Chapter 17
Uncertainty Quantification of Identified Modal Parameters Using the Fisher Information Criterion

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Abstract This paper derives exact results for the maximum achievable accuracy when estimating modal parameters from free vibration and forced vibration signals contaminated by Gaussian white noise. These limits are found through the Cramer-Rao lower bound. The paper proposes a simple yet accurate expression to determine the ratio between the coefficient of variation of damping and frequency. This expression can be useful to compute the variance of the damping using limited number of vibration tests. The paper presents results that verify and validate the derived expressions using laboratory and field experiments.

Keywords Estimation • Cramer-Rao • Fisher information • Uncertainty quantification • Damping

17.1 Introduction

System identification deals with the estimation of parameters in a model that better explain, in some predefined sense, observed data on a system. Due to the presence of noise in the measurements, unmeasured excitations and model class errors, the identified parameters inevitably have error and(or) are not optimal. The type of estimate could range form point estimates to full joint probability density function of the parameters. As an intermediate and common compromise, many researchers have proposed methods that aim to estimate the mean and covariance of the desired parameters. In structural dynamics, it is commonplace to formulate models on the basis of modal parameters, i.e. mode shapes, eigenfrequencies and modal damping. Various algorithms have been proposed to perform system identification [4,7,9] and it is useful to determine bounds on the maximum achievable accuracy or minimum achievable variance that can be achieved on the estimate of the modal parameters.

For applications in inverse structural dynamics problems with modal parameterization we can point out to the work of Gersch [3] as one of the pioneers on the subject. Gersch developed expressions that relate auto-regressive coefficients (AR) of auto-regressive moving average (ARMA) models with eigenvalues of the systems, and thus with modal frequency and damping ratios. He used these expressions together with the Cramer-Rao lower bound (CRLB) theorem to compute the maximum achievable accuracy of the natural frequency and damping ratio, based on the maximum likelihood estimates (MLE) of identified AR coefficients. More recent work on the subject has been carried out by [1, 6, 8]. These recently recommended procedures require the identification of a complete state-space model and subsequent calculations to determine full covariance matrices between the modal parameters. In this paper we aim at a computing simple expressions for systems with well separated modes. We focus on computing a simple and practical expression to determine the relationship between coefficient of variation of damping and the coefficient of variation of the natural frequency.

The paper begins with an examination of the Fisher information matrix and the CRLB. Following this we proceed to derive exact expressions for the CRLB for amplitude, natural frequency, damping and initial conditions under free vibration conditions and arbitrary excitations. The paper proceeds by illustrating the results numerically and verifying them by means of laboratory experiments carried out by the authors [2] and by field experiments reported in the literature [8]. The paper closes with a section summarizing the main findings.

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17.2 Fisher Information

The Fisher information is a measure of the amount of information that an observable random variable X carries about an unknown parameter θ upon which the probability of X depends. The probability function for X, which is also the likelihood function for θ, is a function $f(X;\theta)$; it is the probability mass (or probability density) of the random variable X conditional on the value of θ. The Fisher information can be defined as

$$I(\theta) = -E\left[\frac{\partial^2 \ln f(X,\theta)}{\partial\theta^2}\right] \tag{17.1}$$

or alternatively as

$$I(\theta) = E\left[\left(\frac{\partial \ln f(X,\theta)}{\partial\theta}\right)^2\right] \tag{17.2}$$

An important property of the Fisher information is that, given two independent measurements x and y, the Fisher information is additive, such that

$$I_{x,y}(\theta) = I_x(\theta) + I_y(\theta) \tag{17.3}$$

It can be shown [5] that the Fisher information matrix of a scalar signal $s(t)$ dependent on multiple parameters θ_i and corrupted by additive Gaussian white noise with variance σ^2 is given by

$$I_{ij}(\theta) = \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\frac{\partial s(n;\theta)}{\partial\theta_i}\frac{\partial s(n;\theta)}{\partial\theta_j} \tag{17.4}$$

17.3 Cramer-Rao Lower Bound

If $f(X;\theta)$ satisfies the regularity condition

$$E\left[\frac{\partial \ln(f(X;\theta))}{\partial\theta}\right] = 0 \qquad \forall\theta \tag{17.5}$$

where the expectation is taken with respect to $f(X;\theta)$, then the variance of any unbiased estimator $\hat{\theta}$ must satisfy

$$Var(\hat{\theta}) \geq CRLB = \frac{1}{I(\theta)} \tag{17.6}$$

This lower limit is known as the Cramer-Rao lower bound (CRLB). For the case of multiple parameters, the Fisher information becomes a matrix and the CRLB is given by

$$CRLB = \mathbf{I}(\theta)^{-1} \tag{17.7}$$

the proof of this theorem can be found in [5].

17.4 Maximum Achievable Accuracy Given Free Vibration Measurements

In some cases, noise contaminated free vibration response measurements are available and one wishes to identify modal parameters from these. Free vibration response of a classically damped multi-degree of freedom linear structural system is given as

$$z(x,t) = \sum_i \phi_i(x) y_i(t) \tag{17.8}$$

where

$$y_i(t) = e^{-\xi_i \omega_{n,i} t} \left[\left(\frac{\dot{y}_{o,i} + \xi_i \omega_{n,i} y_{o,i}}{\omega_{d,i}} \right) \sin \omega_{d,i} t + y_{o,i} \cos \omega_{d,i} t \right] \tag{17.9}$$

and $\phi_i(x)$ is the modal amplitude of the ith mode at location x. ξ_i and $\omega_{n,i}$ are the modal damping ratio and modal circular frequency of the ith mode and $\omega_{d,i} = \omega_{n,i}\sqrt{1-\xi_i^2}$. The main objective here is to obtain an expression for the maximum achievable accuracy when estimating modal parameters based on analysis of measurement signals given by

$$s(x,t) = z(x,t) + \nu(t) \tag{17.10}$$

where $\nu(t)$ is a realization of a zero mean white Gaussian random process. This problem has been studied previously, although not with the specific interest in structural dynamics by Wigren and Nehorai [10]. In their study some assumptions were made, however, in this paper we eliminate these assumptions and derive exact expressions for the Cramer-Rao lower bound for modal frequency, damping ratio and initial conditions.

For the special case of (17.10) and for a single degree of freedom system

$$s(t) = \phi e^{-\xi\omega_n t} \left[\left(\frac{\dot{x}_o + \xi\omega_n x_o}{\omega_d} \right) \sin \omega_d t + x_o \cos \omega_d t \right] + \nu(t) \tag{17.11}$$

The exact expressions for the derivatives necessary to compute (17.4) are given by

$$\frac{\partial s}{\partial \phi} = e^{-\xi\omega_n t} \left[\left(\frac{\dot{x}_o + \xi\omega_n x_o}{\omega_d} \right) \sin \omega_d t + x_o \cos \omega_d t \right] \tag{17.12}$$

$$\begin{aligned} \frac{\partial s}{\partial \omega_n} = e^{-\xi\omega_n t}[&-\xi t \left(\frac{\dot{x}_o + \xi\omega_n x_o}{\omega_d} \right) \sin \omega_d t + t \left(\frac{\dot{x}_o + \xi\omega_n x_o}{\omega_d} \right) \sqrt{1-\xi^2} \cos \omega_d t \\ &- \frac{\dot{x}_o}{\omega_n \omega_d} \sin \omega_d t - x_o \xi t \cos \omega_d t - x_o t \sqrt{1-\xi^2} \sin \omega_d t] \end{aligned} \tag{17.13}$$

$$\begin{aligned} \frac{\partial s}{\partial \xi} = e^{-\xi\omega_n t}[&-\omega_n t \left(\frac{\dot{x}_o + \xi\omega_n x_o}{\omega_d} \right) \sin \omega_d t \\ &+ \left(\xi \left(\frac{\dot{x}_o}{\omega_n} + \xi x_o \right) (1-\xi^2)^{-\frac{3}{2}} + \frac{x_o}{\sqrt{1-\xi^2}} \right) \sin \omega_d t \\ &- \left(\frac{\dot{x}_o + \xi\omega_n x_o}{\omega_d} \right) \left(\frac{\omega_n \xi t}{\sqrt{1-\xi^2}} \right) \cos \omega_d t - x_o \omega_n t \cos \omega_d t \\ &+ x_o \left(\frac{\omega_n \xi t}{\sqrt{1-\xi^2}} \right) \sin \omega_d t] \end{aligned} \tag{17.14}$$

$$\frac{\partial s}{\partial \dot{x}_o} = \frac{e^{-\xi\omega_n t}}{\omega_d} \sin \omega_d t \tag{17.15}$$

$$\frac{\partial s}{\partial x_o} = \frac{\xi}{\sqrt{1-\xi^2}} e^{-\xi\omega_n t} \sin \omega_d t + e^{-\xi\omega_n t} \cos \omega_d t \tag{17.16}$$

17.5 Forced Vibration

In the case of forced vibrations, the noise contaminated measured response of the system to an arbitrary input $u(t)$ is given by the convolution integral

$$s(t) = \int_0^t h(t-\tau)u(\tau)d\tau + \nu \tag{17.17}$$

By using the definition in (17.4), the Cramer-Rao lower bound for the variance of the parameters θ that define the impulse response is given by

$$\begin{aligned}\mathbf{I}(\theta) &= \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\frac{\partial s(n;\theta)}{\partial\theta_i}\frac{\partial s(n;\theta)}{\partial\theta_j} \\ &= \frac{1}{\sigma^2}\sum_{n=0}^{N-1}\left(\int_0^{n\Delta t}\frac{\partial h(n\Delta t-\tau)}{\partial\theta_i}u(\tau)d\tau\right)\left(\int_0^{n\Delta t}\frac{\partial h(n\Delta t-\tau)}{\partial\theta_j}u(\tau)d\tau\right)\end{aligned} \tag{17.18}$$

Notice that since the impulse response $h(t)$ of the system is the response due to an initial velocity, the necessary derivatives of the impulse response have already been computed in the previous section and in the case of frequency and damping given by (17.13) and (17.14) with $x_o = 0$ and $\dot{x}_o = 1/m$.

17.6 Numerical Illustration

In this section we proceed to illustrate the theoretically exact CRLB given in (17.4) and (17.7) for frequency and damping based on (17.13) and (17.14) respectively. The simulations are conducted within the frequency range of (0 50] Hz and with the range of damping ratios of (0 0.50]. The total simulation time is 10 s with a $\Delta t = 0.01$ and the measurement noise variance is 0.01. Without loss of generality, the mass of system is taken as unity. Figure 17.1 depicts the ratio between the CoV for damping and the CoV for circular frequency for the case of noise contaminated free vibration measurements. As can be seen, for a fixed value of damping ratio and for natural frequencies less than 40 Hz, the ratio between the coefficients of variation is fundamentally constant.

Figure 17.2 depicts the ratio between the CoV for damping and the CoV for circular frequency for the case of a system excited by Gaussian white noise with unit variance. As can be seen, and in similar fashion to the free vibration case, for a fixed value of damping ratio and for natural frequencies less than 40 Hz, the ratio between the coefficients of variation is fundamentally constant. Moreover it is approximately constant at the same values found for free vibration measurements. Figure 17.3 depicts the ratio between the CoV for damping and the CoV for circular frequency for the case of a system

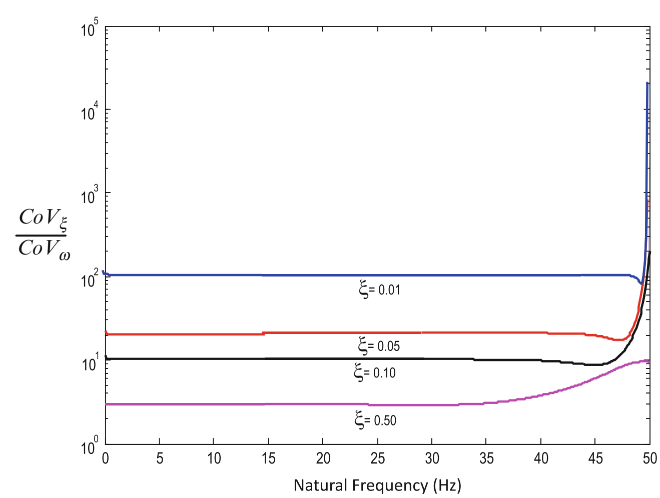

Fig. 17.1 Ratio of the CRLB induced damping CoV to the frequency CoV

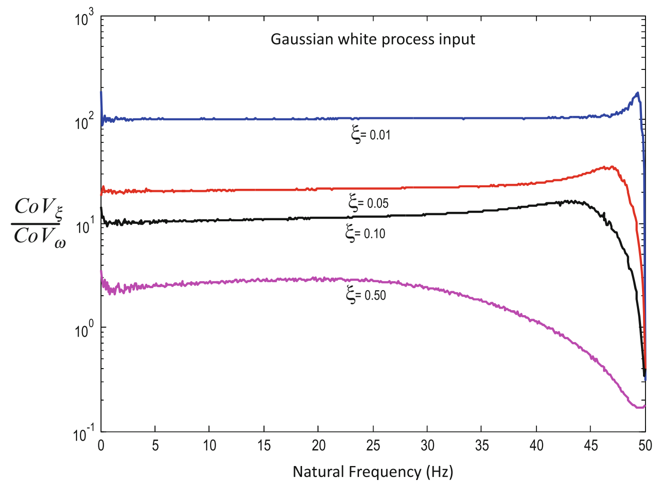

Fig. 17.2 Ratio of the CRLB induced damping CoV to the frequency CoV for the case of Gaussian white noise excitation

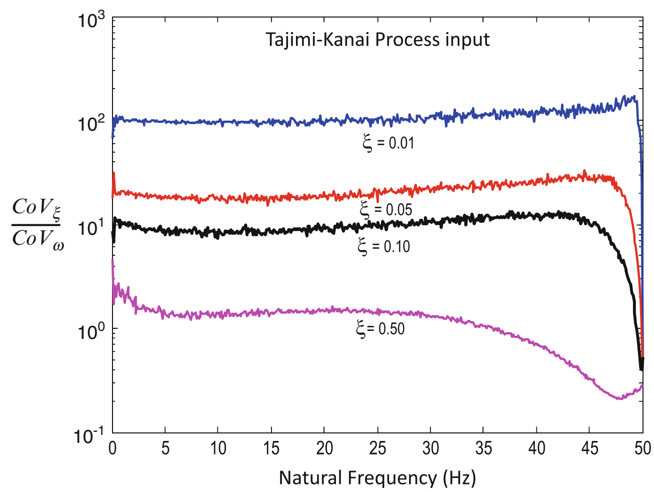

Fig. 17.3 Ratio of the CRLB induced damping CoV to the frequency CoV for the case of a Tajimi-Kanai process excitation

excited by realizations of a Tajimi-Kanai random process, typically used to model synthetic earthquakes. The power spectral density of the Tajimi-Kanai process is given by

$$S(\omega) = \frac{\left[1 + 4\zeta_g^2 \left(\omega/\omega_g\right)^2\right] G_o}{\left[1 - \left(\omega/\omega_g\right)^2\right]^2 + 4\zeta_g^2 \left(\omega/\omega_g\right)^2} \tag{17.19}$$

$$G_o = (1 + 4\zeta_g^2)/(4\zeta_g^2) \tag{17.20}$$

The parameters used to compute Fig. 17.3 were $\zeta_g = 0.6$ and $\omega_g = 4\pi$. To realize the process in time a Gaussian random phase was used. As can be seen, and in similar fashion to the free vibration and the white noise cases, for a fixed value of damping, the ratio is fundamentally constant for natural frequencies less than 40 Hz.

Figure 17.4 depicts the relationship between damping and the ratio between the CoV for damping and the CoV for circular frequency for the case of white noise (which has already been shown to be have almost exactly the same value for free vibration and other type of random process excitations). As can be seen, this presents an almost perfect linear fit in a logarithmic plot.

A best fit can be found by the expression

$$\frac{CoV_\xi}{CoV_\omega} = 0.75\xi^{-1.05} \tag{17.21}$$

This expression was found to be valid independently of the excitation source (free vibration, white noise or other stochastic process) and it is also independent of the level of noise in the signal (since it represents a ratio). The results summarized in Fig. 17.4 agree very well with those found empirically by [3] in the case of Gaussian white input (see Table 2 in [3]).

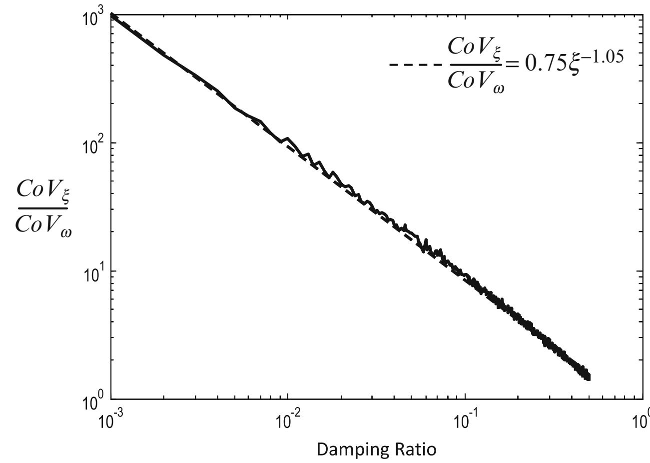

Fig. 17.4 Ratio of the CRLB induced damping CoV to the frequency CoV as a function of the damping ratio

17.7 Verification

In this final section we verify the theoretical findings given in previous sections, specially the fact that under a wide range of excitation conditions the ratio between the coefficient of variation of damping and circular frequency is fixed and only depends on the level of damping and given by the simple expression in (17.21). For this verification we formulate a SDOF system with $m = 1$, $f_n = 10$ Hz and excite it with realizations of a zero mean Gaussian white random process with unit variance. The measurements consist of displacements contaminated with realizations of zero mean Gaussian white noise with variance of 10^{-7}. The ERA-OKID algorithm [4] was used to perform the identification of the system matrices and subsequently the circular frequency and damping ratio were computed. For every selected value of damping ratio the identification was carried out for 1,000 stochastic simulations. For each damping ratio considered, the ratio between the variances were ploted and compared with the theoretical value derived in (17.21). As can be seen in Fig. 17.5 the agreement is very good.

The reason for this close agreement is not coincidental, it is rooted in the fact that the ERA-OKID algorithm operates by first identifying an ARMA model from the data and then proceeds to compute the system matrices, from which the frequencies and damping can be extracted. Since the results given by Gersch [3] already show that in the limit, ARMA models achieve the CRLB, it is expected that the ERA-OKID estimates tend to agree with the absolute accuracy limits derived in this paper.

17.8 Experiments and Validation

Reynders et al. [8] performed subspace system identification of a 30 m steel mast, using three different sensor setups, and presented the results for the identified frequencies and damping ratios with their respective mean and standard deviation. These results for setup 3 are transcribed in Table 17.1. The ratio of the coefficient of variation of the damping with respect to the frequency for each mode is calculated and compared to results of (17.21). As be seen, the proposed expression is consistent with the field data reported by [8].

Furthermore, the authors carried out 55 independent tests on a aluminum cantilever beam previously described in detail in [2]. The ERA-OKID algorithm [4] was used to perform the identification of the frequencies and damping ratios for the first three modes (see Figs. 17.6 and 17.7 for the identified frequency and damping ratios of one of the identified modes). Three flexural modes where identified with frequencies of 6.5, 41.1 and 111.1 Hz. The accelerations and the random input where measured at a sampling frequency of 2,048 Hz. For the implementation of the ERA-OKID the signals where decimated by a factor of 64, 16 and 4 giving a Nyquist band of 32, 128 and 512 Hz for the 1st, 2nd and 3rd mode respectively (each identification was done individually). Results of the experiments are presented in Table 17.2.

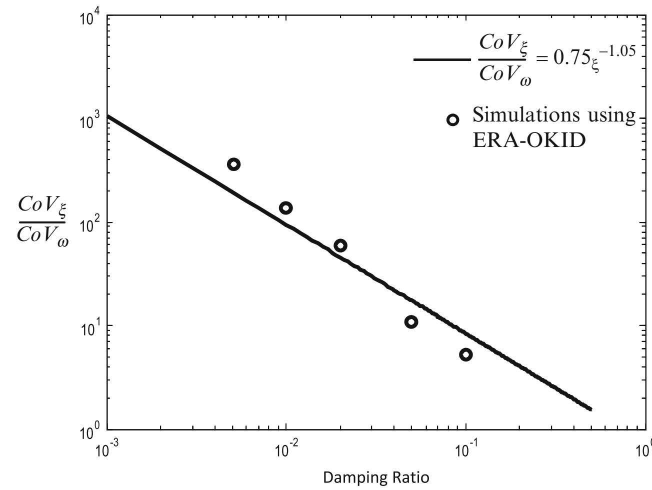

Fig. 17.5 Comparison of ratio of the CRLB induced damping CoV to the frequency CoV as a function of the damping ratio vs. the ratio obtained by performing 1,000 independent random identification results using the ERA-OKID

Table 17.1 Comparison setup 3 of Reynders et al. [8] vs. (17.21)

Mode	μ_ω	σ_ω	μ_ξ	σ_ξ	CoV_ω	CoV_ξ	$\frac{CoV_\xi}{CoV_\omega}$ [8]	$\frac{CoV_\xi}{CoV_\omega} = 0.75\xi^{-1.05}$	$Diff(\%)$
1	1.166	0.0020	0.60	0.170	0.00172	0.28333	165.18	161.44	2.3
2	1.180	0.0025	0.97	0.235	0.00212	0.24227	114.35	97.49	14.7
3	1.951	0.0025	0.47	0.115	0.00128	0.24468	190.95	208.62	−9.3
4	2.601	0.0030	0.35	0.100	0.00115	0.28571	247.71	284.31	−14.8
5	2.711	0.0015	0.19	0.055	0.00055	0.28947	523.18	539.97	−3.2
6	3.684	0.0030	0.27	0.100	0.00081	0.37037	454.81	373.36	17.9
7	4.631	0.0035	0.18	0.075	0.00076	0.41667	551.31	571.51	−3.7

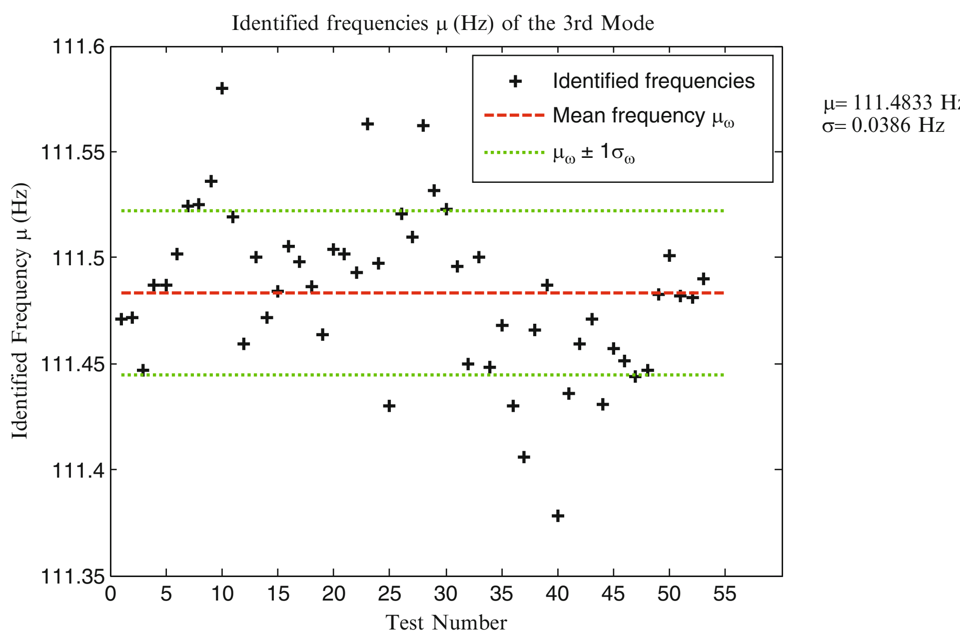

Fig. 17.6 Identified frequency (μ) of the 3rd mode using ERA-OKID with $\mu_\omega = 111.48$ Hz and $\sigma_\omega = 0.0386$ Hz

17.9 Conclusions

The paper derives exact expressions of the Cramer-Rao lower bound for natural frequency, damping ratio, and initial conditions of a single degree of freedom system subject to initial conditions or random excitations. It was shown that noise contaminated signals contain significantly less information about damping in comparison to the information contained about natural frequency. This is independent of the type of random excitation or of the methodology used to perform the system identification. A practical and simple expression was found which allows the computation of the ratio between coefficients

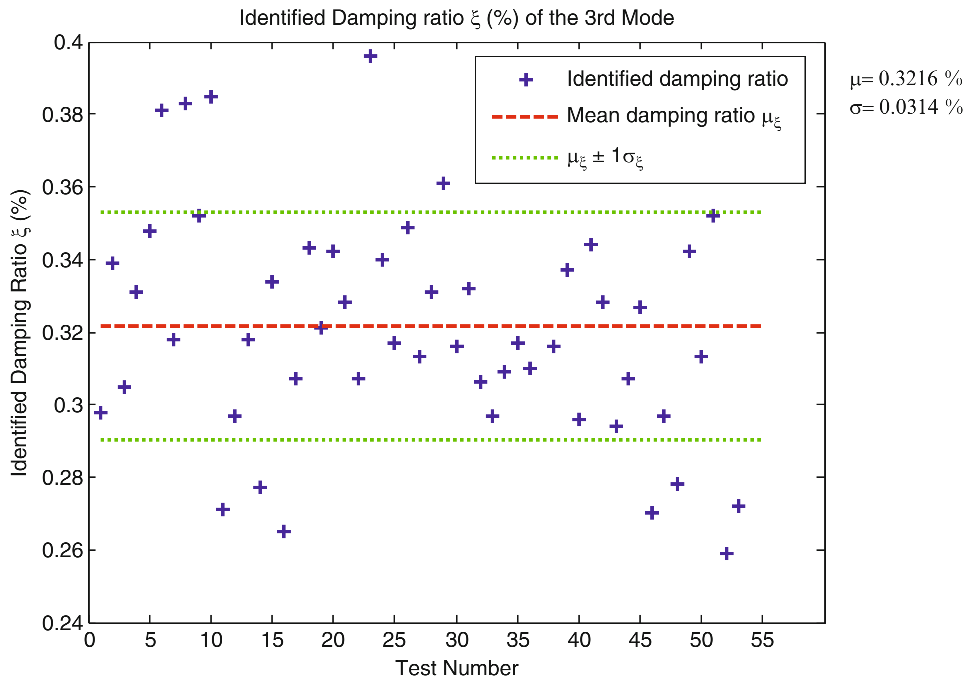

Fig. 17.7 Identified Damping ratio (ξ) of the 3rd mode using ERA-OKID with $\mu_\xi = 0.32\%$ and $\sigma_\xi = 0.0314\%$

Table 17.2 Second order statistical analysis of laboratory tests

Mode	μ_ω	σ_ω	μ_ξ	σ_ξ	CoV_ω	CoV_ξ	$\frac{CoV_\xi}{CoV_\omega}$	$\frac{CoV_\xi}{CoV_\omega} = 0.75\xi^{-1.05}$	$Diff(\%)$
1	6.579	0.0075	0.24	0.112	0.0011	0.4677	412.74	423.67	2.6
2	41.133	0.0110	0.31	0.025	0.0003	0.0810	302.65	319.39	5.2
3	111.483	0.0382	0.32	0.031	0.0003	0.0967	282.41	310.74	9.1

of variation of damping and frequency. Within the limits of this study, this expression was found to be valid independently of the type of excitation or the level of signal-to-noise ratio. The proposed relationship was compared with subspace system identification results performed by independent researchers on an operational structure [8] and also confirmed in 55 laboratory tests on an aluminum cantilever beam carried out by the authors. In both scenarios the accuracy of the proposed expression was satisfactory.

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