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Bifurcation at Isolated Eigenvalues for Some Elliptic Equations on $\mathbb{R}^N$

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Abstract. This paper concerns the bifurcation of bound states $u \in L^2(\mathbb{R}^N)$ for a class of second-order nonlinear elliptic eigenvalue problems that includes cases which are already known to exhibit some surprising behaviour. By treating a larger class of nonlinearities we cover new cases such as a situation where there is no bifurcation at a simple isolated eigenvalue lying at the bottom of the spectrum of the linearization. As an application of recent work on bifurcation for problems that are only Hadamard differentiable, we also establish bifurcation at all isolated eigenvalues of odd multiplicity which are sufficiently far from the essential spectrum.

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1. Introduction

As has already been shown in several earlier contributions [3, 15, 17], the study of bifurcation for bound states $u \in L^2(\mathbb{R}^N)$ of simple looking elliptic equations such as

$$-\Delta u + Vu + \frac{u^3}{\xi^2 + u^2} = \lambda u, \tag{1.1}$$

where $V \in L^\infty(\mathbb{R}^N)$ and $\xi \in L^2(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ with $\xi > 0$ on $\mathbb{R}^N$, reveals a number of surprising phenomena. For example, there are potentials V for which bifurcation can occur at points not belonging to the spectrum of the linearized problem

$$-\Delta u + Vu = \lambda u.$$

On the other hand, as one might expect there is bifurcation at all eigenvalues of $-\Delta + V$ lying below the essential spectrum. However, it is shown in Section 5 below that this is no longer the case for the equation

$$-\Delta u + Vu - \frac{u^3}{\xi^2 + u^2} = \lambda u, \tag{1.2}$$

for some choices of V and ξ and the results in [3, 15, 17] do not apply to (1.2). The occurence of the unusual phenomena mentioned above has nothing to do with a lack of smoothness of the functions V and ξ since the conclusions are the same even if the assumption that V and ξ are infinitely differentiable is added.

The purpose of the present paper is to study bifurcation at isolated eigenvalues of the linearization for a class of equation that includes both (1.1) and (1.2), namely

$$-\Delta u + V(x)u + g(x, u) + h(x, \nabla u) + \xi(x)f(\eta(x)u) = \lambda u \tag{1.3}$$

where V, ξ and $\eta : \mathbb{R}^N \to \mathbb{R}$ and the nonlinear functions $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$, $h : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are such that $g(x, 0) = h(x, 0) = f(0) = 0$ and define terms of order higher than linear near $u \equiv 0$. The precise hypotheses are formulated in Section 3 and the main result is Theorem 4.1. Taking $g \equiv 0, h \equiv 0, \eta = 1/\xi$ and $f(s) = \pm s^3/(1 + s^2)$, we recover (1.1), respectively (1.2). We seek solutions (λ, u) where $\lambda \in \mathbb{R}$ and $u \not\equiv 0$ lies in the usual Sobolev space $H^2(\mathbb{R}^N)$ since any distributional solution $u \in L^2(\mathbb{R}^N)$ of equations (1.1) or (1.2) lies in this space.

Under our hypotheses, the equations (1.1) to (1.3) can all be written in the form $M(u) = \lambda u$ where $M : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is a continuous mapping such that $M(0) = 0$. However, M is not Fréchet differentiable at 0 and consequently the classical results about bifurcation cannot be applied in the cases of interest here. (See parts (2) and (3) of Theorem 3.4.) Nonetheless, M is Gâteaux differentiable at 0 and it is also Lipschitz continuous in an open neighbourhood of 0. These properties imply that M is actually Hadamard differentiable at 0. By exploiting this, new conclusions about bifurcation of bound states for (1.3) are obtained in Theorem 4.1 by using a recent abstract result about bifurcation for such problems proved in [19]. The relevant parts of the abstract theory are set out in Section 2. These results provide information about bifurcation at points which are not too close to the essential spectrum of the linearised operator $-\Delta + V$ and, for such points, the conclusions resemble those for smooth situations.

The other main contribution of this paper is to show that this restriction cannot be avoided without introducing new restrictions on the behaviour of the term $\xi f(\eta u)$ in (1.3). A situation of this kind is treated in Section 5 where we show that there may be no bifurcation at a simple eigenvalue Λ lying at the bottom of the spectrum of $-\Delta + V$ and below its essential spectrum, if Λ is too near the essential spectrum. It is important to note that all the other hypotheses of Theorem 4.1 are satisfied and yet the conclusions (ii) and (iii) fail. Thus this situation serves to show that the restriction involving the distance from the essential spectrum in the abstract result is also necessary since all the other hypotheses are satisfied there too.

Exploiting bifurcation theory for Hadamard differentiable mappings is not the only way to deal with problems like (1.1) to (1.3). Rabier [10, 11] has shown that, for an appropriate class of weights ξ and η, the equations can be treated in weighted Sobolev spaces where Fréchet differentiability of the relevant operators holds. It then follows that bifurcation occurs at every isolated eigenvalue of odd multiplicity of $-\Delta + V$. The situation discussed in Section 5 shows that there are choices of ξ and η for which this method cannot be used.

2. Bifurcation without Fréchet differentiability

For real Banach spaces X and Y,

- $B(X,Y) = \{L : X \to Y : L$ is linear and bounded$\}$
- $Iso(X,Y) = \{L \in B(X,Y) : L : X \to Y$ is an isomorphism$\}$
- $\Phi_0(X,Y) = \{L \in B(X,Y) : L$ is a Fredholm operator of index zero$\}$.

Let X and Y be real Banach spaces and consider the equation $F(\lambda, u) = 0$ where $F : \mathbb{R} \times X \to Y$ with $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}$. Setting $\mathcal{S} = \{(\lambda, u) \in \mathbb{R} \times X : F(\lambda, u) = 0$ and $u \neq 0\}$, λ_0 is called a bifurcation point for the equation $F(\lambda, u) = 0$ if there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathcal{S}$ such that $\lambda_n \to \lambda_0$ and $\|u_n\| \to 0$ as $n \to \infty$. There is continuous bifurcation at λ_0 if there exists a bounded connected subset $\mathcal{C}$ of $\mathcal{S}$ such that $\overline{\mathcal{C}} \cap [\mathbb{R} \times \{0\}] = \{(\lambda_0, 0)\}$. In these statements, $\mathcal{S}$ is treated as a metric space with the metric inherited from $\mathbb{R} \times X$.

In this paper, we only deal with the situation where X and Y are Hilbert spaces with $X \subset Y$ and $F(\lambda, u) = M(u) - \lambda u$ for a mapping $M : X \to Y$.

Let $(H, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a real Hilbert space. For a self-adjoint operator $S : D(S) \subset H \to H$ acting in H, the graph norm of S on $D(S)$ is defined by

$$\|u\|_S = \{\|u\|^2 + \|Su\|^2\}^{1/2} \text{ for } u \in D(S).$$

Recall that since S is closed, the graph space $(D(S), \|\cdot\|_S)$ is a Hilbert space. The following result, which is an easy consequence of the closed graph theorem (see Section 5 of [18]), provides a useful way of identifying the associated topology in concrete situations.

Proposition 2.1. *Let $S : D(S) \subset H \to H$ and $T : D(T) \subset H \to H$ be two self-adjoint operators having the same domain $X = D(S) = D(T)$. Then $\|\cdot\|_S$ and $\|\cdot\|_T$ are equivalent norms on the subspace X and $S, T \in B(X, H)$ for any of these norms.*

For a self-adjoint operator $S : D(S) \subset H \to H$, the spectrum and essential spectrum are denoted by $\sigma(S)$ and $\sigma_e(S)$, respectively. If X denotes the graph space of S then (see, for example, [2])

- $\sigma(S) = \{\lambda \in \mathbb{R} : S - \lambda I \notin Iso(X, H)\}$ and $\Lambda = \inf \sigma(S)$
- $\sigma_e(S) = \{\lambda \in \sigma(S) : S - \lambda I \notin \Phi_0(X, H)\}$ and $\Lambda_e = \inf \sigma_e(S)$
- $S - \lambda I \in \Phi_0(X, H) \Leftrightarrow \lambda \notin \sigma_e(S)$
- $\sigma_d(S) = \sigma(S) \backslash \sigma_e(S)$ consists of isolated eigenvalues of finite multiplicity.

The following result concerning bifurcation for an equation of the form $M(u) = \lambda u$ appears as Corollary 6.6 in [19]. Most of [19] is devoted to the more general equation $F(\lambda, u) = 0$ in the setting of Banach spaces.

Proposition 2.2. *Let $(Y, \langle\cdot,\cdot\rangle, \|\cdot\|)$ be a real Hilbert space and let $(X, \|\cdot\|_X)$ be the graph space of some self-adjoint operator acting in Y. For $\delta > 0$, $B_X(0,\delta) = \{u \in X : \|u\|_X < \delta\}$. Consider the equation $M(u) = \lambda u$ where the function $M : X \to Y$ has the following properties.*

(H1) $M(0) = 0$.

(H2) *M is Gâteaux differentiable at 0 and $M'(0) \in B(X,Y)$ is also a self-adjoint operator acting in Y with domain X.*

(H3) *For some $\delta > 0$, $M = M_1 + M_2$ where $M_1 \in C^1(B_X(0,\delta), Y)$ with $M_1'(0) = M'(0)$ and there exists a constant L such that $\|M_2(u) - M_2(v)\|_Y \le L\|u-v\|_Y$ for all $u, v \in B_X(0,\delta)$. Let*

$$L^Y(M_2) = \lim_{\delta\to 0} \sup_{\substack{u,v\in B_X(0,\delta)\\ u\neq v}} \frac{\|M_2(u) - M_2(v)\|_Y}{\|u-v\|_Y} < \infty.$$

Then, for λ_0 such that $d(\lambda_0, \sigma_e(M'(0))) > L^Y(M_2)$ we have the following conclusions.

(i) *If $\ker\{M'(0) - \lambda_0 I\} = \{0\}$, λ_0 is not a bifurcation point.*

(ii) *If $\dim\ker\{M'(0) - \lambda_0 I\}$ is odd, λ_0 is a bifurcation point and there is continuous bifurcation at λ_0.*

(iii) *If $\ker\{M'(0) - \lambda_0 I\} = \mathrm{span}\{\phi\}$ where $\|\phi\|_Y = 1$, λ_0 is a bifurcation point and, for any sequence $\{(\lambda_n, u_n)\} \subset \mathcal{S}$ such that $\lambda_n \to \lambda_0$ and $\|u_n\|_X \to 0$, we have that $u_n = \langle u_n, \phi\rangle\{\phi + w_n\}$ where $\langle w_n, \phi\rangle = 0$ and $\|w_n\|_X \to 0$.*

Remark. There is an example at the end of Section 6 in [19] in which $X = Y = L^2(0,1)$ and (H1) to (H3) are satisfied with $M_1 = 0$ and $L^Y(M_2) = 1$. The mapping M is the Nemytskii operator defined by $M(u)(x) = u(x)^2/(1+|u(x)|)$ for $u \in Y$ and it is shown that the set of bifurcation points for the equation $Mu = \lambda u$ is $[-1,1]$. Since $M'(0) = 0$, $\sigma(M'(0)) = \sigma_e(M'(0)) = 0$ and so for $\lambda_0 = 1$, we have bifurcation at a point where $\ker(M'(0) - \lambda_0 I) = \{0\}$ and $d(\lambda_0, \sigma_e(M'(0))) = L^Y(M_2)$, showing that the conclusion (i) can fail if $d(\lambda_0, \sigma_e(M'(0))) \not> L^Y(M_2)$. In Corollary 5.2 we see that parts (ii) and (iii) can also fail when (H1) to (H3) are satisfied but $d(\lambda_0, \sigma_e(M'(0))) \not> L^Y(M_2)$.

3. An elliptic equation on $\mathbb{R}^N$

In this section we present and prove our main results concerning bound states $u \in L^2(\mathbb{R}^N)$ of the equation (1.3). In the following subsections we introduce our hypotheses term by term and discuss their main consequences.

3.1. The linear term $-\Delta u + Vu$

Instead of restricting attention to bounded potentials, we deal with a larger class which allows for singularities. We suppose that the potential V belongs to the Kato-Rellich class $T_N(q)$ for some $q \geq 2$ with $q > N/2$. This means that

(V) $V = P + Q$ where $P \in L^\infty(\mathbb{R}^N)$ and $Q \in L^r(\mathbb{R}^N)$ for all $r \in [1, q]$ for some $q \geq 2$ with $q > N/2$.

Clearly (V) is satisfied when $V \in L^\infty(\mathbb{R}^N)$, but $V(x) = |x|^{-\alpha}$ is also allowed provided that $0 \leq \alpha < \min\{2, N/2\}$.

An important consequence of condition (V) is that $S = -\Delta + V : D(S) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is a self-adjoint operator with domain $D(S) = H^2(\mathbb{R}^N)$, see [1, 14, 12] for example. Furthermore, elementary Fourier analysis shows that the graph norm of $S = -\Delta$ is equivalent to the usual Sobolev norm on $H^2(\mathbb{R}^N)$, [16] for example. Then by Proposition 2.1 this is also true for $S = -\Delta + V$ whenever V satisfies the condition (V). In particular, $S \in B(H^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$.

3.2. The term $g(x, u)$

The first nonlinear term in (1.3) is required to satisfy the following condition.

(G) $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that, for all $x \in \mathbb{R}^N$, $g(x, 0) = 0$ and $g(x, \cdot) \in C^1(\mathbb{R})$ with

$$|\partial_s g(x, s)| \leq A\{|s|^\alpha + |s|^\beta\} \text{ for all } (x, s) \in \mathbb{R} \times \mathbb{R}^N$$

for some constant A and exponents α, β satisfying $0 < \alpha \leq \beta < \infty$ for $N \leq 4$ and $0 < \alpha \leq \beta \leq \frac{4}{N-4}$ for $N > 4$.

Theorem 3.1. *Let g satisfy* (G) *and set $G(u)(x) = g(x, u(x))$ for $u \in H^2(\mathbb{R}^N)$. Then $G \in C^1(H^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$ with $DG(u)v = \partial_s g(x, u)v$ for all $u, v \in H^2(\mathbb{R}^N)$. In particular, $G(0) = 0$ and $DG(0) = 0$.*

Proof. The restrictions on α and β in condition (G) ensure that the following intervals A_N and B_N are non-empty:

for $N \leq 4$,

$$A_N = \left(0, \tfrac{\alpha}{2}\right] \cap \left(0, \tfrac{2}{N}\right] \cap \left(0, \tfrac{1}{2}\right) \quad \text{and} \quad B_N = \left(0, \tfrac{\beta}{2}\right] \cap \left(0, \tfrac{2}{N}\right] \cap \left(0, \tfrac{1}{2}\right)$$

and for $N > 4$,

$$A_N = \left[\tfrac{\alpha(N-4)}{2N}, \tfrac{\alpha}{2}\right] \cap \left(0, \tfrac{2}{N}\right] \cap \left(0, \tfrac{1}{2}\right) \quad \text{and} \quad B_N = \left[\tfrac{\beta(N-4)}{2N}, \tfrac{\beta}{2}\right] \cap \left(0, \tfrac{2}{N}\right] \cap \left(0, \tfrac{1}{2}\right).$$

Note that for $N > 4$, $A_N \cap B_N = \emptyset$ if $\alpha < \frac{\beta(N-4)}{N}$. For this reason, we decompose $\partial_s g$ in the following way.

Let $\psi \in C^\infty(\mathbb{R})$ be such that $0 \leq \psi(s) \leq 1$ for all s with $\psi(s) \equiv 1$ for $|s| \leq 1$ and $\psi(s) \equiv 0$ for $|s| \geq 2$.

Set $\gamma_1(x, s) = \psi(s)\partial_s g(x, s)$ and $\gamma_2(x, s) = \{1 - \psi(s)\}\partial_s g(x, s)$ so that $\partial_s g(x, s) = \gamma_1(x, s) + \gamma_2(x, s)$ where

$$|\gamma_1(x, s)| \leq C_1|s|^\alpha \ \text{ and } \ |\gamma_2(x, s)| \leq C_2|s|^\beta \ \text{ for all } \ (x, s) \in \mathbb{R}^N \times \mathbb{R}.$$

Noting that (G) implies that γ_1 and γ_2 are Carathéodory functions, set $\Gamma_i(u)(x) = \gamma_i(x, u(x))$ for $i = 1, 2$.

Choosing p such that $1/p \in A_N$, we have that $p > 2$, $\alpha p \geq 2$ and, for $N > 4$, αp and $2p/(p-2) \leq 2N/(N-4)$.

By the fundamental result concerning Nemytskii operators, we have that $\Gamma_1 : L^{\alpha p}(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$ is a bounded continuous mapping. For $u \in L^{\alpha p}(\mathbb{R}^N)$, Hölder's inequality then shows that $T_1(u)v = \Gamma_1(u)v$ defines a bounded linear operator $T_1(u) : L^{r_1}(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ where $r_1 = \frac{2p}{p-2}$ and that

$$T_1 \in C(L^{\alpha p}(\mathbb{R}^N), B(L^{r_1}(\mathbb{R}^N), L^2(\mathbb{R}^N))).$$

Recalling that $H^2(\mathbb{R}^N)$ is continuously embedded in $L^t(\mathbb{R}^N)$ for $2 \leq t < \infty$ if $N \leq 4$ and for $2 \leq t \leq 2N/(N-4)$ for $N > 4$, this implies that $T_1 \in C(H^2(\mathbb{R}^N), B(H^2(\mathbb{R}^N), L^2(\mathbb{R}^N)))$.

Choosing q such that $1/q \in B_N$, the same arguments show that $\Gamma_2 \in C(L^{\beta q}(\mathbb{R}^N), L^q(\mathbb{R}^N))$ and $T_2 \in C(L^{\beta q}(\mathbb{R}^N), B(L^{r_2}(\mathbb{R}^N), L^2(\mathbb{R}^N)))$ where $T_2(u) = \Gamma_2(u)v$ and $r_2 = \frac{2q}{q-2}$. Hence $T_2 \in C(H^2(\mathbb{R}^N), B(H^2(\mathbb{R}^N), L^2(\mathbb{R}^N)))$.

We now have that $T = T_1 + T_2 \in C(H^2(\mathbb{R}^N), B(H^2(\mathbb{R}^N), L^2(\mathbb{R}^N)))$ where $T(u)v(x) = \partial_s g(x, u(x))v(x)$ for $u, v \in H^2(\mathbb{R}^N)$.

From condition (G), it follows that $|g(x,s)| \leq \frac{A}{\alpha+1}\{|s|^{\alpha+1} + |s|^{\beta+1}\}$ where $1 < \alpha + 1 \leq \beta + 1 < \infty$ for $N \leq 4$ and $1 < \alpha + 1 \leq \beta + 1 \leq \frac{N}{N-4}$ for $N > 4$. Also $g = g_1 + g_2$ where

$$|g_1(x,s)| = |\psi(s)g(x,s)| \leq K_1|s|^{\alpha+1}$$

and

$$|g_2(x,s)| = |\{1 - \psi(s)\}g(x,s)| \leq K_2|s|^{\beta+1},$$

so we have that $G_1 \in C(L^{(\alpha+1)2}(\mathbb{R}^N), L^2(\mathbb{R}^N))$ and $G_2 \in C(L^{(\beta+1)2}(\mathbb{R}^N), L^2(\mathbb{R}^N))$ where $G_1(u)(x) = g_1(x, u(x))$ and $G_2(u)(x) = g_2(x, u(x))$. Hence G_1 and $G_2 \in C(H^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$ and therefore $G = G_1 + G_2 \in C(H^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$.

Now we show that $G : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is Fréchet differentiable at u with $DG(u) = T(u)$ where $T = T_1 + T_2$, so that $DG(u)v = \partial_s g(x, u)v$. For $u, v \in H^2(\mathbb{R}^N)$,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \{G(u+v) - G(u) - T(u)v\}^2 dx \\
&\quad = \int_{\mathbb{R}^N} \left\{ \int_0^1 \frac{d}{dt} g(x, u+tv)\, dt - \partial_s g(x,u)v \right\}^2 dx \\
&\quad = \int_{\mathbb{R}^N} \left\{ \int_0^1 \partial_s g(x, u+tv) - \partial_s g(x,u) dt\, v \right\}^2 dx \\
&\quad \leq \int_{\mathbb{R}^N} \int_0^1 \{\partial_s g(x, u+tv) - \partial_s g(x,u)\}^2 dt\, v^2 dx \\
&\quad = \int_0^1 \|[T(u+tv) - T(u)]v\|_{L^2}^2 dt \leq \int_0^1 \|T(u+tv) - T(u)\|_{B(H^2,L^2)}^2 dt \|v\|_{H^2}^2.
\end{aligned}$$

Since $T \in C(H^2(\mathbb{R}^N), B(H^2(\mathbb{R}^N), L^2(\mathbb{R}^N)))$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $\|T(u+w) - T(u)\|^2_{B(H^2,L^2)} < \varepsilon$ whenever $\|w\|_{H^2} < \delta$. Hence

$$\int_0^1 \|T(u+tv) - T(u)\|^2_{B(H^2,L^2)} dt \leq \varepsilon \text{ when } \|v\|_{H^2} < \delta,$$

proving the Fréchet differentiability of $G : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ at u with $DG(u) = T(u)$. Hence $G \in C^1(H^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$. □

3.3. The term $h(x, \nabla u)$

The nonlinear function of the gradient in (1.3) is required to satisfy the following conditions.

(H) $h : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function such that, for all $x \in \mathbb{R}^N$, $h(x, 0) = 0$ and $h(x, \cdot) \in C^1(\mathbb{R}^N)$ with

$$|\nabla_\xi h(x, \xi)| \leq A\{|\xi|^\alpha + |\xi|^\beta\} \text{ for all } x, \xi \in \mathbb{R}^N$$

for some constant A and exponents α, β satisfying $0 < \alpha \leq \beta < \infty$ for $N \leq 2$ and $0 < \alpha \leq \beta \leq \frac{2}{N-2}$ for $N > 2$.

Theorem 3.2. *Let h satisfy* (H) *and set $H(u)(x) = h(x, \nabla u(x))$ for $u \in H^2(\mathbb{R}^N)$. Then $H \in C^1(H^2(\mathbb{R}^N), L^2(\mathbb{R}^N))$ with $DH(u)v = \nabla_\xi h(x, \nabla u) \cdot \nabla v$ for all $u, v \in H^2(\mathbb{R}^N)$. In particular, $H(0) = 0$ and $DH(0) = 0$.*

Proof. Setting $Ju = \nabla u$, we have that $J \in B(H^2(\mathbb{R}^N), [H^1(\mathbb{R}^N)]^N)$ and $H(u) = N(Ju)$ where $N : [H^1(\mathbb{R}^N)]^N \to L^2(\mathbb{R}^N)$ is defined by $N(w)(x) = h(x, w(x))$ for $w \in W = [H^1(\mathbb{R}^N)]^N$. Hence it is enough to prove that $N \in C^1(W, L^2(\mathbb{R}^N))$ with $DN(w)z = \nabla_\xi h(x, w) \cdot z$ for all $w, z \in W$. This can be done by following the same approach as was used to prove Theorem 3.1 so we need only mention a few crucial points. First of all, $\nabla_\xi h$ is decomposed using a radial cut-off function ψ. For the continuity of the resulting Nemytskii operators from $[L^{\alpha p}(\mathbb{R}^N)]^N$ into $L^p(\mathbb{R}^N)$, see [8] for example. Recall that $H^1(\mathbb{R}^N)$ is continuously embedded in $L^t(\mathbb{R}^N)$ for $2 \leq t < \infty$ if $N \leq 2$ and for $2 \leq t \leq 2N/(N-2)$ for $N > 2$. In the present case, the intervals A_N and B_N are given by

$$A_N = \left(0, \tfrac{\alpha}{2}\right] \cap \left(0, \tfrac{1}{N}\right] \cap \left(0, \tfrac{1}{2}\right) \quad \text{and} \quad B_N = \left(0, \tfrac{\beta}{2}\right] \cap \left(0, \tfrac{1}{N}\right] \cap \left(0, \tfrac{1}{2}\right)$$

for $N \leq 2$ and

$$A_N = \left[\tfrac{\alpha(N-2)}{2N}, \tfrac{\alpha}{2}\right] \cap \left(0, \tfrac{1}{N}\right] \cap \left(0, \tfrac{1}{2}\right) \quad \text{and} \quad B_N = \left[\tfrac{\beta(N-2)}{2N}, \tfrac{\beta}{2}\right] \cap \left(0, \tfrac{1}{N}\right] \cap \left(0, \tfrac{1}{2}\right)$$

for $N > 2$. The restrictions on α and β in (H) ensure that these intervals are non-empty. □

3.4. The term $\xi(x)f(\eta(x)u)$

We now come to the term in (1.3) which is not Fréchet differentiable in many interesting cases. The following basic assumption is assumed to hold throughout the discussion and is sufficient for our main result about bifurcation

(F) (i) $f \in C(\mathbb{R})$ with $\lim_{s\to 0} \frac{f(s)}{s} = 0$ and $|f(s) - f(t)| \leq \ell|s-t|$ for all $s,t \in \mathbb{R}$.
(ii) ξ and η are real-valued measurable functions on $\mathbb{R}^N$ such that $\xi\eta \in L^\infty(\mathbb{R}^N)$.

Under the hypothesis (F), we have that $|\xi(x)f(\eta(x)s)| \leq |\xi(x)|\ell|\eta(x)s| \leq \ell\|\xi\eta\|_{L^\infty}|s|$ for all $x \in \mathbb{R}^N$ and $s \in \mathbb{R}$. Setting $F(u)(x) = \xi(x)f(\eta(x)u(x))$ for $u \in L^2(\mathbb{R}^N)$, it follows that $F(u) \in L^2(\mathbb{R}^N)$ and then in the same way, that

$$F(0) = 0 \text{ and } \|F(u) - F(v)\|_{L^2} \leq \ell\|\xi\eta\|_{L^\infty}\|u - v\|_{L^2} \text{ for all } u, v \in L^2(\mathbb{R}^N).$$

Theorem 3.3. *Let the condition* (F) *be satisfied.*

Then $F : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is Gâteaux differentiable at 0 *with $F'(0) = 0$ and it is also Lipschitz continuous with Lipschitz constant $\ell\|\xi\eta\|_{L^\infty}$. Hence $F : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is also Hadamard differentiable at* 0*.*

'A fortiori', the same conclusions hold for $F : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$.

Proof. For $v \in L^2(\mathbb{R}^N)$ and $t \in \mathbb{R}$, we have that $|F(tv)(x)| \leq \ell\|\xi\eta\|_{L^\infty}|tv(x)|$ and the dominated convergence theorem shows that

$$\left\|\frac{F(tv)}{t}\right\|_{L^2} \to 0 \text{ as } t \to 0.$$

This proves that $F : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is Gâteaux differentiable at 0 with $F'(0) = 0$.

The Lipschitz continuity of F is already established in the remarks following (F). □

The main result about bifurcation only requires F to satisfy the condition (F). As we now show, additional restrictions are required in order to obtain properties of $F : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ such as Fréchet differentiability at 0 or compactness. To facilitate the discussion of these results we formulate some extra properties of the weights ξ and η.

(W1) For some $R > 0$, $\eta \in C^2(|x| > R)$ with

$$\eta(x) > 0 \text{ and } \partial^\alpha\eta/\eta \in L^\infty(|x| > R) \text{ for all multi-indices with } |\alpha| \leq 2.$$

Furthermore

$$\frac{1}{\eta} \in L^2(|x| > R) \quad\text{and}\quad \liminf_{n\to\infty} \int_{|x|>n+1} \xi(x)^2 dx \Big/ \int_{|x|>n} \frac{1}{\eta(x)^2} dx > 0.$$

Here are some typical examples of weights satisfying (F) and (W1)

Examples

(i) For some $R, K > 0$, $\eta(x) = |x|^t$ where $t > N/2$ and $|\xi(x)| \geq K|x|^{-t}$ for $|x| > R$. Note that by (F) we must also have that $|\xi(x)| \leq C|x|^{-t}$ for $|x| > R$.

(ii) For some $R, K > 0$, $\eta(x) = e^{c|x|}$ where $c > 0$ and $|\xi(x)| \geq Ke^{-c|x|}$ for $|x| > R$. By (F) we also have that $|\xi(x)| \leq Ce^{-c|x|}$ for $|x| > R$.

(W2) For some $R > 0$, $\xi \in L^2(|x| > R)$ and there exists $\delta > 0$ such that $|\xi(x)\eta(x)| \geq \delta$ a.e. on $\{x \in \mathbb{R}^N : |x| > R\}$.

Theorem 3.4. *Let the condition* (F) *be satisfied.*

(1) *If* (W1) *is also satisfied, then* $F : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ *is Fréchet differentiable at* 0 *if and only if* $f \equiv 0$.

(2) *If* (W2) *is satisfied and* $\lim_{|s|\to\infty} \frac{f(s)}{s} = A \in \mathbb{R}$ *exists, then* $F : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ *is compact if and only if* $A = 0$. *If* $A \neq 0$, *then* $F : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ *is not Fréchet differentiable at* 0.

(3) *If* $\lim_{|s|\to\infty} \frac{f(s)}{s} = B \in \mathbb{R}$ *and* $|x|^{-N/2}\eta(x) \to \infty$ *as* $|x| \to \infty$, *then* $L(F) \geq |B| \liminf_{|x|\to\infty} |\xi\eta(x)|$, *where* $L(F)$ *is the Lipschitz modulus of* $F : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ *defined by*

$$L(F) = \lim_{\delta\to 0} \sup_{\substack{u,v\in B_{H^2}(0,\delta)\\ u\neq v}} \frac{\|F(u) - F(v)\|_{L^2}}{\|u - v\|_{H^2}} < \infty$$

and 'a fortiori', we have the same lower bound for the L^2*-Lipschitz modulus*

$$L^{L^2}(F) = \lim_{\delta\to 0} \sup_{\substack{u,v\in B_{H^2}(0,\delta)\\ u\neq v}} \frac{\|F(u) - F(v)\|_{L^2}}{\|u - v\|_{L^2}} < \infty,$$

which will be used in applying Proposition 2.2 *to* (1.3).

Remarks. The hypotheses (F)(ii) and (W1) imply that $\xi \in L^2(|x| > R)$ for some $R > 0$. As the proof shows, in part (2) the property $\lim_{s\to 0} \frac{f(s)}{s} = 0$ in (F)(i) can be weakened to $f(0) = 0$.

Combining Theorem 3.3 and part (3) we see that, if (F) holds with

$$\lim_{|s|\to\infty} \frac{f(s)}{s} = \pm\ell, \quad \lim_{|x|\to\infty} |x|^{-N/2}\eta(x) = \infty$$

and

$$\liminf_{|x|\to\infty} |\xi\eta(x)| = \|\xi\eta\|_{L^\infty}, \quad \text{then} \quad L(F) = L^{L^2}(F) = \ell\|\xi\eta\|_{L^\infty}$$

for $F : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$.

Proof. (1) It follows from Theorem 3.3 that, if F is Fréchet differentiable at 0, then $F'(0) = 0$. Suppose that there exists $T \neq 0$ such that $f(T) \neq 0$. It suffices to show that there exists a sequence $\{u_n\} \subset H^2(\mathbb{R}^N)\backslash\{0\}$ such that $\|u_n\|_{H^2} \to 0$ and $\|F(u_n)\|_{L^2}/\|u_n\|_{H^2} \not\to 0$. We construct such a sequence as follows.

Let $\varphi \in C^\infty(\mathbb{R})$ have the following properties:

$$\varphi(s) = 0 \text{ for } s \leq 0, 0 \leq \varphi(s) \leq 1 \text{ for } 0 < s < 1, \varphi(s) = 1 \text{ for } s \geq 1.$$

For $n > R$, where R is the radius given in (W1), set $u_n(x) = \frac{T\varphi(|x|-n)}{\eta(x)}$ for $|x| \geq n$ and $u_n(x) = 0$ for $|x| \leq n$. Then $u_n \in C^2(\mathbb{R}^N)$ and, for $|x| = r > n$,

$$\partial_i u_n(x) = \frac{T}{\eta(x)} \left\{ \frac{\varphi'(r-n)x_i}{r} - \frac{\varphi(r-n)\partial_i \eta(x)}{\eta(x)} \right\}$$

and

$$\begin{aligned} \partial^2_{ij} u_n(x) = &- \frac{T\partial_i \eta(x)}{\eta(x)^2} \left\{ \frac{\varphi'(r-n)x_i}{r} - \frac{\varphi(r-n)\partial_i \eta(x)}{\eta(x)} \right\} \\ &+ \frac{T}{\eta(x)} \left\{ \frac{\varphi''(r-n)x_i x_j}{r^2} + \frac{\varphi'(r-n)\delta_{ij}}{r} - \frac{\varphi'(r-n)x_i x_j}{r^3} \right. \\ &\left. - \frac{\varphi'(r-n)x_i \partial_i \eta(x)}{r\eta(x)} - \frac{\varphi(r-n)\partial^2_{ij}\eta(x)}{\eta(x)} + \frac{\varphi(r-n)\partial_i \eta(x)\partial_j \eta(x)}{\eta(x)^2} \right\}. \end{aligned}$$

Using (W1), these formulae show that there is a constant C such that, for $|\alpha| \leq 2$, $|\partial^\alpha u_n(x)| \leq C|\frac{1}{\eta(x)}|$ for $|x| > R$ and all $n > R$. Therefore it follows from (W1) that $u_n \in H^2(\mathbb{R}^N)$ and there is a constant C such that $\|u_n\|_{H^2} \leq C\|\frac{1}{\eta}\|_{L^2(|x|>n)}$. Hence $\|u_n\|_{H^2} \to 0$ as $n \to \infty$. Furthermore,

$$\|F(u_n)\|^2_{L^2} = \int_{\mathbb{R}^N} \xi^2 f(\eta u_n)^2 dx \geq \int_{|x|>n+1} \xi(x)^2 f(T)^2 dx$$

and so

$$\frac{\|F(u_n)\|^2_{L^2}}{\|u_n\|^2_{H^2}} \geq f(T)^2 \frac{\int_{|x|>n+1} \xi(x)^2 dx}{C^2 \|\frac{1}{\eta}\|^2_{L^2(|x|>n)}}.$$

Thus $\liminf_{n\to\infty} \frac{\|F(u_n)\|_{L^2}}{\|u_n\|_{H^2}} > 0$ by (W1) and $F : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is not Fréchet differentiable at 0.

(2) Suppose first that $A = 0$. Then, for every $\varepsilon > 0$, there exists $A_\varepsilon > 0$ such that $|f(s)| \leq A_\varepsilon + \varepsilon|s|$ for all $s \in \mathbb{R}$. Let $\{u_n\}$ be a bounded sequence in $H^2(\mathbb{R}^N)$. Passing to subsequence, we can suppose that $u_n \rightharpoonup u$ weakly in $H^2(\mathbb{R}^N)$ and we now show that $\|F(u_n) - F(u)\|_{L^2} \to 0$, which establishes the compactness of $F : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$.

For any $r > R$,

$$\int_{|x|>r} F(u_n)^2 dx \leq 2\int_{|x|>r} \xi^2 \{A^2_\varepsilon + \varepsilon^2 \eta^2 u^2_n\} dx \leq 2A^2_\varepsilon \int_{|x|>r} \xi^2 dx + 2\varepsilon^2 \|\xi\eta\|^2_{L^\infty} M^2$$

where $\|u_n\|_{L^2} \leq M$ for all n, and the same estimate hold for $\int_{|x|>r} F(u)^2 dx$. On the other hand,

$$\int_{|x|\leq r} \{F(u_n) - F(u)\}^2 dx \leq \int_{|x|\leq r} \xi^2 \ell^2 \eta^2 (u_n - u)^2 dx \leq \ell^2 \|\xi\eta\|^2_{L^\infty} \int_{|x|\leq r} (u_n - u)^2 dx$$

and hence

$$\begin{aligned}\int_{\mathbb{R}^N} &\{F(u_n) - F(u)\}^2 dx \\ &\leq 2\int_{|x|>r} F(u_n)^2 dx + 2\int_{|x|>r} F(u)^2 dx + \int_{|x|\leq r} \{F(u_n) - F(u)\}^2 dx \\ &\leq 8A_\varepsilon^2 \int_{|x|>r} \xi^2 dx + 8\varepsilon^2 \|\xi\eta\|_{L^\infty}^2 M^2 + \ell^2 \|\xi\eta\|_{L^\infty}^2 \int_{|x|\leq r} (u_n - u)^2 dx.\end{aligned}$$

Since $H^2(|x| < r)$ is compactly embedded in $L^2(|x| < r)$, this shows that

$$\limsup_{n\to\infty} \|F(u_n) - F(u)\|_{L^2}^2 \leq 8A_\varepsilon^2 \int_{|x|>r} \xi^2 dx + 8\varepsilon^2 \|\xi\eta\|_{L^\infty}^2 M^2.$$

But $\int_{|x|>r} \xi^2 dx \to 0$ as $r \to \infty$, so

$$\limsup_{n\to\infty} \|F(u_n) - F(u)\|_{L^2}^2 \leq 8\varepsilon^2 \|\xi\eta\|_{L^\infty}^2 M^2 \text{ for all } \varepsilon > 0,$$

proving that $\|F(u_n) - F(u)\|_{L^2} \to 0$ as required.

Suppose now that $A \neq 0$. Setting $g(s) = f(s) - As$ and then $G(u) = \xi g(\eta u)$, the preceding argument shows that $G : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is compact. Choose $w \in C_0^\infty(\mathbb{R}^N)$ such that $w \not\equiv 0$ and $\operatorname{supp} w \subset B(0, 1/2)$. Consider the sequence defined by $u_n(x) = w(x - ne_1)$ where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^N$. It is easily seen that $u_n \rightharpoonup 0$ weakly in $H^2(\mathbb{R}^N)$ and hence, by the argument just used to prove the compactness of F when $A = 0$, we have $\|G(u_n)\|_{L^2} \to 0$ since $G(0) = 0$. But, for $m, n \geq R + 1$ and $m \neq n$,

$$\|F(u_n) - F(u_m)\|_{L^2} \geq \|A\xi\eta(u_n - u_m)\|_{L^2} - \|G(u_n) - G(u_m)\|_{L^2}$$

where

$$\|A\xi\eta(u_n - u_m)\|_{L^2}^2 \geq A^2\delta^2 \int_{\mathbb{R}^N} (u_n - u_m)^2 dx = 2A^2\delta^2 \int_{\mathbb{R}^N} w^2 dx$$

since $\operatorname{supp} u_n \cap \operatorname{supp} u_m = \emptyset$ and

$$\|G(u_n) - G(u_m)\|_{L^2} \to 0 \text{ as } n, m \to \infty.$$

Thus $\{F(u_n)\}$ has no convergent subsequence and consequently, $F : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is not compact.

Furthermore, since $G(u) = \xi g(\eta u) = F(u) - A\xi\eta u$, it follows from Theorem 3.3 that $G : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is Hadamard differentiable at 0 with $G'(0)u = -A\xi\eta u$. But we have just shown that $\|G'(0)(u_n - u_m)\|_{L^2} \geq \sqrt{2} A\delta \|w\|_{L^2} > 0$ for all $m, n \geq R + 1$ and $m \neq n$, from which it follows that $G'(0) : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is not a compact linear operator. Since the Fréchet derivative of a compact operator is always compact (see [9], for example), this implies that $G : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is not Fréchet differentiable at 0. But the bounded linear operator $u \mapsto A\xi\eta u$ is Fréchet differentiable from $H^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$. Hence $F : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ cannot be Fréchet differentiable at 0.

(3) Choose some $u \in C_0^\infty(\mathbb{R}^N)$ such that $u \geq 0$ on $\mathbb{R}^N$ and $\|u\|_{L^2} = 1$. Then, for $\delta > 0$ and $n \in \mathbb{N}$, define u_n^δ by

$$u_n^\delta(x) = \frac{\delta}{2n^{N/2}} u\left(\frac{x - ne_1}{n}\right) \text{ where } e_1 = (1, 0, \dots, 0).$$

Then, using the change of variable $z = \frac{x-ne_1}{n}$ in the integrals, we have that

$$\|u_n^\delta\|_{L^2} = \frac{\delta}{2}\|u\|_{L^2}, \|\partial_i u_n^\delta\|_{L^2} = \frac{\delta}{2n}\|\partial_i u\|_{L^2}, \|\partial_i\partial_j u_n^\delta\|_{L^2} = \frac{\delta}{2n^2}\|\partial_i\partial_j u\|_{L^2}$$

for $1 \leq i, j \leq N$. Hence $u_n^\delta \in B_{L^2}(0,\delta)$ for all n and there exists n_0 such that $u_n^\delta \in B_{H^2}(0,\delta)$ for all $n \geq n_0$. Also

$$\begin{aligned}
\|F(u_n^\delta)\|_{L^2}^2 &= \int_{\mathbb{R}^N} \xi(x)^2 f\left(\eta(x)\frac{\delta}{2n^{N/2}} u\left(\frac{x-ne_1}{n}\right)\right)^2 dx \\
&= \int_{\mathbb{R}^N} \xi(n[z+e_1])^2 f\left(\eta(n[z+e_1])\frac{\delta}{2n^{N/2}}u(z)\right)^2 n^N dz \\
&= \int_{\{z:u(z)>0\}} \xi(n[z+e_1])^2 \left\{\frac{f(w_n(z))}{w_n(z)}\right\}^2 \eta(n[z+e_1])^2 \left\{\frac{\delta u(z)}{2}\right\}^2 dz
\end{aligned}$$

where

$$w_n(z) \equiv \eta(n[z+e_1])\frac{\delta}{2n^{N/2}}u(z) \to \infty \text{ as } n \to \infty$$

for all $z \neq -e_1$ such that $u(z) > 0$. Hence

$$\begin{aligned}
&\liminf_{n\to\infty} \xi(n[z+e_1])^2 \left\{\frac{f(w_n(z))}{w_n(z)}\right\}^2 \eta(n[z+e_1])^2 \left\{\frac{\delta u(z)}{2}\right\}^2 \\
&\quad \geq B^2 \left\{\frac{\delta u(z)}{2}\right\}^2 \liminf_{|x|\to\infty} (\xi\eta)^2(x)
\end{aligned}$$

for almost all $z \in \mathbb{R}^N$. By Fatou's Lemma,

$$\liminf_{n\to\infty} \|F(u_n^\delta)\|_{L^2}^2 \geq \left[\frac{B\delta}{2}\right]^2 \liminf_{|x|\to\infty} [\xi\eta(x)]^2 \int_{\mathbb{R}^N} u(z)^2 dz$$

and hence

$$\liminf_{n\to\infty} \|F(u_n^\delta)\|_{L^2} \geq \frac{|B|\delta}{2}\|u\|_{L^2} \liminf_{|x|\to\infty} |\xi\eta(x)|.$$

For all $\delta > 0$, this implies that

$$\begin{aligned}
\sup_{\substack{u,v\in B_{H^2}(0,\delta)\\ u\neq v}} \frac{\|F(u)-F(v)\|_{L^2}}{\|u-v\|_{H^2}} &\geq \sup_{n\geq n_0} \frac{\|F(u_n^\delta)\|_{L^2}}{\|u_n^\delta\|_{H^2}} \geq \liminf_{n\to\infty} \frac{\|F(u_n^\delta)\|_{L^2}}{\|u_n^\delta\|_{H^2}} \\
&= \liminf_{n\to\infty} \frac{\|F(u_n^\delta)\|_{L^2}}{\|u_n^\delta\|_{L^2}} \frac{\|u_n^\delta\|_{L^2}}{\|u_n^\delta\|_{H^2}} \geq |B| \liminf_{|x|\to\infty} |\xi\eta(x)|,
\end{aligned}$$

since $\|u_n^\delta\|_{L^2} = \frac{\delta}{2}\|u\|_{L^2}$ and $\frac{\|u_n^\delta\|_{L^2}}{\|u_n^\delta\|_{H^2}} \to 1$ as $n \to \infty$.

Thus $L(F) \geq |B| \liminf_{|x|\to\infty} |\xi\eta(x)|$ for $F : H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$. $\square$

4. Results about bifurcation for (1.3)

Under the hypotheses (V), (G), (H) and (F), we can now treat (1.3) as a special case of Proposition 2.2 and hence of the more general Theorem 6.3 in [19]. For this we choose

$$X = H^2(\mathbb{R}^N), Y = L^2(\mathbb{R}^N) \text{ and } M = S + G + H + F : X \to Y$$

where S, G, H and F are defined in Sections 3.1 to 3.4.

In Subsection 3.1, we have already noted that $S = -\Delta + V : X \subset Y \to Y$ is self-adjoint and that its graph norm on X is equivalent to the usual Sobolev norm for $H^2(\mathbb{R}^N)$.

It follows from Theorems 3.1 to 3.3 that $M \in C(X, Y)$ with $M(0) = 0$ and that $M : X \to Y$ is Gâteaux differentiable at 0 with $M'(0)u = Su$ for all $u \in X$. Setting $M_1 = S+G+H$ and $M_2 = F$, these results also show that $M_1 \in C^1(X,Y)$ with $M_1'(0) = S$ and that $L^Y(M_2) \leq \ell\|\xi\eta\|_{L^\infty}$. Thus the conditions (H1) to (H3) of Proposition 2.2 are satisfied. Furthermore, setting

$$F(\lambda, u) = M(u) - \lambda u \text{ for } (\lambda, u) \in \mathbb{R} \times X,$$

we have that conditions (B1) to (B5) of [19] are satisfied for all $\lambda_0 \notin \sigma_e(S)$. In particular, for $\lambda_0 \notin \sigma_e(S)$, Proposition 2.2 can be applied to (1.3) provided that $d(\lambda_0, \sigma_e(S)) > \ell\|\xi\eta\|_{L^\infty}$. In Section 5 we shall provide examples, see Corollary 5.2 in particular, with $G = H = 0$, where $0 < d(\lambda_0, \sigma(S)) < L^Y(M_2) = \ell\|\xi\eta\|_{L^\infty}$ and the conclusions of Proposition 2.2 fail. These examples also show that the hypothesis (6.1) in Theorem 6.3 of [19] plays an essential role since the other assumptions are satisfied yet the conclusion (6.1) fails.

In the present context, λ_0 is a bifurcation point for the equation $M(u) = \lambda u$ if and only if there exists a sequence $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times H^2(\mathbb{R}^N)$ of solutions of (1.3) with $u_n \not\equiv 0$ such that $\lambda_n \to \lambda_0$ and $\|u_n\|_{H^2} \to 0$.

From the preceding remarks, as an immediate consequence of Proposition 2.2 we obtain the following result. By the methods used in Section 3.2 and 3.3 a nonlinearity of the form $k(x, u(x), \nabla u(x))$ could be treated instead of the separated case $g(x, u) + h(x, \nabla u)$ adopted here.

Theorem 4.1. *Consider the equation* (1.3) *under the hypotheses* (V), (G), (H) *and* (F) *and* λ_0 *such that* $d(\lambda_0, \sigma_e(S)) > \ell\|\xi\eta\|_{L^\infty}$ *where* $S = -\Delta + V$.

(i) *If* $\ker\{S - \lambda_0 I\} = \{0\}$, *then* λ_0 *is not a bifurcation point.*

(ii) *If* $\dim\ker\{S - \lambda_0 I\}$ *is odd, then there is continuous bifurcation at* λ_0.

(iii) *If* $\ker\{S - \lambda_0 I\} = \text{span}\{\phi\}$ *where* $\|\phi\| = 1$ *there is continuous bifurcation at* λ_0 *and, for any sequence* $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times H^2(\mathbb{R}^N)$ *of solutions of* (1.3) *with* $u_n \not\equiv 0$ *such that* $\lambda_n \to \lambda_0$ *and* $\|u_n\|_{H^2} \to 0$, *we have that* $u_n = \langle u_n, \phi\rangle_{L^2}\{\phi + w_n\}$ *where* $\langle w_n, \phi\rangle_{L^2} = 0$ *and* $\|w_n\|_{H^2} \to 0$.

Remarks. If $f \equiv 0$, the result applies to all points $\lambda_0 \notin \sigma_e(S)$ and the conclusions follow from standard bifurcation theory since $M \in C^1(X, Y)$. For $f \not\equiv 0$, previous work deals with the case $g \equiv h \equiv 0$ under much more restrictive assumptions the term F. The following proposition summarises most of the earlier contributions. Its hypotheses imply that (V), (G), (H) and (F) are all satisfied with $G = H = 0$ and $\eta = 1/\xi$. Hence in Proposition 4.2 we are discussing bifurcation for a special case of (1.3), which includes (1.1) but not (1.2). As pointed out in [17], any distributional solution $u \in L^2(\mathbb{R}^N)$ lies in $W^{2,p}(\mathbb{R}^N)$ for all $p \in [2, \infty)$ and bifurcation for (4.1) with respect to the H^2-norm, as is discussed in Theorem 4.1, is equivalent to bifurcation with respect to the L^2-norm.

Proposition 4.2. *Consider the equation*

$$-\Delta u + Vu + \xi f\left(\frac{u}{\xi}\right) = \lambda u \text{ for } u \in H^2\left(\mathbb{R}^N\right) \tag{4.1}$$

under the following hypotheses: $V \in L^\infty(\mathbb{R}^N), \xi \in L^2(\mathbb{R}^N)$ *with* $\xi > 0$ *a.e. and* $f \in C^1(\mathbb{R}^N)$ *is an odd function such that*

(i) $\ell = \sup_{s\in\mathbb{R}} |f'(s)| < \infty$, $f'(0) = 0$ *and* $\left(\frac{f(s)}{s}\right)' > 0$ *for all* $s > 0$,

(ii) *there exists* $A > 0$ *such that* $\sup_{s>0} |As - f(s)| < \infty$.

We have the following conclusions about bifurcation for (4.1), *where* $S = -\Delta + V$. *Recall that* $\Lambda = \inf \sigma(S)$ *and* $\Lambda_e = \inf \sigma_e(S)$.

(a) *If* $\ker(S - \lambda_0 I) = \{0\}$ *and either* $d(\lambda_0, \sigma_e(S)) > A$ *or* $\lambda_0 < \Lambda_e$, *then* λ_0 *is not a bifurcation point.*

(b) *If* $\ker(S - \lambda_0 I) \neq \{0\}$ *and* $\lambda_0 < \Lambda_e$, *then* λ_0 *is a bifurcation point.*

(c) *Suppose that* $N \leq 3$ *and that* $\eta = 1/\xi$ *has the following properties:*

$$\eta \in W^{2,\infty}_{loc}(\mathbb{R}^N),\ \inf \eta > 0 \ \text{ and for}$$
$$\text{some } \ t > 0,\ \partial^\alpha \eta^t \in L^\infty(\mathbb{R}^N) \text{ for } 1 \leq |\alpha| \leq 2.$$

If $\lambda_0 \notin \sigma_e(S)$ *and* $\dim\ker(S - \lambda_0 I)$ *is odd, then* λ_0 *is a bifurcation point.*

(d) *If* $A > \Lambda_e - \Lambda$ *and* $\lambda_0 \in [\Lambda_e, \Lambda + A]$, *then* λ_0 *is a bifurcation point.*

(e) *Suppose that* $V \equiv 0$ *and* $\xi(x) = (1 + |x|^2)^t$ *for some* $t > N/4$. *If* $\lambda_0 > A[1 + \frac{4t-N}{2}]$, *then* λ_0 *is not a bifurcation point and* $\lambda_0 > A = \Lambda + A$ *since* $\Lambda = \Lambda_e = 0$.

Remark 1. Since $f(0) = 0$, it follows from (i) and (ii) that

$$\lim_{s\to 0} \frac{f(s)}{s} = 0 < \frac{f(s)}{s} < A = \lim_{s\to\infty} \frac{f(s)}{s} \text{ for all } s > 0.$$

Also $f'(s) > \frac{f(s)}{s}$ for all $s > 0$, so $\ell \geq A$ and in some case the inequality is strict. For example, $f(s) = |s|^{2\sigma} s/(1 + s^2)^\sigma$ satisfies (i) and (ii) for all $\sigma > 0$, but $\ell = \sup_{s>0} f'(s) = f'(\sqrt{2\sigma + 1}) > 1 = A$. On the other hand, $f(s) = s - \tanh s$ also satisfies (i) and (ii) and in this case $\ell = A = 1$.

Remark 2. With $\eta = 1/\xi$ and $g = h = 0$, we see that (4.1) is a special case of (1.3) satisfying the conditions (V), (G), (H), (F) and (W2). Therefore Theorem 4.1 applies and yields information not contained in the conclusions (a) to (e). Notice however that if f satisfies the hypotheses of Proposition 4.2, $-f$ does not. Of course, $-f$ still satisfies (F) and so Theorem 4.1 can treat this case too, but as is shown in Section 5, the conclusions (b) and (d) of the proposition fail in this case, as does part (c) for $\xi(x) = e^{-\alpha|x|}$ with large positive α. Theorem 4.1 places much weaker restrictions on the weights ξ and η.

Proof. Defining h by $h(0) = 0$ and $h(s) = f(s)/(As)$ for $s \neq 0$, our hypotheses on f imply that h satisfies the conditions (H3) to (H5) of [17] and our equation (4.1) is just (1.1) of [17] with $q = V$. Noting Proposition 2.1 of [17], the conclusions (a), (b) and (d) are consequences of statements (R1) to (R3) in Section 3 of [17], whereas (e) is just a restatement of the example following Theorem 3.1 in that paper.

The hypotheses on η made in part (c) mean that η is a transference weight of order 2 in the sense introduced by P.J. Rabier in [10, 11]. They ensure that $\xi = 1/\eta$ satisfies the condition (H2)* of Section 4 of [17], where the results of [10] are applied to (4.1). In particular, the weighted Sobolev space $W^{2,2}_{\eta}$ is continuously embedded in $H^2(\mathbb{R}^N)$ so bifurcation at λ_0 in $W^{2,2}_{\eta}$ implies that λ_0 is a bifurcation point for (4.1) in the sense of the present paper. Thus part (c) follows from statement (C2) in Section 4.2 of [17]. In fact, as (C2) shows, Rabier's work provides a stronger statement about bifurcation at such points. □

Commentaries on the conclusions

(1) Since $A \leq \ell$, the conclusion (a) is sharper than (i) of Theorem 4.1 for the equation (4.1).
(2) Under the hypotheses of the proposition, consider a potential V such that $\Lambda = \Lambda_e$ and such that there exist $b > a > \Lambda_e$ such that $(a, b) \cap \sigma(S) = \emptyset$. Then choose f with $A > b - \Lambda_e$. We now have that $(a, b) \subset [\Lambda_e, \Lambda + A]$ and hence every $\lambda_0 \in (a, b)$ is a bifurcation point by part (d) despite the fact that $\lambda_0 \notin \sigma(S)$. Note that at these points, $d(\lambda_0, \sigma_e(S)) \leq \lambda_0 - \Lambda_e < b - \Lambda_e < A \leq \ell\|\xi\eta\|_{L^\infty}$ since $\xi\eta \equiv 1$.
(3) In the next section we show that there are functions f satisfying (F) and weights ξ for which statement (b) of the proposition fails even when $\lambda_0 = \Lambda$ is a simple eigenvalue.
(4) The approach devised by Rabier can be used to establish bifurcation for (4.1) at eigenvalues of odd multiplicity of S under much weaker hypotheses on f provided that $\eta = 1/\xi$ is a transference weight. See Section 5 of [10].

5. A case where there is no bifurcation at a simple eigenvalue

In this section we consider a special case of (1.3) in which the hypotheses (V),(G), (H) and (F) are satisfied and $\Lambda = \inf \sigma(S) < \inf \sigma_e(S)$ is a simple eigenvalue of S. Consider the equation

$$-u'' + Vu + e^{-\alpha|x|} f(e^{\alpha|x|}u) = \lambda u \text{ on } \mathbb{R}, \tag{5.1}$$

where α is a positive constant,

(V_0) $V \in C_0(\mathbb{R})$ with $V \leq 0$ but $V \not\equiv 0$ on $\mathbb{R}$, and

(K) $f \in C^1(\mathbb{R})$ is an odd function with $f'(0) = 0$, $(\frac{f(s)}{s})' \le 0$ for $s > 0$ and $\ell \equiv \sup_{s>0} |f'(s)| < \infty$.

Clearly $V \in L^\infty(\mathbb{R})$ and so $V \in T_1(q)$ for all $q \ge 2$. Thus $Su = -u'' + Vu$ defines a self-adjoint operator $S : H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R})$. It follows from (V_0) that $\Lambda \equiv \inf \sigma(S) < 0 = \inf \sigma_e(S)$ and that Λ is a simple eigenvalue of S with an eigenfunction $\phi \in C^2(\mathbb{R})$ which is strictly positive on $\mathbb{R}$ with $\|\phi\|_{L^2} = 1$. Note that $d(\Lambda, \sigma_e(S)) = |\Lambda|$.

For the ensuing calculations it is convenient to write f in the form $f(s) = k(s)s$ with $k(0) = 0$, where (K) ensures that $k \in C(\mathbb{R})$ is an even function having the properties

$$k \in C^1((0,\infty)) \text{ with } k' \le 0 \text{ on } (0,\infty) \text{ and } -L \le k \le 0 \text{ on } \mathbb{R},$$

where $L \equiv -\lim_{s\to\infty} k(s) \in [0, \ell]$. Note that if a function f satisfies the hypotheses of Proposition 4.2, then $-f$ satisfies the condition (K). In particular, $f(s) = -|s|^{2\sigma} s/(1+s^2)^\sigma$ satisfies (K) for all $\sigma > 0$.

Recall that $H^2(\mathbb{R})$ is continuously embedded in $C^1(\mathbb{R})$. Since V and k are continuous, $u \in C^2(\mathbb{R})$ for any solution $(\lambda, u) \in \mathbb{R}\times H^2(\mathbb{R}^N)$ of (5.1). This equation can then be written as

$$-u'' + \{V + k(e^{\alpha|x|}u)\}u = \lambda u. \tag{5.2}$$

Let $Z > 0$ be such that $V(x) = 0$ for $|x| \ge Z$. Note that on $\mathbb{R}\backslash(-Z, Z)$, for $\lambda < 0$, the equation can be written as

$$u'' = \{k(e^{\alpha|x|}u) + |\lambda|\}u. \tag{5.3}$$

Theorem 5.1. *Suppose that the conditions* (V_0) *and* (K) *are satisfied and set* $L \equiv -\lim_{s\to\infty} \frac{f(s)}{s}$. *Note that* $0 \le L \le \ell$.

(i) *If* $|\Lambda| > \ell$, *there is continuous bifurcation at* Λ. *Furthermore, for any sequence* $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times H^2(\mathbb{R})$ *of solutions of* (5.1) *with* $u_n \not\equiv 0$ *such that* $\lambda_n \to \Lambda$ *and* $\|u_n\|_{H^2} \to 0$, *we have that* $u_n = \langle u_n, \phi\rangle_{L^2}\{\phi + w_n\}$ *where* $\langle w_n, \phi\rangle_{L^2} = 0$ *and* $\|w_n\|_{H^2} \to 0$. *Also, for* n *large enough,* $u_n \in C^2(\mathbb{R})$ *has no zeros and so there is a sequence* $\{(\lambda_n, u_n)\} \subset \mathbb{R} \times H^2(\mathbb{R})$ *of solutions of* (5.1) *with* $u_n > 0$ *on* $\mathbb{R}$ *and* $\lambda_n \le \Lambda$ *such that* $\lambda_n \to \Lambda$ *and* $\|u_n\|_{H^2} \to 0$.

(ii) *If* $|\Lambda| < L$ *and* $\alpha > |\Lambda|^{1/2}$, *then* Λ *is not a bifurcation point for* (5.1). *Indeed, setting* $\varepsilon = \min\{(L - |\Lambda|)/2, \alpha^2 - |\Lambda|, |\Lambda|\}$, $u \equiv 0$ *is the only solution of* (5.1) *in* $H^2(\mathbb{R})$ *for* $\lambda \in (\Lambda - \varepsilon, \Lambda + \varepsilon)$.

Remark 1. Inspecting the proof of part (i), we observe that $\lambda_n < \Lambda$ provided that $f(s) < 0$ for all $s > 0$, since

$$\Lambda = \int_{\mathbb{R}} (\phi')^2 + V\phi^2 dx > \int_{\mathbb{R}} (\phi')^2 + W_n\phi^2 dx \ge \inf \sigma(S_n) = \lambda_n,$$

in this case.

Remark 2. In part (ii), the proof in fact shows that $u \equiv 0$ is the only solution with $u(x) \to 0$ as $x \to \infty$ for $\lambda \in (\Lambda-\varepsilon, \Lambda+\varepsilon)$. Note that in Step 1, the monotonicity of J implies that $\lim_{x\to\infty} u'(x)^2$ exists, and hence $u'(x) \to 0$ if $u(x) \to 0$ as $x \to \infty$. The rest of the proof is the same.

Proof. (i) The first part of the conclusion is a special case of Theorem 4.1(iii), so we only need to justify the claims about the signs of u_n and $\lambda_n - \Lambda$. Using the oddness of f, we can suppose that there is a sequence of solutions converging to $(\Lambda, 0)$ in $\mathbb{R} \times H^2(\mathbb{R})$ and, in addition, that $\langle u_n, \phi\rangle_{L^2} > 0$ for all n. Let Z be such that $\operatorname{supp} V \subset [-Z, Z]$. Since $m \equiv \inf_{|x|\le Z} \phi(x) > 0$ and $\|w_n\|_{L^\infty} \to 0$, there exists n_0 such that $\phi + w_n \ge m/2$ on $[-Z, Z]$ for all $n \ge n_0$. By increasing n_0 if necessary, we can also suppose that $\lambda_n < 0$ and $|\lambda_n - \Lambda| < |\Lambda| - \ell$ for all $n \ge n_0$. But, for $|x| \ge Z$, by (5.3) we have that

$$u_n'' = \{k(e^{\alpha|x|}u_n) - \lambda_n\}u_n \le \{-\ell + |\Lambda| - |\lambda_n - \Lambda|\}u_n < 0$$

at points where $u_n < 0$ since $-\ell \le -L \le k(s) \le 0$ for all $s \in \mathbb{R}$. Hence u_n cannot have a negative minimum in the set $(-\infty, -Z] \cup [Z, \infty)$. Since $u_n(-Z) > 0, u_n(Z) > 0$ and $\lim_{|x|\to\infty} u_n(x) = 0$, it follows that $u_n \ge 0$ on $(-\infty, -Z] \cup [Z, \infty)$ and hence on $\mathbb{R}$. Thus any zero of u_n is at least a double zero and the existence of such a value implies that $u \equiv 0$, by the uniqueness of the solution of (5.1) with the conditions $u_n(x_0) = u_n'(x_0) = 0$. Hence we have that $u_n > 0$ on $\mathbb{R}$ for all $n \ge n_0$.

Setting $W_n(x) = V(x) + k(e^{\alpha|x|}u_n(x))$, we see from (5.2) that $u_n \in H^2(\mathbb{R})$ is a positive eigenfunction with eigenvalue λ_n of the operator $S_n u = -u'' + W_n u$. Since $k(s) \ge -L$ for all $s \in \mathbb{R}$, we have that $W_n(x) \ge -L$ for $|x| \ge Z$ and so $\inf \sigma_e(S_n) \ge -L$. On the other hand, $k \le 0$ on $\mathbb{R}$ and hence $\inf \sigma(S_n) \le \inf \sigma(S) = \Lambda < -\ell \le -L$. This implies that $\inf \sigma(S_n)$ is a simple eigenvalue of S_n with a positive eigenfunction and consequently $\lambda_n = \inf \sigma(S_n)$, showing that $\lambda_n \le \Lambda$.

(ii) Let (λ, u) be a non-trivial solution with $\lambda \in (\Lambda-\varepsilon, \Lambda+\varepsilon)$ and $u \in H^2(\mathbb{R})$. We show that this leads to a contradiction.

Step 1, in which we show that u cannot change sign in (Z, ∞).

For $x, s \in \mathbb{R}$, let

$$L(x, s) = e^{-2\alpha x}\Phi(e^{\alpha x}s) \text{ where } \Phi(t) = \int_0^t f(s)ds = \int_0^t k(s)s\,ds.$$

Then $L(\cdot,\cdot) \in C^2(\mathbb{R}^2)$ with

$$\partial_x L(x, s) = \alpha e^{-2\alpha x}\psi(e^{\alpha x}s) \text{ where } \psi(t) = f(t)t - 2\Phi(t)$$

and

$$\partial_s L(x, s) = e^{-\alpha x} f(e^{\alpha x}s) = k(e^{\alpha x}s)s.$$

We observe that

$$\psi(t) = k(t)t^2 - 2\int_0^t k(s)sds = \int_0^t k'(s)s^2 ds \le 0 \text{ for all } t \in \mathbb{R}$$

by (K).

Consider now the function $J : \mathbb{R} \to \mathbb{R}$ defined by

$$J(x) = \frac{1}{2}\{u'(x)^2 + \lambda u(x)^2\} - L(x, u(x)).$$

Clearly, $J \in C^1(\mathbb{R})$ and, for $x > Z$,

$$\begin{aligned}\frac{d}{dx}J(x) &= u'(x)\{u''(x) + \lambda u(x)\} - \partial_x L(x, u(x)) - \partial_s L(x, u(x))u'(x) \\ &= u'(x)\{u''(x) + \lambda u(x) - k(e^{\alpha x}u(x))u(x)\} - \partial_x L(x, u(x)) \\ &= -\partial_x L(x, u(x) = -\alpha e^{-2\alpha x}\psi(e^{\alpha x}u(x)) \geq 0.\end{aligned}$$

We also have that $|f(s)| \leq \ell|s|$ for all $s \in \mathbb{R}$ and so $|\Phi(s)| \leq \frac{1}{2}\ell s^2$ and then $|L(x,s)| \leq \frac{1}{2}\ell s^2$, too.

Since $u \in H^2(\mathbb{R})$ implies that $\lim_{x\to\infty} u(x) = \lim_{x\to\infty} u'(x) = 0$, it follows that $J(x) \to 0$ as $x \to \infty$ and then from the monotonicity of J that $J(x) \leq 0$ for all $x > Z$.

Suppose that $u(x_0) = 0$ for some $x_0 > Z$. Then $0 \geq J(x_0) = \frac{1}{2}u'(x_0)^2$ since $L(x_0, 0) = 0$. By the uniqueness of the solution of (5.1) satisfying the conditions $u(x_0) = u'(x_0) = 0$, this implies that $u \equiv 0$ on $\mathbb{R}$, whereas we have supposed that u is a non-trivial solution. Hence u has no zeros in the interval (Z, ∞).

Step 2, in which we prove that $\lim_{x\to\infty} e^{\alpha x}u(x) = \infty$ or $-\infty$.

In view of step 1 and the oddness of f, we can suppose that $u > 0$ on (Z, ∞). Since $|\lambda| - \alpha^2 = -\lambda - \alpha^2 < (-\Lambda + \varepsilon) - \alpha^2 = |\Lambda| - \alpha^2 + \varepsilon \leq 0$ we can choose $\beta \in (|\lambda|^{1/2}, \alpha)$ and then set $w(x) = ce^{-\beta x}$ where $c = \frac{1}{2}u(R)e^{\beta R}$ and $R = Z + 1$. Then $c > 0$ and we consider the function $z = u - w$. Since $\beta > 0$ and $u \in H^2(\mathbb{R})$, $z(x) \to 0$ as $x \to \infty$. By the choice of c, $z(R) = \frac{1}{2}u(R) > 0$. Let $\Omega = \{x > R : z(x) < 0\}$ and suppose that $\Omega \neq \emptyset$. Then $z \in C^2(\mathbb{R})$ and there exists a point $x_0 \in \Omega$ such that $z(x_0) = \min\{z(x) : x \in \Omega\} < 0$ and $z''(x_0) \geq 0$. But, on Ω,

$$z'' = u'' - w'' = \{k(e^{\alpha x}u) + |\lambda|\}u - \beta^2 w \leq |\lambda|u - \beta^2 u < 0$$

since $k \leq 0$ on $\mathbb{R}$, $w > u > 0$ on Ω and $|\lambda| < \beta^2$. In particular, $z''(x_0) < 0$ contradicting the fact that z attains its minimum at z_0. Hence $\Omega = \emptyset$ and we have proved that $u(x) \geq ce^{-\beta x}$ for all $x > R = Z + 1$. But then, $e^{\alpha x}u(x) \geq ce^{(\alpha-\beta)x}$ for all $x > R$, where $c > 0$ and $\alpha - \beta > 0$. Thus $\lim_{x\to\infty} e^{\alpha x}u(x) = \infty$, as required.

Step 3, in which we obtain a contradiction to the conclusion of Step 1.

As in Step 2, we can assume without loss of generality that $e^{\alpha x}u(x) \to \infty$ as $x \to \infty$ and hence $k(e^{\alpha x}u(x)) \to -L$ as $x \to \infty$. But

$$\begin{aligned}k(e^{\alpha x}u(x)) + |\lambda| &= \{k(e^{\alpha x}u(x)) + L\} - L + |\lambda| \\ &\leq \{k(e^{\alpha x}u(x)) + L\} - L + |\Lambda| + |\lambda - \Lambda| < \{k(e^{\alpha x}u(x)) + L\} - \varepsilon\end{aligned}$$

since $L - |\Lambda| \geq 2\varepsilon$ and $|\lambda - \Lambda| < \varepsilon$. Hence there exists $R_1 > Z + 1$ such that $k(e^{\alpha x}u(x)) + |\lambda| < -\varepsilon/2$ for all $x > R_1$.

Setting $v(x) = \sin\sqrt{\frac{\varepsilon}{2}}x$, we have that $v'' = -\frac{\varepsilon}{2}v$ and the zeros of v are $x_n = \sqrt{\frac{2}{\varepsilon}}n\pi$ for $n \in \mathbb{Z}$. For n even, $v > 0$ on (x_n, x_{n+1}). Now consider an even integer n such that $x_n > R_1$. Then

$$\int_{x_n}^{x_{n+1}} uv'' - u''v dx = uv'|_{x_n}^{x_{n+1}} = -\sqrt{\frac{\varepsilon}{2}}\{u(x_n) + u(x_{n+1})\} < 0.$$

On the other hand,

$$\begin{aligned}\int_{x_n}^{x_{n+1}} uv'' - u''v\,dx &= \int_{x_n}^{x_{n+1}} -u\frac{\varepsilon}{2}v + \{\lambda - k(e^{\alpha x}u)\}uv\,dx \\ &= -\int_{x_n}^{x_{n+1}} \{\frac{\varepsilon}{2} + |\lambda| + k(e^{\alpha x})\}uv\,dx > 0,\end{aligned}$$

since $uv > 0$ on (x_n, x_{n+1}) by step 1 and $k(e^{\alpha x}u(x)) + |\lambda| < -\varepsilon/2$ by the choice of R_1. □

It is natural to look for a result similar to Theorem 5.1 for $N \geq 2$. This can easily be done for part (i), but for part (ii) which is the main point of Theorem 5.1 it is not so clear how to proceed. For the approach used here, the obstacle at present is generalizing Step 1. Steps 2 and 3 can be extended to higher dimensions so one could obtain the conclusion that there is no bifurcation of positive solutions at Λ for potentials V having compact support and for which $\Lambda < \inf \sigma_e(-\Delta + V)$.

Returning to the case $N = 1$, minor modifications of the proof of part (ii) yield the same conclusion for other types of potential. For example (V_0) could be replaced by

(V_1) $V \in L^\infty(\mathbb{R})$ with $V \leq 0$ a.e. on $\mathbb{R}$ and there exists $a < b < Z$ such that $V \in C((a,b))$ with $V(x) < 0$ for $x \in (a,b)$ and $V(x) = 0$ for $|x| > Z$.

or

(V_2) $V \in L^\infty(\mathbb{R})$ with $\lim_{|x|\to\infty} V(x) = 0$ and there exists $Z > 0$ such that $V \in C^1((Z,\infty))$ and $V'(x) \leq 0$ for all $x > Z$.

Unlike (V_0) and (V_1), (V_2) does not ensure that $\inf \sigma(S) < 0$ so this condition has to be added in that case.

Finally, to draw some important information from Theorem 5.1 we specify a class of nonlinearities f which satisfy (K) and for which $L = \ell$.

(Q) $f \in C^1(\mathbb{R})$ is an odd function with $f'(0) = 0$ which is concave on $[0,\infty)$ with $f'(\infty) \equiv \lim_{s\to\infty} f'(s) > -\infty$.

Examples of functions satisfying (Q) are given by $f(s) = \ell\{\arctan s - s\}$ and $f(s) = \ell\{\tanh s - s\}$ for any $\ell > 0$.

As already noted, functions of the form $f(s) = -|s|^{2\sigma}s/(1+s^2)^\sigma$ satisfy (K) for all $\sigma > 0$, but they do not satisfy (Q) since $L = -\lim_{s\to\infty} f'(s) = 1$ and $\ell = \sup_{s\in\mathbb{R}} |f'(s)| = -f'(\sqrt{2\sigma+1}) > 1$ for all $\sigma > 0$. On the other hand, for functions of the form $f(s) = -|s|^\gamma s/(1+|s|^\gamma)$, we find that (K) is satisfied for all $\gamma > 0$ whereas (Q) is satisfied if and only if $0 < \gamma \leq 1$.

The hypothesis (Q) implies that $f' \le 0$ on $\mathbb{R}$ and that $\sup_{s\in\mathbb{R}} |f'(s)| = \ell$ where $\ell = -f'(\infty) = -\lim_{s\to\infty} \frac{f(s)}{s}$. Setting

$$\xi(x) = e^{-\alpha|x|} \text{ and } \eta(x) = e^{\alpha|x|} \text{ and then } F(u) = \xi f(\eta u),$$

it follows that the hypotheses (F), (W1) and (W2) of Section 4 are satisfied and hence from Theorem 3.3 that $F : H^2(\mathbb{R}) \to L^2(\mathbb{R})$ is Hadamard differentiable at 0 with $F'(0) = 0$. However, except in the trivial case $f \equiv 0$, Theorem 3.4 shows that this mapping is not Fréchet differentiable at 0 and it is not compact. Furthermore, $L(F) = L^{L^2}(F) = \ell$.

Corollary 5.2. *Consider the equation* (5.1) *under the hypotheses* (V_0) *and* (Q)*.*

(i) *If $|\Lambda| > \ell$, there is continuous bifurcation at Λ.*
(ii) *If $|\Lambda| < \ell$ and $\alpha > |\Lambda|^{1/2}$, then Λ is not a bifurcation point. Indeed, setting $\varepsilon = \min\{(\ell - |\Lambda|)/2, \alpha^2 - |\Lambda|, |\Lambda|\}$, $u \equiv 0$ is the only solution of* (5.1) *in* $H^2(\mathbb{R})$ *for $\lambda \in (\Lambda - \varepsilon, \Lambda + \varepsilon)$.*

Remark 1. Noting that $d(\Lambda, \sigma_e(S)) = |\Lambda|$ and $L(F) = L^{L^2}(F) = \ell$, we see that $d(\Lambda, \sigma_e(S)) < L(F)$ in part (ii) and that in this case, Λ is not a bifurcation point for (5.1). The other hypotheses of Theorem 4.1 are satisfied in both parts (i) and (ii). Hence, under the assumptions (V_0) and (Q), (5.1) is a special case of (1.3) which satisfies the hypotheses (H1) to (H3) of Proposition 2.2 and so also the conditions (B1) to (B5) of Theorem [19], as discussed at the beginning of Section 4.

Remark 2. Equation (5.1) is just (4.1) with $\xi(x) = e^{-\alpha|x|}$ but the assumption (K) means that f does not satisfy the hypotheses of Proposition 4.2 and (ii) shows that the statement (b) of that proposition does not hold for $\lambda_0 = \Lambda$ in the present situation. Of course, statement (c) also fails for (5.1) but it should be realized that this happens solely because $\eta(x) = 1/\xi(x) = e^{\alpha|x|}$ is not a transference weight. See commentary 4 in Section 4.

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