energy spectrum and quantum localisation of classical chaos. J. Phys. A **22**, 865–878 (1989)
30. G. Casati, F.M. Izrailev, L. Molinari, Scaling properties of the eigenvalue spacing distribution for band random matrices. J. Phys. A **24**, 4755–4762 (1991)
31. J.P. Blocki, A.G. Magner, Chaoticity and shell effects in the nearest-neighbor distributions for an axially-symmetric potential. Phys. Rev. C **85**, 064311 (2012)
32. J. Sakhr, J.M. Nieminen, Poisson-to-Wigner crossover transition in the nearest-neighbor statistics of random points on fractals. Phys. Rev. E **72**, 045204(R) (2005)
33. T. Rudolf, N. Mertig, S. Lock, A. Backer, Consequences of flooding on spectral statistics. Phys. Rev. E **85**, 036213 (2012)
34. M. Vyas, V.K.B. Kota, N.D. Chavda, Transitions in eigenvalue and wavefunction structure in $(1 + 2)$-body random matrix ensembles with spin. Phys. Rev. E **81**, 036212 (2010)
35. N.D. Chavda, V. Potbhare, V.K.B. Kota, Statistical properties of dense interacting Boson systems with one plus two-body random matrix ensembles. Phys. Lett. A **311**, 331–339 (2003)
36. F.S. Stephens et al., Order-to-chaos transition in rotational nuclei. Phys. Rev. Lett. **94**, 042501 (2005)
37. F.S. Stephens et al., Damping, motional narrowing, and chaos in rotational nuclei. Phys. Rev. C **78**, 034303 (2008)
38. H. Alt, C.I. Barbosa, H.D. Gräf, T. Guhr, H.L. Harney, R. Hofferbert, H. Rehfeld, A. Richter, Coupled microwave billiards as a model for symmetry breaking. Phys. Rev. Lett. **81**, 4847–4850 (1998)
39. G.E. Mitchell, E.G. Bilpuch, P.M. Endt, J.F. Shriner Jr., Broken symmetries and chaotic behavior in ^{26}Al. Phys. Rev. Lett. **61**, 1473–1476 (1988)
40. J.F. Shriner Jr., C.A. Grossmann, G.E. Mitchell, Level statistics and transition distributions of ^{30}P. Phys. Rev. C **62**, 054305 (2000)
41. T. Ghur, H.A. Weidenmüller, Isospin mixing and spectral fluctuation properties. Ann. Phys. (N.Y.) **199**, 412–446 (1990)
42. A.A. Adams, G.E. Mitchell, J.F. Shriner Jr., Statistical distribution of reduced transition probabilities in ^{26}Al. Phys. Lett. B **422**, 13–18 (1998)
43. C.I. Barbosa, T. Guhr, H.L. Harney, Impact of isospin on the distribution of transition probabilities. Phys. Rev. E **62**, 1936–1949 (2000)
44. N. Rosenzweig, C.E. Porter, Repulsion of energy levels in complex atomic spectra. Phys. Rev. **120**, 1698–1714 (1960)
45. N. Datta, H. Kunz, A random matrix approach to the crossover of energy-level statistics from Wigner to Poisson. J. Math. Phys. **45**, 870–886 (2004)
46. M.L. Mehta, *Random Matrices*, 3rd edn. (Elsevier, Amsterdam, 2004)
47. A. Abd El-Hady, A.Y. Abul-Magd, M.H. Simbel, Influence of symmetry breaking on the fluctuation properties of spectra. J. Phys. A **35**, 2361–2372 (2002)
48. A.Y. Abul-Magd, H.L. Harney, M.H. Simbel, H.A. Weidenmüller, Statistics of 2^+ levels in even-even nuclei. Phys. Lett. B **579**, 278–284 (2004)
49. A.Y. Abul-Magd, H.L. Harney, M.H. Simbel, H.A. Weidenmüller, Statistical analysis of composite spectra. Ann. Phys. (N.Y.) **321**, 560–580 (2006)
50. A. Al-Sayed, A.Y. Abul-Magd, Level statistics of deformed even-even nuclei. Phys. Rev. C **74**, 037301 (2006)
51. O. Bohigas, R.U. Haq, A. Pandey, Fluctuation properties of nuclear energy levels and widths: comparison of theory and experiment, in *Nuclear Data for Science and Technology*, ed. by K.H. Böckhoff (Reidel, Dordrecht, 1983), pp. 809–813
52. J.D. Garrett, J.Q. Robinson, A.J. Foglia, H.-Q. Jin, Nuclear level repulsion and order vs. chaos. Phys. Lett. B **392**, 24–29 (1997)
53. J. Enders, T. Guhr, N. Huxel, P. von Neumann-Cosel, C. Rangacharyulu, A. Richter, Level spacing distribution of scissors mode states in heavy deformed nuclei. Phys. Lett. B **486**, 273–278 (2000)
54. Z. Bai, J.W. Silverstein, *Spectral Analysis of Large Dimensional Random Matrices*, 2nd edn. (Springer, New York, 2010)

55. L. Pastur, M. Shcherbina, *Eigenvalue Distribution of Large Random Matrices*, Mathematical Surveys and Monographs, vol. 171 (Am. Math. Soc., Providence, 2011)
56. A.M. Tulino, S. Verdú, *Random Matrix Theory and Wireless Communication* (Now Publishers, Hanover, 2004)
57. I.M. Johnstone, On the distribution of the largest eigenvalue in principle components analysis. Ann. Stat. **29**, 295–327 (2001)
58. H.E. Stanley, L.A.N. Amaral, D. Canning, P. Gopikrishnan, Y. Lee, Y. Liu, Econophysics: can physics contribute to the science of economics? Physica A **269**, 156–169 (1999)
59. L. Laloux, P. Cizeau, J.P. Bouchaud, M. Potters, Noise dressing of financial correlation matrices. Phys. Rev. Lett. **83**, 1467–1470 (1999)
60. V. Plerou, P. Gopikrishnan, B. Rosenow, L.A.N. Amaral, T. Guhr, H.E. Stanley, Random matrix approach to cross correlations in financial data. Phys. Rev. E **65**, 066126 (2006)
61. J.B. French, in *Mathematical and Computational Methods in Nuclear Physics*, ed. by J.S. Dehesa, J.M.G. Gomez, A. Polls (Springer, Berlin, 1984), pp. 100–121
62. M.S. Santhanam, P.K. Patrta, Statistics of atmospheric correlations. Phys. Rev. E **64**, 016102 (2001)
63. J.N. Bandyopadhyay, A. Lakshminarayan, Testing statistical bounds on entanglement using quantum chaos. Phys. Rev. Lett. **89**, 060402 (2002)
64. F.J. Dyson, Distribution of eigenvalues for a class of real symmetric matrices. Rev. Mex. Fis. **20**, 231–237 (1971)
65. T.A. Brody, J. Flores, J.B. French, P.A. Mello, A. Pandey, S.S.M. Wong, Random matrix physics: spectrum and strength fluctuations. Rev. Mod. Phys. **53**, 385–479 (1981)
66. A.M. Sengupta, P.P. Mitra, Distributions of singular values for some random matrices. Phys. Rev. E **60**, 3389–3392 (1999)
67. V.K.B. Kota, D. Majumdar, R. Haq, R.J. Leclair, Shell model tests of the bimodal partial state densities in a 2×2 partitioned embedded random matrix ensemble. Can. J. Phys. **77**, 893–901 (1999)
68. V.K.B. Kota, Eigenvalue density for random covariance matrices from a 2×2 partitioned GOE. Adv. Stud. Theor. Phys. **2**(17), 845–854 (2008)
69. D. Bierens de Haan, *Nouvelles Tables: D'Intégrales Définies* (Hafner Publishing Company, New York, 1957), p. 111
70. Z. Pluhař, H.A. Weidenmüller, Approximation for shell-model level densities. Phys. Rev. C **38**, 1046–1057 (1988)
71. T. Nagao, P.J. Forrester, Transitive ensembles of random matrices related to orthogonal polynomials. Nucl. Phys. B **530**, 742–762 (1998)
72. I. Dumitriu, A. Edelman, Matrix models for beta ensembles. J. Math. Phys. **43**, 5830–5847 (2002)
73. K.A. Muttalib, M.E.H. Ismail, Power-law eigenvalue density, scaling and critical random-matrix ensembles. Phys. Rev. E **76**, 051105 (2007)
74. A. Ossipov, I. Rushkin, E. Cuevas, Level-number variance and spectral compressibility in a critical two-dimensional random matrix model. Phys. Rev. E **85**, 021127 (2012)
75. A.C. Bertuola, O. Bohigas, M.P. Pato, Family of generalized random matrix ensembles. Phys. Rev. E **70**, 065102(R) (2004)
76. A.Y. Abul-Magd, Non-extensive random matrix theory—a bridge connecting chaotic and regular dynamics. Phys. Lett. A **333**, 16–22 (2004)
77. A.Y. Abul-Magd, Nonextensive random matrix theory approach to mixed regular-chaotic dynamics. Phys. Rev. E **71**, 066207 (2005)
78. A.Y. Abul-Magd, Random matrix theory within superstatistics. Phys. Rev. E **72**, 066114 (2005)
79. T. Papenbrock, Z. Pluhař, H.A. Weidenmüller, Level repulsion in constrained Gaussian random-matrix ensemble. J. Phys. A **39**, 9709–9726 (2006)
80. S.R. Jain, Random cyclic matrices. Phys. Rev. E **78**, 036213 (2008)
81. M.S. Hussein, M.P. Pato, Description of chaos-order transition with random matrices within the maximum entropy principle. Phys. Rev. Lett. **70**, 1089–1092 (1993)

82. M.S. Hussein, M.P. Pato, Deformed Gaussian orthogonal ensemble description of isospin mixing and spectral fluctuation properties. Phys. Rev. C **47**, 2401–2403 (1993)
83. M.S. Hussein, M.P. Pato, Critical behavior in disordered quantum systems modified by broken time-reversal symmetry. Phys. Rev. Lett. **80**, 1003–1006 (1998)
84. J.X. de Carvalho, M.S. Hussein, M.P. Pato, A.J. Sargeant, Symmetry-breaking study with deformed ensembles. Phys. Rev. E **76**, 066212 (2007)
85. T. Guhr, T. Papenbrock, Spectral correlations in the crossover transition from a superposition of harmonic oscillators to the Gaussian unitary ensemble. Phys. Rev. E **59**, 330–336 (1999)
86. A. Pandey, P. Shukla, Eigenvalue correlations in the circular ensembles. J. Phys. A **24**, 3907–3926 (1991)
87. A. Pandey, S. Ghosh, Skew-orthogonal polynomials and universality of energy-level correlations. Phys. Rev. Lett. **87**, 024102 (2001)
88. S. Ghosh, A. Pandey, Skew-orthogonal polynomials and random-matrix ensembles. Phys. Rev. E **65**, 046221 (2002)
89. A. Pandey, S. Puri, S. Kumar, Long-range correlations in quantum-chaotic spectra. Phys. Rev. E **71**, 066210 (2005)
90. S. Kumar, A. Pandey, Nonuniform circular ensembles. Phys. Rev. E **78**, 026204 (2008)
91. J. Ginibre, Statistical ensembles of complex, quaternion, and real matrices. J. Math. Phys. **6**, 440–449 (1965)
92. H. Markum, R. Pullirsch, T. Wettig, Non-Hermitian random matrix theory and lattics QCD with chemical potential. Phys. Rev. Lett. **83**, 484–487 (1999)
93. P. Shukla, Non-Hermitian random matrices and the Calogero-Sutherland model. Phys. Rev. Lett. **87**, 194102 (2001)
94. P.J. Forrester, T. Nagao, Eigenvalues of the real Ginibre ensemble. Phys. Rev. Lett. **99**, 050603 (2007)
95. F. Haake, *Quantum Signatures of Chaos*, 3rd edn. (Springer, Heidelberg, 2010)
96. P.J. Forrester, *Log-Gases and Random Matrices* (Princeton University Press, Princeton, 2010)
97. M. Kieburg, J.J.M. Verbaarschot, S. Zafeiropoulos, Eigenvalue density of the non-Hermitian Wilson Dirac operator. Phys. Rev. Lett. **108**, 022001 (2012)
98. N. Auerbach, V. Zelevinsky, Super-radiant dynamics, doorways and resonances in nuclei and other open mesoscopic systems. Rep. Prog. Phys. **74**, 106301 (2011)
99. H. Sommers, K. Zyczkowski, Statistical properties of random density matrices. J. Phys. A, Math. Gen. **37**, 8457–8466 (2004)
100. K. Zyczkowski, K.A. Penson, I. Nechita, B. Collins, Generating random density matrices. J. Math. Phys. **52**, 062201 (2011)
101. J.M. Nieminen, Gaussian point processes and two-by-two random matrix theory. Phys. Rev. E **76**, 047202 (2007)
102. J. Sakhr, J.M. Nieminen, Wigner surmises and the two-dimensional homogeneous Poisson point process. Phys. Rev. E **73**, 047202 (2006)
103. Z. Ahmed, S.R. Jain, Gaussian ensemble of 2×2 pseudo-Hermitian random matrices. J. Phys. A, Math. Gen. **36**, 3349–3362 (2003)
104. S. Grossmann, M. Robnik, On level spacing distributions for 2D non-normal Gaussian random matrices. J. Phys. A, Math. Gen. **40**, 409–421 (2007)
105. C. Poli, G.A. Luna-Acosta, H.-J. Stöckmann, Nearest level spacing statistics in open chaotic systems: generalization of the Wigner surmise. Phys. Rev. Lett. **108**, 174101 (2012)
106. H.-J. Stöckmann, P. Šeba, The joint energy distribution function for the Hamiltonian $H = H_0 - iWW^\dagger$ for the one-channel case. J. Phys. A, Math. Gen. **31**, 3439–3449 (1998)
107. V.V. Sokolov, V.G. Zelevinsky, Dynamics and statistics of unstable quantum states. Nucl. Phys. A **504**, 562–588 (1989)
108. G. Shchedrin, V. Zelevinsky, Resonance width distribution for open quantum systems. Phys. Rev. C **86**, 044602 (2012)
109. P.E. Koehler, F. Becvar, M. Krticka, J.A. Harvey, K.H. Guber, Anomalous fluctuations of s-wave reduced neutron widths of $^{192;194}$Pt resonances. Phys. Rev. Lett. **105**, 075502 (2010)

110. E.S. Reich, Nuclear theory nudged. Nature (London) **466**, 1034 (2010)
111. G.L. Celardo, N. Auerbach, F.M. Izrailev, V.G. Zelevinsky, Distribution of resonance widths and dynamics of continuum coupling. Phys. Rev. Lett. **106**, 042501 (2011)
112. H.A. Weidenmüller, Distribution of partial neutron widths for nuclei close to a maximum of the neutron strength function. Phys. Rev. Lett. **105**, 232501 (2010)
113. C.E. Porter, *Statistical Theories of Spectra: Fluctuations* (Academic Press, New York, 1965)
114. Y.Y. Atas, E. Bogomolny, O. Giraud, G. Roux, Distribution of the ratio of consecutive level spacings in random matrix ensembles. Phys. Rev. Lett. **110**, 084101 (2013)
115. V. Oganesyan, D.A. Huse, Localization of interacting fermions at high temperature. Phys. Rev. B **75**, 155111 (2007)
116. A. Pal, D.A. Huse, Many-body localization phase transition. Phys. Rev. B **82**, 174411 (2010)
117. S. Iyer, V. Oganesyan, G. Refael, D.A. Huse, Many-body localization in a quasiperiodic system. Phys. Rev. B **87**, 134202 (2013)
118. J.M.G. Gómez, K. Kar, V.K.B. Kota, R.A. Molina, A. Relaño, J. Retamosa, Many-body quantum chaos: recent developments and applications to nuclei. Phys. Rep. **499**, 103–226 (2011)
119. O. Bohigas, M.P. Pato, Missing levels in correlated spectra. Phys. Lett. B **595**, 171–176 (2004)
120. O. Bohigas, M.P. Pato, Randomly incomplete spectra and intermediate statistics. Phys. Rev. C **74**, 036212 (2006)
121. A. Relaño, J.M.G. Gómez, R.A. Molina, J. Retamosa, E. Faleiro, Quantum chaos and $1/f$ noise. Phys. Rev. Lett. **89**, 244102 (2002)
122. E. Faleiro, J.M.G. Gómez, R.A. Molina, L. Muñoz, A. Relaño, J. Retamosa, Theoretical derivation of $1/f$ noise in quantum chaos. Phys. Rev. Lett. **93**, 244101 (2004)
123. J.M.G. Gómez, A. Relaño, J. Retamosa, E. Faleiro, L. Salasnich, M. Vranicar, M. Robnik, $1/f^{\alpha}$ noise in spectral fluctuations of quantum systems. Phys. Rev. Lett. **94**, 084101 (2005)
124. R.J. Leclair, R.U. Haq, V.K.B. Kota, N.D. Chavda, Power spectrum analysis of the average-fluctuation density separation in interacting particle systems. Phys. Lett. A **372**, 4373–4378 (2008)
125. A. Al-Sayed, Autocorrelation studies for the first 2^{+} nuclear energy levels. Phys. Rev. C **85**, 037302 (2012)
126. K. Roy, B. Chakrabarti, A. Biswas, V.K.B. Kota, S.K. Haldar, Spectral fluctuation and $\frac{1}{f^{\alpha}}$ noise in the energy level statistics of interacting trapped bosons. Phys. Rev. E **85**, 061119 (2012)
127. F. Lup, J. Zhong, Y. Yang, R.H. Scheermann, J. Zhou, Application of random matrix theory to biological networks. Phys. Lett. A **357**, 420–423 (2006)
128. K. Rajan, L.F. Abbott, Eigenvalue spectra of random matrices for neural networks. Phys. Rev. Lett. **97**, 188104 (2006)
129. J.X. de Carvalho, S. Jalan, M.S. Hussein, Deformed Gaussian-orthogonal-ensemble description of small-world networks. Phys. Rev. E **79**, 056222 (2009)
130. A. Pimpinelli, H. Gebremariam, T.L. Einstein, Evolution of terrace-width distributions on vicinal surfaces: Fokker-Planck derivation of the generalized Wigner surmise. Phys. Rev. Lett. **95**, 246101 (2005)
131. P. Carpena, P. Bernaola-Galván, M. Hackenberg, A.V. Coronado, J.L. Oliver, Level statistics of words: finding keywords in literary texts and symbolic sequences. Phys. Rev. E **79**, 035102(R) (2009)
132. A. Khorunzhy, Sparse random matrices and statistics of rooted trees. Adv. Appl. Probab. **33**, 124–140 (2001)
133. O. Khorunzhiy, Rooted trees and moments of large sparse random matrices. Discrete Math. Theor. Comput. Sci. **AC**, 145–154 (2003)
134. N. El Karoui, A rate of convergence result for the largest eigenvalue of complex white Wishart matrices. Ann. Probab. **34**, 2077–2117 (2006)
135. S. Sodin, The Tracy-Widom law for some sparse random matrices. J. Stat. Phys. **136**, 834–841 (2009)

136. C.A. Tracy, H. Widom, Level-spacing distributions and the Airy kernel. Phys. Lett. B **305**, 115–118 (1993)
137. C.A. Tracy, H. Widom, Level-spacing distributions and the Airy kernel. Commun. Math. Phys. **159**, 151–174 (1994)
138. C.A. Tracy, H. Widom, On orthogonal and symplectic matrix ensembles. Commun. Math. Phys. **177**, 727–754 (1996)
139. D.S. Dean, S.N. Majumdar, Large deviations of extreme eigenvalues of random matrices. Phys. Rev. Lett. **97**, 160201 (2006)
140. S.N. Majumdar, M. Vergassola, Large deviations of the maximum eigenvalue of Wishart and Gaussian random matrices. Phys. Rev. Lett. **102**, 060601 (2009)
141. A. Lakshminarayan, S. Tomsovic, O. Bohigas, S.N. Majumdar, Extreme statistics of complex random and quantum chaotic states. Phys. Rev. Lett. **100**, 044103 (2008)
142. S.N. Majumdar, P. Vivo, Number of relevant directions in principal component analysis and Wishart random matrices. Phys. Lev. Lett. **108**, 200601 (2012)
143. M. Fridman, R. Pugatch, M. Nixon, A.A. Friesem, N. Davidson, Measuring maximal eigenvalue distribution of Wishart random matrices with coupled lasers. Phys. Rev. E **85**, 020101(R) (2012)
144. O. Bohigas, J.X. de Carvalho, M.P. Pato, Deformations of the Tracy-Widom distribution. Phys. Rev. E **79**, 031117 (2009)
145. M.R. Zirnbauer, Riemannian symmetric superspaces and their origin in random-matrix theory. J. Math. Phys. **37**, 4986–5018 (1996)
146. D.A. Ivanov, Random-matrix ensembles in p-wave vortices (2001). arXiv:cond-mat/0103089
147. P. Heinzner, A. Huckleberry, M.R. Zirnbauer, Symmetry classes of disordered fermions. Commun. Math. Phys. **257**, 725–771 (2005)
148. J.J.M. Verbaarschot, Spectrum of the QCD Dirac operator and chiral random matrix theory. Phys. Rev. Lett. **72**, 2531–2533 (1994)
149. R.G. Edwards, U.M. Heller, J. Kiskis, R. Narayanan, Quark spectra topology, and random matrix theory. Phys. Rev. Lett. **82**, 4188–4191 (1999)
150. T. Guhr, J.-Z. Ma, S. Meyer, T. Wilke, Statistical analysis and the equivalent of a Thouless energy in lattice QCD Dirac spectra. Phys. Rev. D **59**, 054501 (1999)
151. J.J.M. Verbaarschot, T. Wettig, Random matrix theory and chiral symmetry in QCD. Annu. Rev. Nucl. Part. Sci. **50**, 343–410 (2000)
152. G. Akemann, J. Bloch, L. Shifrin, T. Wettig, Individual complex Dirac eigenvalue distributions from random matrix theory and comparison to quanched lattice QCD with a quark chemical potential. Phys. Rev. Lett. **100**, 032002 (2008)
153. X. Barillier-Pertuisel, O. Bohigas, H.A. Weidenmüller, Random-metric approach to RPA equations. I. Ann. Phys. (N.Y.) **324**, 1855–1874 (2009)
154. T.E.O. Ericson, Fluctuations of nuclear cross sections in the continuum region. Phys. Rev. Lett. **5**, 430–431 (1960)
155. T.E.O. Ericson, A theory of fluctuations in nuclear cross sections. Ann. Phys. (N.Y.) **23**, 390–414 (1963)
156. T.E.O. Ericson, T. Mayer-Kuckuk, Fluctuations in nuclear reactions. Annu. Rev. Nucl. Sci. **16**, 183–206 (1966)
157. T.E.O. Ericson, Structure of amplitude correlations in open quantum systems. Phys. Rev. E **87**, 022907 (2013)
158. G.E. Mitchell, A. Richter, H.A. Weidenmüller, Random matrices and chaos in nuclear physics: nuclear reactions. Rev. Mod. Phys. **82**, 2845–2901 (2010)
159. A. Edelman, N. Raj Rao, Random matrix theory. Acta Numer. **14**, 233–297 (2005)
160. F. Mezzardi, How to generate random matrices from the classical compact groups. Not. Am. Math. Soc. **54**(5), 592–604 (2007)