Chapter 11
Embedded Gaussian Unitary Ensembles: Results from Wigner-Racah Algebra

A long standing question for the embedded ensembles is about their analytical tractability. Amenability to mathematical treatment is one of the four conditions laid down by Dyson [1] for the validity of a random matrix ensemble. To address this issue, in this chapter we will consider embedded unitary ensembles. It is important to recall that out of the three classical ensembles, GUE is mathematically much easier. Simplest embedded unitary ensemble is the embedded Gaussian unitary ensemble of two-body interactions [EGUE(2)] for spinless fermion systems. For m fermions in N sp states, the embedding is generated by the $SU(N)$ algebra. Although EE are known for many years, only recently [2], after the first indications implicit in [3, 4], it is established that the $SU(N)$ Wigner-Racah algebra solves EGUE(2) and also the more general EGUE(k) [as well as EGOE(k)]. These results, with $U(N)$ algebra, extend to BEGUE(k) for spinless bosons in N sp states (see Sects. 11.2 and 11.3 and [5]). For EGUE(2)-**s** for fermions with spin and EGUE(2)-$SU(4)$ for fermions with Wigner's spin-isospin $SU(4)$ symmetry, the embedding algebras, with Ω number of spatial degrees of freedom for a single fermion, are $U(\Omega) \otimes SU(2)$ and $U(\Omega) \otimes SU(4)$ respectively [6, 7]. Similarly, the embedding algebras for BEGUE(2)-F for two-species boson systems with F-spin and BEGUE(2)-$SU(3)$ for spin one bosons are $U(\Omega) \otimes SU(2)$ and $U(\Omega) \otimes SU(3)$ respectively [8, 9]. Again, the Wigner-Racah algebra of these algebras solve the corresponding embedded unitary ensembles. As discussed in Sect. 11.3, all these ensembles can be unified into EGUE(2)-[$U(\Omega) \otimes SU(r)$]. All these results, discussed in some detail in the next seven sections, obtained after more than 30 years of the introduction of embedded ensembles, conclusively establish that two-body random matrix ensembles are amenable to mathematical treatment and thus satisfy Dyson's criterion. Here, Wigner-Racah algebra of the embedding Lie algebras plays the central role.

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11.1 Embedded Gaussian Unitary Ensemble for Spinless Fermions with *k*-Body Interactions: EGUE(*k*)

In this section we deal with EGUE(k), i.e. fermions with a general k-body Hamiltonian although for nuclei, atoms and mesoscopic systems $k=2$ is most important. For a system of m spinless fermions in N sp states, one has the unitary groups $SU(N)$, $U(N_k)$ and $U(N_m)$, $N_r = \binom{N}{r}$, with EGUE(k) invariant under $U(N_k)$ and the embedding in m-particle spaces is defined by $SU(N)$; note that a GUE in m particle spaces is invariant under $U(N_m)$ but not the EGUE(k), $k<m$. Analytical results for EGUE(k) follow from the tensorial decomposition of H with respect to $SU(N)$ and the $SU(N)$ Wigner-Racah algebra; in the end Wigner coefficients disappear as expected [note that the Wigner coefficients involve the sub-algebras of $SU(N)$] and all the expressions for the moments involve only $SU(N)$ Racah coefficients. Firstly, sp creation operator $a_i^\dagger$ for any i-th sp state transforms as the irrep $\{1\}$ of $U(N)$ and similarly a product of r creation operators transform, as we have fermions, as the irrep $\{1^r\}$ in Young tableaux notation. Let us add that a $U(N)$ irrep $\{\lambda_1,\lambda_2,\ldots,\lambda_N\}$ defines the corresponding $SU(N)$ irrep as $\{\lambda_1-\lambda_N,\lambda_2-\lambda_N,\ldots,\lambda_{N-1}-\lambda_N\}$ with $N-1$ rows. The $U(\Omega)\leftrightarrow SU(\Omega)$ correspondence is used throughout and therefore we use $U(\Omega)$ and $SU(\Omega)$ interchangeably. A normalized r-particle creation operator $A^\dagger(f_r\alpha_r)$ behaves as the $SU(N)$ irrep (tensor) $\{1^r\}$. Similarly a r-particle annihilation operator behaves as $\overline{\{1^r\}}=\{1^{N-r}\}$. Tensorial multiplication gives, $\{1^r\}\otimes\overline{\{1^r\}}\to\sum g_\nu\oplus=\sum\{2^\nu 1^{N-2\nu}\}\oplus$, $\nu=0,1,\ldots,r$. Note that $g_0=\{0\}$ for $SU(N)$ and $g_\nu=\overline{g_\nu}$. Also, the ν here is same as the tensorial rank ν used in Chaps. 5 and 6. $SU(N)$ irreducible tensors $B_k(g_\nu\omega_\nu)$ are defined by,

$$B_k(g_\nu\omega_\nu)=\sum_{\alpha_k,\alpha'_k}A^\dagger\big(\{1^k\}\alpha_k\big)A\big(\{1^k\}\alpha'_k\big)\big\langle\{1^k\}\alpha_k\overline{\{1^k\}}\overline{\alpha'_k}\big|g_\nu\omega_\nu\big\rangle, \tag{11.1}$$

where $\langle --\mid --\rangle$'s are $SU(N)$ Wigner coefficients and α's are the other labels for completely specifying the k particle states [they can be specified by any subgroup chain contained in $SU(N)$]. An important property of $B_k(g_\nu\omega_\nu)$ is that they are orthogonal with respect to the traces over k particle spaces. Given a k-body Hamiltonian

$$H(k)=\sum_{v_a,v_b}V_{v_av_b}(k)A^\dagger\big(\{1^k\}v_a\big)A\big(\{1^k\}v_b\big), \tag{11.2}$$

where $V_{v_av_b}(k)$ are matrix elements of $H(k)$ in k-particle space, the $V(k)$ matrix is chosen to be GUE, i.e. $V_{v_av_b}(k)$ are independent Gaussian variables with zero center and variance given by (with bar denoting ensemble average),

$$\overline{V_{v_av_b}(k)V_{v_cv_d}(k)}=\big(\lambda^2/N_k\big)\delta_{v_av_d}\delta_{v_bv_c}. \tag{11.3}$$

Action of $H(k)$ on a given complete set of m-particle basis states will generate EGUE(k) in m-particle spaces. The m-particle matrix elements of $H(k)$ are, with

$s = m - k$,

$$\begin{aligned} &H_{v_m^1 v_m^2}(k) \\ &\quad = \langle \{1^m\} v_m^1 | H(k) | \{1^m\} v_m^2 \rangle \\ &\quad = \binom{m}{k} \sum_{v_a, v_b, v_s} \langle \{1^k\} v_a \{1^s\} v_s | \{1^m\} v_m^1 \rangle^* \langle \{1^k\} v_b \{1^s\} v_s | \{1^m\} v_m^2 \rangle V_{v_a v_b}(k). \end{aligned} \tag{11.4}$$

Unitary decomposition of $H(k)$ in terms of the $SU(N)$ tensors $B_k(g_\nu \omega_\nu)$ is,

$$H(k) = \sum_{g_\nu, \omega_\nu} W_{g_\nu \omega_\nu}(k) B_k(g_\nu \omega_\nu) \tag{11.5}$$

and the W's will be independent Gaussian variables with

$$\overline{W_{g_\nu \omega_\nu}(k) W_{g_\mu \omega_\mu}(k)} = \frac{\lambda^2}{N_k} \delta_{g_\nu g_\mu} \delta_{\omega_\nu \omega_\mu}. \tag{11.6}$$

Using Eqs. (11.1)–(11.5) and the sum-rules for $SU(N)$ Wigner coefficients, the result given by Eq. (11.6) can be proved.

Correlations generated by EGUE(k) in m particle spaces follow from the matrix A of the second moments, i.e.

$$A_{\alpha_m^1 \alpha_m^4; \alpha_m^3 \alpha_m^2} = \overline{\langle \{1^m\} \alpha_m^1 | H(k) | \{1^m\} \alpha_m^2 \rangle \langle \{1^m\} \alpha_m^3 | H(k) | \{1^m\} \alpha_m^4 \rangle}. \tag{11.7}$$

First substituting the $H(k)$ in terms of B_k's as given by Eq. (11.5), then using the Wigner-Eckart theorem for $SU(N)$ and finally applying Eq. (11.6) for carrying out the ensemble average will give

$$\begin{aligned} &\overline{\langle \{1^m\} \alpha_m^1 | H(k) | \{1^m\} \alpha_m^2 \rangle \langle \{1^m\} \alpha_m^3 | H(k) | \{1^m\} \alpha_m^4 \rangle} \\ &\quad = \frac{\lambda^2}{N_k} \sum_{g_\nu \omega_\nu, \nu = 0,1,\ldots,k} \left| \langle \{1^m\} || B_k(g_\nu) || \{1^m\} \rangle \right|^2 \\ &\qquad \times \langle \{1^m\} \alpha_m^1 \overline{\{1^m\} \alpha_m^2} | g_\nu \omega_\nu \rangle \langle \{1^m\} \alpha_m^3 \overline{\{1^m\} \alpha_m^4} | g_\nu \omega_\nu \rangle; \\ &\left| \langle \{1^m\} || B_k(g_\nu) || \{1^m\} \rangle \right|^2 \\ &\quad = \frac{\binom{N}{m}^2 \binom{m}{k}^2}{d(g_\nu) \binom{N}{m-k}} \left[U(\{1^m\}\{1^{N-k}\}\{1^m\}\{1^k\}; \{1^{m-k}\}\{2^\nu 1^{N-2\nu}\}) \right]^2 \\ &\quad = \Lambda^\nu(N, m, m-k), \\ &\Lambda^\nu(N, m, r) = \binom{m-\nu}{r} \binom{N-m+r-\nu}{r}. \end{aligned} \tag{11.8}$$

In Eq. (11.8), $U(- - -)$ are $SU(N)$ Racah coefficients, $\langle -- \,||\, -- \,||\, -- \rangle$ are $SU(N)$ reduced matrix elements and $d(g_\nu) = d(\nu) = \binom{N}{\nu}^2 - \binom{N}{\nu-1}^2$. In the final step used is the formula given in [10] for $SU(N)$ U-coefficients. An alternative expression for the covariance in Eq. (11.7) follows from the Biedenharn-Elliott sum rule for $SU(N)$ [2, 11, 12],

$$\begin{aligned}
&\overline{\langle\{1^m\}\alpha_m^1|H(k)|\{1^m\}\alpha_m^2\rangle\langle\{1^m\}\alpha_m^3|H(k)|\{1^m\}\alpha_m^4\rangle} \\
&\quad= \sum_{g_\mu\omega_\mu,\mu=0,1,\ldots,m-k} \frac{\lambda^2}{N_k}\Lambda^\mu(N,m,k) \\
&\qquad\times \langle\{1^m\}\alpha_m^1\overline{\{1^m\}\alpha_m^4}|g_\mu\omega_\mu\rangle\langle\{1^m\}\alpha_m^3\overline{\{1^m\}\alpha_m^2}|g_\mu\omega_\mu\rangle. \qquad (11.9)
\end{aligned}$$

To derive Eq. (11.9), the two $SU(N)$ Wigner coefficients in Eq. (11.8) are first transformed into the two Wigner coefficients appearing in Eq. (11.9) multiplied by a $SU(N)$ Racah coefficient by a Racah transform. This new Racah coefficient multiplied by the two Racah coefficients in Eq. (11.8) is then reduced to the square of a Racah coefficient using Biedenharn-Elliott sum rule. Then the final Racah coefficient [see Eq. (11.10) below] is simplified using the formulas in [10]. Equation (11.9) gives the eigenvalue decomposition of the matrix of second moments with the first part in the sum giving eigenvalues E_μ and the product of the two Wigner coefficients giving eigenvectors. The eigenvalues E_μ are given by,

$$E_\mu = \frac{\lambda^2}{N_k}\Lambda^\mu(N,m,k) = \frac{\lambda^2}{N_k}\frac{(N_m)^2\binom{m}{k}^2}{d(g_\mu)(N_k)}\big[U(f_m f_{N-m+k} f_m f_{m-k};\, f_k g_\mu)\big]^2. \qquad (11.10)$$

Equations (11.8) and (11.9) lead to remarkably simple expressions for the variance and the excess parameter for the eigenvalue density. Obviously, ensemble averaged centroid is zero and the variance is

$$\overline{\langle H^2\rangle^m} = \frac{1}{N_m}\sum_{v_m^i,v_m^j}\overline{H_{v_m^i v_m^j}H_{v_m^j v_m^i}} = \frac{\lambda^2}{N_k}\Lambda^0(N,m,k). \qquad (11.11)$$

This result follows easily from (11.9) and the sum rule $\sum_{v_m^i}\langle\{1^m\}v_m^i\overline{\{1^m\}v_m^i} \mid g_\mu\omega_\mu\rangle = \sqrt{N_m}\,\delta_{\mu,0}$. Now the fourth moment, dropping λ^2/N_k factor, is

$$\begin{aligned}
&\overline{\langle H^4\rangle^m} \\
&\quad= \frac{1}{N_m}\sum_{v_m^i,v_m^j,v_m^{k'},v_m^l}\overline{H_{v_m^i v_m^j}H_{v_m^j v_m^{k'}}H_{v_m^{k'} v_m^l}H_{v_m^l v_m^i}} \\
&\quad= \frac{1}{N_m}\sum_{v_m^i,v_m^j,v_m^{k'},v_m^l}\Bigg\{2\Bigg[\sum_{g_\nu,\omega_\nu}\langle f_m v_m^i|B_k(g_\nu\omega_\nu)|f_m v_m^j\rangle\langle f_m v_m^j|B_k(g_\nu\omega_\nu)|f_m v_m^{k'}\rangle\Bigg] \\
&\qquad\times\Bigg[\sum_{g_\mu,\omega_\mu}\langle f_m v_m^{k'}|B_k(g_\mu\omega_\mu)|f_m v_m^l\rangle\langle f_m v_m^l|B_k(g_\mu\omega_\mu)|f_m v_m^i\rangle\Bigg]
\end{aligned}$$

$$+\left[\sum_{g_\nu,\omega_\nu}\left\langle f_m v_m^i\middle|B_k(g_\nu\omega_\nu)\middle|f_m v_m^j\right\rangle\left\langle f_m v_m^{k'}\middle|B_k(g_\nu\omega_\nu)\middle|f_m v_m^l\right\rangle\right]$$

$$\times\left[\sum_{g_\mu,\omega_\mu}\left\langle f_m v_m^j\middle|B_k(g_\mu\omega_\mu)\middle|f_m v_m^{k'}\right\rangle\left\langle f_m v_m^l\middle|B_k(g_\mu\omega_\mu)\middle|f_m v_m^i\right\rangle\right]\Bigg\}. \tag{11.12}$$

Here we have used Eqs. (11.5) and (11.6) and applied Wigner Eckart theorem. Now, formula for the excess parameter follows easily by using both Eqs. (11.8) and (11.9) together with the orthonormal properties of $SU(N)$ Wigner coefficients. The final formula is [2],

$$\begin{aligned}\gamma_2(N,m,k) &= \frac{\overline{\langle H^4\rangle^m}}{[\overline{\langle H^2\rangle^m}]^2}-3\\ &= \left[(N_m)^{-1}\sum_{\nu=0}^{min\{k,m-k\}}\frac{\Lambda^\nu(N,m,m-k)\Lambda^\nu(N,m,k)d(g_\nu)}{[\Lambda^0(N,m,k)]^2}\right]-1.\end{aligned} \tag{11.13}$$

In the dilute limit Eq. (11.13) reduces to the binary correlation result given by Eq. (4.32). Thus EGUE(k) generates Gaussian densities. For a complete proof, higher order cumulants should be studied. In principle, the formalism given above applies to k_6 but the exact formula is not yet derived. At this stage it is useful to remark that for EGOE(k),

$$\overline{V_{v_a v_b}(k)V_{v_c v_d}(k)} = \left(\lambda^2/N_k\right)\{\delta_{v_a v_d}\delta_{v_b v_c}+\delta_{v_a v_c}\delta_{v_b v_d}\}, \tag{11.14}$$

and in the dilute limit EGUE(k) result for γ_2 reduces to that of EGOE(k); see [3] for details.

Going beyond the lower order moments of the state density, it is also possible to derive formulas for the lower order moments

$$\Sigma_{rr}\left(m,m'\right)=\overline{\left\langle H^r\right\rangle^m\left\langle H^r\right\rangle^{m'}}-\overline{\left\langle H^r\right\rangle^m}\,\overline{\left\langle H^r\right\rangle^{m'}} \tag{11.15}$$

with $r=1$ and 2, of the two-point correlation function,

$$S^{m,m'}\left(E,E'\right)=\overline{\rho^m(E)\rho^{m'}\left(E'\right)}-\overline{\rho^m(E)}\,\overline{\rho^{m'}\left(E'\right)}. \tag{11.16}$$

The final formulas are [13],

$$\hat{\Sigma}_{11}\left(m,m'\right)=\frac{\Sigma_{11}(m,m')}{\sqrt{\overline{\langle H^2\rangle^m}\,\overline{\langle H^2\rangle^{m'}}}}=\sqrt{\frac{\Lambda^0(N,m,m-k)}{N_m\Lambda^0(N,m,k)}\frac{\Lambda^0(N,m',m'-k)}{N_{m'}\Lambda^0(N,m',k)}}, \tag{11.17}$$

and

$$\hat{\Sigma}_{22}(m, m') = \frac{\Sigma_{22}(m, m')}{\overline{\langle H^2 \rangle^m} \; \overline{\langle H^2 \rangle^{m'}}}$$
$$= \frac{2}{N_m N_{m'}} \sum_{\nu=0}^{k} \frac{\Lambda^\nu(N, m, m-k)\Lambda^\nu(N, m', m'-k)}{\Lambda^0(N, m, k)\Lambda^0(N, m', k)} d(\nu). \quad (11.18)$$

The result for $\overline{\langle H \rangle^m \langle H \rangle^{m'}}$ and hence for $\hat{\Sigma}_{11}$, follows easily from the simple trace formula $\langle H(k) \rangle^m = \binom{m}{k} \langle H(k) \rangle^k$ or alternatively by applying Eq. (11.8) and using the fact that only $\nu = 0$ terms will contribute to $\langle H \rangle^m$. Similarly, Σ_{22} formula has been derived using

$$\overline{\langle H^2 \rangle^m \langle H^2 \rangle^{m'}} = [N_m N_{m'}]^{-1} \sum_{a,b,c,d} \overline{\left|H_{a,b}(m)\right|^2 \left|H_{c,d}(m')\right|^2}$$
$$= \overline{\langle H^2 \rangle^m} \, \overline{\langle H^2 \rangle^{m'}} + 2[N_m N_{m'}]^{-1} \sum_{a,b,c,d} \left\{\overline{H_{a,b}(m) H_{c,d}(m')}\right\}^2 \quad (11.19)$$

where $H_{a,b}(m) = \langle m, a|H|m, b\rangle$ is a m-particle matrix element. Note that we have used $\overline{x^2 y^2} = \overline{x^2}\;\overline{y^2} + 2(\overline{xy})^2$. Applying Eq. (11.8) to the second term in the second equality and using orthonormal properties of $SU(N)$ Wigner coefficients will give finally the formula for $\hat{\Sigma}_{22}(m, m')$. The formulas for $\hat{\Sigma}_{rr}(m, m)$, $r = 1, 2$ were derived first in [2, 3]. It is important to remind that Σ_{rr} is the (rr)-th bivariate moment of the two point function. Before turning to EGUE/EGOE with spin degree of freedom, it is important to mention that in the standard applications of GUE/GOE, correlations between levels with different m will be zero [i.e. $\hat{\Sigma}_{11}(m, m') = 0$ and $\hat{\Sigma}_{22}(m, m') = 0$] as independent GUE/GOE description for levels with different m has to be used. Therefore results given by Eqs. (11.17)–(11.19) provide useful signatures for EGUE/EGOE and in Chap. 12 this will be discussed in more detail.

11.2 Embedded Gaussian Unitary Ensemble for Spinless Boson Systems: BEGUE(k)

For spinless bosons in N sp states with a general k-body Hamiltonian, we have BEGUE(k). As pointed out in [2], it is striking that all the EGUE(k) results of Sect. 11.1 translate directly to those of BEGUE(k) by applying the well known $N \to -N$ symmetry [14, 15], i.e. in the fermion results replace N by $-N$ and then take the absolute value of the final result. For example, the m boson space dimension N_m^B is

$$N_m^B = \left|\binom{-N}{m}\right| = \binom{N+m-1}{m}. \quad (11.20)$$

More importantly the eigenvalues E_μ of the matrix of the second moments follow from Eq. (11.10) by using $N \to -N$ symmetry,

$$\Lambda_B^\nu(N,m,k) \to \left|\binom{m-\nu}{k}\binom{-N-m+k-\nu}{k}\right| = \binom{m-\nu}{k}\binom{N+m+\nu-1}{k}. \tag{11.21}$$

This result was explicitly derived in [5]. Moreover, for bosons $\{k\}\otimes\{k^{N-1}\} \to g_\nu = \{2\nu, \nu^{N-2}\}$, $\nu = 0,1,\ldots,k$. Also, the $N \to -N$ symmetry and Eq. (11.20) will give $d^B(g_\nu) = \{(N+\nu-1)_\nu\}^2 - \{(N+\nu-2)_{\nu-1}\}^2$ and this is same as Eq. (15) of [5]. Similarly Eqs. (11.11), (11.13), (11.17) and (11.18) for $\langle H^2\rangle$, $\gamma_2(N,m,k)$, Σ_{11} and Σ_{22} respectively extend directly to BEGUE(k) with $\Lambda^\nu(N,m,k)$ replaced by $\Lambda_B^\nu(N,m,k)$ defined in Eq. (11.21) and similarly replacing N_m by N_m^B and $d(g_\nu)$ by $d^B(g_\nu)$. Detailed derivations given in [5] are in agreement with these. In addition, for fermions to bosons there is also a $m \leftrightarrow N$ symmetry and this connects fermion results (say for M_p and Σ_{pq}) in dilute limit to boson results in dense limit as discussed in Sect. 9.4 and [14].

11.3 EGUE(2)-$SU(r)$ Ensembles: General Formulation

Consider a system of m fermion or bosons in Ω number of sp levels each r-fold degenerate. Then the SGA is $U(r\Omega)$ and it is possible to consider $U(r\Omega) \supset U(\Omega)\otimes SU(r)$ algebra. Now, for random two-body Hamiltonians preserving $SU(r)$ symmetry, one can introduce embedded GUE with $U(\Omega)\otimes SU(r)$ embedding and this ensemble is called EGUE(2)-$SU(r)$. Ensembles with $r=2$ and 4 for fermions correspond to fermions with spin (or isospin [16]) and spin-isospin $SU(4)$ symmetry [17–19] respectively. Similarly, for bosons $r=2,3$ are of interest. Also $r=1$ gives back EGUE(2) and BEGUE(2) both. It is important to note that the distinction between fermions and bosons is in the $U(\Omega)$ irreps that need to be considered. Now, we will give a formulation in terms of $SU(\Omega)$ Wigner-Racah algebra that is valid for any $r \geq 1$ [20].

Let us begin with the normalized two-particle states $|f_2F_2; v_2\beta_2\rangle$ where the $U(r)$ irreps $F_2 = \{1^2\}$ and $\{2\}$ and the corresponding $U(\Omega)$ irreps f_2 are $\{2\}$ (symmetric) and $\{1^2\}$ (antisymmetric) respectively for fermions and $\{1^2\}$ (antisymmetric) and $\{2\}$ (symmetric) respectively for bosons. Similarly v_2 are additional quantum numbers that belong to f_2 and β_2 belong to F_2. As f_2 uniquely defines F_2, from now on we will drop F_2 unless it is explicitly needed and also we will use the $f_2 \leftrightarrow F_2$ equivalence whenever needed. With $A^\dagger(f_2v_2\beta_2)$ and $A(f_2v_2\beta_2)$ denoting creation and annihilation operators for the normalized two particle states, a general two-body Hamiltonian operator $\widehat{H}$ preserving $SU(r)$ symmetry can be written as

$$\widehat{H} = \widehat{H}_{\{2\}} + \widehat{H}_{\{1^2\}} = \sum_{f_2,v_2^i,v_2^f,\beta_2;\, f_2=\{2\},\{1^2\}} H_{f_2v_2^iv_2^f}(2)A^\dagger\big(f_2v_2^f\beta_2\big)A\big(f_2v_2^i\beta_2\big). \tag{11.22}$$

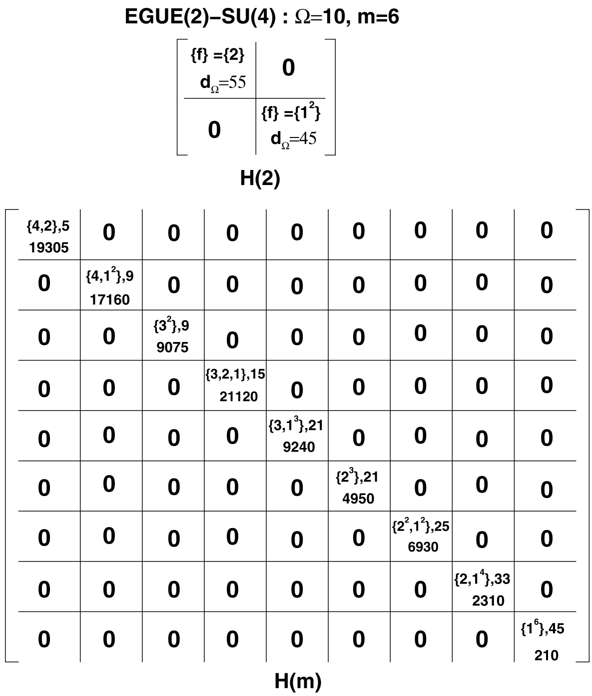

Fig. 11.1 (**a**) EGUE(2)-$SU(4)$ ensemble for fermions in the defining space. (**b**) Decomposition of the H matrix in ($\Omega = 10$, $m = 6$) space into direct sum of matrices with fixed $SU(\Omega)$ irrep f_m. There is a EGUE(2)-$SU(4)$ ensemble in each f_m space corresponding to each diagonal block in the figure. Shown also next to each f_m in the figure, is the eigenvalue $\langle \hat{C}_2(SU(4))\rangle^{f_m}$ of the quadratic Casimir invariant of $SU(4)$. Similarly, below each f_m shown is the matrix dimension

In Eq. (11.22), $H_{f_2 v_2^i v_2^f}(2) = \langle f_2 v_2^f \beta_2 \mid H \mid f_2 v_2^i \beta_2\rangle$ independent of the β_2's. The uniform summation over β_2 in Eq. (11.22) ensures that $\widehat{H}$ is $SU(r)$ scalar and therefore it will not connect states with different f_2's. However, $\widehat{H}$ is not a $SU(r)$ invariant operator. Just as the two particle states, we can denote the m particle states by $|f_m v_m^f \beta_m^F\rangle$; $F_m = \widetilde{f_m}$ for fermions and $F_m = f_m$ for bosons. Action of $\widehat{H}$ on these states generates states that are degenerate with respect to β_m^F but not v_m^f. Therefore for a given f_m, there will be $d_\Omega(f_m)$ number of levels each with $d_r(\tilde{f}_m)$ number of degenerate states. Formula for the dimension $d_\Omega(f_m)$ is [21],

$$d_\Omega(f_m) = \prod_{i<j=1}^{\Omega} \frac{f_i - f_j + j - i}{j - i}, \tag{11.23}$$

where $f_m = \{f_1, f_2, \ldots\}$. Equation (11.23) also gives $d_r(F_m)$ with the product ranging from $i = 1$ to r and replacing f_i by F_i. As $\widehat{H}$ is a $SU(r)$ scalar, the m particle H matrix will be a direct sum of matrices with each of them labeled by the f_m's with dimension $d_\Omega(f_m)$. Thus

$$H(m) = \sum_{f_m} H_{f_m}(m) \oplus . \tag{11.24}$$

Figure 11.1 shows an example for Eq. (11.24) with $r = 4$ for fermions. As seen from Eq. (11.22), the H matrix in two particle spaces is a direct sum of the two matrices

$H_{f_2}(2)$, one in the $f_2 = \{2\}$ space and the other in $\{1^2\}$ space. Similarly, for the 6 particle example shown in Fig. 11.1 there are 9 f_m's and therefore the H matrix is a direct sum of 9 matrices. It should be noted that the matrix elements of $H_{f_m}(m)$ matrices receive contributions from both $H_{\{2\}}(2)$ and $H_{\{1^2\}}(2)$.

Embedded random matrix ensemble EGUE(2)-$SU(r)$ for a m fermion or boson systems with a fixed f_m, i.e. $\{H_{f_m}(m)\}$, is generated by the ensemble of H operators given in Eq. (11.22) with $H_{\{2\}}(2)$ and $H_{\{1^2\}}(2)$ matrices replaced by independent GUE ensembles of random matrices,

$$\{H(2)\} = \{H_{\{2\}}(2)\}_{\text{GUE}} \oplus \{H_{\{1^2\}}(2)\}_{\text{GUE}}. \tag{11.25}$$

In Eq. (11.25), $\{--\}$ denotes ensemble. Random variables defining the real and imaginary parts of the matrix elements of $H_{f_2}(2)$ are independent Gaussian variables with zero center and variance given by (with bar representing ensemble average),

$$\overline{H_{f_2 v_2^1 v_2^2}(2) H_{f_2' v_2^3 v_2^4}(2)} = \delta_{f_2 f_2'} \delta_{v_2^1 v_2^4} \delta_{v_2^2 v_2^3} (\lambda_{f_2})^2. \tag{11.26}$$

Also, the independence of the $\{H_{\{2\}}(2)\}$ and $\{H_{\{1^2\}}(2)\}$ GUE ensembles imply,

$$\begin{aligned} &\overline{\left[H_{\{2\} v_2^1 v_2^2}(2)\right]^P \left[H_{\{1^2\} v_2^3 v_2^4}(2)\right]^Q} \\ &= \left\{\overline{\left[H_{\{2\} v_2^1 v_2^2}(2)\right]^P}\right\}\left\{\overline{\left[H_{\{1^2\} v_2^3 v_2^4}(2)\right]^Q}\right\} \quad \text{for } P \text{ and } Q \text{ even}, \\ &= 0 \quad \text{for } P \text{ or } Q \text{ odd}. \end{aligned} \tag{11.27}$$

Action of $\widehat{H}$ defined by Eq. (11.22) on m particle basis states with a fixed f_m, along with Eqs. (11.26)–(11.27) generates EGUE(2)-$SU(r)$ ensemble $\{H_{f_m}(m)\}$; it is labeled by the $U(\Omega)$ irrep f_m with matrix dimension $d_\Omega(f_m)$.

As discussed before for EGUE(k) for fermions in Sect. 11.1 and similarly for bosons in Sect. 11.2, tensorial decomposition of $\widehat{H}$ with respect to the embedding algebra $U(\Omega) \otimes SU(r)$ plays a crucial role in generating analytical results; as before $U(\Omega)$ and $SU(\Omega)$ are used interchangeably. As $\widehat{H}$ preserves $SU(r)$, it transforms as the irrep $\{0\}$ with respect to the $SU(r)$ algebra. However with respect to $SU(\Omega)$, the tensorial characters, in Young tableaux notation, for $f_2 = \{2\}$ are $\mathbf{F}_\nu = \{0\}$, $\{21^{\Omega-2}\}$ and $\{42^{\Omega-2}\}$ with $\nu = 0$, 1 and 2 respectively. Similarly for $f_2 = \{1^2\}$ they are $\mathbf{F}_\nu = \{0\}$, $\{21^{\Omega-2}\}$ and $\{2^2 1^{\Omega-4}\}$ with $\nu = 0, 1, 2$ respectively. Note that $\mathbf{F}_\nu = f_2 \times \overline{f_2}$ where $\overline{f_2}$ is the irrep conjugate to f_2 and the $\times$ denotes Kronecker product. Young tableaux for the $\mathbf{F}_\nu$ are same as those in Figs. 9.2 and 5.1b for $f_2 = \{2\}$ and $\{1^2\}$ respectively with N replaced by Ω in the figures. Now, we can define unitary tensors B's that are scalars in $SU(r)$ space,

$$\begin{aligned} B(f_2 \mathbf{F}_\nu \omega_\nu) = &\sum_{v_2^i, v_2^f, \beta_2} A^\dagger(f_2 v_2^f \beta_2) A(f_2 v_2^i \beta_2) \langle f_2 v_2^f \overline{f_2}\, \overline{v_2^i} | \mathbf{F}_\nu \omega_\nu\rangle \\ &\times \langle F_2 \beta_2 \overline{F_2}\, \overline{\beta_2} \mid 00\rangle. \end{aligned} \tag{11.28}$$

In Eq. (11.28), $\langle f_2 - - - \rangle$ are $SU(\Omega)$ Wigner coefficients and $\langle F_2 - - - \rangle$ are $SU(r)$ Wigner coefficients. The expansion of $\widehat{H}$ in terms of B's is,

$$\widehat{H} = \sum_{f_2,\mathbf{F}_\nu,\omega_\nu} W(f_2\mathbf{F}_\nu\omega_\nu)B(f_2\mathbf{F}_\nu\omega_\nu). \tag{11.29}$$

The expansion coefficients W's follow from the orthogonality of the tensors B's with respect to the traces over fixed f_2 spaces. Then we have the most important relation needed for all the results given ahead,

$$\overline{W(f_2\mathbf{F}_\nu\omega_\nu)W\left(f_2'\mathbf{F}_\nu'\omega_\nu'\right)} = \delta_{f_2 f_2'}\delta_{\mathbf{F}_\nu\mathbf{F}_\nu'}\delta_{\omega_\nu\omega_\nu'}(\lambda_{f_2})^2 d_r(F_2). \tag{11.30}$$

This is derived starting with Eq. (11.29) and using Eqs. (11.25)–(11.28). Also used are the sum rules for Wigner coefficients appearing in Eq. (11.28).

Turning to m particle H matrix elements, first we denote the $U(\Omega)$ and $U(r)$ irreps by f_m and F_m respectively. Correlations generated by EGUE(2)-$SU(r)$ between states with (m, f_m) and $(m', f_{m'})$ follow from the covariance between the m-particle matrix elements of H. Now using Eqs. (11.29) and (11.30) along with the Wigner-Eckart theorem applied using $SU(\Omega) \otimes SU(r)$ Wigner-Racah algebra (see for example [22]) will give

$$\begin{aligned}
&\overline{H_{f_m v_m^i v_m^f} H_{f_{m'} v_{m'}^i v_{m'}^f}} \\
&\quad = \overline{\left\langle f_m F_m v_m^f \beta \middle| H \middle| f_m F_m v_m^i \beta\right\rangle\left\langle f_{m'} F_{m'} v_{m'}^f \beta' \middle| H \middle| f_{m'} F_{m'} v_{m'}^i \beta'\right\rangle} \\
&\quad = \sum_{f_2,\mathbf{F}_\nu,\omega_\nu} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} \sum_{\rho,\rho'} \langle f_m | \| B(f_2\mathbf{F}_\nu) \| | f_m\rangle_\rho \langle f_{m'} | \| B(f_2\mathbf{F}_\nu) \| | f_{m'}\rangle_{\rho'} \\
&\qquad \times \left\langle f_m v_m^i \mathbf{F}_\nu\omega_\nu \mid f_m v_m^f\right\rangle_\rho \left\langle f_{m'} v_{m'}^i \mathbf{F}_\nu\omega_\nu \mid f_{m'} v_{m'}^f\right\rangle_{\rho'}; \\
&\langle f_m | \| B(f_2\mathbf{F}_\nu) \| | f_m\rangle_\rho = \sum_{f_{m-2}} F(m) \frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}} \frac{U(f_m \overline{f_2} f_m f_2; f_{m-2}\mathbf{F}_\nu)_\rho}{U(f_m \overline{f_2} f_m f_2; f_{m-2}\{0\})}.
\end{aligned} \tag{11.31}$$

Here the summation in the last equality is over the multiplicity index ρ and this arises as $f_m \times \mathbf{F}_\nu$ gives in general more than once the irrep f_m. In Eq. (11.31), $F(m) = -m(m-1)/2$, $d_\Omega(f_m)$ is dimension with respect to $U(\Omega)$ as given by Eq. (11.23) and $\langle \ldots \mid \ldots \rangle$ and $U(\ldots)$ are $SU(\Omega)$ Wigner and Racah coefficients respectively. Similarly, $\mathcal{N}_{f_m}$ is dimension with respect to the S_m group,

$$\mathcal{N}_{f_m} = \frac{m! \prod_{i<k=1}^{r} (\ell_i - \ell_k)}{\ell_1!\ell_2! \cdots \ell_r!}; \quad \ell_i = f_i + r - i. \tag{11.32}$$

Note that r denotes total number of rows in the Young tableaux for f_m.

Lower order cross correlations between states with different (m, f_m) are given by the normalized bivariate moments $\hat{\Sigma}_{rr}(m, f_m : m', f_{m'})$, $r = 1, 2$ of the two-point function S^ρ where, with $\rho^{m,f_m}(E)$ defining fixed-(m, f_m) density of states,

$$\begin{aligned}
S^{mf_m:m'f_{m'}}(E, E') &= \overline{\rho^{m,f_m}(E)\rho^{m',f_{m'}}(E')} - \overline{\rho^{m,f_m}(E)}\,\overline{\rho^{m',f_{m'}}(E')};\\
\hat{\Sigma}_{11}(m, f_m : m', f_{m'}) &= \overline{\langle H\rangle^{m,f_m}\langle H\rangle^{m',f_{m'}}}\Big/\sqrt{\overline{\langle H^2\rangle^{m,f_m}}\;\overline{\langle H^2\rangle^{m',f_{m'}}}},\\
\hat{\Sigma}_{22}(m, f_m : m', f_{m'}) &= \overline{\langle H^2\rangle^{m,f_m}\langle H^2\rangle^{m',f_{m'}}}\Big/\Big[\overline{\langle H^2\rangle^{m,f_m}}\;\overline{\langle H^2\rangle^{m',f_{m'}}}\Big] - 1.
\end{aligned} \tag{11.33}$$

In Eq. (11.33), $\overline{\langle H^2\rangle^{m,f_m}}$ is the second moment (or variance) of the eigenvalue density $\overline{\rho^{m,f_m}(E)}$ and its centroid $\overline{\langle H\rangle^{m,f_m}} = 0$ by definition. We begin with $\overline{\langle H\rangle^{m,f_m}\langle H\rangle^{m',f_{m'}}}$. As $\langle H\rangle^{m,f_m}$ is the trace of H (divided by dimensionality) in (m, f_m) space, only $\mathbf{F}_\nu = \{0\}$ will generate this. Then trivially,

$$\begin{aligned}
\overline{\langle H\rangle^{m,f_m}\langle H\rangle^{m',f_{m'}}} &= \sum_{f_2} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} P^{f_2}(m, f_m) P^{f_2}(m', f_{m'});\\
P^{f_2}(m, f_m) &= F(m) \sum_{f_{m-2}} [\mathcal{N}_{f_{m-2}}/\mathcal{N}_{f_m}].
\end{aligned} \tag{11.34}$$

In terms of m particle H matrix elements, $\overline{\langle H^2\rangle^{m,f_m}}$ is

$$\overline{\langle H^2\rangle^{m,f_m}} = \big[d(f_m)\big]^{-1} \sum_{v_m^1, v_m^2} \overline{H_{f_m v_m^1 v_m^2} H_{f_m v_m^2 v_m^1}}.$$

Applying Eq. (11.31) and the orthonormal properties of the $SU(\Omega)$ Wigner coefficients lead to

$$\overline{\langle H^2\rangle^{m,f_m}} = \sum_{f_2} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} \sum_{\nu=0,1,2} \mathcal{Q}^\nu(f_2 : m, f_m) \tag{11.35}$$

where

$$\mathcal{Q}^\nu(f_2 : m, f_m) = \big[F(m)\big]^2 \sum_{f_{m-2}, f'_{m-2}} \frac{\mathcal{N}_{f_{m-2}}}{\mathcal{N}_{f_m}} \frac{\mathcal{N}_{f'_{m-2}}}{\mathcal{N}_{f_m}} X_{UU}\big(f_2; f_{m-2}, f'_{m-2}; \mathbf{F}_\nu\big). \tag{11.36}$$

The X_{UU} function involves $SU(\Omega)$ Racah coefficients,

$$\begin{aligned}
&X_{UU}\big(f_2; f_{m-2}, f'_{m-2}; \mathbf{F}_\nu\big)\\
&\quad = \sum_\rho \frac{U(f_m, \overline{f_2}, f_m, f_2; f_{m-2}, \mathbf{F}_\nu)_\rho U(f_m, \overline{f_2}, f_m, f_2; f'_{m-2}, \mathbf{F}_\nu)_\rho}{U(f_m, \overline{f_2}, f_m, f_2; f_{m-2}, \{0\}) U(f_m, \overline{f_2}, f_m, f_2; f'_{m-2}, \{0\})}.
\end{aligned} \tag{11.37}$$

Summation over the multiplicity index ρ in Eq. (11.37) arises naturally in applications to physical problems as all the physically relevant results should be independent of ρ which is a label for equivalent $SU(\Omega)$ irreps. Let us add that,

$$\mathscr{Q}^{\nu=0}(f_2:m,f_m)=\left[P^{f_2}(m,f_m)\right]^2. \tag{11.38}$$

Equations (11.34)–(11.36) and Table 4 of [7] will allow one to calculate covariances $\hat{\Sigma}_{11}$ in energy centroids. For the covariances $\hat{\Sigma}_{22}$ in spectral variances, the formula is [7]

$$\begin{aligned}
\hat{\Sigma}_{22}(m,f_m;m',f_{m'}) &= \frac{X_{\{2\}}+X_{\{1^2\}}+4X_{\{1^2\}\{2\}}}{\overline{\langle H^2\rangle^{m,f_m}}\,\overline{\langle H^2\rangle^{m',f_{m'}}}};\\
X_{f_2} &= \frac{2(\lambda_{f_2})^4}{[d_\Omega(f_2)]^2}\sum_{\nu=0,1,2}\left[d(\mathbf{F}_\nu)\right]^{-1}\mathscr{Q}^\nu(f_2:m,f_m)\mathscr{Q}^\nu\left(f_2:m',f_{m'}\right),\\
X_{\{1^2\}\{2\}} &= \frac{\lambda^2_{\{2\}}\lambda^2_{\{1^2\}}}{d_\Omega(\{2\})d_\Omega(\{1^2\})}\sum_{\nu=0,1}\left[d(\mathbf{F}_\nu)\right]^{-1}\mathscr{R}^\nu(m,f_m)\mathscr{R}^\nu\left(m',f_{m'}\right).
\end{aligned} \tag{11.39}$$

Here $d(\mathbf{F}_\nu)$ are dimension of the irrep $\mathbf{F}_\nu$, and we have $d(\{0\})=1$, $d(\{2,1^{\Omega-2}\})=\Omega^2-1$, $d(\{4,2^{\Omega-2}\})=\Omega^2(\Omega+3)(\Omega-1)/4$, and $d(\{2^2,1^{\Omega-4}\})=\Omega^2(\Omega-3)(\Omega+1)/4$. Note that $\mathscr{Q}^\nu(f_2:m,f_m)$ are defined by Eq. (11.36). The function $\mathscr{R}^\nu(m,f_m)$ also involve $SU(\Omega)$ U-coefficients,

$$\begin{aligned}
\mathscr{R}^\nu(m,f_m) &= \left[F(m)\right]^2\sum_{f_{m-2},f'_{m-2}}\frac{\mathscr{N}_{f_{m-2}}}{\mathscr{N}_{f_m}}\frac{\mathscr{N}_{f'_{m-2}}}{\mathscr{N}_{f_m}}Y_{UU}\left(f_{m-2},f'_{m-2};\mathbf{F}_\nu\right);\\
&Y_{UU}\left(f_{m-2},f'_{m-2};\mathbf{F}_\nu\right)\\
&= \sum_\rho\frac{U(f_m,\{1^{\Omega-2}\},f_m,\{1^2\};f_{m-2},\mathbf{F}_\nu)_\rho U(f_m,\{2^{\Omega-1}\},f_m,\{2\};f'_{m-2},\mathbf{F}_\nu)_\rho}{U(f_m,\{1^{\Omega-2}\},f_m,\{1^2\};f_{m-2},\{0\})U(f_m,\{2^{\Omega-1}\},f_m,\{2\};f'_{m-2},\{0\})}.
\end{aligned} \tag{11.40}$$

In $Y_{UU}(f_{m-2},f'_{m-2};\mathbf{F}_\nu)$, f_{m-2} comes from $f_m\otimes\{1^{\Omega-2}\}$ and f'_{m-2} comes from $f_m\otimes\{2^{\Omega-1}\}$. Similarly, the summation is over $\nu=0$ and 1 only as $\nu=2$ parts for $f_2=\{2\}$ and $\{1^2\}$ are different. It is useful to note that,

$$\mathscr{R}^{\nu=0}(m,f_m)=P^{\{2\}}(m,f_m)P^{\{1^2\}}(m,f_m). \tag{11.41}$$

Formulas for X_{UU} and Y_{UU} are given in [7] and they are simplified version of the formulas given in [23]. For illustration, some of these results are collected in Table 11.1. These and Eqs. (11.33)–(11.41) will allow one to derive analytical/numerical results for spectral variances and covariances in energy centroids and spectral variances for any EGUE(2)-$SU(r)$ for fermion or boson systems.

Table 11.1 Formulas for $X_{UU}(f_2; f_{m-2}, f'_{m-2}; \mathbf{F}_\nu)$ and $Y_{UU}(f_{m-2}, f'_{m-2}; \mathbf{F}_\nu)$ with $\nu = 1, 2$

$\{f_{m-2}\}\{f'_{m-2}\}$	$X_{UU}(\{1^2\}; f_{m-2}, f'_{m-2}; \{2^\nu, 1^{\Omega-2\nu}\})$
$\{f(ab)\}\{f(ab)\}$	$\frac{\Omega}{(\Omega-2)}\{\delta_{\nu,2} + \frac{(\Omega-1)(\Omega-2)}{2\Pi_a^{(b)}\Pi_b^{(a)}}\delta_{\nu,2} + (3-2\nu)\frac{(\Omega-1)}{2}$ $\times [(1+\frac{1}{\tau_{ab}})\frac{1}{\Pi_b^{(a)}} + (1-\frac{1}{\tau_{ab}})\frac{1}{\Pi_a^{(b)}} - \frac{4}{\Omega}\delta_{\nu,1}]\}$
$\{f(ab)\}\{f(ac)\}$	$\frac{\Omega(\Omega-1)}{2(\Omega-2)}\{\frac{2}{(\Omega-1)}\delta_{\nu,2} - \frac{4}{\Omega}\delta_{\nu,1} + (3-2\nu)\frac{1}{\Pi_a^{(bc)}}\}$
$\{f_{m-2}\}\{f'_{m-2}\}$	$X_{UU}(\{2\}; f_{m-2}, f'_{m-2}; \{2\nu, \nu^{\Omega-2}\})$
$\{f(ab)\}\{f(ab)\}$	$\frac{\Omega(\Omega+1)}{2}\{\frac{1}{\Pi_a^{(b)}\Pi_b^{(a)}}\delta_{\nu,2} + \frac{2}{(\Omega+1)(\Omega+2)}\delta_{\nu,2}$ $+ (3-2\nu)\frac{1}{(\Omega+2)}[\frac{(\tau_{ab}-1)^2}{\tau_{ab}(\tau_{ab}+1)}\frac{1}{\Pi_b^{(a)}} + \frac{(\tau_{ab}+1)^2}{\tau_{ab}(\tau_{ab}-1)}\frac{1}{\Pi_a^{(b)}} - \frac{4}{\Omega}\delta_{\nu,1}]\}$
$\{f(aa)\}\{f(aa)\}$	$\frac{\Omega}{(\Omega+2)}\{\delta_{\nu,2} + (3-2\nu)\frac{2(\Omega+1)}{\Pi'_a} + \frac{(\Omega+1)(\Omega+2)}{2\Pi''_a}\delta_{\nu,2} - \frac{2(\Omega+1)}{\Omega}\delta_{\nu,1}\}$
$\{f(aa)\}\{f(bb)\}$	$-\frac{2(\Omega+1)}{(\Omega+1)}\delta_{\nu,1} + \frac{\Omega}{(\Omega+2)}\delta_{\nu,2}$
$\{f(aa)\}\{f(ab)\}$	$\frac{\Omega}{(\Omega+2)}\{\delta_{\nu,2} + (3-2\nu)\frac{(\Omega+1)(\tau_{ab}+1)}{(\tau_{ab}-1)\Pi_a^{(b)}} - \frac{2(\Omega+1)}{\Omega}\delta_{\nu,1}\}$
$\{f_{m-2}\}\{f'_{m-2}\}$	$Y_{UU}(f_{m-2}, f'_{m-2}; \{2, 1^{\Omega-2}\})$
$\{f(ab)\}\{f(ab)\}$	$-\frac{\Omega}{2}[\frac{(\Omega^2-1)}{(\Omega^2-4)}]^{1/2}\{(1+\frac{1}{\tau_{ab}})\frac{1}{\Pi_a^{(b)}} + (1-\frac{1}{\tau_{ab}})\frac{1}{\Pi_b^{(a)}} - \frac{4}{\Omega}\}$
$\{f(ab)\}\{f(ac)\}$	$-\frac{\Omega}{2}[\frac{(\Omega^2-1)}{(\Omega^2-4)}]^{1/2}\{(1+\frac{1}{\tau_{ac}})\frac{1}{\Pi_a^{(b)}} - \frac{4}{\Omega}\}$
$\{f(ab)\}\{f(aa)\}$	$-\Omega[\frac{(\Omega^2-1)}{(\Omega^2-4)}]^{1/2}\{\frac{1}{\Pi_a^{(b)}} - \frac{2}{\Omega}\}$

11.3.1 Results for BEGUE(2): r = 1

Simplest of the EGUE(2)-$SU(r)$ are the EGUEs with $r = 1$ and they corresponds to EGUE(2) and BEGUE(2) depending on totally antisymmetric or symmetric f_m one considers. Also they correspond to $k = 2$ in Sects. 11.1 and 11.2 respectively. For illustration we consider BEGUE(2) in some detail. For this ensemble, in order to apply the formulas for $\langle H^2\rangle$, $\hat{\Sigma}_{11}$ and $\hat{\Sigma}_{22}$, first we need the formulas for X_{UU} and Y_{UU}. Some of these, taken from Tables 4 and 7 of [7], are given in Table 11.1. For applying these formulas, we need the 'axial distances' τ_{ij} for the boxes i and j in a given Young tableaux. Given a $f_m = \{f_1, f_2, \ldots, f_\Omega\}$ we have,

$$\tau_{ij} = f_i - f_j + j - i. \tag{11.42}$$

In terms of τ_{ij} the functions $\Pi_a^{(b)}$, $\Pi_b^{(a)}$, $\Pi_a^{(bc)}$, Π'_a and Π''_a are defined as,

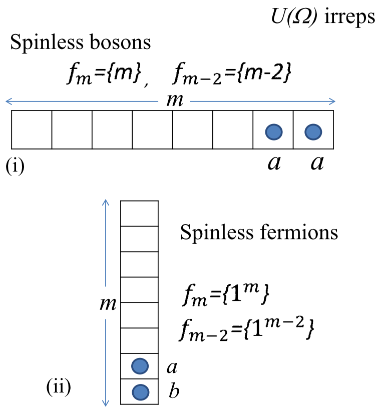

Fig. 11.2 Young tableaux denoting the $SU(\Omega)$ irreps $f_m = \{m\}$ and $\{1^m\}$ as appropriate for (**i**) spinless boson and (**ii**) spinless fermion systems. Removal of two boxes generating $m-2$ particle irreps f_{m-2} for these systems are also shown in the figure. For (**i**) only the irrep $f_2 = \{2\}$ will apply and similarly for (**ii**) only $\{1^2\}$ will apply. Figure is taken from [20] with permission from American Institute of Physics (Color figure online)

$$\begin{aligned}
\Pi_a^{(b)} &= \prod_{i=1,2,\ldots,\Omega;i\neq a,i\neq b} (1-1/\tau_{ai}) \\
\Pi_b^{(a)} &= \prod_{i=1,2,\ldots,\Omega;i\neq a,i\neq b} (1-1/\tau_{bi}) \\
\Pi_a^{(bc)} &= \prod_{i=1,2,\ldots,\Omega;i\neq a,i\neq b,i\neq c} (1-1/\tau_{ai}); \quad a\neq b\neq c, \\
\Pi'_a &= \prod_{i=1,2,\ldots,\Omega;i\neq a} (1-1/\tau_{ai}) \\
\Pi''_a &= \prod_{i=1,2,\ldots,\Omega;i\neq a} (1-2/\tau_{ai}).
\end{aligned} \tag{11.43}$$

With these we can calculate X_{UU} and Y_{UU}; see [7] for full discussion. For BEGUE(2), the algebra $U(\Omega)\otimes SU(r)$ with $r=1$ reduces to just $U(\Omega)$ or $SU(\Omega)$. Similarly, f_m is the totally symmetric irrep $\{m\}$ and $f_{m-2}=\{m-2\}$. Therefore to generate f_{m-2} only the action of removal of $\{2\}$ from f_m is allowed. Denoting the last two boxes of f_m by a and a (note that we can remove only boxes from the right end to get proper Young tableaux and also boxes in a given row must have the same symbol to apply the results in Table 11.1) as shown in Fig. 11.2, we have

$$\begin{aligned}
\tau_{ai} &= m+i-1, \\
\Pi'_a &= \frac{m}{m+\Omega-1}, \\
\Pi''_a &= \frac{m(m-1)}{(m+\Omega-1)(m+\Omega-2)}.
\end{aligned} \tag{11.44}$$

Similarly $\mathcal{N}_{f_m} = 1$ and $\mathcal{N}_{f_{m-2}} = 1$ as both are symmetric irreps. Now the formulas in Table 11.1 will give X_{UU} and there by $\mathcal{Q}^\nu$ in Eq. (11.36),

$$
\begin{aligned}
\mathcal{Q}^{\nu=0}\big(\{2\}; m, \{m\}\big) &= \frac{m^2(m-1)^2}{4}, \\
\mathcal{Q}^{\nu=1}\big(\{2\}; m, \{m\}\big) &= \frac{m^2(m-1)^2}{4} \frac{2(\Omega+m)(\Omega^2-1)}{m(\Omega+2)}, \\
\mathcal{Q}^{\nu=2}\big(\{2\}; m, \{m\}\big) &= \frac{m^2(m-1)^2}{4} \frac{\Omega^2(\Omega-1)(\Omega+m)(\Omega+m+1)}{2(\Omega+2)m(m-1)}.
\end{aligned}
\tag{11.45}
$$

These and Eq. (11.35) will give,

$$
\big\langle H^2 \big\rangle^{\{m\}} = \lambda_{\{2\}}^2 \binom{m}{2} \binom{\Omega+m-1}{2} = \lambda_{\{2\}}^2 \Lambda_B^0(\Omega, m, 2). \tag{11.46}
$$

This agrees with the result stated in Sect. 11.2. As $P^{\{2\}}(m, \{m\}) = -m(m-1)/2$, we have easily,

$$
\begin{aligned}
&\hat{\Sigma}_{11}\big(\{m\}, \{m'\}\big) \\
&\quad = \frac{2\sqrt{m(m-1)(m')(m'-1)}}{\Omega(\Omega+1)\sqrt{(\Omega+m-1)(\Omega+m-2)(\Omega+m'-1)(\Omega+m'-2)}}.
\end{aligned}
\tag{11.47}
$$

Again, this agrees with the result stated in Sect. 11.2. Further, $\hat{\Sigma}_{22}$ is determined only by $X_{\{2\}}$ defined in Eq. (11.39) and then, using Eq. (11.45), we have

$$
\begin{aligned}
&\hat{\Sigma}_{22}\big(\{m\}, \{m'\}\big) \\
&\quad = \frac{2}{36\binom{\Omega+2}{3}^2(\Omega+3)\binom{\Omega+m-1}{2}\binom{\Omega+m'-1}{2}} \\
&\qquad \times \bigg[4\Omega^2(\Omega-1)\binom{\Omega+m+1}{2}\binom{\Omega+m'+1}{2} \\
&\qquad + 4(\Omega+2)^2(\Omega+3)\binom{m}{2}\binom{m'}{2} \\
&\qquad + 4\big(\Omega^2-1\big)(\Omega+3)(m-1)(\Omega+m) \\
&\qquad \times \big(m'-1\big)\big(\Omega+m'\big)\bigg].
\end{aligned}
\tag{11.48}
$$

For $m = m'$, it can be verified that Eq. (11.48) reduces to

$$\hat{\Sigma}_{22}\big(\{m\},\{m\}\big) = \frac{2}{(\Omega^B_m)^2} \sum_{\nu=0}^{2} \frac{[\Lambda^\nu_B(\Omega,m,m-2)]^2 d^B(g_\nu)}{[\Lambda^0_B(\Omega,m,2)]^2} \tag{11.49}$$

as expected from Sect. 11.2; Eq. (11.49) agrees with the result given for BEGUE(k) in [5]. Finally, it is useful to mention that in the $m \longrightarrow \infty$ and N finite limit we have,

$$\begin{aligned}
\hat{\Sigma}_{11}\big(\{m\},\{m\}\big) &= \frac{2}{\Omega(\Omega+1)}, \\
\hat{\Sigma}_{22}\big(\{m\},\{m\}\big) &= 8\frac{\Omega^2(\Omega-1)+(\Omega+2)^2(\Omega+3)+4(\Omega^2-1)(\Omega+3)}{\Omega^2(\Omega+1)^2(\Omega+2)^2(\Omega+3)}.
\end{aligned} \tag{11.50}$$

Non-vanishing of $\hat{\Sigma}_{11}$ and $\hat{\Sigma}_{22}$ for finite N in the $m \longrightarrow \infty$ is interpreted in [5, 24] as non-ergodicity of BEGUE ensembles. See the discussion in Chap. 9 for the resolution of this problem.

In the next four sections we will consider specific $SU(r)$'s and present results that are appropriate for some physical systems.

11.4 Embedded Gaussian Unitary Ensemble for Fermions with Spin: EGUE(2)-$SU(2)$ with $r = 2$

Embedded Gaussian Unitary Ensemble for fermions with spin $\mathbf{s} = \frac{1}{2}$ degree of freedom corresponds to $r = 2$ in Sect. 11.3 and this ensemble, applicable to mesoscopic systems with mobile electrons carrying spin degree of freedom, is denoted by EGUE(2)-$SU(2)$ or EGUE(2)-$\mathbf{s}$. For this ensemble, the $U(\Omega)$ irreps for m fermion systems with spin S are $f_m = \{2^p 1^q\}$ where $m = 2p + q$ and $S = q/2$. Formulas for $\langle H^2\rangle^{m,S}$ and the normalized bivariate moments $\hat{\Sigma}_{rr}(m, S : m', S')$, $r = 1, 2$ of the two-point correlation function $S^{mS:m'S'}(E, E')$ follow from the formulation given in Sect. 11.3. It is easily seen that with $\langle S^2\rangle = S(S+1)$,

$$\begin{aligned}
&\overline{\langle H\rangle^{m,S}\langle H\rangle^{m',S'}} = \sum_{f_2(s_2)} \frac{(\lambda_{f_2})^2}{d_\Omega(f_2)} P^{s_2}(m,S)P^{s_2}(m',S'); \\
&P^{s_2}(m,S) = \big[(2s_2+1)m(m-4s_2+2)+4(2s_2-1)\langle S^2\rangle\big]\big/8, \quad s_2 = 0, 1.
\end{aligned} \tag{11.51}$$

To proceed further we need X_{UU} and Y_{UU}. The f_{m-2} irreps obtained by removing $\{2\}$ or $\{1^2\}$ from f_m follow from Fig. 11.3. Note that all three choices (i)–(iii) shown in the figure will apply for $\{1^2\}$ and only (i) will apply to $\{2\}$. Using the formulas in Table 11.1, the final formula for $\overline{\langle H^2\rangle^{(m,S)}}$, in terms of $m^x = (\Omega - \frac{m}{2})$ is

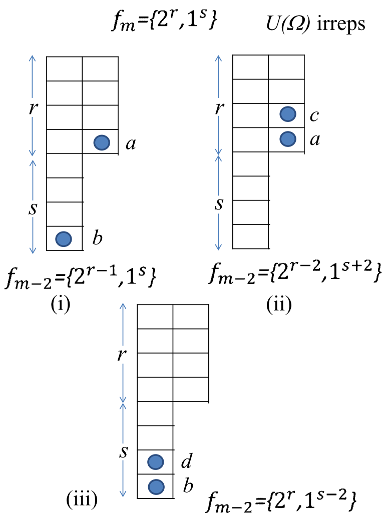

Fig. 11.3 Young tableaux denoting the two-column $SU(\Omega)$ irreps $f_m = \{2^r 1^s\}$ appropriate for EGUE(2)-$SU(2)$. Removal of two boxes generating $m-2$ particle irreps f_{m-2} are also shown in the figure. For (**i**) both the irreps $f_2 = \{2\}$ and $\{1^2\}$ will apply while for (**ii**) and (**iii**) only $\{1^2\}$ will apply (Color figure online)

$$
\begin{aligned}
\overline{\langle H^2\rangle^{m,S}} &= \sum_{f_2} \frac{(\lambda_{f_2})^2}{d(f_2)} \sum_{\nu=0,1,2} \mathscr{Q}^\nu(f_2 : m, S);\\
\mathscr{Q}^0\big(\{2\} : m, S\big) &= \big[P^0(m, S)\big]^2,\\
\mathscr{Q}^1\big(\{2\} : m, S\big) &= \big[(\Omega+1)P^0(m, S)/2\big]\big[m^x(m+2)/2 + \langle S^2\rangle\big],\\
\mathscr{Q}^2\big(\{2\} : m, S\big) &= \big[\Omega(\Omega+3)P^0(m, S)/4\big]\big[m^x(m^x+1) - \langle S^2\rangle\big],\\
\mathscr{Q}^0\big(\{1^2\} : m, S\big) &= \big[P^1(m, S)\big]^2,\\
\mathscr{Q}^1\big(\{1^2\} : m, S\big) &= \frac{(\Omega-1)}{16(\Omega-2)}\big[8(\Omega+2)P^1(m, S)P^2(m, S)\\
&\quad + 8\Omega(m-1)(\Omega-2m+4)\langle S^2\rangle\big],\\
\mathscr{Q}^2\big(\{1^2\} : m, S\big) &= \frac{\Omega}{8(\Omega-2)}\big[\big(3\Omega^2 - 7\Omega + 6\big)\big(\langle S^2\rangle\big)^2\\
&\quad + 3m(m-2)m^x\big(m^x-1\big)(\Omega+1)(\Omega+2)/4\\
&\quad + \langle S^2\rangle\big\{-mm^x(5\Omega-3)(\Omega+2)\\
&\quad + \Omega(\Omega-1)(\Omega+1)(\Omega+6)\big\}\big];\\
P^2(m, S) &= 3m^x(m-2)/2 - \langle S^2\rangle.
\end{aligned}
\tag{11.52}
$$

Further, Eqs. (11.51) and (11.52) will give $\hat{\Sigma}_{11}$ for any (m, S, m', S', Ω). For $\hat{\Sigma}_{22}$ the only unknowns are $\mathscr{R}^\nu$ and they are given by

$$
\begin{aligned}
\mathscr{R}^0\big(\{2\}\{1^2\} : mS\big) &= P^0(m, S)P^1(m, S),\\
\mathscr{R}^1\big(\{2\}\{1^2\} : mS\big) &= -\frac{1}{2}\sqrt{\frac{(\Omega^2-1)(\Omega+2)}{(\Omega-2)}}\,P^0(m, S)P^2(m, S).
\end{aligned}
\tag{11.53}
$$

Finally, let us consider the excess parameter $\overline{\gamma_2(m,S)} = \{\overline{\langle H^4\rangle^{m,S}}/[\overline{\langle H^2\rangle^{m,S}}]^2\} - 3$ and this is the most important (as $\overline{\langle H^3\rangle^{m,S}} = 0$) lower order shape parameter for fixed-(m,S) density of states $\overline{\rho^{m,S}(E)}$. General expression, derived using $SU(\Omega)$ algebra given in [11], for the fourth moment $\overline{\langle H^4\rangle^{m,S}}$ in terms of U-coefficients involves the multiplicity labels ρ's. However, for the physically interesting situation with $S=0$ (i.e. $f_m = \{2^r\}, r = m/2$), all the multiplicity labels will be unity and then $\overline{\gamma_2(m,S=0)}$ is given by [6],

$$\begin{aligned}\left[\overline{\gamma_2(m,S=0)}+1\right] = {} & \left[\overline{\langle H^2\rangle^{m,S=0}}\right]^{-2} \sum_{f_2^a, f_2^b} \frac{(\lambda_{f_2^a})^2(\lambda_{f_2^b})^2}{d_\Omega(f_2^a)d_\Omega(f_2^b)} \\ & \times \sum_{\nu_1,\nu_2} \frac{d_\Omega(f_m)}{\sqrt{d_\Omega(F_{\nu_1})d_\Omega(F_{\nu_2})}} \left|\langle f_m|\left\| B\left(f_2^a F_{\nu_1}\right)\right\| |f_m\rangle\right|^2 \\ & \times \left|\langle f_m|\left\| B\left(f_2^b F_{\nu_2}\right)\right\| |f_m\rangle\right|^2 U(f_m \overline{f_m} f_m f_m; F_{\nu_1} F_{\nu_2}).\end{aligned} \tag{11.54}$$

In Eq. (11.54), $f_2^a = \{2\}, \{1^2\}$ and similarly f_2^b. This expression is pleasing and it is possible to obtain the triple barred coefficients using the tables in [23] and Eq. (11.31). But still we need $U(f_m \overline{f_m} f_m f_m; F_{\nu_1} F_{\nu_2})$ coefficient and deriving a formula for this needs further advances in $SU(N)$ Racah algebra. Thus, our present knowledge of $SU(N)$ Wigner-Racah algebra will not allow us to go too far in analytically solving EGUE(2)-**s** and even the simpler EGUE(2).

11.5 Embedded Gaussian Unitary Ensemble for Fermions with Wigner's Spin-Isospin $SU(4)$ Symmetry: EGUE(2)-$SU(4)$ with $r = 4$

Wigner introduced in 1937 [17] the spin-isospin $SU(4)$ supermultiplet scheme for atomic nuclei. There is good evidence for the goodness of this symmetry in some parts of the periodic table [25] and also more recently there is new interest in $SU(4)$ symmetry for heavy $N \sim Z$ nuclei [18, 19]. Therefore it is clearly of importance to study embedded Gaussian unitary ensemble of random matrices generated by random two-body interactions with $SU(4)$ symmetry and this corresponds to EGUE(2)-$SU(4)$ with $r = 4$ in Sect. 11.3. Before giving some analytical results for EGUE(2)-$SU(4)$, we will first turn to a brief discussion of the $SU(4)$ algebra.

Let us consider a system with m nucleons distributed in Ω number of orbits each with spin ($\mathbf{s} = \frac{1}{2}$) and isospin ($\mathbf{t} = \frac{1}{2}$) degrees of freedom. Then the total number of sp states is $N = 4\Omega$ and the spectrum generating algebra is $U(4\Omega)$. The sp states in uncoupled representation are $a^\dagger_{i,\alpha}|0\rangle = |i,\alpha\rangle$ with $i = 1,2,\ldots,\Omega$ denoting the spatial orbits and $\alpha = 1,2,3,4$ are the four spin-isospin states $|m_\mathbf{s}, m_\mathbf{t}\rangle = |\frac{1}{2},\frac{1}{2}\rangle$, $|\frac{1}{2},-\frac{1}{2}\rangle$, $|-\frac{1}{2},\frac{1}{2}\rangle$ and $|-\frac{1}{2},-\frac{1}{2}\rangle$ respectively. The $(4\Omega)^2$ number of operators $C_{i\alpha;j\beta}$

generate $U(4\Omega)$ algebra. For m fermions, all states belong to the $U(4\Omega)$ irrep $\{1^m\}$. In uncoupled notation, $C_{i\alpha;j\beta} = a^\dagger_{i,\alpha} a_{j,\beta}$. Similarly $U(\Omega)$ and $U(4)$ algebras are generated by A_{ij} and $B_{\alpha\beta}$ respectively, where $A_{ij} = \sum_{\alpha=1}^{4} C_{i\alpha;j\alpha}$ and $B_{\alpha\beta} = \sum_{i=1}^{\Omega} C_{i\alpha;i\beta}$. The number operator $\hat{n}$, the spin operator $\hat{S} = S^1_\mu$, the isospin operator $\hat{T} = T^1_\mu$ and the Gamow-Teller operator $\sigma\tau = (\sigma\tau)^{1,1}_{\mu,\mu'}$ of $U(4)$ in spin-isospin coupled notation are [26],

$$
\begin{aligned}
&\hat{n} = 2\sum_i \mathscr{A}^{0,0}_{ii;0,0}, \qquad S^1_\mu = \sum_i \mathscr{A}^{1,0}_{ii;\mu,0}, \qquad T^1_\mu = \sum_i \mathscr{A}^{0,1}_{ii;0,\mu}, \\
&(\sigma\tau)^{1,1}_{\mu,\mu'} = \sum_i \mathscr{A}^{1,1}_{ii;\mu,\mu'}; \qquad \mathscr{A}^{s,t}_{ij;\mu_\mathbf{s},\mu_\mathbf{t}} = (a^\dagger_i \tilde{a}_j)^{s,t}_{\mu_\mathbf{s},\mu_\mathbf{t}}.
\end{aligned}
\tag{11.55}
$$

Note that $\tilde{a}_{j;\mu_\mathbf{s},\mu_\mathbf{t}} = (-1)^{1+\mu_\mathbf{s}+\mu_\mathbf{t}} a_{j;-\mu_\mathbf{s},-\mu_\mathbf{t}}$. These 16 operators form $U(4)$ algebra. Dropping the number operator, we have $SU(4)$ algebra. For the $U(4)$ algebra, the irreps are characterized by the partitions $\{F\} = \{F_1, F_2, F_3, F_4\}$ with $F_1 \geq F_2 \geq F_3 \geq F_4 \geq 0$ and $m = \sum_{i=1}^{4} F_i$. Note that F_α are the eigenvalues of $B_{\alpha\alpha}$. Due to the antisymmetry constraint on the total wavefunction, the $U(\Omega)$ irrep $\{f\} = \{\widetilde{F}\}$ which is obtained by changing rows to columns in $\{F\}$; note that $F_i \leq \Omega$ and $f_i \leq 4$. Before proceeding further, let us examine the quadratic Casimir invariants of $U(\Omega)$, $U(4)$ and $SU(4)$ algebras. For example,

$$
\begin{aligned}
C_2\big[U(\Omega)\big] &= \sum_{i,j} A_{ij} A_{ji} = \hat{n}\Omega - \sum_{i,j,\alpha,\beta} a^\dagger_{i,\alpha} a^\dagger_{j,\beta} a_{j,\alpha} a_{i,\beta}, \\
C_2\big[U(4)\big] &= \sum_{\alpha,\beta} B_{\alpha,\beta} B_{\beta,\alpha} \quad \Rightarrow \quad C_2\big[U(\Omega)\big] + C_2\big[U(4)\big] = \hat{n}(\Omega + 4).
\end{aligned}
\tag{11.56}
$$

Also, in terms of spin, isospin and Gamow-Teller operators, $C_2[SU(4)] = S^2 + T^2 + (\sigma\tau)\cdot(\sigma\tau)$ and

$$
\big\langle C_2\big[U(4)\big]\big\rangle^{\{F\}} = \sum_{i=1}^{4} F_i(F_i + 5 - 2i) = \left\langle C_2\big[SU(4)\big] + \frac{\hat{n}^2}{4} \right\rangle^{\{F\}}. \tag{11.57}
$$

The space exchange or Majorana operator $\widetilde{M}$ that exchanges the spatial coordinates of the particles (the index i) and leaves the spin-isospin quantum numbers unchanged allow us to understand the significance of $SU(4)$ symmetry,

$$
\widetilde{M}\big|i,\alpha,\alpha'; j,\beta,\beta'\big\rangle = \big|j,\alpha,\alpha'; i,\beta,\beta'\big\rangle, \tag{11.58}
$$

where α, β are labels for spin and α', β' are labels for isospin. As $|i,\alpha,\alpha'; j,\beta,\beta'\rangle = a^\dagger_{i,\alpha,\alpha'} a^\dagger_{j,\beta,\beta'}|0\rangle$, Eqs. (11.58), (11.56) and (11.57) in that order will give,

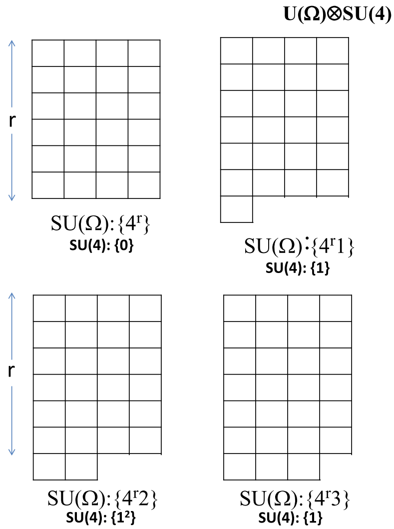

Fig. 11.4 Young tableaux denoting the special $SU(\Omega)$ irreps $f_m^{(p)} = \{4^r, p\}$, $p = 0, 1, 2, 3$ considered in EGUE(2)-$SU(4)$ analysis with $U(\Omega) \otimes SU(4)$ embedding algebra. The corresponding $SU(4)$ irreps are also given in the figure (Color figure online)

Table 11.2 $P^{f_2}(m, f_m)$ for $f_m = \{4^r, p\}$; $p = 0, 1, 2$ and 3 and $\{f_2\} = \{2\}, \{1^2\}$

f_m	$P^{f_2}(m, f_m)$	
	$f_2 = \{2\}$	$f_2 = \{1^2\}$
$\{4^r\}$	$-3r(r+1)$	$-5r(r-1)$
$\{4^r, 1\}$	$-\frac{3r}{2}(2r+3)$	$-\frac{5r}{2}(2r-1)$
$\{4^r, 2\}$	$-(3r^2+6r+1)$	$-5r^2$
$\{4^r, 3\}$	$-\frac{3}{2}(r+2)(2r+1)$	$-\frac{5r}{2}(2r+1)$

$$
\begin{aligned}
2\kappa \widetilde{M} &= 2\kappa \sum_{i,j,\alpha,\beta,\alpha',\beta'} \left(a^\dagger_{j,\alpha,\alpha'} a^\dagger_{i,\beta,\beta'}\right)\left(a^\dagger_{i,\alpha,\alpha'} a^\dagger_{j,\beta,\beta'}\right)^\dagger \\
&= \kappa \left\{C_2\left[U(\Omega)\right] - \Omega \hat{n} = 4\hat{n} - C_2\left[U(4)\right]\right\} \\
&= 2\kappa \left\{2\hat{n}\left(1 - \frac{\hat{n}}{16}\right) - \frac{1}{2} C_2\left[SU(4)\right]\right\}.
\end{aligned}
\tag{11.59}
$$

The preferred $U(\Omega)$ irrep for the ground state of a m nucleon system is the most symmetric one. Therefore $\langle C_2[U(\Omega)]\rangle$ should be maximum for the ground state irrep. This implies, as seen from Eq. (11.59), the strength κ of $\widetilde{M}$ must be negative. As a consequence, as follows from the last equality in Eq. (11.59), the ground states are labeled by $SU(4)$ irreps with smallest eigenvalue for the quadratic Casimir invariant consistent with a given (m, T_z), $T = |T_z|$. Therefore, for $N = Z$ even-even, $N = Z$ odd-odd and $N = Z \pm 1$ odd-A nuclei the $U(\Omega)$ irreps for the gs are $\{4^r\}$, $\{4^r, 2\}$, $\{4^r, 1\}$ and $\{4^r, 3\}$ with spin-isospin structure being $(0, 0)$, $(1, 0) \oplus (0, 1)$, $(\frac{1}{2}, \frac{1}{2})$, and $(\frac{1}{2}, \frac{1}{2})$ respectively. For convenience, the gs $U(\Omega)$ irreps are denoted by

Table 11.3 $\overline{\langle H^2\rangle^{m,f_m}}$, $\mathcal{Q}^{\nu=1,2}(f_2:m,f_m)$ and $\mathcal{R}^{\nu=1}(m,f_m)$ for some examples

f_m	$\overline{\langle H^2\rangle^{m,f_m}}$
$\{4^r\}$	$\frac{r(\Omega-r+4)}{2}[\lambda^2_{\{2\}}3(r+1)(\Omega-r+3)+\lambda^2_{\{1^2\}}5(r-1)(\Omega-r+5)]$
$\{4^r,1\}$	$\frac{r(\Omega-r+4)}{4}[\lambda^2_{\{2\}}\{6r(\Omega-r+1)+9\Omega+15\}$ $+\lambda^2_{\{1^2\}}5\{2r(\Omega-r+5)-\Omega-9\}]$
$\{4^r,2\}$	$\lambda^2_{\{2\}}\frac{1}{2}[3r^4-6(\Omega+2)r^3+(3\Omega^2+6\Omega-5)r^2$ $+(\Omega+2)(6\Omega+17)r+\Omega(\Omega+1)]$ $+\lambda^2_{\{1^2\}}\frac{5r}{2}(\Omega-r+4)\{(\Omega+4)r-r^2-3\}$
$\{4^r,3\}$	$\frac{1}{4}[\lambda^2_{\{2\}}3(r+2)(\Omega-r+2)(2r\Omega-2r^2+6r+\Omega+1)$ $+\lambda^2_{\{1^2\}}5r(\Omega-r+4)(2r\Omega-2r^2+6r+\Omega-1)]$

f_m	f_2	ν	$\mathcal{Q}^{\nu}(f_2:m,f_m)$
$\{4^r\}$	$\{2\}$	1	$\frac{9r(r+1)^2(\Omega-r)(\Omega+1)(\Omega+4)}{2(\Omega+2)}$
		2	$\frac{3r\Omega(r+1)(\Omega-r+1)(\Omega-r)(\Omega+4)(\Omega+5)}{4(\Omega+2)}$
	$\{1^2\}$	1	$\frac{25r(r-1)^2(\Omega-r)(\Omega-1)(\Omega+4)}{2(\Omega-2)}$
		2	$\frac{5r\Omega(r-1)(\Omega+3)(\Omega+4)(\Omega-r)(\Omega-r-1)}{4(\Omega-2)}$

f_m	$\mathcal{R}^{\nu=1}(m,f_m)$
$\{4^r\}$	$-\frac{15r}{2}\sqrt{\frac{\Omega^2-1}{\Omega^2-4}}(r^2-1)(\Omega-r)(\Omega+4)$

$f_m^{(p)}$ where

$$f_m^{(p)}=\{4^r,p\};\quad m=4r+p \text{ and } p=\text{mod}(m,4). \tag{11.60}$$

For the special $SU(\Omega)$ irreps in Eq. (11.60), and shown in Fig. 11.4, analytical formulas are much simpler than for a general $SU(\Omega)$ irrep [7].

The formalism given in Sect. 11.3 was applied in detail in [7]. For example, formulas for $P^{f_2}(m,f_m)$ are given in Table 11.2 for $\{f_m^{(p)}\}$ irreps. Evaluating all the $\mathcal{Q}$'s as given in detail in [7], analytical formulas for $\mathcal{Q}^{\nu}(f_2:m,f_m)$ and also for $\overline{\langle H^2\rangle^{m,f_m}}$ are obtained for $\{f_m^{(p)}\}$ irreps. Some of these results are given in Table 11.3. Equations (11.34)–(11.36) and Tables 4 and 7 of [7] will allow us to calculate covariances $\hat{\Sigma}_{11}$ in energy centroids for any irrep. On the other hand, the results in Tables 11.2 and 11.3 will give formulas for $\hat{\Sigma}_{11}$ for $\{f_m^{(p)}\}$ irreps. Similarly, the $\mathcal{R}$ formula given in Table 11.3 will us to calculate $\hat{\Sigma}_{22}$ for the irrep $\{4^r\}$. Note that, $\mathcal{Q}^{\nu=0}(f_2:m,f_m)=[P^{f_2}(m,f_m)]^2$ and $\mathcal{R}^{\nu=0}(m,f_m)=P^{\{2\}}(m,f_m)P^{\{1^2\}}(m,f_m)$.

11.6 Embedded Gaussian Unitary Ensemble for Bosons with F-Spin: BEGUE(2)-$SU(2)$ with $r=2$

For two species boson systems with F-spin, following the discussion in Chap. 10, we have BEGUE(2)-$SU(2)$ or BEGUE(2)-F. For this ensemble, results in Sect. 11.3 with $r=2$ will be applicable. For such a m boson system, the $SU(\Omega)$ irreps will be two rowed denoted by $f_m=\{m-r,r\}$ with $F=\frac{m}{2}-r$. With this, there are three allowed f_{m-2} irreps as shown in Fig. 11.5. The irreps in (i) and (iii) in the figure can be obtained by removing $f_2=\{2\}$ from f_m. However for (ii) in the figure both $\{2\}$ and $\{1^2\}$ will apply. For $f_{m-2}=\{m-r-2,r\}$ irrep [this corresponds to (i) in Fig. 11.5] we have

$$\begin{aligned}
\tau_{a2} &= m-2r+1,\\
\tau_{ai} &= m-r+i-1; \quad i=3,4,\ldots,\Omega,\\
\Pi'_a &= \frac{(m-2r)(m-r+1)}{(m-2r+1)(m-r+\Omega-1)},\\
\Pi''_a &= \frac{(m-2r-1)(m-r)(m-r+1)}{(m-2r+1)(m-r+\Omega-1)(m-r+\Omega-2)}.
\end{aligned} \tag{11.61}$$

Similarly for $f_{m-2}=\{m-r,r-2\}$ irrep [this corresponds to (iii) in Fig. 11.5] we have

$$\begin{aligned}
\tau_{b1} &= 2r-m-1,\\
\tau_{bi} &= r+i-2, \quad i=3,4,\ldots,\Omega\\
\Pi'_a &= \frac{(r)(2r-m-2)}{(2r-m-1)(r+\Omega-2)},\\
\Pi''_a &= \frac{(2r-m-3)(r)(r-1)}{(2r-m-1)(r+\Omega-2)(r+\Omega-3)}.
\end{aligned} \tag{11.62}$$

Finally, for $f_{m-2}=\{m-r-1,r-1\}$ irrep [this corresponds to (ii) in Fig. 11.5] we have

$$\begin{aligned}
\tau_{ab} &= m-2r+1=2F+1,\\
\tau_{ai} &= m-r+i-1, \qquad \tau_{bi}=r+i-2; \quad i=3,4,\ldots,\Omega,\\
\Pi_a^{(b)} &= \frac{(m-r+1)}{(m-r+\Omega-1)},\\
\Pi_b^{(a)} &= \frac{(r)}{(r+\Omega-2)}.
\end{aligned} \tag{11.63}$$

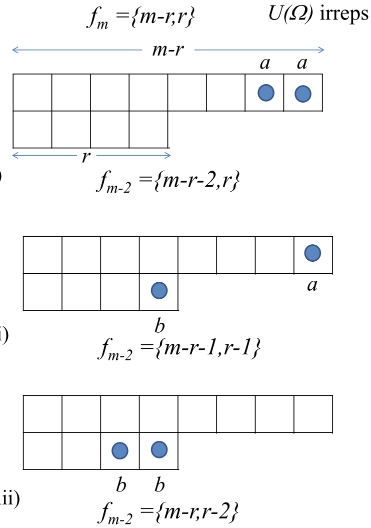

Fig. 11.5 Young tableaux denoting the two-rowed $SU(\Omega)$ irreps $f_m = \{m-r, r\}$ appropriate for BEGUE(2)-$SU(2)$. Removal of two boxes generating $m-2$ particle irreps f_{m-2} are also shown in the figure. For (**ii**) both the irreps $f_2 = \{2\}$ and $\{1^2\}$ will apply and for (**i**) and (**iii**) only $\{2\}$ will apply. Figure is taken from [20] with permission from American Institute of Physics (Color figure online)

These and $\mathcal{N}_{f_{m-2}}/\mathcal{N}_{f_m}$ will give the formulas for the lower order moments of one and two point functions as described in Sect. 11.3. The dimension ratios are,

$$\begin{aligned}\frac{\mathcal{N}_{\{m-r-2,r\}}}{\mathcal{N}_{\{m-r,r\}}} &= \frac{(m-r)(m-r+1)(m-2r-1)}{m(m-1)(m-2r+1)}, \\ \frac{\mathcal{N}_{\{m-r-1,r-1\}}}{\mathcal{N}_{\{m-r,r\}}} &= \frac{r(m-r+1)}{m(m-1)}, \\ \frac{\mathcal{N}_{\{m-r,r-2\}}}{\mathcal{N}_{\{m-r,r\}}} &= \frac{r(r-1)(m-2r+3)}{m(m-1)(m-2r+1)}.\end{aligned} \tag{11.64}$$

Using Eqs. (11.61)–(11.64) and the expressions in Table 11.1, it is possible to derive analytical formulas for the P's, $\mathcal{Q}$'s and $\mathcal{R}$'s that define $\langle H^2\rangle$, $\hat{\Sigma}_{11}$ and $\hat{\Sigma}_{22}$. The final formulas (obtained in [20] using MATHEMATICA) are, with (m, F) defining f_m,

$$\begin{aligned}P^{\{2\}}(m,F) &= \frac{1}{8}\big[3m(m-2)+4F(F+1)\big], \\ P^{\{1^2\}}(m,F) &= \frac{1}{8}\big[m(m+2)-4F(F+1)\big], \\ \mathcal{Q}^{\nu=0}\big(\{2\}:m,F\big) &= \big[P^{\{2\}}(m,F)\big]^2, \\ \mathcal{Q}^{\nu=0}\big(\big\{1^2\big\}:m,F\big) &= \big[P^{\{1^2\}}(m,F)\big]^2, \\ \mathcal{Q}^{\nu=1}\big(\{2\}:m,F\big) &= \frac{(\Omega+1)}{16(\Omega+2)} \\ &\quad\times\big[2(\Omega-2)P^{\{2\}}(m,F)\big\{3(2\Omega+m)(m-2)+4F(F+1)\big\} \\ &\quad+8\Omega(m-1)(\Omega+2m-4)F(F+1)\big],\end{aligned}$$

$$
\begin{aligned}
\mathscr{Q}^{\nu=1}\big(\{1^2\}:m,F\big) &= \frac{(\Omega-1)P^{\{1^2\}}(m,F)}{8}\big[(2\Omega+m)(m+2)-4F(F+1)\big],\\
\mathscr{Q}^{\nu=2}\big(\{2\}:m,F\big) &= \frac{(\Omega)}{8(\Omega+2)}\bigg[\big(3\Omega^2+7\Omega+6\big)\big[F(F+1)\big]^2\\
&\quad+\frac{3}{16}m(m-2)(2\Omega+m)(2\Omega+m+2)(\Omega-1)(\Omega-2)\\
&\quad+\frac{F(F+1)}{2}\big\{m(2\Omega+m)(5\Omega+3)(\Omega-2)\\
&\quad+2\Omega\big(\Omega^2-1\big)(\Omega-6)\big\}\bigg],\\
\mathscr{Q}^{\nu=2}\big(\{1^2\}:m,F\big) &= \frac{\Omega(\Omega-3)P^{\{1^2\}}(m,F)}{16}\\
&\quad\times\big[(2\Omega+m)(2\Omega+m-2)-4F(F+1)\big],\\
\mathscr{R}^{\nu=0}(m,F) &= P^{\{2\}}(m,F)P^{\{1^2\}}(m,F),\\
\mathscr{R}^{\nu=1}(m,F) &= \sqrt{\frac{\Omega^2-1}{\Omega^2-4}}\,\frac{(2-\Omega)P^{\{1^2\}}(m,F)}{8}\big\{4\big[F(F+1)-3\Omega\big]\\
&\quad+3m(2\Omega+m-2)\big\}.
\end{aligned}
\tag{11.65}
$$

Note that Eq. (11.65) is closely related to the BEGOE(2)-F results given by Eq. (10.7). More importantly, they are related to the EGUE(2)-$SU(2)$ results by $\Omega \to -\Omega$ transformation.

11.7 Embedded Gaussian Unitary Ensemble for Spin One Bosons: BEGUE(2)-$SU(3)$ with $r = 3$

Spin one boson systems, as discussed in Chap. 10, posses $U(3\Omega) \supset U(\Omega) \otimes [SU(3) \supset SO(3)]$ symmetry. For these systems, it is possible to consider interactions preserving the $SU(3)$ symmetry. This gives, for the GUE version, BEGUE(2)-$SU(3)$ that corresponds to $r = 3$ in Sect. 11.3. As $U(3)$ irreps will have, in Young tableaux representation, maximum 3 rows, the $U(\Omega)$ irrep also will have maximum three rows. Given m bosons in Ω number of sp levels, the allowed $U(\Omega)$ irreps are $\{f_1, f_2, f_3, f_4, \ldots, f_\Omega\} = \{f_1, f_2, f_3\}$ with $f_1 + f_2 + f_3 = m$, $f_1 \geq f_2 \geq f_3 \geq 0$ and $f_i = 0$ for $i = 4, 5, \ldots, \Omega$. For $f_2 = 0$ and $f_3 = 0$, we have totally symmetric irreps with $\{f_1\} = \{m\}$ and for these irreps all the results derived in Sect. 11.3.1 will apply directly. Similarly, for $f_2 \neq 0$ and $f_3 = 0$, all the results of Sect. 11.6 will apply. Thus, the non-trivial irreps for BEGUE(2)-$SU(3)$ are the m-boson irreps $f_m = \{f_1, f_2, f_3\}$ with $f_3 \neq 0$. Given a f_m, in general there will be six f_{m-2}

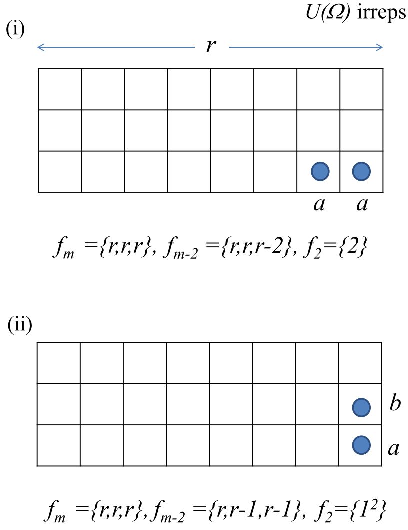

Fig. 11.6 Young tableaux denoting the three-rowed $SU(\Omega)$ irreps $f_m = \{r, r, r\}$, $m = 3r$ appropriate for BEGUE(2)-$SU(3)$. Removal of two boxes generating $m-2$ particle irreps f_{m-2} are also shown in the figure. For (**i**) only the irrep $f_2 = \{2\}$ will apply while for (**ii**) only $\{1^2\}$ will apply. Figure is taken from [20] with permission from American Institute of Physics (Color figure online)

and they are $\{f_1-2, f_2, f_3\}$, $\{f_1, f_2-2, f_3\}$, $\{f_1, f_2, f_3-2\}$, $\{f_1-1, f_2-1, f_3\}$, $\{f_1-1, f_2, f_3-1\}$, $\{f_1, f_2-1, f_3-1\}$. Therefore, as seen from Sect. 11.3, deriving analytical formulas for P's, $\mathscr{Q}$'s and $\mathscr{R}$'s that determine $\langle H^2\rangle$, $\hat{\Sigma}_{11}$ and $\hat{\Sigma}_{22}$ will be cumbersome. One situation that is amenable to analytical treatment is for the irreps $\{n+p, n, n\}$ where $m = 3n + p$ with $p = 0$, 1 and 2 [these are similar to the $\{4^r, p\}$ irreps considered for EGUE(2)-$SU(4)$]. Here we will present the results for $p = 0$ and for others see [20]. For this class of irreps, the f_{m-2} are simple as shown in Fig. 11.6. For $f_{m-2} = \{n, n, n-2\}$, Π'_a and Π''_a are needed and they are given by,

$$\Pi'_a = \frac{3n}{\Omega+n-3}, \qquad \Pi''_a = \frac{6n(n-1)}{(\Omega+n-3)(\Omega+n-4)}. \tag{11.66}$$

Similarly, for $f_{m-2} = f_{n,n-1,n-1}$ we need τ_{ab}, $\Pi_a^{(b)}$ and $\Pi_b^{(a)}$ and they are,

$$\tau_{ab} = -1, \qquad \Pi_a^{(b)} = \frac{3n}{2(\Omega+n-3)}, \qquad \Pi_b^{(a)} = \frac{2(n+1)}{(\Omega+n-2)}. \tag{11.67}$$

In addition, ratio of the S_Ω dimensions needed are,

$$\frac{\mathscr{N}_{n,n,n-2}}{\mathscr{N}_{n,n,n}} = \frac{2(n-1)}{(3n-1)}, \qquad \frac{\mathscr{N}_{n,n-1,n-1}}{\mathscr{N}_{n,n,n}} = \frac{n+1}{(3n-1)}. \tag{11.68}$$

With these, carrying out simplification of the formulas given in Table 11.1 will give the following formulas (with $\pi = 1$ for $\{2\}$ and -1 for $\{1^2\}$) ,

$$\begin{aligned}
&P^{f_2}\big(m,\{n,n,n\}\big) = -\frac{6}{3-\pi} n(n-\pi),\\
&\mathscr{Q}^{\nu=0}\big(f_2 : m,\{n,n,n\}\big) = \big[P^{f_2}\big(m,\{n,n,n\}\big)\big]^2,\\
&\mathscr{Q}^{\nu=1}\big(f_2 : m,\{n,n,n\}\big) = \frac{3(3+\pi)^2(\Omega+\pi)(\Omega-3)n(n-\pi)^2(\Omega+n)}{8(\Omega+2\pi)},\\
&\mathscr{Q}^{\nu=2}\big(f_2 : m,\{n,n,n\}\big)\\
&\quad = \frac{3(3+\pi)\Omega(\Omega-3+\pi)(\Omega-3)n(n-\pi)(\Omega+n)(\Omega+n+\pi)}{16(\Omega+2\pi)},\\
&\mathscr{R}^{\nu=0}\big(m,\{n,n,n\}\big) = P^{\{2\}}\big(m,\{n,n,n\}\big)P^{\{1^2\}}\big(m,\{n,n,n\}\big),\\
&\mathscr{R}^{\nu=1}\big(m,\{n,n,n\}\big) = -\sqrt{\frac{\Omega^2-1}{\Omega^2-4}}\,3(\Omega-3)n\big(n^2-1\big)(\Omega+n).
\end{aligned} \tag{11.69}$$

Using these equations one can calculate the variances $\langle H^2\rangle$ and the covariances $\hat{\Sigma}_{11}$ and $\hat{\Sigma}_{22}$ for irreps of the type $\{n,n,n\}$. For example, Eq. (11.35) can be simplified to give a compact formula for spectral variances,

$$\begin{aligned}
\big\langle H^2\big\rangle^{m,\{n,n,n\}} &= \lambda_{\{2\}}^2\left[\frac{3}{2}n(n-1)(\Omega+n-3)(\Omega+n-4)\right]\\
&\quad + \lambda_{\{1^2\}}^2\left[\frac{3}{4}n(n+1)(\Omega+n-2)(\Omega+n-3)\right].
\end{aligned} \tag{11.70}$$

Using the tables in [7] and the results in Sect. 11.3, one can calculate numerically $\hat{\Sigma}_{11}$ and $\hat{\Sigma}_{22}$ for any f_m. Applications of this will be discussed in Chap. 12.

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