Chapter 5
The Mathematical Concept of Measuring Risk

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One of the key tasks in risk management is the quantification of risk implied by uncertain future scenarios which then has to be interpreted with respect to certain risk management decisions. Mathematically, the usual tool for doing so is a quantitative risk measure. The financial industry standard risk measure Value-at-Risk exhibits some serious deficiencies and a vital research activity has been ongoing to search for better alternatives. In this chapter we give an introduction to the general theory of monetary, convex, and coherent risk measures and present illustrating and motivating examples.

Keywords Risk measures · Acceptance sets · Robust representation · Risk sharing

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The Facts

- Quantitative risk measures are key tools in financial risk management. The most prominent examples are the Value-at-Risk and the Average Value-at-Risk risk measures, see Sect. 2.
- Due to their importance in modern risk assessment, there is a vivid research activity, both in practice as well as in academia, on the topic of classifying suitable risk measures. This has amongst others led to the development of the theory of convex monetary risk measures.
- We present three basic approaches to defining such risk measures, one purely axiomatic, and two more constructive ones. The axiomatic approach classifies

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the monetary risk measures as functions with certain properties, see Sect. 3.1. In the two other approaches convex monetary risk measures are constructed by specifying either sets of acceptable portfolios (Sect. 3.2) or by considering the worst case given a set of probability models (Sect. 3.3).
- The exposition is completed by illustrating examples throughout the text.

1 Introduction

Mathematically, the possibility of different, uncertain outcomes of future states of the world can be modeled by a function $X : \Omega \to \mathbb{R}$ where Ω denotes a fixed set of scenarios; i.e. each possible scenario $\omega \in \Omega$ is represented by a real number $X(\omega)$. For example, X could represent the uncertain (discounted) values of a portfolio of financial assets at a future point in time. If there exists an idea about how likely the realizations of the possible future scenarios are, then this is typically modeled by further assuming X to be a random variable; that is one considers a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where the σ-algebra $\mathcal{F}$ denotes the set of all events and $\mathbb{P}$ is a probability measure that determines the probability of each event. Probability theory, which is a concise and axiomatic translation of our intuition about randomness into a mathematical theory, was initiated and developed in the ground breaking work of the Russian mathematician Kolmogorov in 1933 [16].

We note that in the above model approach there exists risk or uncertainty at two levels. When X is assumed to be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, which is the usual approach in quantitative risk management, then uncertainty about the future realization of X is described by the assumed probabilistic structure implied by the probability model $\mathbb{P}$. This type of 'measurable' uncertainty is often referred to as *risk*. However, the choice of a specific probability model $\mathbb{P}$ is a disputed approach since in general there is not enough knowledge to make a reliable choice of one specific probability structure $\mathbb{P}$. This controversial discussion has been nurtured again by the recent financial crisis. There is thus a second level of (model) uncertainty which concerns the 'immeasurable' risk of not knowing the correct probability model and which is referred to as *Knightian uncertainty*. The distinction between measurable risk as opposed to immeasurable uncertainty was established first by Frank Knight in his work 'Risk, Uncertainty, and Profit' [15]. It is an important challenge to extend risk management approaches by mathematical tools that deal with Knightian uncertainty, and, as we will see, the theory of (convex) risk measures is contributing to this research objective.

A *risk measure* helps the risk manager to measure and quantify the risk implied by the uncertainty about the future realization of X. Such quantitative risk measures are usually obtained by applying a certain functional ρ to X which yields a real number $\rho(X)$ that indicates the risk level which then has to be interpreted in terms of risk management decisions. The definition and theory of quantitative risk measures has been initiated and closely influenced by the need for quantitative risk management in the financial and insurance industry. In these areas, the outcome $\rho(X)$ of a risk measure may be interpreted as required capital reserve to hedge against the risk of future losses, as management tool for limiting the amount of risk a unit within an institution may take (for example to constrain a single trader's portfolio), or as insurance premium required to compensate the insurance company for bearing the risk of the insured claims.

Historically, the first one to systematically consider the return of an investment portfolio in relation to its risk was Markowitz in 1952 [18]. He modeled the value of a portfolio by a random variable X and used the *standard deviation* (or *variance*) of the distribution as risk measure. By determining the so-called efficient frontier the portfolio manager could then optimize the return for a given risk level. In 1973, Black, Scholes, and Merton developed the famous *Black-Scholes-Merton formula* to determine the price of a European call option which had enormous impact on the development of financial derivatives markets [8, 19]. This price can be interpreted as a risk measure to hedge against the risk of selling such an option, where the risk measure is the expectation under the so called risk-neutral probability measure. In the early 1990s, the financial industry, public sector, and academia alike started to recognize the need for systematic risk management of the enormous increase in so-called off-balance-sheet products like derivatives. At the investment bank JP Morgan, for instance, the introduction of the so-called Weatherstone 4.15 report asked for the daily assessment of the firms market risk measured in terms of Value-at-Risk (VaR). VaR, which quickly has been established as the industry standard and most prominent risk measure, is a certain quantile of the loss distribution of a portfolio (see Sect. 2.1 for more details). In response to a sequence of disasters in particular over the last two decades, like the ruin of the Barings Bank caused by the single trader Nick Leason in 1995, the fall of the hedge fund Long-Term Capital Management in 1998, or the most serious recent financial crisis that started in 2007, a road to systematic regulation of the banking and insurance industry has evolved. The currently applicable regulation guidelines, which in Germany are implemented by the BAFIN (Bundesanstalt für Finanzdienstleistungsaufsicht), are formulated in the so-called Basel II Accord for the banking sector and Solvency II Accord for the insurance sector. In these guidelines one of the main regulation principles is to require sufficient capital reserves of a firm as to hedge against the risk exposure of future losses using the risk measure VaR.

Despite the status as the industry standard, VaR is often criticized mainly by academics for some fundamental deficiencies. In particular, in certain scenarios VaR is punishing the pooling (or diversification) of risk and is encouraging the accumulation of shortfall risk, which is the opposite of what our intuition about good properties of risk measures would be. Also, while VaR considers the probability that a loss occurs it is not concerned about the size of possible losses. This criticism about VaR has initiated a vital research activity aiming at specifying desirable axioms for risk measures. The outcome of this research has been the axiomatic definitions of the families of *monetary*, *convex*, and *coherent* risk measures; see Artzner et al. [1], Föllmer and Schied [2], and Frittelli and Rosazza-Gianin [5]. Further, the characterization of convex risk measures presented in Sect. 3.3 will reveal that the concept of convex risk measures takes Knightian uncertainty into account. Despite the academic advances and warnings concerning VaR, more appropriate risk measures like Average Value-at-Risk (AVaR), also referred to as Expected Shortfall ($\mathbb{ES}$) (see Sect. 2.2 for more details), have not yet been incorporated into the regulation guidelines. However, the experiences gained during the financial crisis, which revealed the deficiencies in the use of VaR as proposed in the regulation guidelines of Basel II, have initiated a further reformation of regulation mechanisms that takes place under the notion Basel III.

The objective of this chapter is to present the general mathematical concept of monetary risk measures. As described above, the theory of monetary risk measures has been developed in the environment of financial and insurance markets. We will remain in that framework and assume in the following that X models the (discounted) value of a portfolio and a risk measure $\rho(X)$ measures the risk in terms of capital. However, the fundamental concept of (financial) risk management has potential to also be applied to other fields where it is necessary to quantify risk exposure in a concise way. We start by describing in more detail the two most prominent monetary risk measures VaR and AVaR in Sect. 2. In Sect. 3 we then provide the general axiomatic definitions of monetary, convex, and coherent risk measures as well as presenting two alternative methods of constructing convex monetary risk measure. The theory is illustrated by several concrete examples. In Sect. 4 we discuss the optimal risk sharing problem as an example of a typical research question in this field before we conclude in Sect. 5 with some food for thoughts.

2 Two Prominent Examples: VaR and AVaR

2.1 Value-at-Risk

The most common quantitative risk measure in use is the so-called *Value-at-Risk* (VaR_α) at level $\alpha \in (0,1)$. We model the risk of a discounted portfolio at a future point in time by a random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $F(x) := \mathbb{P}(X \le x)$, $x \in \mathbb{R}$, the distribution function of the risk X. Note that the risk manager is concerned about the *downside risk* of X (i.e. small values of X which imply losses). The risk measure VaR_α measures the minimal amount of cash $m \in \mathbb{R}$ that has to be added to the portfolio X such that the probability of a loss of $X + m$ is less than α. In other words,

$$\mathrm{VaR}_\alpha(X) = \inf\bigl\{m \in \mathbb{R} \mid \mathbb{P}(X + m < 0) \le \alpha\bigr\}. \tag{2.1}$$

Depending on the risk management situation, typical values for α are 0.05 (5 %), 0.01 (1 %), or 0.001 (0.1 %). VaR_α can thus be interpreted as the required capital reserve such that the probability of losses at a given future point in time is less than α.

Example 2.1 As mentioned in the introduction, the risk measure stipulated by the Basel II regulations to compute a bank's required capital reserve is VaR_α. More precisely, in Basel II the probability distribution of X is assumed to be normal, that is

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(y-\mu)^2}{\sigma^2}} dy \quad \text{for } x \in \mathbb{R}. \tag{2.2}$$

The mean parameter μ and variance parameter σ^2 have to be estimated from historical data. The computation of VaR_α for a general normal distribution with mean μ and variance σ^2 can be reduced to the computation of the inverse of the standard normal distribution function:

$$\mathrm{VaR}_\alpha(X) = -\mu - \sigma\Phi^{-1}(\alpha),$$

where Φ denotes the standard normal distribution function (i.e. F as in (2.2) with $\mu = 0$ and $\sigma^2 = 1$). Indeed, since the normal distribution function in (2.2) is continuous and strictly increasing it is sufficient by the definition of VaR_α in (2.1) to observe that

$$\begin{aligned}\mathbb{P}\big(X + \big(-\mu - \sigma\Phi^{-1}(\alpha)\big) < 0\big) &= \mathbb{P}\big(X \le \mu + \sigma\Phi^{-1}(\alpha)\big) \\ &= \mathbb{P}\left(\frac{X-\mu}{\sigma} \le \Phi^{-1}(\alpha)\right) \\ &= \Phi\big(\Phi^{-1}(\alpha)\big) = \alpha,\end{aligned}$$

where the last equality comes from the fact that $\frac{X-\mu}{\sigma}$ is standard normal distributed. VaR_α for normally distributed random variables is thus easily computed. However, the assumption that the risk X is normally distributed is very critical since empirical data analysis indicates that the normal distribution strongly underestimates the probability of extreme events in most situations (see also [10], Chap. 6).

Despite its popularity, the use of VaR_α exhibits some serious deficiencies. In particular, it has been fundamentally criticized because of the following two major problems:

(i) VaR_α only considers the probability of encountering losses but not the size of the potential loss in case a loss scenario occurs. Hence, optimizing portfolios under constraints on the risk given by VaR_α may result in portfolios which indeed have a low probability of loss, i.e. less than the specified level α, but which may produce (extremely) high losses if a loss scenario occurs. These effects have been observed, for instance, during the last financial crisis, and it seems evident that in such situations a sound measuring of risk should not only take into account the likeliness of a loss scenarios, but also depend on the quantity that might be lost.

(ii) As we will see in Example 3.3, VaR_α is a positively homogeneous, monetary but not a convex risk measure (see definitions in Sect. 3.1). In particular, one can construct realistic examples of two risks X and Y such that:

$$\mathrm{VaR}_\alpha\left(\frac{1}{2}(X+Y)\right) > \frac{1}{2}\mathrm{VaR}_\alpha(X) + \frac{1}{2}\mathrm{VaR}_\alpha(Y).$$

So $\mathrm{VaR}_\alpha(X+Y)$ of a merged portfolio is not necessarily bounded above by the sum of the VaR_α's of the individual portfolios. But this means that measuring risk with VaR_α may penalize diversification instead of encouraging it. Further, decentralization of risk management might be difficult using VaR_α because one cannot be sure that by aggregating the VaR_α levels of different portfolios (or different units) one will obtain a bound for the overall risk.

2.2 Average Value-at-Risk

The fundamental criticism about VaR_α has led to the search for better alternatives. Before we present the general axiomatic approach which has been the outcome of these efforts in the next section, we will introduce another popular risk measure which attempts to solve problems (i) and (ii) described above. Instead of just considering the quantile corresponding to some specified level $\alpha \in (0, 1)$, one averages over all quantiles less than the given level α. One thus takes into account not only the likeliness of a loss scenario but also the quantity that might be lost, which addresses problem (i) above. The corresponding risk measure is called the *Average Value-at-Risk* (AVaR_α) and is given by

$$\mathrm{AVaR}_\alpha(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\lambda(X)d\lambda, \tag{2.3}$$

where we assume that the expectation is finite, i.e. $\mathbb{E}(|X|) < \infty$, in order to have the right hand side of (2.3) well-defined. Obviously, like VaR_α, also AVaR_α only depends on the probability distribution of X and we have $\mathrm{AVaR}_\alpha \geq \mathrm{VaR}_\alpha$. In Fig. 1 $\mathrm{VaR}_\alpha(X)$ and $\mathrm{AVaR}_\alpha(X)$ of a normal distributed random variable X are plotted against the respective normal density.

But contrary to VaR_α, we will see in Example 3.4 that AVaR_α is convex and even *coherent* (see the definition in Sect. 3.1), which addresses problem (ii) above. One can actually show that AVaR_α is the best approximation of VaR_α in the class of convex risk measures which only depend on the distribution of the portfolio (see [4]). Sometimes, AVaR_α is also referred to as Expected Shortfall ES_α. This is motivated by the following alternative representation which is valid when X has a continuous distribution function:

$$\mathrm{AVaR}_\alpha(X) = \mathbb{E}\big[-X \,|\, -X \geq \mathrm{VaR}_\alpha(X)\big]. \tag{2.4}$$

For general distribution functions the equality in (2.4) turns into a greater-or-equal inequality.

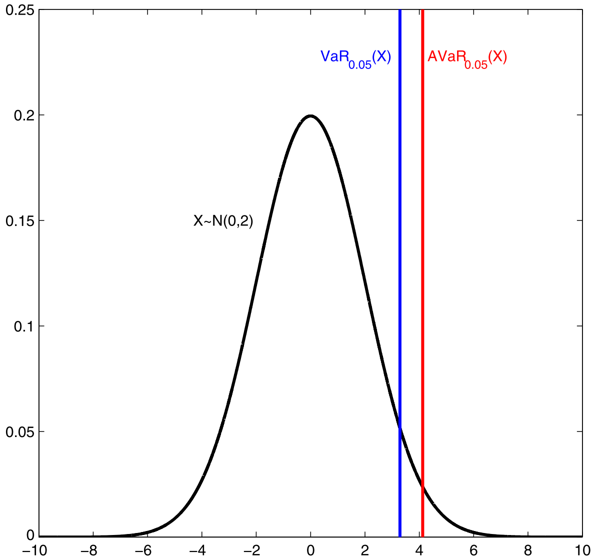

Fig. 1 $\mathrm{VaR}_\alpha(X)$ vs. $\mathrm{AVaR}_\alpha(X)$ at level $\alpha = 5\ \%$ of a normally distributed r.v. X. Indeed $\mathrm{AVaR}_\alpha(X) > \mathrm{VaR}_\alpha(X)$

Example 2.2 Consider again the example of a normally distributed X with mean μ and variance σ^2. Then the distribution function is continuous and we can use representation (2.4) to compute AVaR_α for a given level $\alpha \in (0, 1)$ as

$$\mathrm{AVaR}_\alpha(X) = -\mu + \sigma \frac{\phi(\Phi^{-1}(\alpha))}{\alpha},$$

where ϕ is the density and Φ is the distribution function of a standard normal distribution. Indeed, using representation (2.4) we observe that

$$\mathrm{AVaR}_\alpha(X) = -\mu + \sigma \mathbb{E}\left[-\frac{X-\mu}{\sigma} \,\middle|\, -\frac{X-\mu}{\sigma} \geq \mathrm{VaR}_\alpha\left(\frac{X-\mu}{\sigma}\right)\right],$$

which reduces the problem to the computation of AVaR_α for the standard normal random variable $\frac{X-\mu}{\sigma}$. Again by (2.4) we get

$$\begin{aligned}\mathrm{AVaR}_\alpha\left(\frac{X-\mu}{\sigma}\right) &= -\frac{1}{\alpha}\int_{-\infty}^{\Phi^{-1}(\alpha)} x\phi(x)dx \\ &= \frac{1}{\alpha}\big[\phi(x)\big]_{-\infty}^{\Phi^{-1}(\alpha)} = \frac{\phi(\Phi^{-1}(\alpha))}{\alpha}.\end{aligned}$$

3 Monetary, Convex, and Coherent Risk Measures

Motivated by the examples above, we now introduce the general mathematical theory and characterization of monetary risk measures. More precisely, we give three alternative approaches to monetary risk measures, and we will observe that they are basically equivalent. Namely, in Sect. 3.1 we will undertake an axiomatic approach, in Sect. 3.2 we will construct monetary risk measures by means of a set of acceptable portfolios, whereas in Sect. 3.3 the construction incorporates ideas on how to deal with the uncertainty about the right probabilistic model—the Knightian uncertainty—mentioned in the introduction. The axiomatic approach to (coherent) monetary risk measures goes back to [1] and was further extended by [2, 5, 12]. For an exhaustive treatment of the subject, and in particular for the mathematical details, we refer to [4], for a survey to [3] and [17], Chap. 4.

3.1 What Properties Should a Risk Measure Have?

Consider a set of portfolios $\mathcal{X}$ which we assume to be of sufficiently nice mathematical structure in order to allow for the mathematical analysis which follows; see [4] for the details. We assume that the portfolios are discounted which allows us to compare values in the future with cash amounts today. In the following we are concerned with depicting basic properties that any monetary risk measure ρ should satisfy in order to be suited to measure the risk in terms of cash amounts needed to secure a given portfolio $X \in \mathcal{X}$. One undoubted feature of any risk evaluation is that more is better than less, that is if the payoff of a portfolio X is higher than the payoff of another portfolio Y, then the measured risk $\rho(X)$ of X should be lower than the measured risk $\rho(Y)$ of Y. This property is referred to as monotonicity of the risk measure ρ. Another property of monetary risk measures is based on the observation that cash amounts have no intrinsic risk in the sense that facing a sure loss m, we know that we have to have a corresponding security of $-m$ in order to be able to meet future payments. Therefore, it seems natural to assess the risk of a certain amount m as $-m$ or, more generally, if we add a certain amount m to a given portfolio X then the risk $\rho(X+m)$ of $X+m$ as compared to the risk $\rho(X)$ of X should be increased or decreased by m—depending on whether m is a loss or a gain—and thus corresponds to $\rho(X)-m$. Mathematically these basic features of a monetary risk measure are expressed in the following way:

Definition 3.1 $\rho : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$ is called a *monetary risk measure* if $\rho(0) < \infty$ and ρ satisfies the following conditions for all $X, Y \in \mathcal{X}$:

- *Monotonicity*: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$;
- *Cash invariance*: If $m \in \mathbb{R}$, then $\rho(X+m) = \rho(X) - m$.

Since $\rho(X + \rho(X)) = 0$, we may interpret $\rho(X)$ as a capital requirement, i.e. as the minimal amount of capital that must be added to or can be withdrawn from X in order to obtain zero risk and thus make X acceptable from the point of view of a supervising agency.

Example 3.2 Let X be a square integrable random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The *standard deviation* (or *volatility*) σ_X of X

$$\sigma_X := \sqrt{\mathbb{E}\big[\big(X - \mathbb{E}[X]\big)^2\big]},$$

which first was systematically introduced by Markowitz in the risk analysis [18] of portfolio choices and still is a widely used risk indicator, is neither cash invariant nor monotone and thus not a monetary risk measure. The lack of monotonicity is mainly due to the fact that the standard deviation considers risk symmetric in the sense that the risk of gains is assessed in the same way as the risk of losses, whereas a risk manager is usually only concerned with the risk of losses. For example, let $X > 0$ be a random variable with positive support. Then $Y := aX > X$ for any constant $a > 1$, and also $\sigma_Y = a\sigma_X > \sigma_X$. Thus monotonicity is not fulfilled. Augmenting σ_X to

$$\mathrm{mv}(X) := \mathbb{E}[-X] + \sigma_X,$$

we obtain the so-called *mean-variance risk measure* which is cash invariant, but still does not satisfy monotonicity. The lack of monotonicity is the reason why the very popular asset pricing based on optimizing a portfolio under σ_X or mv has drawbacks such as producing negative prices in some cases.

It is often required that the risk measure ρ should favor diversification: Consider two portfolios X, Y and the possibilities to either invest in X or in Y or in a fraction $\lambda X + (1 - \lambda)Y$, $\lambda \in [0, 1]$, of both. Favoring diversification means that the risk of the diversified investment $\lambda X + (1 - \lambda)Y$ should not exceed the risks of both X and Y, thereby accounting for the fact that the downside risk, in particular the risk of default, is lower in the diversified investment $\lambda X + (1 - \lambda)Y$ as compared to the most risky of X and Y. Formally this property is know as quasi-convexity of the risk measure ρ:

- *Quasi Convexity*: $\rho(\lambda X + (1 - \lambda)Y) \leq \max(\rho(X), \rho(Y))$, for $0 \leq \lambda \leq 1$.

If the risk measure ρ satisfies cash invariance then it can indeed be shown that quasi-convexity is equivalent to convexity, i.e.

- *Convexity*: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$, for $0 \leq \lambda \leq 1$.

The latter property is very desirable from an analytic point of view since it allows for an analysis of convex monetary risk measures by means of tools from the field of convex analysis and optimization. This field provides comprehensive toolboxes for dealing with optimization problems that naturally occur in the financial risk context, like e.g. portfolio optimization under constraints on the portfolio risk given by some convex monetary risk measure.

Recall the Value-at-Risk discussed in Sect. 2.1. It follows immediately that VaR_α is a monetary risk measure in the above sense. However, as we will show in the following example and as was already mentioned in Sect. 2.1, VaR_α is not convex (and thus not quasi convex either).

Example 3.3 Consider two portfolios X, Y which are independent and identically distributed. Let X and Y take the value 100 with probability 0.99 and -100 with probability 0.01. Then the convex combination $\frac{1}{2}(X+Y)$ assumes the values 100, 0, and -100 with probabilities 0.9801, 0.0198, and 0.0001 respectively. Hence, $\mathrm{VaR}_{0.01}(X) = \mathrm{VaR}_{0.01}(Y) = -100$, whereas $\mathrm{VaR}_{0.01}(\frac{1}{2}(X+Y)) = 0$.

As mentioned in Sect. 2.2 the Average Value-at-Risk indeed satisfies convexity and thus favors diversification. Moreover, it inherits a scaling invariance property from the Value-at-Risk which is known as positive homogeneity:

- *Positive Homogeneity*: $\rho(\lambda X) = \lambda\rho(X)$, for $\lambda \geq 0$.

It might be debated whether this latter property is reasonable in every setting as it implies a linear dependence of risk with respect to the amount invested into a portfolio. Especially for large multipliers $\lambda > 0$ this might not be very realistic, and one should instead have $\rho(\lambda X) > \lambda\rho(X)$, and thus pure convexity, to penalize a concentration of risk. The idea here is that if the maximal loss of a portfolio is for instance 1 Euro, then this might not be seen as very risky, whereas the risk of a million times the very same portfolio, and thus a possible loss of a million Euros, may be viewed as providing much more risk than simply a million times the very low risk of the initial portfolio.

Nevertheless, many risk measures exhibit the positive homogeneity property and the positively homogeneous convex monetary risk measures form an important subclass called *coherent risk measures*.

Example 3.4 The Average Value-at-Risk AVaR_α as presented in Sect. 2.2 is a coherent risk measure. Indeed, cash invariance, monotonicity, and positive homogeneity follow immediately from the properties of VaR_α. The crucial property of convexity can easily be deduced from the following representation of AVaR_α:

$$\mathrm{AVaR}_\alpha(X) = \lim_{n\to\infty} \frac{\sum_{i=1}^{[n\alpha]}(-X_{i,n})}{[n\alpha]},$$

where $[n\alpha]$ is the integer part of $n\alpha$, $X_1, \dots, X_n$ is a sequence of independent random variables which have the same distribution as X, and $X_{1,n} \geq \cdots \geq X_{n,n}$ is the order statistics of $(X_1, \dots, X_n)$.

3.2 Constructing Risk Measures via Acceptance Sets

As an alternative to the axiomatic approach, one could define a monetary risk measure on $\mathcal{X}$ by fixing a class of portfolios $\mathcal{A} \subset \mathcal{X}$ which are acceptable in the sense that they do not require additional capital in order to secure their risk. Now the risk of any portfolio $X \in \mathcal{X}$ is evaluated as the minimal amount of cash m that has to be added to X such that $X + m$ is acceptable, that means $X + m \in \mathcal{A}$. The risk measure $\rho_{\mathcal{A}}$ induced by $\mathcal{A}$ in the described way has a formal representation as follows

$$\rho_{\mathcal{A}}(X) = \inf\{m \in \mathbb{R} \mid m + X \in \mathcal{A}\}. \tag{3.1}$$

Assuming that $\mathcal{A}$ satisfies certain properties such as convexity and a monotonicity property ($X \in \mathcal{A}_\rho$, $Y \in \mathcal{X}$, $Y \geq X$, then $Y \in \mathcal{A}_\rho$), one can prove that $\rho_{\mathcal{A}}$ is a convex monetary risk measure as defined in Sect. 3.1; see [4], Proposition 4.7.

Furthermore, one can show that the converse is also true: every convex risk measure in the sense of Sect. 3.1 is induced by an acceptance set $\mathcal{A}$ as in (3.1), and thus the two approaches to defining a risk measure are equivalent. To see this, consider any monetary risk measure ρ and let

$$\mathcal{A}_\rho = \big\{X \in \mathcal{X} \mid \rho(X) \leq 0\big\} \tag{3.2}$$

be the set of portfolios that are acceptable under ρ, the so called acceptance set of ρ. Then, $\rho = \rho_{\mathcal{A}_\rho}$.

In the following we provide some prominent examples of this approach.

Example 3.5 (Monetary Risk Measure Induced by Expected Utility) In economic theory the preferences of some agent are often modeled by a utility function that is a strictly concave and strictly increasing function $u : \mathbb{R} \to \mathbb{R}$, which quantifies how utile the agent considers payoffs compared to each other. The property of being increasing encodes the fact that more is better. The concavity of u is due to the fact that a typical agent is very sensitive to losses, fairly sensitive to gains, but not that sensitive to very large gains as compared to slightly smaller gains, simply because above a certain level she has reached an amount of wealth above which her consumption cannot be significantly improved. Such agents are usually assumed to assess the utility of a portfolio $X \in \mathcal{X}$ by taking the expected utility value of X, i.e. $\mathbb{E}[u(X)]$. Hence, a natural way to obtain a reasonable acceptance set is to call acceptable the set of portfolios $X \in \mathcal{X}$ such that the expected utility exceeds a certain threshold c, that is $X \in \mathcal{A}$ if and only if

$$\mathbb{E}\big[u(X)\big] \geq c.$$

The corresponding acceptance set defines a convex monetary risk measure via (3.1).

Example 3.6 (Shortfall Risk) Recall Example 3.5 If the focus is more on the losses, instead of considering a utility function u and a lower bound c on the expected utility as in the previous example, it is more natural to replace u by a loss function $l : \mathbb{R} \to \mathbb{R}$ which is assumed to be convex and increasing and non-constant. Then the corresponding acceptance set is given by

$$\mathcal{A} := \big\{X \in \mathcal{X} \mid \mathbb{E}\big[l(-X)\big] \leq \tilde{c}\big\} \tag{3.3}$$

where $\tilde{c} \in \mathbb{R}$ is an upper bound on the *shortfall risk* $\mathbb{E}[l(-X)]$ of a portfolio X. This acceptance set defines a convex monetary risk measure $\rho_{\mathcal{A}}$ as in (3.1). If

$l(x) = -u(-x)$, then the acceptance set $\mathcal{A}$ equals the acceptance set in Example 3.5, and the associated risk measures coincide. Note, however, that the loss function l is only required to be increasing, so it may indeed be flat on some interval $(-\infty, a]$. In particular l may vanish on $(-\infty, 0]$, which means that the corresponding acceptability criterion in (3.3) only depends on the possible losses of a portfolio X.

3.3 A Robust Approach to Measuring Risk

Consider any portfolio $X \in \mathcal{X}$. A very natural way to assess its risk is looking at the expected value $\mathbb{E}_{\mathbb{P}}[X]$ of X under some probability measure $\mathbb{P}$. It can be easily verified that $\mathbb{E}_{\mathbb{P}}[-X]$ is indeed a coherent risk measure as introduced in Sect. 3.1. But computing the expected value means that we need full information about the probability distribution of X, that is we need to know the right probability measure $\mathbb{P}$ describing the real world. However, in many cases we don't know which probability model is appropriate to describe the true distribution of X. In other words, there is ambiguity on the right probability measure under which we should take the expectation. We may attempt to overcome this problem by specifying not just one but a class $\mathcal{Q}$ of probability measures that one considers as possible descriptions of reality and then to taking the expectation under each probability measure and to looking at the worst case. The corresponding risk measure is a coherent risk measure and is given by the following expression:

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} \mathbb{E}_{\mathbb{Q}}[-X], \quad X \in \mathcal{X}. \tag{3.4}$$

One can even go one step further and penalize each probability measure $\mathbb{Q} \in \mathcal{Q}$ according to how likely the corresponding probability model appears to be. This is achieved by introducing a penalizing function $\alpha : \mathcal{Q} \to \mathbb{R}$ which assigns to each $\mathbb{Q} \in \mathcal{Q}$ a certain penalization $\alpha(\mathbb{Q})$. The corresponding risk measure is

$$\rho(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} \big(\mathbb{E}_{\mathbb{Q}}[-X] - \alpha(\mathbb{Q})\big), \quad X \in \mathcal{X}, \tag{3.5}$$

which is a convex monetary risk measure. Clearly, letting $\alpha(\mathbb{Q}) = 0$ for all $\mathbb{Q} \in \mathcal{Q}$ we obtain (3.4).

Conversely, one can prove by means of tools from convex analysis that basically every convex monetary risk measure can be represented as in (3.5) where the penalizing function α is known as the dual function of the risk measure ρ. Hence, we observe that the three approaches to defining a risk measure presented throughout Sects. 3.1, 3.2, and 3.3 in principle are equivalent and lead to the same class of risk measures, these are the convex monetary risk measures. Moreover, we note that representation (3.5) reveals a robust structure with respect to model uncertainty and thus the capability of monetary convex risk measures to deal with Knightian uncertainty as discussed in the introduction. For further details, we refer to [4, 9, 14].

Example 3.7 (Entropic Risk Measure) Consider some reference probability measure $\mathbb{P}$ which we believe is the best description of the likeliness of any future events. In presence of ambiguity about the true probability measure, we decide to take into account all probability measures $\mathbb{Q}$ which are consistent with $\mathbb{P}$ in the sense that there are no events that are likely under $\mathbb{Q}$ but have zero probability under our reference probability measure $\mathbb{P}$. We say that such a probability measure $\mathbb{Q}$ is *absolutely continuous* with respect to $\mathbb{P}$, written as $\mathbb{Q} \ll \mathbb{P}$. If $\mathbb{Q}$ strongly deviates from $\mathbb{P}$, it should not play the same role in our risk analysis as probability measures which are just slight modifications of $\mathbb{P}$. This is realized by penalizing each $\mathbb{Q}$ by a kind of distance to $\mathbb{P}$ which is known as the (relative) entropy $H(\mathbb{Q} \mid \mathbb{P})$ of $\mathbb{Q}$ with respect to $\mathbb{P}$. The formal definition is

$$H(\mathbb{Q} \mid \mathbb{P}) := \mathbb{E}_{\mathbb{Q}}\left[\log \frac{d\mathbb{Q}}{d\mathbb{P}}\right]$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is the density of $\mathbb{Q}$ with respect to $\mathbb{P}$. Indeed we have that $H(\mathbb{Q} \mid \mathbb{P}) \geq 0$ and that $H(\mathbb{Q} \mid \mathbb{P}) = 0$ if and only if $\mathbb{Q} = \mathbb{P}$. Taking the worst case over all probability models penalized with the relative entropy as in (3.5) yields the following convex monetary risk measure

$$e_\beta(X) = \sup_{\mathbb{Q} \ll \mathbb{P}} \left(\mathbb{E}_{\mathbb{Q}}[-X] - \frac{1}{\beta} H(\mathbb{Q} \mid \mathbb{P}) \right), \quad X \in \mathcal{X}, \tag{3.6}$$

where we allow for a parameter $\beta > 0$ determining the impact of the weighting. Solving the variational problem appearing on the right hand side of (3.6) we obtain that

$$e_\beta(X) = \frac{1}{\beta} \log \mathbb{E}\left[e^{-\beta X}\right], \quad X \in \mathcal{X}. \tag{3.7}$$

This is the so called *entropic risk measure*.

Example 3.8 (Acceptability Floor) Consider a set $\mathcal{Q}$ of probability measures, and let $\gamma : \mathcal{Q} \to \mathbb{R}$ be such that $\sup_{\mathbb{Q} \in \mathcal{Q}} \gamma(\mathbb{Q}) < \infty$. The function γ specifies an acceptability floor in the sense that a portfolio X is considered to be acceptable if and only if

$$\mathbb{E}_{\mathbb{Q}}[X] \geq \gamma(\mathbb{Q}) \quad \text{for all } \mathbb{Q} \in \mathcal{Q}.$$

The corresponding acceptance set $\mathcal{A}$ defines a convex monetary risk measure $\rho_{\mathcal{A}}$ as in (3.1) which also has the following representation

$$\rho_{\mathcal{A}}(X) = \sup_{\mathbb{Q} \in \mathcal{Q}} \left(\mathbb{E}_{\mathbb{Q}}[-X] + \gamma(\mathbb{Q}) \right), \quad X \in \mathcal{X}.$$

4 A Case Study: Optimal Risk Sharing

So far we have been concerned with specifying a class of risk measures—the (convex) monetary risk measures—which satisfy certain conditions that make them apt for risk management of financial portfolios. However, the choice of an appropriate risk measure within the vast class of convex monetary risk measures, where being *appropriate* depends on factors such as the business structure or stability under optimization, is a non-trivial problem. And even after solving that problem, this is by far not the end of the story. There are a lot of issues arising beyond the level of specifying an appropriate risk measure and simply applying it to quantify the risk of some portfolios. In what follows we present a typical problem arising in risk management when more than one agents are involved. In that case it is very natural to look for cooperation opportunities from which all agents benefit in the sense that the individual risk of each agent is reduced by mutual protection. In other words the agents seek an optimal risk sharing:

Consider n agents with initial portfolios W_i, $i = 1, \dots, n$, who assess the risk of any portfolio offered to them by individual convex monetary risk measures $\rho_i : \mathcal{X} \to \mathbb{R}$, $i = 1, \dots, n$. The aggregate portfolio is $W = W_1 + \cdots + W_n$. An (re-)allocation of W is any $(X_1, \dots, X_n) \in \mathcal{X}^n$ such that $\sum_{i=1}^n X_i = W$. Denote by $\mathbb{A}(W)$ the set of all reallocations of W. We assume that the agents are allowed to exchange risks without changing the aggregate portfolio, that is the agents may agree on exchanging the initial allocation $(W_1, \dots, W_n)$ for some other allocation $(X_1, \dots, X_n)$ of W. The optimal risk sharing problem is to find a reallocation of W amongst the n agents such that the total risk is minimized, that is find $(\bar{X}_1, \dots, \bar{X}_n) \in \mathbb{A}(W)$ such that

$$\sum_{i=1}^n \rho_i(\bar{X}_i) = \inf\left\{ \sum_{i=1}^n \rho_i(Y_i) \;\middle|\; (Y_1, \dots, Y_n) \in \mathbb{A}(W) \right\}. \tag{4.1}$$

Note that, due to cash-invariance, if $(\bar{X}_1, \dots, \bar{X}_n)$ is a solution to (4.1), then we have that for every $(c_1, \dots, c_n) \in \mathbb{R}^n$ such that $\sum_{i=1}^n c_i = 0$ the allocation $(\bar{X}_1 + c_1, \dots, \bar{X}_n + c_n)$ is a solution to (4.1) too. Hence, provided that (4.1) allows for a solution, it is always possible to find a solution which respects the individual rationality constraints of the agents, that is a solution $(\bar{X}_1, \dots, \bar{X}_n)$ to (4.1) such that all agents are better off: $\rho_i(\bar{X}_i) \le \rho_i(W_i)$, $i = 1, \dots, n$.

Consider the following function

$$\square \rho_i(W) := \inf\left\{ \sum_{i=1}^n \rho_i(Y_i) \;\middle|\; (Y_1, \dots, Y_n) \in \mathbb{A}(W) \right\}, \quad W \in \mathcal{X}. \tag{4.2}$$

Problem (4.1) is equivalent to finding $(\bar{X}_1, \dots, \bar{X}_n) \in \mathbb{A}(W)$ such that

$$\rho_1(\bar{X}_1) + \cdots + \rho_n(\bar{X}_n) = \square \rho_i(W). \tag{4.3}$$

Provided that $\square_i \rho_i > -\infty$ it can be shown that $\square_i \rho_i$ is again a convex monetary risk measure which is interpreted as the risk measure of the market or the representative agent. This has a particularly nice interpretation in the case of a company, e.g. an insurer, with different business units/entities possibly being exposed to different kinds of risks. Suppose that the risk of each unit is measured by a convex monetary risk measure, which depends on the business structure of that particular unit. Thus we may view each unit as an agent in the above sense. Then the stand alone risk of unit i with business W_i is $\rho_i(W_i)$. In this situation there are two major questions to be answered. First of all, given the structure of risk measurement for the units, what is a sound monetary risk measure for the whole company as such? Secondly, keeping in mind the typical situation that the measured risks correspond to e.g. solvency capital requirements, what is the advantage of each unit of being member of a group in the sense of being part of the company? One should expect that the risk profile of the unit should profit from the fact that the company may to some extent cover potential losses in that unit with gains from another. If this is the case, this obviously implies a competitive advantage, at least over competitors with similar business plans, but without comparable backup. Otherwise, if this *diversification effect* is not observed, that is if the risk of the unit would simply remain its stand alone risk $\rho_i(W_i)$, then from a risk perspective there is no reason to stay within the company. In that case the shareholders might for instance be tempted to sell off that unit. However, according to the results above, assuming that problem (4.1) admits a solution $(\bar{X}_1, \ldots, \bar{X}_n)$, it is very natural to consider the convex monetary risk measure (4.2) as the risk measure of the company and the optimal allocation $(\bar{X}_1, \ldots, \bar{X}_n)$ as the businesses of the units after an optimal mutual reinsurance. Since we may assume that $(\bar{X}_1, \ldots, \bar{X}_n)$ respects the individual rationality constraints we have that

$$d_i := \rho_i(W_i) - \rho_i(\bar{X}_i) \geq 0 \quad \text{and} \quad D := \sum_{i=1}^{n} d_i = \sum_{i=1}^{n} \big(\rho_i(W_i) - \rho_i(\bar{X}_i)\big) \geq 0 \tag{4.4}$$

where d_i is the diversification effect for unit i, and D is the diversification effect of the company. This gives an elegant answer to the posed questions.

Apparently, the assumption that the risk sharing in (4.1) is over all possible allocations of the aggregate risk W may seem far from reality. Hence, it appears that in a next step one should allow for constraints on the set of allocations $\mathbb{A}(W)$. However, in many cases the optimal allocation coming from solving the unconstrained problem (4.1) indeed exhibits structures which are very often traded, such as linear sharing of the aggregate portfolio or reinsurance by means of stop-loss contracts; see [6, 7, 11, 13]. Moreover, when developing new kinds of (reinsurance) contracts, knowledge of the structure of solutions to the unconstrained problem (4.1) might be of advantage. For a detailed discussion of the optimal risk sharing problem when agents apply convex monetary risk measures we refer to [6, 7, 11, 13].

5 Food for Thoughts

In this section we give an overview of possible research topics in the field of monetary risk measures that are directly related to our presented examples and case studies.

5.1 Appropriate Risk Measures

The class of monetary risk measures is quite broad. Even though monetary risk measures, and in particular convex monetary risk measures, exhibit basic properties that from a certain point of view any risk measure should satisfy, this class of risk measures still includes functions that are not reasonable in application. For instance the worst case risk measure

$$\rho_{worst}(X) := -\inf_{\omega \in \Omega} X(\omega)$$

is a coherent risk measure. But measuring risks by means of ρ_{worst} implies taking no risk, and thus not at all taking an active part in the economy, at least if the markets are assumed to be arbitrage-free. Hence, an important task is, given some specific setting, e.g. a specific field of business or risk profile, to find a suitable convex monetary risk measure for that setting. This involves understanding what suitable in some given setting means, depicting additional requirements that a convex monetary risk measure for that setting should satisfy, and studying and testing the corresponding class of monetary risk measures. In that context, apart from describing certain risk averseness or matching observed structures like e.g. the behavior of risks of large, highly diversified portfolios in certain markets, also numerical issues like stability in optimization play an important role.

5.2 Risk Sharing

The optimal risk sharing problem was outlined in Sect. 4 above. The tasks are to prove the existence of solutions to (4.3), to explicitly characterize these solutions given certain classes of convex monetary risk measures, and to study the problem under additional constraints on the set of feasible allocations.

5.3 Optimization Under Convex Monetary Risk Measures

Many applications of convex monetary risk measures lead to a convex optimization problem which is very often not easily solved analytically. Hence numerical methods have to be applied. However, since convex monetary risk measures are highly non-linear and non-smooth structures, they very often behave poorly in optimization. The field of convex optimization provides a lot of tools to deal with even these kind of problems. The challenge here is to spot the right methods, and maybe to develop new ones which are particularly suited in case of optimization under some convex monetary risk measure, taking advantage of the properties like cash invariance and monotonicity.

6 Summary

The purpose of any risk modeling is, in a first step, to understand what is risk associated to some random outcome and what are the major sources of risk for that outcome. Then, in a second step, the aim is to apply the gained knowledge in order to quantify the risk, thereby opening for the possibility to compare risks of different outcomes and to seek to some extent protection against risk. In case of a financial portfolio two major sources of risk are depicted as being the uncertainty of the exact outcome given multiple scenarios, and the ambiguity about the right probabilistic model for the likeliness of the different scenarios. These sources of risk are accounted for in the theory of convex monetary risk measures which quantify the risk in terms of a cash amount that has to be added to the analyzed portfolio in order to make it acceptable. As, with the increasing complexity of financial products and in the aftermath of the financial crisis, risk analysis and quantification rapidly gains importance, there is a vivid ongoing research activity in the field of (convex) monetary risk measures. Apparently, the developed risk measuring machinery may also be adopted to other than merely financial risks.

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