# Subgraphs Satisfying MSO Properties on $z$-Topologically Orderable Digraphs

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**Abstract.** We introduce the notion of $z$-topological orderings for digraphs. We prove that given a digraph $G$ on $n$ vertices admitting a $z$-topological ordering, together with such an ordering, one may count the number of subgraphs of $G$ that at the same time satisfy a monadic second order formula $\varphi$ and are the union of $k$ **directed** paths, in time $f(\varphi, k, z) \cdot n^{O(k \cdot z)}$. Our result implies the polynomial time solvability of many natural counting problems on digraphs admitting $z$-topological orderings for constant values of $z$ and $k$. Concerning the relationship between $z$-topological orderability and other digraph width measures, we observe that any digraph of **directed** path-width $d$ has a $z$-topological ordering for $z \leq 2d+1$. On the other hand, there are digraphs on $n$ vertices admitting a $z$-topological order for $z = 2$, but whose directed path-width is $\Theta(\log n)$. Since graphs of bounded **directed** path-width can have both arbitrarily large **undirected** tree-width and arbitrarily large clique width, our result provides for the first time a suitable way of partially transposing metatheorems developed in the context of the monadic second order logic of graphs of constant **undirected** tree-width and constant clique width to the realm of digraph width measures that are closed under taking subgraphs and whose constant levels incorporate families of graphs of arbitrarily large undirected tree-width and arbitrarily large clique width.

**Keywords:** Slice Theory, Digraph Width Measures, Monadic Second Order Logic of Graphs, Algorithmic Meta-theorems.

## 1 Introduction

Two cornerstones of parametrized complexity theory are Courcelle's theorem [13] stating that monadic second order logic properties may be model checked in linear time in graphs of constant undirected tree-width, and its subsequent generalization to counting given by Arnborg, Lagergren and Seese [2]. The importance of such metatheorems stem from the fact that several NP-complete problems such as Hamiltonicity, colorability, and their respective #P-hard counting counterparts, can be modeled in terms of $\text{MSO}_2$ sentences and thus can be efficiently solved in graphs of constant **undirected** tree-width.

In this work we introduce the notion of $z$-topological orderings for digraphs and provide a suitable way of partially transposing the metatheorems in [13,2] to digraphs admitting *$z$-topological orderings* for constant values of $z$. In order to state our main

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result we will first give a couple of easy definitions: Let $G = (V, E)$ be a directed graph. For subsets of vertices $V_1, V_2 \subseteq V$ we let $E(V_1, V_2)$ denote the set of edges with one endpoint in $V_1$ and another endpoint in $V_2$. We say that a linear ordering $\omega = (v_1, v_2, ..., v_n)$ of the vertices of $V$ is a $z$-topological ordering of $G$ if for every **directed** simple path $p = (V_p, E_p)$ in $G$ and every $i$ with $1 \leq i \leq n$, we have that $|E_p \cap E(\{v_1..., v_i\}, \{v_{i+1}, ..., v_n\})| \leq z$. In other words, $\omega$ is a $z$-topological ordering if every **directed** simple path of $G$ bounces back and forth at most $z$ times along $\omega$. The terminology $z$-topological ordering is justified by the fact that any topological ordering of a DAG $G$ according to the usual definition, is a 1-topological ordering according to our definition. Conversely if a digraph admits a 1-topological ordering, then it is a DAG. We denote by $MSO_2$ the monadic second order logic of graphs with edge set quantification. An edge-weighting function for a digraph $G = (V, E)$ is a function $w : E \to \Omega$ where $\Omega$ is a finite commutative semigroup of size polynomial in $|V|$ whose elements are totally ordered. The weight of a subgraph $H = (V', E')$ of $G$ is defined as $w(H) = \sum_{e \in E'} w(e)$. A maximal-weight subgraph of $G$ satisfying a given property $\varphi$ is a subgraph $H = (V', E')$ such that $H \models \varphi$ and such that for any other subgraph $H' = (V'', E'')$ of $G$ for which $H' \models \varphi$ we have $w(H) \geq w(H')$. Now we are in a position to state our main theorem:

**Theorem 1.1 (Main Theorem).** *For each $MSO_2$ formula $\varphi$ and each positive integers $k, z \in \mathbf{N}$ there exists a computable function $f(\varphi, z, k)$ such that: Given a digraph $G = (V, E)$ on $n$ vertices, a $z$-topological ordering $\omega$ of $G$, an edge-weighting function $w : E \to \Omega$, and a positive integer $l \leq n$, we can count in time $f(\varphi, z, k) \cdot n^{O(z \cdot k)}$ the number of subgraphs $H$ of $G$ simultaneously satisfying the following four properties:*

*(i)* $H \models \varphi$
*(ii)* *$H$ is the union of $k$ directed paths*[1]
*(iii)* *$H$ has $l$ vertices*
*(iv)* *$H$ has maximal weight*

Our result implies the polynomial time solvability of many natural counting problems on digraphs admitting $z$-topological orderings for constant values of $z$ and $k$. We observe that graphs admitting $z$-topological orderings for constant values of $z$ can already have simultaneously unbounded tree-width and unbounded clique-width, and therefore the problems that we deal with here cannot be tackled by the approaches in [13,2,15]. For instance any DAG is 1-topologically orderable. In particular, the $n \times n$ directed grid in which all horizontal edges are directed to the left and all vertical edges oriented up is 1-topologically orderable, while it has both undirected tree-width $\Omega(n)$ and clique-width $\Omega(n)$.

## 2 Applications

To illustrate the applicability of Theorem 1.1 with a simple example, suppose we wish to count the number of Hamiltonian cycles on $G$. Then our formula $\varphi$ will express that

[1] A digraph $H$ is the union of $k$ directed paths if $H = \cup_{i=1}^{k} p_i$ for not necessarily vertex-disjoint nor edge-disjoint directed paths $p_1, ..., p_k$.

the graphs we are aiming to count are cycles, namely, connected graphs in which each vertex has degree precisely two. Such a formula can be easily specified in $MSO_2$ . Since any cycle is the union of two directed paths, we have $k = 2$. Since we want all vertices to be visited our $l = n$. Finally, the weights in this case are not relevant, so it is enough to set the semigroup $\Omega$ to be the one element semigroup $\{1\}$, and the weights of all edges to be 1. In particular the total weight of any subgraph of $G$ according to this semigroup will be 1. By Theorem 1.1 we can count the number of Hamiltonian cycles in $G$ in time $f(\varphi, k, z) \cdot n^{2z}$. We observe that Hamiltonicity can be solved within the same time bounds for other directed width measures, such as directed tree-width [33].

Interestingly, Theorem 1.1 allow us to count structures that are much more complex than cycles. And in our opinion it is rather surprising that counting such complex structures can be done in XP. For instance, we could choose to count the number of maximal Hamiltonian subgraphs of $G$ which can be written as the union of $k$ directed paths. We can repeat this trick with virtually any natural property that is expressible in $MSO_2$ . For instance we can count the number of maximal weight 3-colorable subgraphs of $G$ that are the union of $k$-paths. Or the number of subgraphs of $G$ that are the union of $k$ directed paths and have di-cuts of size $k/10$. Observe that, as it should be expected, our framework does not allow one to find in polynomial time a maximal di-cut of the whole graph $G$ nor to determine in polynomial time whether the whole graph $G$ is 3-colorable since these problems are already NP-complete for DAGs, i.e., for $z = 1$.

If $H = (V, E)$ is a digraph, then the underlying undirected graph of $H$ is obtained from $H$ by forgetting the direction of its edges. A very interesting application of Theorem 1.1 consists in counting the number of maximal-weight subgraphs of $G$ which are the union of $k$ paths and whose underlying undirected graph satisfy some structural property, such as, connectedness, planarity, bounded genus, bipartiteness, etc.

**Corollary 2.1.** *Let $G = (V, E)$ be a digraph on $n$ vertices and $w : E \to \Omega$ be an edge weighting function. Then given a $z$-topological ordering $\omega$ of $G$ one may count in time $O(n^{k \cdot z})$ the number of maximal-weight subgraphs of $G$ that are the union of $k$ directed paths and whose underlying undirected graph satisfy any combination of the following properties:* 1) *Connectedness,* 2) *Being a forest,* 3) *Bipartiteness,* 4) *Planarity,* 5) *Constant Genus $g$,* 6) *Outerplanarity,* 7) *Being Series Parallel,* 8) *Having Constant Treewidth $t$* 9) *Having Constant Branchwidth $b$,* 10) *Satisfy any minor closed property.*

Another family of applications for Theorem 1.1 arises from the fact that the monadic second order alternation hierarchy is infinite [37]. Additionally, each level $r$ of the polynomial hierarchy has a very natural complete problem, the $r$-round-3-coloring problem, that also belongs to the $r$-th level of the monadic second order hierarchy (Theorem 11.4 of [1]). Thus by Theorem 1.1 we may count the number of $r$-round-3-colorable subgraphs of $G$ that are the union of $k$ directed paths in time $f(\varphi_r, z, k) \cdot n^{O(z \cdot k)}$.

We observe that the condition that the subgraphs we consider are the union of $k$ directed paths is not as restrictive as it might appear at a first glance. For instance one can show that for any $a, b \in N$ the $a \times b$ undirected grid is the union of 4 directed paths. Additionally these grids have zig-zag number $O(\min\{a, b\})$. Therefore counting the number of maximal grids of height $O(z)$ on a digraph of zig-zag number $z$ is a neat example of problem which can be tackled by our techniques but which cannot be formulated as a linkage problem, namely, the most successful class of problems that has been shown to be solvable in polynomial time for constant values of several digraph width measures [33].

## 3 Overview of the Proof of Theorem 1.1

We will prove Theorem 1.1 within the framework of regular slice languages, which was originally developed by the author to tackle several problems within the partial order theory of concurrency [17,18]. The main steps of the proof of Theorem 1.1 are as follows. To each regular slice language $\mathcal{L}$ we associate a possibly infinite set of digraphs $\mathcal{L}_{\mathcal{G}}$. In Section 6 we will define the notion of $z$-dilated-saturated regular slice language and show that given any digraph $G$ together with a $z$-topological ordering $\omega = (v_1, v_2, ..., v_n)$ of $G$, and any $z$-dilated-saturated slice language $\mathcal{L}$, one may efficiently count the number of subgraphs of $G$ that are isomorphic to some digraph in $\mathcal{L}_{\mathcal{G}}$ ( Theorem 6.2). Then in Section 7 we will show that given any monadic second order formula $\varphi$ and any natural numbers $z, k$ one can construct a $z$-dilated-saturated regular slice language $\mathcal{L}(\varphi, z, k)$ representing the set of all digraphs that at the same time satisfy $\varphi$ and are the union of $k$ directed paths (Theorem 7.1). The construction of $\mathcal{L}(\varphi, z, k)$ is done once and for all for each $\varphi$,$k$ and $z$, and is completely independent from the digraph $G$. Finally, the proof of Theorem 1.1 will follow by plugging Theorem 7.1 into Theorem 6.2. Proofs of intermediate results omitted for a matter of clarity or due to lack of space can be found in the full version of this work [19].

## 4 Comparison with Existing Work

Since the last decade, the possibility of lifting the metatheorems in [13,2] to the directed setting has been an active line of research. Indeed, following an approach delineated by Reeds [39] and Johnson, Robertson, Seymour and Thomas [33], several digraph width measures have been defined in terms of the number of cops needed to capture a robber in a certain evasion game on digraphs. From these variations we can cite for example, directed tree-width [39,33], DAG width [6], D-width [40,29], directed path-width [4], entanglement [7,8], Kelly width [32] and Cycle Rank [21,31]. All these width measures have in common the fact that DAGs have the lowest possible constant width (0 or 1 depending on the measure). Other width measures in which DAGs do not have necessarily constant width include DAG-depth [24], and Kenny-width [24].

The introduction of the digraph width measures listed above was often accompanied by algorithmic implications. For instance, certain linkage problems that are NP-complete for general graphs, e.g. Hamiltonicity, can be solved efficiently in graphs of constant directed tree-width [33]. The winner of certain parity games of relevance to the theory of $\mu$-calculus can be determined efficiently in digraphs of constant DAG width [6], while it is not known if the same can be done for general digraphs. Computing disjoint paths of minimal weight, a problem which is NP-complete in general digraphs, can be solved efficiently in graphs of bounded Kelly width. However, except for such sporadic successful algorithmic implications, researchers have failed to come up with an analog of Courcelle's theorem for graph classes of constant width for any of the digraph width measures described above. It turns out that there is a natural barrier against this goal: It can be shown that unless all the problems in the polynomial hierarchy have sub-exponential algorithms, which is a highly unlikely assumption, $MSO_2$ model checking is intractable in any class of graphs that is closed under taking subgraphs and whose undirected tree-width is poly-logarithmic unbounded [34,35]. An analogous result can be proved with respect to model checking of $MSO_1$ properties if we assume a non-uniform version of the extended exponential time hypothesis [26,25]. All classes of digraphs of constant width with respect to the directed measures described above are closed under subgraphs and have poly-logarithmically unbounded tree-width, and thus fall into the impossibility theorem of [34,35]. It is worth noting that Courcelle, Makowsky and Rotics have shown that $MSO_1$ model checking is tractable in classes of graphs of constant clique-width [15,16], and that these classes are poly-logarithmic unbounded, but they are not closed under taking subgraphs.

We define the *zig-zag number* of a digraph $G$ to be the minimum $z$ for which $G$ has a $z$-topological ordering, and denote it by $zn(G)$. The zig-zag number is a digraph width measure that is closed under taking subgraphs, poly-logarithmically unbounded and that has interesting connections with some of the width measures described above. In particular we can prove the following theorem stating that families of graphs of constant zig-zag number are strictly richer than families of graphs of constant directed path-width.

**Theorem 4.1.** *Let $G$ be a digraph of directed path-width $d$. Then $G$ has zig-zag number $z \leq 2d+1$. Furthermore, given a directed path decomposition of $G$ one can efficiently derive a $z$-topological ordering of $G$. On the other hand, there are digraphs on $n$ vertices whose zig-zag number is $2$ but whose directed path-width is $\Theta(\log n)$.*

Theorem 4.1 legitimizes the algorithmic relevance of Theorem 1.1 since path decompositions of graphs of constant directed path-width can be computed in polynomial time [42]. The same holds with respect to the cycle rank of a graph since constant cycle-rank decompositions[2] can be converted into constant directed-path decompositions in polynomial time [30]. Therefore all the problems described in Section 1 can be solved efficiently in graphs of constant directed path-width and in graphs of constant cycle rank. We should notice that our main theorem circumvents the impossibility results of [34,35,26,25] by confining the monadic second order logic properties to subgraphs that are the union of $k$ *directed* paths.

A pertinent question consists in determining whether we can eliminate either $z$ or $k$ from the exponent of the running time $f(\varphi, k, z) \cdot n^{O(k \cdot z)}$ stated in Theorem 1.1. The following two theorems say that under strongly plausible parameterized complexity assumptions [20], namely that $W[2] \neq FPT$ and $W[1] \neq FTP$, the dependence of both $k$ and $z$ in the exponent of the running time is unavoidable.

**Theorem 4.2 (Lampis-Kaouri-Mitsou[36]).** *Determining whether a digraph $G$ of cycle rank $z$ has a Hamiltonian circuit is $W[2]$ hard with respect to $z$.*

Since by Theorem 4.1 constant cycle rank is less expressive than constant zig-zag number, the hardness result stated in Theorem 4.2 also works for zig-zag number. Given a sequence of $2k$ not necessarily distinct vertices $\sigma = (s_1, t_1, s_2, t_2, ..., s_k, t_k)$, a $\sigma$-linkage

[2] By cycle-rank decomposition we mean a direct elimination forest[30].

is a set of internally disjoint directed paths $p_1, p_2, ..., p_k$ where each $p_i$ connects $s_i$ to $t_i$.

**Theorem 4.3 (Slivkins[41]).** *Given a DAG G, determining whether G has a $\sigma$-linkage $\sigma = (s_1, t_1, s_2, t_2, ..., s_k, t_k)$ is hard for $W[1]$.*

It is not hard to see that $\sigma$-linkages are expressible in $MSO_2$ . Additionally, since a $\sigma$-linkage is clearly union of $k$-paths, Theorem 4.3 implies that the dependence on $k$ in the exponent of the running time in Theorem 1.1 is necessary even if $z$ is fixed to be 1.

Below we compare the zig-zag number with several other digraph width measures. If $G$ is a digraph, we write $dtw(G)$ for its directed tree-width [33], $Dw(G)$ for its *D-width* [30], $dagw(G)$ for its DAG-width [6], $dpw(G)$ for its directed path-width [4], $kellyw(G)$ for its Kelly-width [24], $ddp(G)$ for its DAG-depth [24], $Kw(G)$ for its K-width [24], $s(G)$ for its weak separator number [30] and $r(G)$ for its cycle rank [30]. We write $A \precnsim B$ to indicate that there are graphs of constant width with respect to the measure $A$ but unbounded width with respect to the measure $B$. We write $A \preceq B$ to express that $A$ is not asymptotically greater than $B$.

$$zn(G) \precnsim dpw(G) \overset{[30]}{\precnsim} cr(G) \overset{[25]}{\precnsim} \begin{cases} Kw(G) \\ ddp(G) \end{cases} \qquad \frac{cr(G)}{\log n} \overset{[30]}{\preceq} s(G) \tag{1}$$

$$\frac{zn(G)}{\log n} \preceq s(G) \overset{[30]}{\preceq} Dw(G) \overset{[30]}{\preceq} dagw(G) \overset{[6]}{\preceq} dpw(G) \tag{2}$$

$$\sqrt{\frac{zn(G)}{\log n}} \preceq dtw(G) \overset{[32]}{\preceq} kellyw(G) \qquad \sqrt{Dw(G)} \overset{[23]}{\preceq} dtw(G) \overset{[23]}{\preceq} Dw(G) \tag{3}$$

The numbers above $\precnsim$ and $\preceq$ point to the references in which these relations where established. The only new relations are $zn(G) \precnsim dpw(G)$, $zn(G)/\log n \preceq s(G)$ and $\sqrt{zn(G)/\log n} \preceq dtw(G)$ which follow from Theorem 4.1 together with the already known relations between directed path-width and the other digraph width measures.

## 5 Regular Slice Languages

A slice $\mathbf{S} = (V, E, l, s, t)$ is a digraph comprising a set of vertices $V$, a set of edges $E$, a vertex labeling function $l : V \to \Gamma$ for some set of symbols $\Gamma$, and functions $s, t : E \to V$ which respectively associate to each edge $e \in E$, a source vertex $e^s$ and a target vertex $e^t$. We notice that an edge might possibly have the same source and target ($e^s = e^t$). The vertex set $V$ is partitioned into three disjoint subsets: an in-frontier $I \subseteq V$ a center $C \subseteq V$ and an out-frontier $O \subseteq V$. Additionally, we require that each frontier-vertex in $I \cup O$ is the endpoint of exactly one edge in $E$ and that no edge in $E$ has both endpoints in the same frontier. The frontier vertices in $I \cup O$ are labeled by $l$ with numbers from the set $\{1, ..., q\}$ for some natural number $q \geq \max\{|I|, |O|\}$ in such a way that no two vertices in the same frontier receive the same number. Vertices belonging to different frontiers may on the other hand be labeled with the same number. The center vertices in $C$ are labeled by $l$ with elements from $\Gamma \backslash \{1, ..., q\}$. We say that a slice $\mathbf{S}$ is normalized if $l(I) = \{1, ..., |I|\}$ and $l(O) = \{1, ..., |O|\}$. Non-normalized slices will play an important role in Section 6 when we define the notion of sub-slice. Since we will deal with weighted graphs, we will also allow the edges of a slice to be weighted by a function $w : E \rightarrow \Omega$ where $\Omega$ is a finite commutative semigroup.

A slice $\mathbf{S}_1$ with frontiers $(I_1, O_1)$ can be glued to a slice $\mathbf{S}_2$ with frontiers $(I_2, O_2)$ provided $l_1(O_1) = l_2(I_2)$ and that for each $i \in l(O_1)$ there exist edges $e_1 \in \mathbf{S}_1$ and $e_2 \in \mathbf{S}_2$ such that either $e_1^t \in O_1$, $e_2^s \in I_2$ and $l_1(e_1^t) = l_2(e_2^s)$ or $e_1^s \in O_1$, $e_2^t \in I_2$ and $l_1(e_1^s) = l_2(e_2^t)$. In this case the glueing gives rise to the slice $\mathbf{S}_1 \circ \mathbf{S}_2$ with frontiers $(I_1, O_2)$ which is obtained by fusing each such pair of edges $e_1, e_2$. The fusion of $e_1$ with $e_2$ proceeds as follows. First we create an edge $e_{12}$. If $e_1^t \in O_1, e_2^s \in I_2$, we set $e_{12}^s = e_1^s, e_{12}^t = e_2^t$ and delete $e_1, e_2, e_1^t, e_2^s$. Otherwise, if $e_1^s \in O_1, e_2^t \in I_2$, we set $e_{12}^s = e_2^s, e_{12}^t = e_1^t$ and delete $e_1, e_2, e_1^s, e_2^t$. Thus in the process of gluing two slices, the vertices in the glued frontiers disappear. If $\mathbf{S}_1$ and $\mathbf{S}_2$ are weighted by functions $w_1$ and $w_2$, then we add the requirement that the glueing of $\mathbf{S}_1$ with $\mathbf{S}_2$ can be performed if the weights of the edges touching the out-frontier of $\mathbf{S}_1$ agree with the weights of their corresponding edges touching the in-frontier of $\mathbf{S}_2$. In the opposite direction, any slice can be decomposed into a sequence of atomic parts which we call *unit slices*, namely, slices with at most one vertex on its center. Thus slices may be regarded as a graph theoretic analog of the knot theoretic braids [3], in which twists are replaced by vertices. Within automata theory, slices may be related to several formalisms such as graph automata [43,11], graph rewriting systems [12,5,22], and others [28,27,10,9]. In particular, slices may be regarded as a specialized version of the multi-pointed graphs defined in [22] but subject to a slightly different composition operation.

The width of a slice $\mathbf{S}$ with frontiers $(I, O)$ is defined as $w(\mathbf{S}) = \max\{|I|, |O|\}$. In the same way that letters from an alphabet may be concatenated by automata to form infinite languages of strings, we may use automata or regular expressions over alphabets of slices of a bounded width to define infinite families of digraphs. Let $\Sigma_{\mathbb{S}}^{c,q}$ denote the set of all unit slices of width at most $c$ and whose frontier vertices are numbered with numbers from $\{1, ..., q\}$ for $q \geq c$. We say that a slice is *initial* if its in-frontier is empty and *final* if its out-frontier is empty. A slice with empty center is called a *permutation slice*. Due to the restriction that each frontier vertex of a slice must be connected to precisely one edge, we have that each vertex in the in-frontier of a permutation slice is necessarily connected to a unique vertex in its out-frontier. The empty slice, denoted by $\varepsilon$, is the slice with empty center and empty frontiers. We regard the empty slice as a permutation slice. A subset $\mathcal{L}$ of the free monoid $(\Sigma_{\mathbb{S}}^{c,q})^*$ generated by $\Sigma_{\mathbb{S}}^{c,q}$ is a slice language if for every sequence of slices $\mathbf{S}_1\mathbf{S}_2...\mathbf{S}_n \in \mathcal{L}$ we have that $\mathbf{S}_1$ is an initial slice, $\mathbf{S}_n$ a final slice and $\mathbf{S}_i$ can be glued to $\mathbf{S}_{i+1}$ for each $i \in \{1, ..., n-1\}$. We should notice that at this point the operation of the monoid in consideration is just the concatenation $\mathbf{S}_1\mathbf{S}_2$ of slice symbols $\mathbf{S}_1$ and $\mathbf{S}_2$ and should not be confused with the composition $\mathbf{S}_1 \circ \mathbf{S}_2$ of slices. The unit of the monoid is just the empty symbol $\lambda$ and not the empty slice, thus the elements of $\mathcal{L}$ are simply sequences of slices, regarded as dumb letters. To each slice language $\mathcal{L}$ over $\Sigma_{\mathbb{S}}^{c,q}$ we associate a graph language $\mathcal{L}_{\mathcal{G}}$ consisting of all digraphs obtained by composing the slices in each string in $\mathcal{L}$.

$$\mathcal{L}_{\mathcal{G}} = \{\mathbf{S}_1 \circ \mathbf{S}_2 ... \circ \mathbf{S}_n | \mathbf{S}_1\mathbf{S}_2...\mathbf{S}_n \in \mathcal{L}\} \tag{4}$$

However we observe that a set $\mathcal{L}_G$ of digraphs may be represented by several different slice languages, since a digraph in $\mathcal{L}_G$ may be decomposed in several ways as a string of unit slices. We will use the term *unit decomposition* of a digraph $H$ to denote any sequence of unit slices $\mathbf{U} = \mathbf{S}_1\mathbf{S}_2...\mathbf{S}_n$ whose composition $\mathbf{S}_1 \circ \mathbf{S}_2 \circ ... \circ \mathbf{S}_n$ yields $H$. We say that the unit decomposition $\mathbf{U}$ is *dilated* if it contains permutation slices, including possibly the empty slice. The slice-width of $\mathbf{U}$ is the minimal $c$ for which $\mathbf{U} \in (\Sigma_{\mathbb{S}}^{c,q})^*$ for some $q$. In other words, the slice width of a unit decomposition is the width of the widest slice appearing in it.

A slice language is regular if it is generated by a finite automaton or regular expressions over slices. We notice that since any slice language is a subset of the free monoid generated by a slice alphabet $\Sigma_{\mathbb{S}}^{c,q}$, we do not need to make a distinction between regular and rational slice languages. Therefore, by Kleene's theorem, every slice language generated by a regular expression can be also generated by a finite automaton. Equivalently, a slice language is regular iff it can be generated by the slice graphs defined below [17]:

**Definition 5.1 (Slice Graph).** *A* slice graph *over a slice alphabet $\Sigma_{\mathbb{S}}^{c,q}$ is a labeled directed graph $\mathcal{SG} = (\mathcal{V}, \mathcal{E}, \mathcal{S}, \mathcal{I}, \mathcal{T})$ possibly containing loops but without multiple edges where $\mathcal{I} \subseteq \mathcal{V}$ is a set of initial vertices, $\mathcal{T} \subseteq \mathcal{V}$ a set of final vertices and the function $\mathcal{S} : \mathcal{V} \rightarrow \Sigma_{\mathbb{S}}^{c,q}$ satisfies the following conditions:*

- $\mathcal{S}(\mathfrak{v})$ *is a initial slice for every vertex* $\mathfrak{v}$ *in* $\mathcal{I}$,
- $\mathcal{S}(\mathfrak{v})$ *is final slice for every vertex* $\mathfrak{v}$ *in* $\mathcal{T}$ *and,*
- $(\mathfrak{v}_1, \mathfrak{v}_2) \in \mathcal{E}$ *implies that* $\mathcal{S}(\mathfrak{v}_1)$ *can be glued to* $\mathcal{S}(\mathfrak{v}_2)$.

We say that a slice graph is *deterministic* if none of its vertices has two out-neighbors labeled with the same slice and if there is no two initial vertices labeled with the same slice. In other words, in a deterministic slice graph no two distinct walks are labeled with the same sequence of slices. We denote by $\mathcal{L}(\mathcal{SG})$ the slice language generated by $\mathcal{SG}$, which we define as the set of all sequences slices $\mathcal{S}(\mathfrak{v}_1)\mathcal{S}(\mathfrak{v}_2)\cdots\mathcal{S}(\mathfrak{v}_n)$ where $\mathfrak{v}_1\mathfrak{v}_2\cdots\mathfrak{v}_n$ is a walk on $\mathcal{SG}$ from an initial vertex to a final vertex. We write $\mathcal{L}_G(\mathcal{SG})$ for the language of digraphs derived from $\mathcal{L}(\mathcal{SG})$.

## 6 Counting Subgraphs Specified by a Slice Language

A sub-slice of a slice $\mathbf{S}$ is a subgraph of $\mathbf{S}$ that is itself a slice. If $\mathbf{S}'$ is a sub-slice of $\mathbf{S}$ then we consider that the numbering in the frontiers of $\mathbf{S}'$ are inherited from the numbering of the frontiers of $\mathbf{S}$. Therefore, even if $\mathbf{S}$ is normalized, its sub-slices might not be. If $\mathbf{U} = \mathbf{S}_1\mathbf{S}_2...\mathbf{S}_n$ is a unit decomposition of a digraph $G$, then a sub-unit-decomposition of $\mathbf{U}$ is a unit decomposition $\mathbf{U}' = \mathbf{S}'_1\mathbf{S}'_2...\mathbf{S}'_n$ of a subgraph $H$ of $G$ such that $\mathbf{S}'_i$ is a sub-slice of $\mathbf{S}_i$ for $1 \leq i \leq n$. We observe that sub-unit-decompositions may be padded with empty slices. A unit decomposition $\mathbf{U} = \mathbf{S}_1\mathbf{S}_2...\mathbf{S}_n$ may have exponentially many sub-unit-decompositions of a given slice-width $c$. However, as we will state in Lemma 6.1 the set of all such sub-unit decompositions of $\mathbf{U}$ may be represented by a slice graph of size polynomial in $n$. A normalized unit decomposition is a unit decomposition $\mathbf{U} = \mathbf{S}_1\mathbf{S}_2...\mathbf{S}_n$ such that $\mathbf{S}_i$ is a normalized slice for each $i \in \{1, ..., n\}$.

A slice language is normalized if all unit decompositions in it are normalized. A slice-graph is normalized if all slices labeling its vertices are normalized. We notice that a regular slice language is normalized if and only if it is generated by a normalized slice graph.

**Lemma 6.1.** *Let $G$ be a digraph with $n$ vertices, $\mathbf{U} = \mathbf{S}_1\mathbf{S}_2...\mathbf{S}_n$ be a normalized unit decomposition of $G$ of slice-width $q$, and let $c \in \boldsymbol{N}$ be such that $c \leq q$. Then one can construct in time $n \cdot q^{O(c)}$ an acyclic and deterministic slice graph $\mathcal{SUB}^c(\mathbf{U})$ on $n \cdot q^{O(c)}$ vertices whose slice language $\mathcal{L}(\mathcal{SUB}^c(\mathbf{U}))$ consists of all sub-unit-decompositions of $\mathbf{U}$ of slice-width at most $c$.*

Let $\omega = (v_1, v_2, ..., v_n)$ be a linear ordering of the vertices of a digraph $H$. We say that a dilated unit decomposition $\mathbf{U} = \mathbf{S}_1\mathbf{S}_2...\mathbf{S}_m$ of $H$ is compatible with $\omega$ if $v_i$ is the center vertex of $\mathbf{S}_{j_i}$ for each $i \in \{1, ..., n\}$ and if $j_i > j_{i-1}$ for each $i \in \{1, ..., n-1\}$ (observe that we need to use the subindex $j_i$ instead of simply $i$ because $\mathbf{U}$ is dilated and therefore some slices in $\mathbf{U}$ have no center vertex). Notice that for each ordering $\omega$ there might exist several unit decompositions of $H$ that are compatible to $\omega$. If $\omega$ is a $z$-topological ordering of a digraph $G$ and if $\mathbf{U}$ is a dilated unit decomposition of $G$ that is compatible with $\omega$, then we say that $\mathbf{U}$ has zig-zag number $z$. The zig-zag number of a slice language $\mathcal{L}$ is the maximal zig-zag number of a unit decomposition in $\mathcal{L}$. If a dilated unit decomposition $\mathbf{U}$ has zig-zag number $z$ then any of its sub-unit decompositions has zig-zag number at most $z$.

**Proposition 6.1.** *Let $\mathbf{U}$ be a unit decomposition of zig-zag number $z$. Then any sub-unit-decomposition in $\mathcal{L}(\mathcal{SUB}^c(\mathbf{U}))$ has zig-zag number at most $z$.*

A slice language $\mathcal{L}$ is $z$-dilated-saturated, if $\mathcal{L}$ has zig-zag number at most $z$ and if for every digraph $H \in \mathcal{L}_{\mathcal{G}}$, every $z$-topological ordering $\omega$ of $H$ and every dilated unit decomposition $\mathbf{U}$ of $H$ that is compatible with $\omega$ we have that $\mathbf{U} \in \mathcal{L}$. We should **emphasize** that the intersection of the graph languages generated by two slice graphs is not in general reflected by the intersection of their slice languages. Indeed, it is easy to define slice languages $\mathcal{L}, \mathcal{L}'$ for which $\mathcal{L}_{\mathcal{G}} = \mathcal{L}'_{\mathcal{G}}$ but for which $\mathcal{L} \cap \mathcal{L}' = \emptyset$! Additionally, a reduction from the Post correspondence problem [38] established by us in [17] implies that even determining whether the intersection of the graph languages generated by slice languages is empty, is undecidable. However this is not an issue if at least one of the intersecting languages is $z$-dilated-saturated, as stated in the next proposition.

**Proposition 6.2.** *Let $\mathcal{L}$ and $\mathcal{L}'$ be two slice languages over $\Sigma_{\mathbb{S}}^{c,q}$, such that $\mathcal{L}$ has zig-zag number $z$ and such that $\mathcal{L}'$ is $z$-saturated. If we let $\mathcal{L}^{\cap} = \mathcal{L} \cap \mathcal{L}'$, then $\mathcal{L}^{\cap}_{\mathcal{G}} = \mathcal{L}_{\mathcal{G}} \cap \mathcal{L}'_{\mathcal{G}}$.*

If $\mathbf{S}$ is a normalized slice in $\Sigma_{\mathbb{S}}^{c,q}$ with in-frontier $I$ and out-frontier $O$ then a *$q$-numbering* of $\mathbf{S}$ is a pair of functions $in : I \rightarrow \{1, ..., q\}$, $out : O \rightarrow \{1, ..., q\}$ such that for each two vertices $v, v' \in I$, $l(v) < l(v')$ implies that $in(l(v)) < in(l(v'))$ and, for each two vertices $v, v' \in O$, $l(v) < l(v')$ implies that $out(l(v)) < out(l(v'))$. We let $(\mathbf{S}, in, out)$ denote the slice obtained from $\mathbf{S}$ by renumbering each frontier vertex $v \in I$ with $in(l(v))$ and each out frontier vertex $v \in O$ with the $out(l(v))$. The $q$-numbering-expansion of a normalized slice $\mathbf{S}$ is the set $\mathcal{N}(\mathbf{S})$ of all $q$-numberings of $\mathbf{S}$.

Let $\mathcal{SG} = (\mathcal{V}, \mathcal{E}, \mathcal{S}, \mathcal{I}, \mathcal{T})$ be a slice graph over $\Sigma_{\mathbb{S}}^{c,q}$. Then the $q$-numbering expansion of $\mathcal{SG}$ is the slice graph $\mathcal{N}^q(\mathcal{SG}) = (\mathcal{V}', \mathcal{E}', \mathcal{S}', \mathcal{I}', \mathcal{T}')$ defined as follows. For each vertex $\mathfrak{v} \in \mathcal{V}$ and each slice $(\mathbf{S}(\mathfrak{v}), in, out) \in \mathcal{N}^q(\mathcal{S}(\mathfrak{v}))$ we create a vertex $\mathfrak{v}_{in,out}$ in $\mathcal{V}'$ and label it with $(\mathbf{S}, in, out)$. Subsequently we connect $\mathfrak{v}_{in,out}$ to $\mathfrak{v}'_{in',out'}$ if there was an edge $(\mathfrak{v}, \mathfrak{v}') \in \mathcal{E}$ and if $(\mathbf{S}, in, out)$ can be glued to $(\mathbf{S}', in', out')$.

**Theorem 6.1.** *Let $G$ be digraph, $\mathbf{U} = \mathbf{S}_1\mathbf{S}_2...\mathbf{S}_n$ be a normalized unit decomposition of $G$ of slice-width $q$ and zig-zag number $z$, $\mathcal{SG}$ be a normalized $z$-dilated-saturated slice graph over $\Sigma_{\mathbb{S}}^{c,q}$ and $\mathcal{N}^q(\mathcal{SG})$ be the $q$-numbering expansion of $\mathcal{SG}$. Then the set of all sub-unit-decompositions of $\mathbf{U}$ of slice-width at most $c$ whose composition yields a graph isomorphic to some graph in $\mathcal{L}_{\mathcal{G}}(\mathcal{SG})$ is represented by the regular slice language $\mathcal{L}(\mathcal{SUB}^c(\mathbf{U})) \cap \mathcal{L}(\mathcal{N}^q(\mathcal{SG}))$.*

Let $\mathcal{SG} = (\mathcal{V}, \mathcal{E}, \mathcal{S}, \mathcal{I}, \mathcal{T})$ be a slice graph and $(\Omega, +)$ be a finite commutative semi-group with an identity element $0$. Then the *$\Omega$-weight expansion* of $\mathcal{SG}$ is the slice graph $\mathcal{W}^{\Omega}(\mathcal{SG}) = (\mathcal{V}', \mathcal{E}', \mathcal{S}', \mathcal{I}', \mathcal{T}')$ defined as follows: For each vertex $\mathfrak{v} \in \mathcal{V}$ labeled with the slice $\mathcal{S}(\mathfrak{v}) = (V, E, l)$, we add the set of vertices $\{\mathfrak{v}_{w,tot}\}_w$ to $\mathcal{V}'$ where $w$ ranges over all weighting functions $w : E \to \Omega$ and $tot$ ranges over $\Omega$. We label each $\mathfrak{v}_{w,tot}$ with the tuple $(\mathcal{S}(\mathfrak{v}), w, tot)$. Then we add an edge $(\mathfrak{v}_{w,tot}, \mathfrak{v}'_{w',tot'})$ to $\mathcal{E}'$ if and only if $(\mathfrak{v}, \mathfrak{v}') \in \mathcal{E}$, if the slice $(\mathcal{S}(\mathfrak{v}), w)$ can be glued to the slice $(\mathcal{S}(\mathfrak{v}'), w')$ and if $tot' = tot + \sum_{e \in E'^{out}} w(e)$. The set of final vertices $\mathcal{T}'$ consists of all vertices in $\mathcal{V}'$ which are labeled with a triple $(\mathbf{S}, w, tot)$ where $\mathbf{S}$ is a final slice. The set of initial vertices $\mathcal{I}'$ consists of all vertices in $\mathcal{V}'$ which are labeled with a triple $(\mathbf{S}, w, 0)$ where $\mathbf{S}$ is an initial slice. Intuitively if $\mathcal{SG}$ generates a language of graphs $\mathcal{L}_{\mathcal{G}}$, then $\mathcal{W}^{\Omega}(\mathcal{SG})$ generates the language $\mathcal{L}'_{\mathcal{G}}$ of all possible weighted versions of graphs in $\mathcal{L}_{\mathcal{G}}(\mathcal{SG})$. In Theorem 6.2 below $q$ is the cut-width of $G$ and therefore it can be as large as $O(n^2)$. The parameter $c$ on the other hand is the slice-width of the subgraphs that are being counted.

**Theorem 6.2 (Subgraphs in a Saturated Slice Language).** *Let $G = (V, E)$ be a digraph of cut-width $q$ with respect to a $z$-topological ordering $\omega = (v_1, v_2, ..., v_n)$ of its vertices, and let $\mathcal{SG}$ be a deterministic normalized $z$-dilated-saturated slice graph over $\Sigma_{\mathbb{S}}^{c,q}$ on $r$ vertices. Let $w : E \to \Omega$ be an weighting function on $E$ and $l \leq n$ be a positive integer. Then we may count in time $r^{O(1)} \cdot n^{O(c)} \cdot q^{O(c)}$ the number of subgraphs of $G$ of size $l$, that are isomorphic to some subgraph in $\mathcal{L}_{\mathcal{G}}(\mathcal{SG})$ and have maximal weight.*

*Proof.* Let $\mathbf{U} = \mathbf{S}_1\mathbf{S}_2...\mathbf{S}_n$ be any normalized unit decomposition that is compatible with $\omega$, i.e., such that $v_i$ is the center vertex of $\mathbf{S}_i$ for $i = 1, ..., n$. Clearly such a unit decomposition can be constructed in polynomial time in $n$. Since $\mathcal{SG}$ is dilated saturated, by Theorem 6.1 the set of all subgraphs of $G$ that are isomorphic to some digraph in $\mathcal{L}_{\mathcal{G}}(\mathcal{SG})$ is represented by the regular slice language $\mathcal{L}(\mathcal{SUB}^c(U)) \cap \mathcal{L}(\mathcal{W}^{\Omega}(\mathcal{N}^q(\mathcal{SG})))$. By Lemma 6.1 $\mathcal{SUB}^c(U)$ has $n \cdot q^{O(c)}$ vertices and can be constructed within the same time bounds. The numbering expansion $\mathcal{N}^q(\mathcal{SG})$ of $\mathcal{SG}$ has $\binom{q}{O(c)} \cdot r = r \cdot q^{O(c)}$ vertices and can be constructed within the same time bounds. The $\Omega$-expansion $\mathcal{W}^{\Omega}(\mathcal{N}^q(\mathcal{SG}))$ of $\mathcal{N}^q(\mathcal{SG})$ has $|\Omega|^{O(c)} \cdot r \cdot q^{O(c)} = n^{O(c)} \cdot r \cdot q^{O(c)}$ vertices and can be constructed within the same time bounds. Let $\mathcal{SG}^{\cap} = \mathcal{W}^{\Omega}(\mathcal{N}^q(\mathcal{SG})) \cap \mathcal{SUB}^c(\mathbf{U})$. Since $\mathcal{SG}^{\cap}$ can be obtained by a product construction, it has $r \cdot n^{O(c)} \cdot q^{O(c)}$ vertices. Since $\mathcal{SUB}^c(\mathbf{U})$ is acyclic, $\mathcal{SG}^{\cap}$ is also acyclic. Therefore counting the subgraphs in $G$ isomorphic to some graph in $\mathcal{L}_G(\mathcal{SG})$ amounts to counting the number of simple directed paths from an initial to a final vertex in $\mathcal{SG}^{\cap}$. Since we are only interested in counting subgraphs with $l$ vertices, we can intersect this acyclic slice graph with the slice graph $\mathcal{SG}^l$ generating all unit decomposition over $\Sigma_{\mathbb{S}}^{c,q}$ containing precisely $l$ unit slices that are not permutation slices. Again the slice graph $\mathcal{SG}^{\cap} \cap \mathcal{SG}^l$ will be acyclic. Finally since we are only interested in counting maximal-weight subgraphs, we delete from $\mathcal{T}'$ those vertices labeled with triples $(\mathbf{S}, w, tot)$ in which $tot$ is not maximal. The label of each path from an initial to a final vertex in this last slice graph identifies unequivocally a subgraph of $G$ of size $l$ and maximal weight. By standard dynamic programming we can count the number of paths in a DAG from a set of initial vertices to a set of final vertices in time polynomial on the number of vertices of the DAG. Thus we can determine the number of $l$-vertex maximal-weight subgraphs of $G$ which are isomorphic to some digraph in $\mathcal{L}(\mathcal{SG})$ in time $r^{O(1)}n^{O(c)}q^{O(c)}$. □

## 7 Subgraphs Satisfying a Given MSO Property

In this section we will only give the necessary definitions to state Lemma 7.1 and Theorem 7.1, which are crucial steps towards the proof of Theorem 1.1. For an extensive account on the monadic second order logic of graphs we refer the reader to the treatise [14] (in special Chapters 5 and 6). As it is customary, we will represent a digraph $G$ by a relational structure $G = (V, E, s, t, l_V, l_E)$ where $V$ is a set of vertices, $E$ a set of edges, $s, t \subseteq E \times V$ are respectively the source and target relations, $l_V \subseteq V \times \Sigma_V$ and $l_E \subseteq V \times \Sigma_E$ are respectively the vertex-labeling and edge-labeling relations. We give the following semantics to these relations: $s(e, v)$ and $t(e, v')$ are true if $v$ and $v'$ are respectively the source and the target of the edge $e$; $l_V(v, a)$ is true if $v$ is labeled with the symbol $a \in \Sigma_V$ while $l_E(e, b)$ is true if $e$ is labeled with the symbol $b \in \Sigma_E$. We always assume that $e$ is oriented from its source to its target. Let $\{x, y, z, z_1, y_1, z_1, ...\}$ be an infinite set of first order variables and $\{X, Y, Z, X_1, Y_1, Z_1, ...\}$ be an infinite set of second order variables. Then the set of $MSO_2$ formulas is the smallest set of formulas containing:

- the atomic formulas $x \in X$, $V(x)$, $E(x)$, $s(x, y)$, $t(x, y)$, $l_V(x, a)$ for each $a \in \Sigma_V$, $l_E(x, b)$ for each $b \in \Sigma_E$,
- the formulas $\varphi \wedge \psi$, $\varphi \vee \psi$, $\neg\varphi$, $\exists x.\varphi(x)$ and $\exists X.\varphi(X)$, where $\varphi$ and $\psi$ are $MSO_2$ formulas.

If $\mathcal{X}$ is a set of second order variables, and $G = (V, E)$ is a graph, then an interpretation of $\mathcal{X}$ over $G$ is a function $M : \mathcal{X} \to 2^V$ that assigns to each variable in $\mathcal{X}$ a subset of vertices of $V$. The semantics of a formula $\varphi(\mathcal{X})$ over free variables $\mathcal{X}$ being true on a graph $G$ under interpretation $M$ is the usual one. A sentence is a formula $\varphi$ without free variables. For a sentence $\varphi$ and a graph $G$, if it is the case that $\varphi$ is true in $G$, then we say that $G$ satisfies $\varphi$ and denote this by $G \models \varphi$. Now we are in a position to state a crucial Lemma towards the proof of Theorem 1.1. Intuitively it states that for any MSO$_2$ formula $\varphi$ the set of all unit decompositions of a fixed width of digraphs satisfying $\varphi$ forms a regular set.

**Lemma 7.1.** *For any MSO$_2$ sentence $\varphi$ over digraphs and any $c \in \boldsymbol{N}$, the set $\mathcal{L}(\varphi, \Sigma^c_{\mathbb{S}})$ of all slice strings $\mathbf{S}_1\mathbf{S}_2...\mathbf{S}_k$ over $\Sigma^c_{\mathbb{S}}$ such that $\mathbf{S}_1 \circ \mathbf{S}_2 \circ ... \circ \mathbf{S}_k = G$ and $G \models \varphi$ is a regular subset of $(\Sigma^c_{\mathbb{S}})^*$.*

Lemma 7.1 gives a slice theoretic analog of Courcelle's model checking theorem: In order to verify whether a digraph $G$ of existential slice-width at most $c$ satisfies a given MSO property $\varphi$, one just needs to find a slice decomposition $\mathbf{U} = \mathbf{S}_1\mathbf{S}_2...\mathbf{S}_n$ of $G$ and subsequently verify whether the deterministic finite automaton (or slice graph) accepting $\mathcal{L}(\varphi, \Sigma^c_{\mathbb{S}})$ accepts $\mathbf{U}$. However the goal of the present work is to make a rather different use of Lemma 7.1. Namely, next in Theorem 7.1 we will restrict Lemma 7.1 in such a way that it concerns only $z$-dilated-saturated regular slice languages, so that it can be coupled to Theorem 6.2, yielding in this way a proof of Theorem 1.1.

**Theorem 7.1.** *For any MSO$_2$ formula $\varphi$ and any $k, z \in \boldsymbol{N}$, one may effectively construct a $z$-dilated-saturated slice graph $\mathcal{SG}(\varphi, k, z)$ over the slice alphabet $\Sigma^{k\cdot z}_{\mathbb{S}}$ whose graph language $\mathcal{L}_{\mathcal{G}}(\mathcal{SG}(\varphi, k, z))$ consists precisely of the digraphs of zig-zag number at most $z$ that satisfy $\varphi$ and that are the union of $k$ directed paths.*

Finally we are in a position to prove Theorem 1.1. The proof will follow from a combination of Theorems 7.1 and 6.2.

**Proof of Theorem 1.1** Given a monadic second order formula $\varphi$, and positive integers $k$ and $z$, first we construct the dilated-saturated slice graph $\mathcal{SG}(\varphi, z, k)$ over $\Sigma^{k\cdot z}_{\mathbb{S}}$ as in Theorem 7.1. Since the slice-width of a digraph is at most $O(n^2)$ if we plug $q = O(n^2)$, $r = |\mathcal{SG}(\varphi, z, k)|$ and $\mathcal{SG}(\varphi, z, k)$ into Theorem 6.2, and if we let $f(\varphi, z, k) = r^{O(1)}$, then we get an overall upper bound of $f(\varphi, z, k) \cdot n^{O(k\cdot z)}$ for computing the number of subgraphs of $G$ that satisfy $\varphi$ and that are the union of $k$-directed paths. □

## 8 Final Comments

In this work we have employed slice theoretic techniques to obtain the polynomial time solvability of many natural combinatorial questions on digraphs of constant directed path-width, cycle rank, K-width and DAG-depth. We have done so by using the zig-zag number of a digraph as a point of connection between these directed width measures, regular slice languages and the monadic second order logic of graphs. Thus our results shed new light into a field that has resisted algorithmic metatheorems for more than a decade. More precisely, we showed that despite the severe restrictions imposed by the impossibility results in [34,35,26,25], it is still possible to develop logic-based algorithmic metatheorems for digraph width measures that are able to solve in polynomial time a considerable variety of interesting problems.

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