# On Existence of Total Input-Output Pairs of Abstract Time Systems

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**Abstract.** We consider a class of abstract mathematical models called blocks which generalize some input-output system models which are frequently used in system theory, cybernetics, control theory, signal processing. A block can be described by a multifunction which maps a collection of input signals (input signal bunch) to a non-empty set of collections of output signals (set of output signal bunches). The input and output signal bunches are defined on a subset of a continuous time domain.

We investigate and provide methods for proving the existence of a pair of corresponding input and output signal bunches of a given block, both components of which are defined on the entire time domain (a total input-output pair), and the existence of a total output signal bunch corresponding to a given total input signal bunch.

**Keywords:** input-output system, abstract time system, dynamical system, signal transformer, semantics.

## 1 Introduction

An abstract view of a computing system as a transformation of data, a function, or an input-output relation is rather common in computer science. In fact, this view is rooted in foundations of computing and is notable in the works of A. Turing and A. Church.

Nevertheless, a large amount of computing systems used today act not as pure data transformers, but as agents interacting with physical processes. Such systems are now frequently called cyber-physical systems [1, 2]. Examples include automotive systems, robotics, process control, medical devices, etc. [3].

As was stressed in [4], an important aspect that cyber-physical systems must take into account is the passage of (physical) time. The actions of such systems must be properly timed. Besides, the computational aspect of a system must be understood and modeled in a close relation with physical processes. However, this is not taken into account when a system is viewed as an input-output relation on data.

A simple way to resolve this is to view a system as an input-output relation on time-varying quantities (signals). A view of this kind is extensively used in signal processing and control theory [5, 6]. However, the kinds of mathematical

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models of systems usually considered in these fields (e.g. systems of linear or non-linear difference or differential equations, transfer function representation of linear systems, etc.) do not provide a high-level abstraction of processes that take place in cyber-physical systems [1].

A high-level treatment of systems as input-output relations (or as relations in general) can be found in mathematical systems theory. During the second half of the XX century a large number of works that dealt with general mathematical theory of systems were published by L. Zadeh [7, 8], R. Kalman [9], M. Arbib [10], G. Klir [11], W. Wymore [12], M. Mesarovic [13], B.P. Zeigler [14], V.M. Matrosov [15], and others [16–18].

Many of these works were inspired and influenced by the General Systems Theory by L. Bertalanffy, Cybernetics introduced by N. Wiener, information theory introduced by C. Shannon, circuit theory in electrical engineering, automata theory, control theory. A historical account on the mutual influence between these fields is given in [19, 11]. In particular, the approach developed by M. Mesarovic [13] is based on formalization of a system as a relation on objects. Other approaches such as those developed by M. Arbib [10], W. Wymore [12], B.P. Zeigler [14] resulted from unification of the theory of systems described by differential equations and automata theory.

With regard to the input-output view, many of the mentioned works introduce some kind of abstract view of a system as an input-output relation on time-varying quantities (e.g. a general time system [13, Section 2.5], external behavior of a dynamical system [9, Section 1.1], oriented abstract object [7, Chapter 1, Paragraph 4], I/O observational frame [14, Section 5.3]) and consider such a relation as a mathematical representation of an observable behavior of a real-world system. The most basic example is the definition of a Mesarovic time system [13] as a binary relation $S \subseteq I \times O$, where $I$ and $O$ are sets of input and output functions on a time domain $T$ ($I \subseteq A^T$, $O \subseteq B^T$).

However, one aspect that is not sufficiently investigated in works on mathematical systems theory with regard to time systems is partiality of input and/or output signals as functions of time. This aspect becomes most important, when the input-output relation describing a real-world system results not from a direct observation of its behavior, but as an abstraction of a lower level mathematical model of this system. The reason is that a lower level model (e.g. a system of differential equations, a hybrid system, etc. [20]) may describe the behavior of a real-world system only on a bounded time interval, after which the model's behavior becomes undefined. This can indicate a real phenomenon (e.g. termination or destruction of the real system), or a failure of the model [21].

For example, a phenomenon of a finite time blow-up is well known in the theory of differential equations and applied mathematics [21, 22]. It is characterized by the unbounded growth of the value of one or several system variables during a bounded time interval. A simple example is a (non-zero) solution $x(t) = 1/(c-t)$, $c = const$ of the equation $x'(t) = x(t)^2$, for which $|x(t)| \to +\infty$, when $t \to c$. A survey of the respective results and applications can be found in [21–23].

Another situation when a mathematical model defines a system's behavior on a bounded time interval is a Zeno behavior [24, 25] which arises in hybrid (discrete-continuous) systems [20]. In this case, a hybrid system performs an infinite sequence of discrete steps during a bounded total time, but each step takes a non-zero time. A simple example in which this behavior arises is a hybrid automaton [20] which models a bouncing ball [24].

It should be noted that the problems of detection of finite time blow-up or Zeno behaviors, their physical interpretation, and if necessary, adjustment of a model to avoid such behaviors are non-trivial, so one cannot assume that any available and useful model of real-world system would be free of them.

This dictates that when using an input-output abstraction of a real system based on an available mathematical model of this system, one must take into account partial input and outputs.

In the previous work [26] we introduced a class of input-output abstractions of real-world systems which we called a class of *blocks*. A block can be thought of as a generalization of a Mesarovic time system which takes into account partiality of inputs and outputs as functions of time.

Basically, a block is a multifunction which maps a collection of input signals (*input signal bunch*) to a (non-empty) set of collections of output signals (a set of *output signal bunches*), and a signal is a partial function on a continuous time domain. The operation of a block can be described by a set of input-output pairs (I/O pairs) $(i, o)$, where $i$ is an input signal bunch and $o$ is a corresponding output signal bunch. The main aspects captured by this notion are nondeterminism (multiple possible output signal bunches corresponding to one input signal bunch), continuous time, partiality of inputs and output signals.

In the work [26] we studied the notions of causality (nonanticipation), refinement, and composition for blocks.

In this work we continue to investigate properties of blocks and consider the following questions.

We introduced the notion of a block to take into account the possibility of partial inputs and outputs, or, in other words, to allow I/O pairs $(i, o)$, where $dom(i)$ and $dom(o)$ may not cover the whole time domain (note that $dom(i) \neq dom(o)$ is also possible). In the work [26] we gave the following interpretation to this partiality: the case $dom(o) \subset dom(i)$ for an I/O pair $(i, o)$ means that a block receives an input signal bunch $i$, but does not process it completely. It outputs $o$ and terminates abnormally. On the other hand, the case $dom(o) = dom(i)$ means that a block processes the input signal bunch $i$ completely, outputs $o$, and terminates. In particular, if $T$ is a time domain, $(i, o)$ is an I/O pair, and $dom(i) = T$, then if $dom(o) = T$, the block processes the input completely and normally and outputs $o$, otherwise, it outputs $o$ and terminates abnormally at some time moment. In the former case, $(dom(i) = dom(o) = T)$ we say that $(i, o)$ is a total I/O pair.

One can say that in models such as Mesarovic time system all I/O pairs are total. But in blocks total I/O pairs form only a subset of the set of all I/O pairs.

In this chapter we consider the following question about total I/O pairs:

**(A)** How can one prove that a given block $B$ has a total I/O pair (if $B$ indeed has a total I/O pair) ?

Using the same techniques which will used to answer this question, in this chapter we will also give an answer to the following question:

**(B)** How can one prove that for a given input signal bunch $i$, where $dom(i) = T$, there exists an I/O pair $(i, o)$ with $dom(o) = T$ ?

That is, a block admits a total output for a given total input.

In this chapter we will consider (A) and (B) for strongly nonanticipative blocks [26] only. As we argued in [26], strongly nonanticipative blocks are sufficient for modeling physically realizable (causal) real-world systems.

The practical significance of the questions (A) and (B) follows from the interpretation of the case $dom(o) \subset dom(i)$ as an abnormal termination of a block on the input $i$. More specifically, the methods used for solving (B) can interpreted as methods of proving that it is possible for a block to process a given input normally (without errors) and can be rather straightforwardly linked with such domains as viability theory, control synthesis, real-time software verification, etc.

The chapter is organized in the following way. In Section 2 we recall the definition of a block and other related definitions and facts from [26]. In Section 3 we show that each strongly nonanticipative block has a representation in the form of an abstract dynamical system of a special kind called Nondeterministic Complete Markovian System (NCMS) [27]. In Section 4 we use the facts about existence of global-in-time trajectories of NCMS which were shown in [27] and the representation given in Section 3 in order to prove criteria which answer the questions (A) and (B).

## 2 Preliminaries

### 2.1 Notation

We will use the following notation: $\mathbb{N} = \{1, 2, 3, ...\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}_+$ is the set of nonnegative real numbers, $f : A \to B$ is a total function from $A$ to $B$, $f : A \tilde{\to} B$ is a partial function from $A$ to $B$, $2^A$ is the power set of a set $A$, $f|_X$ is the restriction of a function $f$ to a set $X$. If $A, B$ are sets, then $B^A$ denotes the set of all total functions from $A$ to $B$ and ${}^A B$ denotes the set of all partial function from $A$ to $B$. For a function $f : A \tilde{\to} B$ the symbol $f(x) \downarrow$ ($f(x) \uparrow$) means that $f(x)$ is defined (respectively undefined) on the argument $x$.

We denote the domain and range of a function as $dom(f) = \{x \,|\, f(x) \downarrow\}$ and $range(f) = \{y \mid \exists x \ f(x) \downarrow \wedge \ y = f(x)\}$ respectively. We will use the the same notation for the domain and range of a binary relation: if $R \subseteq A \times B$, then $dom(R) = \{x \,|\, \exists\, y \ (x, y) \in R\}$ and $range(R) = \{y \,|\, \exists\, x \ (x, y) \in R\}$.

We will use the notation $f(x) \cong g(x)$ for the strong equality (where $f$ and $g$ are partial functions): $f(x) \downarrow$ iff $g(x) \downarrow$ and $f(x) \downarrow$ implies $f(x) = g(x)$.

The symbol $\circ$ denotes a functional composition: $(f \circ g)(x) \cong g(f(x))$.

The notation $X \mapsto y$, where $X$ is a given set and $y$ is a given value, means a constant function defined on $X$ which takes the value $y$.

By $T$ we denote the (positive real) time scale $[0, +\infty)$. We assume that $T$ is equipped with a topology induced by the standard topology on $\mathbb{R}$.

Additionally, we define the following class of sets:

$$\mathcal{T}_0 = \{\emptyset, T\} \cup \{[0, x) \,|\, x \in T \backslash \{0\}\} \cup \{[0, x] \,|\, x \in T\}$$

i.e. the set of (possibly empty, bounded or unbounded) intervals with left end 0.

The symbols $\neg$, $\vee$, $\wedge$, $\Rightarrow$, $\Leftrightarrow$ denote the logical operations of negation, disjunction, conjunction, implication, and equivalence respectively.

### 2.2 Multi-valued Functions

A multi-valued function [28] assigns one or more resulting values to each argument value. An application of a multi-valued function to an argument is interpreted as a nondeterministic choice of a result.

**Definition 1 ([28]).** *A (total) multi-valued function from a set $A$ to a set $B$ (denoted as $f : A \xrightarrow{tm} B$) is a function $f : A \to 2^B \backslash \{\emptyset\}$.*

Thus the inclusion $y \in f(x)$ means that $y$ is a possible value of $f$ on $x$.

### 2.3 Named Sets

We will use a simple notion of a named set to formalize an assignment of values to variable names in program and system semantics.

**Definition 2.** *([28]) A named set is a partial function $f : V \tilde{\to} W$ from a non-empty set of names $V$ to a set of values $W$.*

A named set can be considered as special case of a more general notion of nominative data [28] which reflects hierarchical data organizations.

We will use a special notation for the set of named sets: ${}^{V}W$ denotes the set of all named sets $f : V \tilde{\to} W$ (this notation just emphasises that $V$ is interpreted as a set of names). We consider named sets equal, if their graphs are equal.

An expression of the form $[n_1 \mapsto a_1, n_2 \mapsto a_2, ...]$ (where $n_1, n_2, ...$ are distinct names) denotes a named set $d$ such that the graph of $d$ is $\{(n_1, a_1), (n_2, a_2), ...\}$. A nowhere-defined named set is called an *empty named set* and is denoted as $[]$.

For any named sets $d_1, d_2$ we write $d_1 \subseteq d_2$ (*named set inclusion*), if the graph of a function $d_1$ is a subset of the graph of $d_2$.

### 2.4 Signals, Signal Bunches, and Blocks

Informally, a block is an abstract model of a system which receives input signals and produces output signals (Fig. 1). The input signals can be thought of as certain time-varying characteristics (attributes) of the external environment of the system which are relevant for (the operation of) this system. Each instance of an input signal has a certain time domain on which it is defined (present). An input signal bunch can be thought of as a collection of instances of input signals of the system. Each input signal bunch $i$ has an associated domain of existence ($dom(i)$) which is a superset of the union of the domains of signals contained in $i$. The domain of an input signal bunch can be thought of as a time span of the existence of the external environment of the system. The output signals can be considered as effects (results) of the system's operation. An output signal bunch, or simply an output of the block, can be thought of as a collection of instances of output signals of the system. The output signals have domains of definition (presence) and each output signal bunch $o$ has an associated domain of existence ($dom(o)$) which is a superset of the union of the domains of signals contained in $o$. The domain of an output signal bunch can be thought of as a time span during which the system operates. It is assumed that for an output signal bunch $o$ which corresponds to a given input signal bunch $i$ the inclusion $dom(o) \subseteq dom(i)$ holds (i.e. the system does not operate when the environment does not exist). However, in the general case, the presence of a given input signal at a given time does not imply the presence of a certain output signal at the same or any other time moment.

A block can operate nondeterministically, i.e. for one input signal bunch it may choose an output signal bunch from a set of possible variants. However, for any input signal bunch there exists at least one corresponding output signal bunch (although the values of all signals in it may be absent at all times, which means that the block does not produce any output values).

Normally, a block processes the whole input signal bunch, and does or does not produce output values. However, in certain cases a block may not process the whole input signal bunch and may terminate at some time moment before its end. This situation is interpreted as an abnormal termination of a block.

Let $W$ be a fixed non-empty set of values.

**Definition 3** (**[26]**)

*(1) A signal is a partial function from $T$ to $W$ ($f : T \tilde{\to} W$).*
*(2) A $V$-signal bunch (where $V$ is a set of names) is a function $sb : T \tilde{\to} {}^{V} W$ such that $dom(sb) \in \mathcal{T}_0$. The set of all $V$-signal bunches is denoted as $Sb(V, W)$.*
*(3) A signal bunch is a $V$-signal bunch for some $V$.*
*(4) A signal bunch $sb$ is trivial, if $dom(sb) = \emptyset$ and is total, if $dom(sb) = T$. A trivial signal bunch is denoted as $\perp$.*
*(5) For a given signal bunch $sb$, a signal corresponding to a name $x$ is a partial function $t \mapsto sb(t)(x)$. This signal is denoted as $sb[x]$.*
*(6) A signal bunch $sb_1$ is a prefix of a signal bunch $sb_2$ (denoted as $sb_1 \preceq sb_2$), if $sb_1 = sb_2|_A$ for some $A \in \mathcal{T}_0$.*

The relation $\preceq$ on $V$-signal bunches is a partial order (for an arbitrary $V$).

It can be generalized to pairs as follows: for any signal bunches $sb_1, sb_2, sb_1', sb_2'$ denote $(sb_1, sb_2) \preceq^2 (sb_1', sb_2')$ iff there exists $A \in \mathcal{T}_0$ such that $sb_1 = sb_1'|_A$ and $sb_2 = sb_2'|_A$. The relation $\preceq^2$ is a partial order on pairs of signal bunches.

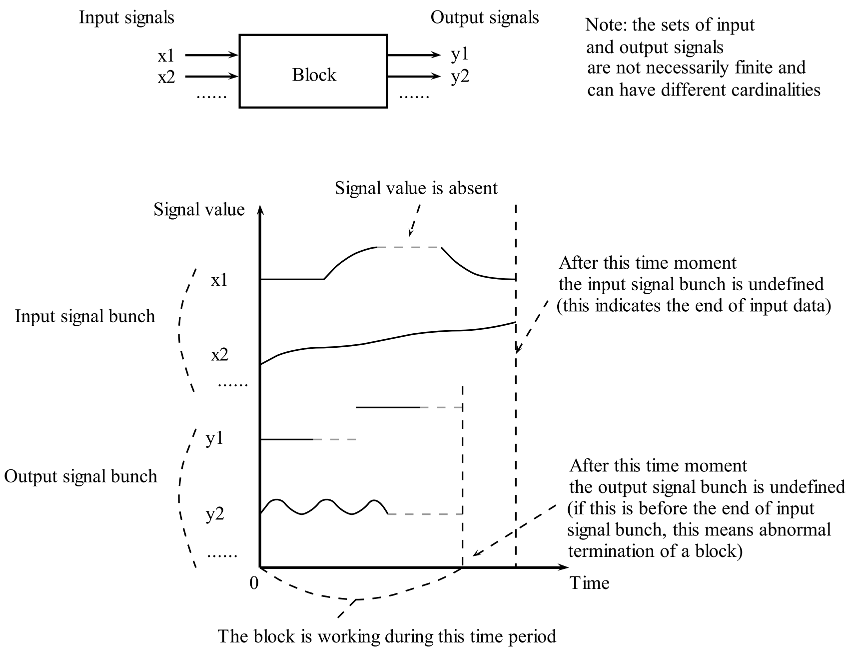


**Fig. 1.** An illustration of a block with the input signals $x_1$, $x_2$, ... and the output signals $y_1$, $y_2$, ... . The input and output signals are lumped into an input and output signal bunch respectively. Solid curves represent (present) signal values. Dashed horizonal segments denote absence of a signal value. Dashed vertical lines indicate the right boundaries of the domains of signal bunches.

A block has a syntactic aspect (e.g. a description in a specification language) and a semantic aspect – a partial multi-valued function on signal bunches.

**Definition 4.** *(1) A block is an object $B$ (syntactic aspect) together with an associated set of input names $In(B)$, a set of output names $Out(B)$, and a total multi-valued function $Op(B) : Sb(In(B), W) \xrightarrow{tm} Sb(Out(B), W)$ (operation, semantic aspect) such that $o \in Op(B)(i)$ implies $dom(o) \subseteq dom(i)$.*

*(2) Two blocks $B_1$, $B_2$ are semantically identical, if $In(B_1) = In(B_2)$, $Out(B_1) = Out(B_2)$, and $Op(B_1) = Op(B_2)$.*

*(3) An I/O pair of a block $B$ is a pair of signal bunches $(i, o)$ such that $o \in Op(B)(i)$. The set of all I/O pairs of $B$ is denoted as $IO(B)$ and is called the input-output (I/O) relation of $B$.*

An inclusion $o \in Op(B)(i)$ means that $o$ is a possible output of a block $B$ on the input $i$. For each input $i$ there is some output $o$. The domain of $o$ is a subset of the domain of $i$. If $o$ becomes undefined at time $t$, but $i$ is still defined at $t$, we interpret this as an (unrecoverable) error during the operation of the block $B$.

### 2.5 Causal and Strongly Nonanticipative Blocks

**Definition 5.** *A block $B$ is deterministic, if $Op(B)(i)$ is a singleton set for each $In(B)$-signal bunch $i$.*

**Definition 6.** *A deterministic block $B$ is causal iff for all signal bunches $i_1, i_2$ and $A \in \mathcal{T}_0$, $o_1 \in Op(B)(i_1)$, $o_2 \in Op(B)(i_2)$, the equality $i_1|_A = i_2|_A$ implies $o_1|_A = o_2|_A$.*

This means that the value of the output signal bunch at time $t$ can depend only on the values of the input signal at times $\leq t$.

**Definition 7.** *A block $B$ is a sub-block of a block $B'$ (denoted as $B \trianglelefteq B'$), if $In(B) = In(B')$, $Out(B) = Out(B')$, and $IO(B) \subseteq IO(B')$.*

**Definition 8.** *A block $B$ is strongly nonanticipative, if for each $(i, o) \in IO(B)$ there exists a deterministic causal sub-block $B' \trianglelefteq B$ such that $(i, o) \in IO(B')$.*

Informally, the operation of a strongly nonanticipative block $B$ can be interpreted as a two-step process:

1. Before receiving the input signals, the block $B$ (nondeterministically) chooses a deterministic causal sub-block $B' \trianglelefteq B$ (response strategy).
2. The block $B'$ receives input signals of $B$ and produces the corresponding output signals (response) which become the output signals of $B$.

### 2.6 Nondeterministic Complete Markovian Systems (NCMS)

The notion of a NCMS was introduced in [27] as a special kind of abstract dynamical systems for the purpose of studying the relation between existence of global and local trajectories of dynamical systems. This notion is close to the notion of a *solution system* introduced by O. Hájek in [29], but there are some more and less important differences which will be described below.

Let $T = \mathbb{R}_+$ be the positive real time scale. Denote by $\mathfrak{T}$ the set of all connected subsets of $T$ (i.e. bounded and unbounded intervals) with cardinality greater than one.

Let $Q$ be a set (a state space) and $Tr$ be some set of functions of the form $s : A \to Q$, where $A \in \mathfrak{T}$. Let us call its elements (partial) trajectories.

**Definition 9 ([27]).** *A set of trajectories $Tr$ is closed under proper restrictions (CPR), if $s|_A \in Tr$ for each $s \in Tr$ and $A \in \mathfrak{T}$ such that $A \subseteq dom(s)$.*

**Definition 10 ([27])**

*(1) A trajectory $s_1 \in Tr$ is a subtrajectory of $s_2 \in Tr$ (denoted as $s_1 \sqsubseteq s_2$), if $dom(s_1) \subseteq dom(s_2)$ and $s_1 = s_2|_{dom(s_1)}$.*
*(2) A trajectory $s_1 \in Tr$ is a proper subtrajectory of $s_2 \in Tr$ (denoted as $s_1 \sqsubset s_2$), if $s_1 \sqsubseteq s_2$ and $s_1 \neq s_2$.*
*(3) Trajectories $s_1, s_2 \in Tr$ are incomparable, if neither $s_1 \sqsubseteq s_2$, nor $s_2 \sqsubseteq s_1$.*

The set $(Tr, \sqsubseteq)$ is a (possibly empty) partially ordered set (poset).

**Definition 11 ([27]).** *A CPR set of trajectories $Tr$ is*

*(1) Markovian (see Fig. 2 below), if for each $s_1, s_2 \in Tr$ and $t \in T$ such that $t = \sup dom(s_1) = \inf dom(s_2)$, $s_1(t) \downarrow$, $s_2(t) \downarrow$, and $s_1(t) = s_2(t)$, the following function $s$ belongs to $Tr$:*

$$s(t) = \begin{cases} s_1(t), & t \in dom(s_1) \\ s_2(t), & t \in dom(s_2) \end{cases}$$

*(2) complete, if each non-empty chain in $(Tr, \sqsubseteq)$ has a supremum.*

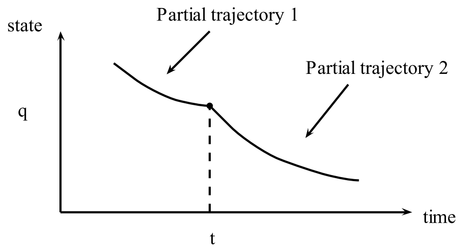


**Fig. 2.** Markovian property for nondeterministic systems with partial trajectories. If one partial trajectory ends and another begins in the state $q$ at time $t$ (both are defined at $t$), then their concatenation is a partial trajectory.

**Definition 12 ([27]).** *A nondeterministic complete Markovian system (NCMS) is a triple $(T, Q, Tr)$, where $Q$ is a set (state space) and $Tr$ (trajectories) is a set of functions $s : T \tilde{\to} Q$ such that $dom(s) \in \mathfrak{T}$, which is CPR, complete, and Markovian.*

The main similarities and differences between a NCMS and a solution system [29] are:

- The time domain $T$ and the set of states $Q$ of NCMS correspond to the time domain $R$ and the phase-space $P$ of a solution system. For simplicity we assume that $T$ is fixed to be $\mathbb{R}_+$, while in [29] $R$ can can be any subset of $\mathbb{R}$.
- Trajectories of NCMS correspond to the members of a solution system (solutions). However, their domains cannot be singleton sets, while solutions can have singleton time domains.
- CPR property of NCMS basically corresponds to the Partialization property of solution systems. The difference is that Partialization allows restrictions on singleton sets, while CPR does not include this.
- Markovian property of NCMS basically corresponds to the Concatenation property of solution systems. By themselves these properties are not equivalent: Markovian property is weaker in the sense that it does not allow one to make a union of two trajectories, if the intersection of their domains is

not singleton. But using both CPR and Markovian properties, one can make a union of two trajectories even if their domains have non-singleton intersection. The term "Markovian" is meant to indicate that if a system is in a given state, the set of its possible futures does not depend in its past.
– In the stand-alone (process-independent) definition of a solution system [29, Definition 2.1], there is no assumption which corresponds / is analogous to the Completeness property of NCMS.

### 2.7 LR Representation of NCMS

In this subsection we will describe a convenient general representation of NCMS in terms of certain predicates.

**Definition 13 ([27]).** *Let $s_1, s_2 : T \tilde{\to} Q$. Then $s_1$ and $s_2$ coincide:*

*(1) on a set $A \subseteq T$, if $s_1|_A = s_2|_A$ (this is denoted as $s_1 \doteq_A s_2$);*
*(2) in a left neighborhood of $t \in T$, if either $t = 0$ and $s_1(0) = s_2(0)$, or $t > 0$ and there exists $t' \in [0, t)$, such that $s_1 \doteq_{(t',t]} s_2$ (this is denoted as $s_1 \doteq_{t-} s_2$);*
*(3) in a right neighborhood of $t \in T$, if there exists $t' > t$, such that $s_1 \doteq_{[t,t')} s_2$ (this is denoted as $s_1 \doteq_{t+} s_2$).*

Let $Q$ be a set. Denote by $ST(Q)$ the set of pairs $(s, t)$ where $s : A \to Q$ for some $A \in \mathfrak{T}$ and $t \in A$.

**Definition 14 ([27]).** *A predicate $p : ST(Q) \to Bool$ is called*

*(1) left-local, if $p(s_1, t) \Leftrightarrow p(s_2, t)$ whenever $\{(s_1, t), (s_2, t)\} \subseteq ST(Q)$ and $s_1 \doteq_{t-} s_2$, and, moreover, $p(s, t)$ whenever $t$ is the least element of $dom(s)$;*
*(2) right-local, if $p(s_1, t) \Leftrightarrow p(s_2, t)$ whenever $\{(s_1, t), (s_2, t)\} \subseteq ST(Q)$, $s_1 \doteq_{t+} s_2$, and, moreover, $p(s, t)$ whenever $t$ is the greatest element of $dom(s)$.*

Let us denote by $LR(Q)$ the set of all pairs $(l, r)$, where $l : ST(Q) \to Bool$ is a left-local predicate and $r : ST(Q) \to Bool$ is a right-local predicate.

**Definition 15.** *A pair $(l, r) \in LR(Q)$ is called a LR representation of a NCMS $\Sigma = (T, Q, Tr)$, if*

$$Tr = \{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A \; l(s, t) \wedge r(s, t))\}.$$

**Theorem 1.** *(About LR representation)*

*(1) Each pair $(l, r) \in LR(Q)$ is a LR representation of a NCMS with the set of states $Q$.*
*(2) Each NCMS has a LR representation.*

The proof follows immediately from [27, Theorem 1] and is omitted here.

## 3 Representation of Strongly Nonanticipative Blocks

In this section we will introduce a representation of a strongly nonanticipative block in the form of an abstract dynamical system of a special kind (initial I/O NCMS). Although we mainly concentrate on the mere existence of such a representation, the proofs of lemmas given below actually describe a method for constructing a particular concrete representation of a block.

Let $W$ denote a fixed non-empty set of values.

**Definition 16.** *An input-output (I/O) NCMS is an NCMS* $(T, Q, Tr)$ *such that* $Q$ *has a form* ${}^{I}W \times X \times {}^{O}W$ *for some sets* $I$ *(set of input names),* $X \neq \emptyset$ *(set of internal states), and* $O$ *(set of output names). The* ${}^{I}W$ *is called an input data set and* ${}^{O}W$ *is called an output data set.*

Informally, an I/O NCMS describes possible evolutions of triples $(d_{in}, x, d_{out})$ of input data ($d_{in} \in {}^{I}W$), internal state ($x \in X$), and output data ($d_{out} \in {}^{O}W$).

**Lemma 1.** *Each I/O NCMS* $(T, Q, Tr)$ *has a unique set of input names, internal states, and output names.*

The proof follows immediately from the definitions.

For a I/O NCMS $\Sigma$ we will denote as $In(\Sigma)$ its unique set of input names, as $Out(\Sigma)$ its set of output names, and as $IState(\Sigma)$ its internal state space.

For any I/O NCMS $\Sigma = (T, Q, Tr)$ and a state $q \in Q$ we will denote as $in(q)$, $istate(q)$, $out(q)$ the projections of $q$ on the first, second, and third coordinate respectively. Correspondingly, for any $s \in Tr$, $in \circ s$, $istate \circ s$, $out \circ s$, denote a composition of the respective projection map with a trajectory.

For each $i \in Sb(In(\Sigma), W)$ let us denote

- $S(\Sigma, i) = \{s \in Tr \mid dom(s) \in \mathcal{T}_0 \ \wedge in \circ s \preceq i\}$;
- $S_{max}(\Sigma, i)$ is the set of all $\sqsubseteq$-maximal (i.e. non-continuable) trajectories from $S(\Sigma, i)$;
- $S_{init}(\Sigma, i) = \{s(0) \mid s \in S(\Sigma, i)\}$;
- $S_{init}(\Sigma) = \{s(0) \mid s \in Tr \wedge dom(s) \in \mathcal{T}_0\}$.

For each $Q' \subseteq Q$ let us denote:

$$Sel_{1,2}(Q', d, x) = \{q \in Q' \mid \exists d' \ q = (d, x, d')\},$$

i.e., selection of states from $Q'$ by the value of the first and second component.

For each $Q' \subseteq Q$ and $i \in Sb(In(\Sigma), W)$ let us denote:

$$o_{all}(\Sigma, Q', i) = \begin{cases} \{\bot\}, & Q' = \emptyset \text{ or } i = \bot; \\ \{\{0\} \mapsto out(q) \mid q \in Q'\}, & Q' \neq \emptyset \text{ and} \\ & dom(i) = \{0\}; \\ \{out \circ s \mid s \in S_{max}(\Sigma, i) \wedge s(0) \in Q'\} \cup & Q' \neq \emptyset \text{ and} \\ \quad \cup \{\{0\} \mapsto out(q) \mid q \in Q' \backslash S_{init}(\Sigma, i)\}, & \{0\} \subset dom(i), \end{cases}$$

where $\{0\} \mapsto out(q)$ denotes a signal bunch defined on $\{0\}$ which takes a value $out(q)$. Also, for each $Q_0 \subseteq Q$ let us denote:

$$O_{all}(\Sigma, Q_0, i) = \begin{cases} \{\bot\}, & dom(i) = \emptyset; \\ \bigcup_{x \in IState(\Sigma)} o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i), & dom(i) \neq \emptyset. \end{cases}$$

**Definition 17.** *An initial I/O NCMS is a pair* $(\Sigma, Q_0)$*, where* $\Sigma = (T, Q, Tr)$ *is a I/O NCMS and* $Q_0$ *is a set (admissible initial states) such that* $S_{init}(\Sigma) \subseteq Q_0 \subseteq Q$.

**Definition 18.** *A NCMS representation of a block* $B$ *is an initial I/O NCMS* $(\Sigma, Q_0)$ *such that*

*(1)* $In(B) = In(\Sigma)$ *and* $Out(B) = Out(\Sigma)$*;*
*(2)* $Op(B)(i) = O_{all}(\Sigma, Q_0, i)$ *for all* $i \in Sb(In(B), W)$.

Informally, the operation of a block $B$ represented by an initial I/O NCMS $(\Sigma, Q_0)$ on an input signal bunch $i$ can be described as follows:

(1) If $i(0)$ is undefined, then $B$ stops (the output signal bunch is $\bot$).
(2) Otherwise, $B$ chooses an arbitrary internal state $x \in IState(\Sigma)$.
(3) If there is no admissible initial state $q \in Q_0$ with $in(q) = i(0)$ and $istate(q) = x$ (i.e. $Sel_{1,2}(Q_0, i(0), x) = \emptyset$), then $B$ stops.
(4) Otherwise, $B$ chooses an arbitrary $q \in Q_0$ such that $in(q) = i(0)$ and $istate(q) = x$ (i.e. $q \in Sel_{1,2}(Q_0, i(0), x)$).
(5) If $dom(i) = \{0\}$ or there is no trajectory $s$ which starts in $q$ and is defined on some interval (of positive length) from $\mathcal{T}_0$, then $B$ outputs $out(q)$ at time 0 and stops.
(6) Otherwise, $B$ chooses an arbitrary maximal trajectory $s$ defined on an interval from $\mathcal{T}_0$ such that $s(0) = q$ and $in \circ s \preceq i$ and outputs the signal bunch $out \circ s$.

The main result of this section is the following:

**Theorem 2.** *(About representation of a strongly nonanticipative block) Each strongly nonanticipative block has a NCMS representation.*

In the rest of the section we will prove several helper lemmas, and finally, give a proof of this theorem.

**Lemma 2.** *Let* $(T, Q, Tr)$ *be a NCMS,* $Q'$ *be a set,* $f : Q \to Q'$ *be an injective function, and* $Tr' = \{f \circ s \mid s \in Tr\}$*. Then* $(T, Q', Tr')$ *is a NCMS.*

The proof follows immediately from the definition of NCMS and is omitted here.

**Lemma 3.** *Let* $(T, Q^j, Tr^j)$*,* $j \in J$ *be an indexed family of NCMS such that* $Q^j \cap Q^{j'} = \emptyset$*, if* $j \neq j'$*. Let* $Q = \bigcup_{j \in J} Q^j$ *and* $Tr = \bigcup_{j \in J} Tr^j$*. Then* $(T, Q, Tr)$ *is a NCMS.*

The proof follows immediately from the definition of NCMS and is omitted here.

**Lemma 4.** *Let $\Sigma$ be a I/O NCMS, $i \in Sb(In(\Sigma), W)$, and $s \in S(\Sigma, i)$. Then there exists $s' \in S_{max}(\Sigma, i)$ such that $s \sqsubseteq s'$.*

*Proof.* Consider a set $G = \{s'' \in S(\Sigma, i) \mid s \sqsubseteq s''\}$. From the completeness property of NCMS it follows that each non-empty $\sqsubseteq$-chain of elements of $G$ has a least upper bound in the poset $(Tr, \sqsubseteq)$ which belongs to $G$. Moreover, $G \neq \emptyset$. Then Zorn's lemma implies that $G$ has some $\sqsubseteq$-maximal element $s'$. Then $s' \in S_{max}(\Sigma, i)$ and $s \sqsubseteq s'$. □

**Lemma 5.** *If $\Sigma = (T, Q, Tr)$ is a I/O NCMS, $Q' \subseteq Q$, $i \in Sb(In(\Sigma), W)$, then*

*(1) $o_{all}(\Sigma, Q', i) \subseteq Sb(Out(\Sigma), W)$;*
*(2) $dom(o) \subseteq dom(i)$ for each $o$ in $o_{all}(\Sigma, Q', i)$;*
*(3) $o_{all}(\Sigma, Q', i) \neq \emptyset$.*

The proof follows from the definitions and Lemma 4 and is omitted here.

**Lemma 6.** *Each initial I/O NCMS is a NCMS representation of a unique (up to semantic identity) block.*

*Proof.* Uniqueness up to semantic identity is obvious from Definition 18. Let us prove that if $(\Sigma, Q_0)$ is an initial I/O NCMS, where $\Sigma = (T, Q, Tr)$, then it is a NCMS representation of some block.

Let $i \in Sb(In(\Sigma), W)$. Let us show that $O_{all}(\Sigma, Q_0, i)$ is a non-empty subset of $Sb(Out(\Sigma), W)$ and $dom(o) \subseteq dom(i)$ for all $o \in O_{all}(\Sigma, Q_0, i)$. This is obvious, if $dom(i) = \emptyset$. Consider the case when $dom(i) \neq \emptyset$. Then $O_{all}(\Sigma, Q_0, i) = \bigcup_{x \in IState(\Sigma)} o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i)$. For any $x \in IState(\Sigma)$ we have $Sel_{1,2}(Q_0, i(0), x) \subseteq Q_0 \subseteq Q$. Besides, $IState(\Sigma) \neq \emptyset$. Then Lemma 5 implies that $O_{all}(\Sigma, Q_0, i) \in 2^{Sb(Out(\Sigma), W)} \backslash \{\emptyset\}$ and $dom(o) \subseteq dom(i)$ for all $o \in O_{all}(\Sigma, Q_0, i)$. Thus $(\Sigma, Q_0)$ is a NCMS representation of a block. □

**Lemma 7.** *Let $B$ be a deterministic causal block. Then $B$ has a NCMS representation.*

*Proof (Sketch).* Let $X = \{i \in Sb(In(B), W) \mid \exists\ t \in T\ dom(i) = [0, t]\}$ and $Q = {}^{In(B)}W \times X \times {}^{Out(B)}W$. Then $X \neq \emptyset$. Let *in*, *istate*, *out* denote projection maps from $Q$ on the first, second, and third coordinate respectively. Let $Tr$ be the set of all functions of the form $s : A \to Q$, where $A \in \mathfrak{T}$, such that

(a) for each $t \in dom(s) \backslash \{0\}$ we have $dom(istate(s(t))) = [0, t]$ and $istate(s(t))(t) = in(s(t))$ and if $s(0) \downarrow$, then $istate(s(t))(0) = in(s(0))$;
(b) for each $t \in dom(s)$ we have $t \in dom(o)$ and $out(s(t)) = o(t)$, where $o$ is a unique member of $Op(B)(istate(s(t))$;
(c) if $t_1, t_2 \in dom(s) \backslash \{0\}$ and $t_1 \leq t_2$, then $istate(s(t_1)) \sqsubseteq istate(s(t_2))$.

It is straightforward to check that $\Sigma = (T, Q, Tr)$ is a NCMS.

Let $i \in Sb(In(B), W)$ and $o \in Op(B)(i)$. Let us show that for each $s \in S(\Sigma, i)$, $out \circ s = o|_{dom(s)}$, and if $s \in S_{max}(\Sigma, i)$, then $out \circ s = o$.

Let $s \in S(\Sigma, i)$. Then $dom(s) \in \mathcal{T}_0$ and $in \circ s \preceq i$ by definition of $S(\Sigma, i)$ and $istate(s(t))(t) = in(s(t)) = i(t)$ for all $t \in dom(s)\backslash\{0\}$ by (a). If $t', t \in dom(s)$ and $0 < t' \leq t$, then $i(t') = istate(s(t'))(t') = istate(s(t))(t')$ by (c). Moreover, we have $s(0) \downarrow$, whence $istate(s(t))(0) = in(s(0)) = i(0)$ for each $t \in dom(s)\backslash\{0\}$ by (a). Then for each $t \in dom(s)\backslash\{0\}$, $istate(s(t)) = i|_{[0,t]}$, because $dom(istate(s(t))) = [0, t]$. Then $Op(B)(istate(s(t))) = Op(B)(i|_{[0,t]}) = \{o|_{[0,t]}\}$, because $B$ is deterministic and causal. Then $out(s(t)) = (o|_{[0,t]})(t)$ and $t \in dom(o)$ for each $t \in dom(s)$ by (b). This implies that $dom(s) \subseteq dom(o)$ and for all $t \in dom(s)$, $out(s(t)) = o(t)$. Thus $out \circ s = o|_{dom(s)}$. We have $\{0\} \subset dom(s) \subseteq dom(o)$, so $dom(o) \in \mathfrak{T}$. Because $in(s(0)) = i(0)$ and $out(s(0)) = o(0)$, it is easy to see that a function $s' : dom(o) \to Q$ such that $s'(0) = s(0)$ and $s'(t) = (i(t), i|_{[0,t]}, o(t))$ for all $t \in dom(o)\backslash\{0\}$ satisfies (a), (b), and (c). Moreover, $s' \in Tr$, $dom(s') \in \mathcal{T}_0$, and $in \circ s' = i|_{dom(o)} \preceq i$. Then $s' \in S(\Sigma, i)$. Besides, $s'|_{dom(s)} = s$. Then if $s \in S_{max}(\Sigma, i)$, then $s' = s$ and $out \circ s = o$.

Let us denote $Q_0 = \{(d_{in}, x, d_{out}) \in Q \mid \exists (i, o) \in IO(B)\ \{0\} \in dom(o) \wedge d_{in} = i(0) \wedge d_{out} = o(0)\}$. It is straightforward to show that $S_{init}(\Sigma) \subseteq Q_0$. Thus $(\Sigma, Q_0)$ is an initial I/O NCMS. Obviously, $In(\Sigma) = In(B)$, $Out(\Sigma) = Out(B)$.

It is easy to check that $Q_0$ satisfies the following property:

(d) if $(i, o) \in IO(B)$, $q \in Q_0$, $i \neq \perp$, $in(q) = i(0)$, then $o \neq \perp$ and $out(q) = o(0)$.

Now let us show that $(\Sigma, Q_0)$ is a NCMS representation of $B$. It is sufficient to show that $Op(B)(i) = O_{all}(\Sigma, Q_0, i)$ for all $i \in Sb(In(B), W)\backslash\{\perp\}$.

Let $i \in Sb(In(B), W)\backslash\{\perp\}$ and $o \in Op(B)(i)$. Consider the following cases:

(1) $Sel_{1,2}(Q_0, i(0), x) = \emptyset$ for some $x \in IState(\Sigma)$. Then there is no $(i', o') \in IO(B)$ such that $i'(0) = i(0)$ and $o'(0) \downarrow$. Then $o = \perp$ and $Sel_{1,2}(Q_0, i(0), x) = \emptyset$ for all $x \in IState(\Sigma)$. Then $O_{all}(\Sigma, Q_0, i) = \{\perp\} = Op(B)(i)$.

(2) $Sel_{1,2}(Q_0, i(0), x) \neq \emptyset$ for all $x \in IState(\Sigma)$ and $dom(i) = \{0\}$. Then $o(0) \downarrow$ and $out(q) = o(0)$ for each $q \in Sel_{1,2}(Q_0, i(0), x) \subseteq Q_0$ by the property (d). Then $o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i) = \{\{0\} \mapsto o(0)\} = \{o\}$ for all $x \in IState(\Sigma)$, whence $O_{all}(\Sigma, Q_0, i) = Op(B)(i)$.

(3) $Sel_{1,2}(Q_0, i(0), x) \neq \emptyset$ for all $x \in IState(\Sigma)$, $\{0\} \subset dom(i)$, and $dom(o) \subseteq \{0\}$. If $in(q) = i(0)$ for some $q \in S_{init}(\Sigma, i)$, then $q = s(0)$ for some $s \in S(\Sigma, i)$, whence $out \circ s = o|_{dom(s)}$ as we have shown above, but this is impossible, because $\{0\} \subset dom(s)$ and $dom(o) \subseteq \{0\}$. Thus $in(q) \neq i(0)$ for each $q \in S_{init}(\Sigma, i)$. Then for each $x \in IState(\Sigma)$, $s(0) \notin Sel_{1,2}(Q_0, i(0), x)$ for all $s \in S_{max}(\Sigma, i)$ and $Sel_{1,2}(Q_0, i(0), x) \cap S_{init}(\Sigma, i) = \emptyset$.
Then $o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i) = \{\{0\} \mapsto out(q) \mid q \in Sel_{1,2}(Q_0, i(0), x)\}$ for each $x \in IState(\Sigma)$, whence $O_{all}(\Sigma, Q_0, i) = \{\{0\} \mapsto out(q) \mid q \in Q_0 \wedge in(q) = i(0)\} \neq \emptyset$. Because $in(q) = i(0)$ for some $q \in Q_0$, by the property (d) we have $0 \in dom(o)$ and $O_{all}(\Sigma, Q_0, i) = \{\{0\} \mapsto o(0)\} = \{o\} = Op(B)(i)$.

(4) $Sel_{1,2}(Q_0, i(0), x) \neq \emptyset$ for all $x \in IState(\Sigma)$ and $\{0\} \subset dom(o)$. We have $dom(o) \in \mathfrak{T}$. Let $x \in IState(\Sigma)$ and $q \in Sel_{1,2}(Q_0, i(0), x)$. Then $in(q) = i(0)$ and have $out(q) = o(0)$ by the property (d). It is easy to see that a function $s' : dom(o) \to Q$ such that $s'(0) = q$ and $s'(t) = (i(t), i|_{[0,t]}, o(t))$ for all $t \in dom(o)\backslash\{0\}$ satisfies (a), (b), and (c). Moreover, $s' \in Tr$, $dom(s') \in$

$\mathcal{T}_0$, and $in \circ s' = i|_{dom(o)} \preceq i$. Then $s' \in S(\Sigma, i)$. Then $s'(0) = q \in S_{init}(\Sigma, i)$. Because $q \in Sel_{1,2}(Q_0, i(0), x)$ is arbitrary, we have $Sel_{1,2}(Q_0, i(0), x) \subseteq S_{init}(\Sigma, i)$. Then $o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x), i) = \{out \circ s \mid s \in S_{max}(\Sigma, i) \wedge s(0) \in Sel_{1,2}(Q_0, i(0), x)\} = \{o\}$ for each $x \in IState(\Sigma)$, because $out \circ s = o$ for any $s \in S_{max}(\Sigma, i)$ as we have show above and $Sel_{1,2}(Q_0, i(0), x) \neq \emptyset$. Then $O_{all}(\Sigma, Q_0, i) = \{o\} = Op(B)(i)$, because $IState(\Sigma) \neq \emptyset$.

In all cases (1)-(4) we have $O_{all}(\Sigma, Q_0, i) = Op(B)(i)$. Thus $(\Sigma, Q_0)$ is a NCMS representation of the block $B$. □

Let $\Sigma_1 = (T, Q_1, Tr_1)$ and $\Sigma_2 = (T, Q_2, Tr_2)$ be I/O NCMS such that $In(\Sigma_1) = In(\Sigma_2)$ and $Out(\Sigma_1) = Out(\Sigma_2)$.

Let us introduce the following notions.

**Definition 19.** *(1) a state embedding from $\Sigma_1$ to $\Sigma_2$ is a function $f : Q_1 \to Q_2$ such that $\{f \circ s \mid s \in Tr_1\} = \{s \in Tr_2 \mid \exists t \in dom(s)\ \exists q \in Q_1\ s(t) = f(q)\}$ and there exists an injective function $g : IState(\Sigma_1) \to IState(\Sigma_2)$ such that for all $q \in Q_1$, $f(q) = (in(q), g(istate(q)), out(q))$.*

*(2) A state embedding from an initial I/O NCMS $(\Sigma_1, Q_0^1)$ to an initial I/O NCMS $(\Sigma_2, Q_0^2)$ is a state embedding $f$ from $\Sigma_1$ to $\Sigma_2$ such that for each $q \in Q_1$, $q \in Q_0^1$ iff $f(q) \in Q_0^2$.*

Note that it follows immediately from this definition that a state embedding from $\Sigma_1$ to $\Sigma_2$ is an injective function.

**Lemma 8.** *Let $\Sigma_1 = (T, Q_1, Tr_1)$ and $\Sigma_2 = (T, Q_2, Tr_2)$ be I/O NCMS, $In(\Sigma_1) = In(\Sigma_2)$ and $Out(\Sigma_1) = Out(\Sigma_2)$, and $f$ be a state embedding from $\Sigma_1$ to $\Sigma_2$. Let $i \in Sb(In(\Sigma_1), W)$. Then $S_{max}(\Sigma_2, i) \supseteq \{f \circ s \mid s \in S_{max}(\Sigma_1, i)\}$ and $\{q \in S_{init}(\Sigma_2, i) \mid \exists q' \in Q_1\ q = f(q')\} = \{f(q'') \mid q'' \in S_{init}(\Sigma_1, i)\}$.*

The proof follows immediately from the definitions and is omitted here.

**Lemma 9.** *For $j = 1, 2$ let $(\Sigma_j, Q_0^j)$ be a NCMS representation of a block $B_j$. Assume that $In(\Sigma_1) = In(\Sigma_2)$ and $Out(\Sigma_1) = Out(\Sigma_2)$ and there exists a state embedding $f$ from $(\Sigma_1, Q_0^1)$ to $(\Sigma_2, Q_0^2)$. Then $B_1 \trianglelefteq B_2$.*

*Proof (Sketch).* Assume that $\Sigma_1 = (T, Q_1, Tr_1)$ and $\Sigma_2 = (T, Q_2, Tr_2)$. We have $In(\Sigma_1) = In(\Sigma_2)$ and $Out(\Sigma_1) = Out(\Sigma_2)$. Because $f$ is a state embedding, there exists an injective function $g : IState(\Sigma_1) \to IState(\Sigma_2)$ such that $f(q) = (in(q), g(istate(q)), out(q))$ for all $q \in Q$.

Let $i \in Sb(In(B), W)$. Then for $j = 1, 2$, $Op(B_j)(i) = O_{all}(\Sigma_j, Q_0^j, i)$.

Let us show that $O_{all}(\Sigma_2, Q_0^2, i) \supseteq O_{all}(\Sigma_1, Q_0^1, i)$. This is obvious, if $i = \perp$, so assume that $i \neq \perp$. Let us fix some $x_1 \in IState(\Sigma_1)$. Denote $Q_1' = Sel_{1,2}(Q_0^1, i(0), x_1)$ and $Q_2' = Sel_{1,2}(Q_0^2, i(0), g(x_1))$. Because $g$ is injective and $Q_0^2 \supseteq \{f(q) \mid q \in Q_0^1\}$, it is straightforward to show that $Q_2' \supseteq \{f(q) \mid q \in Q_1'\}$ and $Q_2' \neq \emptyset$ iff $Q_1' \neq \emptyset$.

Let us show that $o_{all}(\Sigma_2, Q_2', i) \supseteq o_{all}(\Sigma_1, Q_1', i)$. This is obvious, if $Q_1' = \emptyset$ or $Q_2' = \emptyset$, so assume that $Q_1' \neq \emptyset$ and $Q_2' \neq \emptyset$.

Consider the case when $dom(i) = \{0\}$. Because $Q'_2 \supseteq \{f(q) \mid q \in Q'_1\}$, it is easy to check that $o_{all}(\Sigma_2, Q'_2, i) = o_{all}(\Sigma_1, Q'_1, i)$.

Consider the case when $\{0\} \subset dom(i)$. By Lemma 8 we have $S_{max}(\Sigma_2, i) \supseteq \{f \circ s \mid s \in S_{max}(\Sigma_1, i)\}$ and $\{q \in S_{init}(\Sigma_2, i) \mid \exists q' \in Q_1 \; q = f(q')\} = \{f(q'') \mid q'' \in S_{init}(\Sigma_1, i)\}$. Because $f$ is injective and $Q'_2 \supseteq \{f(q) \mid q \in Q'_1\}$,

$$\{out \circ s \mid s \in S_{max}(\Sigma_2, i) \wedge s(0) \in Q'_2\} \supseteq \{out \circ (f \circ s) \mid s \in S_{max}(\Sigma_1, i) \wedge \\ \wedge f(s(0)) \in Q'_2\} \supseteq \{out \circ s \mid s \in S_{max}(\Sigma_1, i) \wedge s(0) \in Q'_1\}.$$

Because $Q'_1 \subseteq Q_1$ and $f$ is injective, it is straightforward to check that $Q'_2 \backslash S_{init}(\Sigma_2, i) \supseteq \{f(q) \mid q \in Q'_1 \backslash S_{init}(\Sigma_1, i)\}$. Then from the definition of $o_{all}$ it follows that $o_{all}(\Sigma_2, Q'_2, i) \supseteq o_{all}(\Sigma_1, Q'_1, i)$.

Because $x_1 \in IState(\Sigma_1)$ is arbitrary, it easily follows that $Op(B_2)(i) \supseteq Op(B_1)(i)$. We conclude that $B_1 \trianglelefteq B_2$. □

**Definition 20.** *A disjoint union of an indexed family of initial I/O NCMS* $((\Sigma_j, Q_0^j))_{j \in J}$*, where* $J \neq \emptyset$ *and* $\Sigma_j = (T, Q_j, Tr_j)$ *for each* $j \in J$*, is a pair* $(\Sigma, Q_0)$*, where* $\Sigma = (T, Q, Tr)$ *and*

*(1)* $Q = {}^{IN}W \times (\bigcup_{j \in J} \{j\} \times IState(\Sigma_j)) \times {}^{OUT}W$*, where* $IN = \bigcup_{j \in J} In(\Sigma_j)$*, and* $OUT = \bigcup_{j \in J} Out(\Sigma_j)$*;*
*(2)* $Tr = \{f_j \circ s \mid j \in J \wedge s \in Tr_j\}$*;*
*(3)* $Q_0 = \{f_j(q) \mid j \in J \wedge q \in Q_0^j\}$*;*

*where for each* $j \in J$*,* $f_j : Q_j \to Q$ *is a function such that*

$$f_j(q) = (in(q), (j, istate(q)), out(q)), \;\; q \in Q_j.$$

**Lemma 10.** *Let* $(\Sigma, Q_0)$ *be a disjoint union of an indexed family of initial I/O NCMS* $((\Sigma_j, Q_0^j))_{j \in J}$*, where* $J \neq \emptyset$*. Then* $(\Sigma, Q_0)$ *is an initial I/O NCMS.*

The proof is straightforward and is omitted here.

**Definition 21.** *(1) A complete set of sub-blocks of a block* $B$ *is a set* $\mathcal{B}$ *of sub-blocks of* $B$ *such that* $IO(B) = \bigcup_{B' \in \mathcal{B}} IO(B')$*.*
*(2) A complete indexed family of sub-blocks of a block* $B$ *is an indexed family* $(B_j)_{j \in J}$ *such that* $\{B_j \mid j \in J\}$ *is a complete set of sub-blocks of* $B$*.*

**Definition 22.** *A state-restriction of a NCMS* $\Sigma = (T, Q, Tr)$ *on a set* $Q'$*, denoted as* $\Sigma|_{Q'}$*, is a triple* $(T, Q \cap Q', \{s \in Tr \mid \forall t \in dom(s) \; s(t) \in Q'\})$*.*

**Lemma 11.** $\Sigma|_{Q'}$ *is a NCMS for each NCMS* $\Sigma = (T, Q, Tr)$ *and set* $Q'$*,*

The proof follows immediately from the definition of NCMS and is omitted here.

**Lemma 12.** *Let* $(B_j)_{j \in J}$ *be a complete indexed family of sub-blocks of a block* $B$*, where* $J \neq \emptyset$*. Assume that for each* $j \in J$*,* $B_j$ *has a NCMS representation* $(\Sigma_j, Q_0^j)$*. Let* $(\Sigma, Q_0)$ *be a disjoint union of* $((\Sigma_j, Q_0^j))_{j \in J}$*. Then* $(\Sigma, Q_0)$ *is a NCMS representation of* $B$*.*

*Proof (Sketch).* Assume that $\Sigma_j = (T, Q_j, Tr_j)$ for each $j \in J$ and $\Sigma = (T, Q, Tr)$.

By Lemma 10, $(\Sigma, Q_0)$ is an initial I/O NCMS, whence by Lemma 6, there exists a block $B'$ (unique up to semantic identity) such that $(\Sigma, Q_0)$ is a NCMS representation of $B'$.

For each $j \in J$ we have $In(\Sigma_j) = In(B_j) = In(B)$ and $Out(\Sigma_j) = Out(B_j) = Out(B)$, because $B_j \trianglelefteq B$. Because $J \neq \emptyset$, $In(B') = In(\Sigma) = \bigcup_{j \in J} In(\Sigma_j) = In(B)$ and $Out(B') = Out(\Sigma) = \bigcup_{j \in J} Out(\Sigma_j) = Out(B)$.

For each $j \in J$, let $g_j : IState(\Sigma_j) \to IState(\Sigma)$ and $f_j : Q_j \to Q$ be functions such that $g_j(x) = (j, x)$ for all $x \in IState(\Sigma_j)$ and $f_j(q) = (in(q), g_j(istate(q)), out(q))$ for all $q \in Q_j$.

Using Lemma 9 it is not difficult to show that $B \trianglelefteq B'$.

Let us show that $B' \trianglelefteq B$. Let $(i, o) \in IO(B')$. Then $o \in O_{all}(\Sigma, Q_0, i)$. If $i = \perp$, then $o = \perp$ and $(i, o) \in IO(B)$ (because $B$ is a block). Consider the case when $i \neq \perp$. Then there exists $x^* \in IState(\Sigma) = \bigcup_{j \in J} \{j\} \times IState(\Sigma_j)$ such that $o \in o_{all}(\Sigma, Sel_{1,2}(Q_0, i(0), x^*), i)$. Then there exists $j \in J$ and $x_j^* \in IState(\Sigma_j)$ such that $x^* = (j, x_j^*)$.

Let $Q'_j = {}^{In(\Sigma)}W \times (\{j\} \times IState(\Sigma_j)) \times {}^{Out(\Sigma)}W$, and $\Sigma'_j = \Sigma|_{Q'_j}$. By Lemma 11, $\Sigma'_j$ is a NCMS. We will denote by $Tr'_j$ the set of trajectories of $\Sigma'_j$. Moreover, $Q'_j$ is the set of states of $\Sigma'_j$ and $In(\Sigma'_j) = In(\Sigma)$, $Out(\Sigma'_j) = Out(\Sigma)$. Besides, $\Sigma'_j$ is an I/O NCMS and $S_{init}(\Sigma'_j) = S_{init}(\Sigma|_{Q'_j}) \subseteq S_{init}(\Sigma) \cap Q'_j \subseteq Q_0 \cap Q'_j \subseteq Q'_j$, because $(\Sigma, Q_0)$ is an initial I/O NCMS. Denote $Q'_{0,j} = Q_0 \cap Q'_j$. Then $(\Sigma'_j, Q'_{0,j})$ is an initial I/O NCMS. Moreover, $x^* \in IState(\Sigma'_j)$.

It is straightforward to show that $o \in O_{all}(\Sigma'_j, Q'_{0,j}, i)$. By Lemma 6, there exists a block $B'_j$ such that $(\Sigma'_j, Q'_{0,j})$ is a NCMS representation of $B'_j$. Let $g : IState(\Sigma'_j) \to IState(\Sigma_j)$ and $f : Q'_j \to Q_j$ be functions such that $g((j, x)) = x$ for $x \in IState(\Sigma'_j)$ and $f(q) = (in(q), g(istate(q)), out(q))$ for all $q \in Q'_j$. Obviously, $In(\Sigma'_j) = In(\Sigma_j)$, $Out(\Sigma'_j) = Out(\Sigma_j)$, and $g$ is injective. Moreover, $f$ is an inverse of $f_j$, whence $\{f \circ s \mid s \in Tr'_j\} = Tr_j$. Because for any $s \in Tr_j$, $dom(s) \neq \emptyset$ and for each $t \in dom(s)$, $s(t) = f(f_j(s(t)))$, where $f_j(s(t)) \in Q'_j$, we have $\{f \circ s \mid s \in Tr'_j\} = \{s \in Tr_j \mid \exists t \in dom(s)\ \exists q \in Q'_j\ s(t) = f(q)\}$. Then $f$ is a state embedding from $\Sigma'_j$ to $\Sigma_j$. Moreover, for each $q \in Q'_j$, $q \in Q'_{0,j} = Q_0 \cap Q'_j$ iff $q = f_j(q')$ for some $q' \in Q_0^j$ iff $f(q) \in Q_0^j$. Then $f$ is a state embedding from $(\Sigma'_j, Q'_{0,j})$ to $(\Sigma_j, Q_0^j)$. Then $B'_j \trianglelefteq B_j$ by Lemma 9. Because $o \in O_{all}(\Sigma'_j, Q'_{0,j}, i) = Op(B'_j)(i)$, we have $o \in Op(B_j)(i)$, whence $(i, o) \in IO(B)$. Thus $B' \trianglelefteq B$. We conclude that $B \trianglelefteq B'$ and $B' \trianglelefteq B$, so $B$ and $B'$ are semantically identical and $(\Sigma, Q_0)$ is a NCMS representation of $B$. □

Now we can prove Theorem 2.

*Proof (of Theorem 2).*

Let $B$ be a strongly nonanticipative block. Let us show that $B$ has a NCMS representation. Let $\mathcal{R}$ be the set of all relations $R \subseteq IO(B)$ such that $R$ is an I/O relation of a deterministic causal block. For each $R \in \mathcal{R}$ let us define a block $B_R$ such that $IO(B_R) = R$, $In(B_R) = In(B)$, $Out(B_R) = Out(B)$. Then $B_R$ is a deterministic causal block for each $R \in \mathcal{R}$ and $IO(B) = \bigcup_{R \in \mathcal{R}} IO(B_R)$, because $B$ is strongly nonanticipative. Then $(B_R)_{R \in \mathcal{R}}$ is a complete indexed family of sub-blocks of $B$ and $\mathcal{R} \neq \emptyset$. By Lemma 7, for each $R \in \mathcal{R}$ there exists an initial I/O NCMS $(\Sigma_R, Q_0^R)$ which is a NCMS representation of $B_R$. Let $(\Sigma, Q_0)$ be a disjoint union of $((\Sigma_R, Q_0^R))_{R \in \mathcal{R}}$. Then by Lemma 12, $(\Sigma, Q_0)$ is a NCMS representation of $B$. □

# 4 Existence of Total I/O Pairs of Strongly Nonanticipative Blocks

## 4.1 Using NCMS Representation

The following theorems show that the questions (A) and (B) formulated in Section 1 can be reduced to the problem of existence of total trajectories of NCMS.

**Theorem 3.** *Let $B$ be a strongly nonanticipative block and $(\Sigma, Q_0)$ be its NCMS representation, where $\Sigma = (T, Q, Tr)$. Then $B$ has a total I/O pair iff there exists $s \in Tr$ such that $dom(s) = T$.*

*Proof.* Let us prove the "If" part. Assume that $s \in Tr$ and $dom(s) = T$. Let $q_0 = s(0)$, $x = istate(q_0)$, $i = in \circ s$, $o = out \circ s$, and $Q' = Sel_{1,2}(Q_0, i(0), x)$. Then $q_0 \in S_{init}(\Sigma) \subseteq Q_0$, whence $q_0 \in Q'$, so $Q' \neq \emptyset$. Besides, $s \in S_{max}(\Sigma, i)$, because $dom(s) = T$ and $in \circ s = i \preceq i$. Then because $s(0) \in Q'$, we have $o = out \circ s \in o_{all}(\Sigma, Q', i)$ by the definition of $o_{all}$. Then $o \in O_{all}(\Sigma, Q_0, i) = Op(B)(i)$, because $i \neq \perp$ and $(\Sigma, Q_0)$ is a NCMS representation of $B$. Then $(i, o) \in IO(B)$ and $dom(i) = dom(o) = T$. Thus $B$ has a total I/O pair.

Now let us prove the "Only if" part. Assume that $B$ has a total I/O pair $(i, o) \in IO(B)$. Because $(\Sigma, Q_0)$ is a NCMS representation of $B$ and $i \neq \perp$, we have $o \in O_{all}(\Sigma, Q_0, i)$. Then there is $x \in IState(\Sigma)$ such that $o \in o_{all}(\Sigma, Q', i)$, where $Q' = Sel_{1,2}(Q_0, i(0), x)$. Then $o = out \circ s$ for some $s \in S_{max}(\Sigma, i)$ such that $s(0) \in Q'$, because $dom(o) = T$. Then $s \in Tr$, $dom(s) = T$. □

**Theorem 4.** *Let $B$ be a strongly nonanticipative block and $(\Sigma, Q_0)$ be its NCMS representation, where $\Sigma = (T, Q, Tr)$.*

*Let $i \in Sb(In(B), W)$ and $dom(i) = T$. Let $(l, r)$ be a LR representation of $\Sigma$ and $l' : ST(Q) \to Bool$ and $r' : ST(Q) \to Bool$ be predicates such that*

$$l'(s, t) \Leftrightarrow l(s, t) \wedge (\min dom(s) \downarrow = t \vee in(s(t)) = i(t)).$$

$$r'(s, t) \Leftrightarrow r(s, t) \wedge (\max dom(s) \downarrow = t \vee in(s(t)) = i(t)).$$

*Then*

*(1) $(l', r') \in LR(Q)$;*
*(2) If $(l', r')$ is a LR representation of a NCMS $\Sigma' = (T, Q, Tr')$, then $\{o \in Op(B)(i) \mid dom(o) = T\} \neq \emptyset$ iff there exists $s \in Tr'$ such that $dom(s) = T$.*

*Proof (Sketch).*

(1) It is straightforward to check that $l'$ is left-local and $r'$ is right-local.
(2) Assume that $(l', r')$ is a LR representation of a NCMS $\Sigma' = (T, Q, Tr')$. Then $Tr' = \{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A\ l'(s, t) \wedge r'(s, t))\}$.

It is straightforward to show that $\{s \in Tr' \mid dom(s) \in \mathcal{T}_0\} = S(\Sigma, i)$.
Let us show that $\{o \in Op(B)(i) \mid dom(o) = T\} \neq \emptyset$ iff there exists $s \in Tr'$ such that $dom(s) = T$.

**"If"** Assume that $s \in Tr'$ and $dom(s) = T$. Then $s \in S(\Sigma, i)$. Let $q_0 = s(0)$, $x = istate(q_0)$, $o = out \circ s$, and $Q' = Sel_{1,2}(Q_0, i(0), x)$. Then $q_0 \in S_{init}(\Sigma') \subseteq S_{init}(\Sigma) \subseteq Q_0$, whence $q_0 \in Q'$, so $Q' \neq \emptyset$. Besides, $s \in S_{max}(\Sigma, i)$, because $dom(s) = T$. Then because $s(0) \in Q'$, we have $o = out \circ s \in o_{all}(\Sigma, Q', i)$ by the definition of $o_{all}$. Then $o \in O_{all}(\Sigma, Q_0, i) = Op(B)(i)$, because $i \neq \bot$ and $(\Sigma, Q_0)$ is a NCMS representation of $B$. Besides, $dom(o) = T$. Thus $\{o \in Op(B)(i) \mid dom(o) = T\} \neq \emptyset$.

**"Only if"** Assume that $o \in Op(B)(i)$ and $dom(o) = T$. Because $(\Sigma, Q_0)$ is a NCMS representation of $B$ and $i \neq \bot$, we have $o \in O_{all}(\Sigma, Q_0, i)$. Then there is $x \in IState(\Sigma)$ such that $o \in o_{all}(\Sigma, Q', i)$, where $Q' = Sel_{1,2}(Q_0, i(0), x)$. Then $o = out \circ s$ for some $s \in S_{max}(\Sigma, i)$ such that $s(0) \in Q'$, because $dom(o) = T$. Then $s \in S(\Sigma, i)$, whence $s \in Tr'$ and $dom(s) = T$.

□

Now we will focus on the problem of existence of total trajectories of a NCMS.

### 4.2 Existence of Globally Defined Trajectories of NCMS

An obvious method for proving existence of a total trajectory of a NCMS with a given LR representation $(l, r)$ is just guessing a function $s : T \to Q$ such that $\forall t \in T\ l(s,t) \wedge r(s,t)$.

As an alternative to guessing an entire trajectory one can try to find/guess for each $t$ a partial trajectory $s_t$ defined in a neighborhood of $t$ which satisfies $l(s_t, t) \wedge r(s_t, t)$ in such a way that all $s_t$, $t \in T$ can be glued together into a total function. An important aspect here is that the admissible choices of $s_t$, $s_{t'}$ for distant time moments $t, t' \in T$ (i.e. such that $s_t$, $s_{t'}$ appear as subtrajectories of some total trajectory) can be dependent.

However, this method can be generalized: instead of guessing an exact total trajectory or its exact locally defined subtrajectories, one can guess some "region" (subset of trajectories) which presumably contains a total trajectory and has some convenient representation. It is desirable that for this region the proof of existence of a total trajectory can be accomplished by finding/guessing locally defined trajectories in a neighborhood of each time moment independently, or at least so that when choosing a local trajectory in a neighborhood of a time moment $t$ one does not need to care about a choice of a local trajectory in a neighborhood of a distant time moment.

We formalize the described generalized method of proving existence of total trajectories of a NCMS as follows.

Let $\Sigma = (T, Q, Tr)$ be a fixed NCMS.

**Definition 23.** *$\Sigma$ satisfies*

*(1) local forward extensibility (LFE) property, if for each $s \in Tr$ of the form $s : [a,b] \to Q$ $(a < b)$ there exists a trajectory $s' : [a,b'] \to Q$ such that $s' \in Tr$, $s \sqsubseteq s'$ and $b' > b$.*
*(2) global forward extensibility (GFE) property, if for each trajectory $s$ of the form $s : [a,b] \to Q$ there exists a trajectory $s' : [a,+\infty) \to Q$ such that $s \sqsubseteq s'$.*

**Theorem 5.** *Let $(l,r)$ be a LR representation of $\Sigma$. Then $\Sigma$ has a total trajectory iff there exists a pair $(l',r') \in LR(Q)$ such that*

*(1) $l'(s,t) \Rightarrow l(s,t)$ and $r'(s,t) \Rightarrow r(s,t)$ for all $(s,t) \in ST(Q)$;*
*(2) $\forall t \in [0,\epsilon]$ $l'(s,t) \wedge r'(s,t)$ for some $\epsilon > 0$ and a function $s : [0,\epsilon] \to Q$;*
*(3) if $(l',r')$ is a LR representation of a NCMS $\Sigma'$, then $\Sigma'$ satisfies GFE.*

*Proof (Sketch).* Let us prove the "If" part. Assume that (1)-(3) hold. By (2) there exists $\epsilon > 0$ and $s : [0,\epsilon] \to Q$ such that $l'(s,t) \wedge r'(s,t)$ for all $t \in [0,\epsilon]$. Let $\Sigma' = (T,Q,Tr')$ be a NCMS such that $(l',r')$ is a LR representation of $\Sigma'$ (which exists, because $(l',r') \in LR(Q)$). Then by (3), $\Sigma'$ satisfies GFE. Besides, $s \in Tr'$. Then there exists $s' : [0,+\infty) \to Q$ such that $s' \in Tr'$ and $s \sqsubseteq s'$. Then $l'(s,t) \wedge r'(s,t)$ for all $t \in T$, whence $s' \in Tr$, because of (1), so $\Sigma$ has a total trajectory.

Now let us prove the "Only if" part. Assume that $\Sigma$ has a total trajectory $s^* \in Tr$. Let $l' : ST(Q) \to Bool$ and $r' : ST(Q) \to Bool$ be predicates such that

$$l'(s,t) \Leftrightarrow l(s,t) \wedge (\min dom(s) \downarrow= t \vee s(t) = s^*(t)).$$

$$r'(s,t) \Leftrightarrow r(s,t) \wedge (\max dom(s) \downarrow= t \vee s(t) = s^*(t)).$$

It is easy to check that $(l',r') \in LR(Q)$ and $(l',r')$ satisfies (1)-(3). □

Theorem 5 means that existence of a total trajectory of a NCMS $\Sigma$ with LR representation $(l,r)$ can be proved using the following approach:

(1) Choose/guess a pair $(l',r') \in LR(Q)$, where $l'(s,t) \Rightarrow l(s,t)$ and $r'(s,t) \Rightarrow r(s,t)$ for all $(s,t) \in ST(Q)$.
By Definition 15 and Theorem 1, this pair is a LR representation of the NCMS $\Sigma' = (T,Q,Tr')$, where

$$Tr' = \{s : A \to Q \mid A \in \mathfrak{T} \wedge (\forall t \in A\ l'(s,t) \wedge r'(s,t))\} \subseteq Tr.$$

The set $Tr' \subseteq Tr$ plays the role of a region which presumably contains a total trajectory.
(2) If it is possible to find a function $s$ on a small segment $[0,\epsilon]$ which satisfies $l'(s,t) \wedge r'(s,t)$ for $t \in [0,\epsilon]$ (i.e. $s$ is a trajectory of $\Sigma'$) and prove that $\Sigma'$ satisfies GFE, then $\Sigma$ has a total trajectory.

To complete this method of proving existence of a total trajectory, in the next section we will show that the GFE property of a NCMS can be proven by proving existence of certain locally defined trajectories independently in a neighborhood of each time moment.

### 4.3 Reduction of the GFE Property to the LFE Property

As above, let $\Sigma = (T, Q, Tr)$ be a fixed NCMS.

**Definition 24 ([27]).** *A right dead-end path (in $\Sigma$) is a trajectory $s : [a, b) \to Q$, where $a, b \in T$, $a < b$, such that there is no $s' : [a, b] \to Q$, $s \in Tr$ such that $s \sqsubset s'$ (i.e. $s$ cannot be extended to a trajectory on $[a, b]$).*

**Definition 25 ([27]).** *An escape from a right dead-end path $s : [a, b) \to Q$ (in $\Sigma$) is a trajectory $s' : [c, d) \to Q$ (where $d \in T \cup \{+\infty\}$) or $s' : [c, d] \to Q$ (where $d \in T$) such that $c \in (a, b)$, $d > b$, and $s(c) = s'(c)$. An escape $s'$ is called infinite, if $d = +\infty$.*

**Definition 26 ([27]).** *A right dead-end path $s : [a, b) \to Q$ in $\Sigma$ is called strongly escapable, if there exists an infinite escape from $s$.*

**Lemma 13.** *If $s : [a, b) \to Q$ is a right dead-end path and $c \in (a, b)$, then $s|_{[c,b)}$ is a right dead-end path.*

The proof follows immediately from the CPR and Markovian properties of $\Sigma$.

**Lemma 14.** *$\Sigma$ satisfies GFE iff $\Sigma$ satisfies LFE and each right dead-end path is strongly escapable.*

The proof is analogous to the proof of Lemma 3 in [27] and is omitted here.

**Definition 27**

*(1) A right extensibility measure is a function $f^+ : \mathbb{R} \times \mathbb{R} \tilde{\to} \mathbb{R}$ such that $A = \{(x, y) \in T \times T \mid x \leq y\} \subseteq dom(f^+)$, $f(x, y) \geq 0$ for all $(x, y) \in A$, $f^+|_A$ is strictly decreasing in the first argument and strictly increasing in the second argument, and for each $x \geq 0$, $f^+(x, x) = x$ and $\lim_{y \to +\infty} f^+(x, y) = +\infty$.*

*(2) A right extensibility measure $f^+$ is called normal, if $f^+$ is continuous on $\{(x, y) \in T \times T \mid x \leq y\}$ and there exists a function $\alpha$ of class $K_\infty$ (i.e. the function $\alpha : [0, +\infty) \to [0, +\infty)$ is continuous, strictly increasing, and $\lim_{x \to +\infty} \alpha(x) = +\infty$, $\alpha(0) = 0$) such that $\alpha(y) < y$ for all $y > 0$ and the function $y \mapsto f^+(\alpha(y), y)$ is of class $K_\infty$.*

Let us fix a right extensibility measure $f^+$.

**Definition 28.** *A right dead-end path $s : [a, b) \to Q$ is called $f^+$-escapable (Fig. 3), if there exists an escape $s' : [c, d] \to Q$ from $s$ such that $d \geq f^+(c, b)$.*

An example of a right extensibility measure is $f^+(x, y) = 2y - x$ $(x \leq y)$. In this case, for a right dead-end path to be $f^+$-escapable it is necessary that there exists an escape $s' : [c, d] \to Q$ with $d - b \geq b - c$.

**Theorem 6.** *Assume that $f^+$ is a normal right extensibility measure and $\Sigma$ satisfies LFE. Then each right dead-end path is strongly escapable iff each right dead-end path is $f^+$-escapable.*

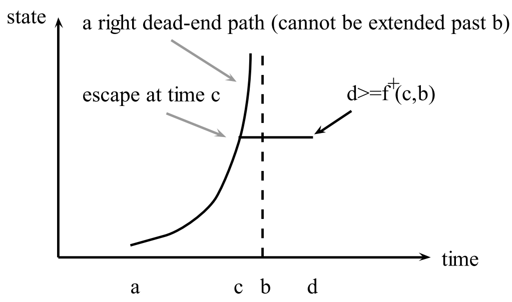


**Fig. 3.** An $f^+$-escapable right dead-end path $s : [a, b) \to Q$ (shown as a curve) and a corresponding escape $s' : [c, d] \to Q$ (shown as a horizonal segment) with $d \geq f^+(c, b)$

*Proof (Sketch).* The statement of this theorem is similar to the statement of [27, Theorem 2] with the difference that here it is assumed that $\Sigma$ satisfies LFE instead of a stronger condition called weak local extensibility (WLE) [27] (which is used in the proof of [27, Lemma 15]) and the right extensibility measure is assumed to be normal. However, it is straightforward to check that the proof given in [27] is valid for the statement formulated here. □

**Theorem 7 (A criterion for the GFE property).** *Let $(l, r)$ be an LR representation of a NCMS $\Sigma$ and $f^+$ be a normal right extensibility measure. Then $\Sigma$ satisfies GFE iff for each $t > 0$ there exists $\epsilon \in (0, t]$ such that for each $t_0 \in [t - \epsilon, t)$ and $s : [t_0, t] \to Q$:*

*(1)* $(\forall \tau \in [t_0, t]\ l(s, \tau) \wedge r(s, \tau)) \Rightarrow \exists t_1 > t$
$\exists s' : [t, t_1] \to Q\ s'(t) = s(t) \wedge (\tau \in dom(s')\ l(s', \tau) \wedge r(s', \tau))$;
*(2)* $(\forall \tau \in [t_0, t)\ l(s, \tau) \wedge r(s, \tau)) \wedge \neg l(s, t) \Rightarrow \exists t_1 \in (t_0, t)$
$\exists s' : [t_1, f^+(t_1, t)] \to Q\ s'(t_1) = s(t_1) \wedge (\tau \in dom(s')\ l(s', \tau) \wedge r(s', \tau))$.

*Proof.* Let us prove the "If" part. Assume that for each $t > 0$ there exists $\epsilon \in (0, t]$ such that (1) and (2) hold for each $t_0 \in [t - \epsilon, t)$ and $s : [t_0, t] \to Q$.

Firstly, let us show that $\Sigma$ satisfies LFE. Let $\bar{s} : [a, b] \to Q$ be a trajectory of $\Sigma$. Then $b > a \geq 0$. Then for $t = b$ there exists $\epsilon \in (0, t]$ such that (1) holds for each $t_0 \in [t - \epsilon, t)$ and $s : [t_0, t] \to Q$. Let $t_0 = \max\{a, t - \epsilon\}$ and $s = \bar{s}|_{[t_0, t]}$. Then $s \in Tr$ by the CPR property and $l(s, \tau) \wedge r(s, \tau)$ for all $\tau \in [t_0, t]$, and by (1) there exists $t_1 > t = b$ and $s' : [t, t_1] \to Q$ such that $s'(t) = s(t) = \bar{s}(t)$ and $l(s', \tau) \wedge r(s', \tau)$ for all $\tau \in dom(s')$. Then $s' \in Tr$. Let us define $s'' : [a, t_1] \to Q$ as follows: $s''(\tau) = \bar{s}(\tau)$, if $\tau \in [a, b]$ and $s''(\tau) = s'(\tau)$, if $\tau \in [b, t_1]$. Then $s'' \in Tr$ by the Markovian property. Also, $\bar{s} \sqsubseteq s''$ and $t_1 > b$. So $\Sigma$ satisfies LFE.

Secondly, let us show that each right dead-end path in $\Sigma$ is $f^+$-escapable. Let $\bar{s} : [a, b) \to Q$ be a right dead-end path in $\Sigma$ . Then $b > 0$. Then for $t = b$ there exists $\epsilon \in (0, t]$ such that (2) holds for each $t_0 \in [t - \epsilon, t)$ and $s : [t_0, t] \to Q$. Let $t_0 = \max\{a, t - \epsilon\}$ and $s$ be some continuation of $\bar{s}|_{[t_0, t)}$ on $[t_0, t]$. Then $s|_{[t_0, t)} \in Tr$ by the CPR property and $l(s, \tau) \wedge r(s, \tau)$ for all $\tau \in [t_0, t)$. Besides,

$\neg l(s,t)$, because $\bar{s}$ is a dead-end path and $r(s,t)$ holds. Then by (2) there exists $t_1 \in (t_0, t)$ and $s' : [t_1, f^+(t_1,t)] \to Q$ such that $s'(t_1) = s(t_1)$ and $l(s',\tau) \wedge r(s',\tau)$ for all $\tau \in dom(s')$. Then $s' \in Tr$. Moreover, $t_1 \in (a,b)$, $s'(t_1) = s(t_1) = \bar{s}(t_1)$, and $\max dom(s') \geq f^+(t_1, b)$. Thus $s'$ is an escape from $\bar{s}$ and $\bar{s}$ is $f^+$-escapable.

Now by Theorem 6, each right dead-end path in $\Sigma$ is strongly escapable. Then by Lemma 14, $\Sigma$ satisfies GFE.

Now let us prove the "Only if" part. Assume that $\Sigma$ satisfies GFE. Let $t > 0$. Let us choose an arbitrary $\epsilon \in (0, t]$. Assume that $t_0 \in [t-\epsilon, t)$ and $s : [t_0, t] \to Q$.

Let us show (1). Assume that $l(s,\tau) \wedge r(s,\tau)$ for all $\tau \in [t_0, t]$. Then $s \in Tr$ and by GFE there exists $s_1 : [t_0, +\infty] \to Q$ such that $s_1 \in Tr$ and $s \sqsubseteq s_1$. Let $t_1 = t + 1$ and $s' = s|_{[t,t_1]}$. Then $s' \in Tr$ by the CPR property and $s'(t) = s(t)$ and $l(s',\tau) \wedge r(s',\tau)$ for all $\tau \in dom(s')$.

Let us show (2). Assume that $l(s,\tau) \wedge r(s,\tau)$ for all $\tau \in [t_0, t)$. Then $s|_{[t_0,t)} \in Tr$. Firstly, consider the case when $s|_{[t_0,t)}$ is a right dead-end path in $\Sigma$. Then by Lemma 14 it is strongly escapable, so there exists $t_1 \in (t_0, t)$ and $s_1 : [t_1, +\infty) \to Q$ such that $s_1(t_1) = s(t)$ and $s_1 \in Tr$. Let $s' = s_1|_{[t_1, f^+(t_1,t)]}$. Then $s' \in Tr$ by the CPR property and $s'(t_1) = s(t_1)$ and $l(s',\tau) \wedge r(s',\tau)$ for all $\tau \in dom(s')$.

Now assume that $s|_{[t_0,t)}$ is not a right dead-end path. Then there exists $s_0 : [t_0, t] \to Q$ such that $s_0 \in Tr$ and $s|_{[t_0,t)} \sqsubseteq s_0$. Then by GFE there exists $s_1 : [t_0, +\infty] \to Q$ such that $s_1 \in Tr$ and $s_0 \sqsubseteq s_1$. Let us choose any $t_1 \in (t_0, t)$ and define $s' = s_1|_{[t_1, f^+(t_1,t)]}$. Then $s' \in Tr$ by the CPR property, and $s'(t_1) = s_1(t_1) = s_0(t_1) = s(t_1)$ and $l(s',\tau) \wedge r(s',\tau)$ for all $\tau \in dom(s')$. □

This theorem means that to prove the GFE property, it is sufficient to prove the existence of certain locally defined trajectories in a neighborhood of each $t \in T$.

## 5 Conclusion

We have considered the questions of the existence of total I/O pairs of a given strongly nonanticipative block and the existence of a total output signal bunch for a given total input signal bunch. We have reduced them to the problem of the existence of total trajectories of NCMS using a NCMS representation. For the latter problem we have proposed a criterion which can be reduced to the problem of checking the existence of certain locally defined trajectories.

## References


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