

Chapter 4

Pure Spinor Superfields: An Overview

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Maximally supersymmetric theories do not allow off-shell superspace formulations with traditional superfields containing a finite set of auxiliary fields. It has become clear that off-shell supersymmetric action formulations of such models can be achieved by the introduction of pure spinors. In this talk, an overview of this formalism is given, with emphasis on $D = 10$ super-Yang–Mills theory and $D = 11$ supergravity. This is a somewhat expanded version of a talk presented at the workshop “Breaking of supersymmetry and ultraviolet divergences in extended supergravity” (BUDS), Laboratori Nazionali di Frascati, March 25–28, 2013.

4.1 Introduction

The search for formalisms treating maximally supersymmetric models in a “covariant” way—covariance here taken in the sense of manifestly exhibiting Lorentz symmetry as well as the full supersymmetry—has a long history. To a large extent it has been pursued in terms of first-quantised particle (or string) theories, with the purpose of then applying second quantisation to obtain a covariant field theory. Let us remind how the problem arises, first in a particle or string theory, and then in field theory.

The Brink–Schwarz superparticle [1, 2], where the fermions are Lorentz spinors, exhibits a problematic mixture of first and second class constraints, as does the Green–Schwarz superstring [3]. That this must be the case is realised already from a counting of the fermionic degrees of freedom describing massless supermultiplets, i.e., from the 1/2-BPS property of a massless (short) supermultiplet. There is half a spinor of first class constraint and half a spinor of second class constraints [4–6], and these can not be separated in a Lorentz-covariant manner. The first class constraints generate

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the so called κ -symmetry [4]. Some attempts to a direct covariant treatment of the κ -symmetry have appeared (see e.g. [7, 8]), but most of the proposed solutions to the problem have involved drastic changes of variables, such as twistor [9] methods.

Supertwistors solve the problem of covariant quantisation of superparticles in 3, 4, 6 and 10 dimensions [10–15], and make manifest not only super-Poincaré but the whole superconformal symmetry (except, of course in $D = 10$). We mention the supertwistor track here partly since it has similarities with our main focus of attention, pure spinors, in that both twistors and pure spinors are bosonic spinors (i.e., of “wrong” statistics), and partly since twistor methods (of a different flavor) have been of revived interest later and used for amplitude calculations [16–22]. Some works seems to point towards a deeper relation between pure spinors and twistors [23]. It should be mentioned that, although some attempts have been made [24, 25], twistor transform methods seem less powerful in string theory than in particle theory, due to the massive spectrum.

The corresponding problem is of course seen also in field theory. There, the natural way of manifesting supersymmetry is to use superfields, that depend not only on the bosonic coordinates x^m , but also on some fermions θ^μ , that together form a (Wess–Zumino) superspace [26]. If the field theory in question is a gauge theory [27], the superfield formulation will be a gauge theory on superspace [28–32], and if it contains gravity [33–37], it will be described as superspace geometry [26, 37–44]. In both cases, the maximally supersymmetric models (which means 16 supercharges for super-Yang–Mills theory (SYM) and 32 for supergravity (SG)) only have on-shell formulations in superspace. This can be stated in a couple of equivalent ways. The supersymmetry transformations close only modulo the equations of motion. In a component formalism, there is no set of auxiliary (non-dynamical) fields, that can be added so that the bosonic and fermionic numbers of fields agree off-shell and fill a representation of supersymmetry. We will come back to the superspace formulations of some maximally supersymmetric models later, and examine them in more detail, because it is precisely the traditional superspace theories that form the basis of the pure spinor superfield formalism.

Pure spinors are interesting objects from a mathematical point of view. The original definition by E. Cartan [45, 46] is valid in even dimensions. A Cartan pure spinor is a spinor annihilated by half-dimensional isotropic (light-like) subspace. If the dimension is $D = 2n$, then this can be expressed as $\gamma^{+i} \lambda = 0$, $i = 1, \dots, n$, for a suitable choice of basis (depending on the pure spinor λ). Here, we think of the signature of space-time as split. For euclidean signature, take the γ -matrices with holomorphic indices. Modulo a complex scale, the pure spinor space is isomorphic to the space of isotropic n -planes, which is $SO(2n)/U(n)$. This condition can be translated into certain bilinear conditions on the spinor. The first case where the pure spinor condition is non-trivial is $n = 4$. Up to $n = 6$, the pure spinors form the only non-trivial orbit of the rotation group in between the full orbit of unconstrained spinors and the trivial orbit of 0, but for higher n there are more orbits [47–49], of which the pure spinor is the most constrained.

The “pure spinors” we will use sometimes coincide with Cartan pure spinors, sometimes not. The canonical example of $D = 10$ SYM is an example where they are identical. The important and defining property, that we will give a geometric interpretation, is a bilinear identity $(\lambda\gamma^\mu\lambda) = 0$, which in $D = 10$ coincides with the constraint on a Cartan pure spinor. Even if Cartan pure spinors are uninteresting in $D < 8$, we will encounter non-trivial “pure spinor” constraints e.g. in $D = 6$ and $D = 3$, essentially due to the presence of R-symmetry. We will also use the bilinear constraint in odd dimensions, notably $D = 11$.

We are mainly concerned with field theories, including supergravity, and will not say much about the use of pure spinors in superstring theory. From investigations of the superspace formulation of maximally supersymmetric theories, it was early recognised that pure spinors might have a rôle to play in an off-shell formulation [50–52]. The discovery of the precise rôle of pure spinors came from two independent (but in retrospect clearly related) lines of research. One, the covariant quantisation of the superstring, provided a valid set of ghost variables for a covariant superstring, and thereby also for its massless sector [53, 54]. The other was the systematic search for higher-derivative terms in maximally supersymmetric theories, where revisiting the structure of the superspace constraints revealed a cohomological structure of the deformations [55–57], which later was realised to be equivalent to that of the pure spinor BRST operator. The latter formalism led to results on deformations of SYM [55, 58, 59] (e.g. the full form of the terms related to F^4) as well as SG [60–65] models.

Pure spinor superfield models have been given for SYM [54, 55, 58, 59, 66–68] for $D = 11$ supergravity [69, 70] and for $D = 3$ superconformal models [71–73]. It is quite clear that the method applies to any maximally supersymmetric model that does not contain selfdual fields.

The wide breakthrough of the use of pure spinors in connection with supersymmetry came with the realisation of Berkovits that they provide a good set of variables for covariant quantisation of the superstring [53, 74, 75]. The formalism has been extensively used in superstring theory, see e.g. [76–99]. Applications to supermembrane theory have also been attempted, but with less clear results [100–102].

This presentation takes its starting point in the traditional superspace formulation of supersymmetric field theories. In Sect. 4.2 we explain why the basics of the pure spinor superfield formalism is (almost) inherent in the superspace formalism. We derive the BRST operator of the linearised models. Section 4.3 deals with the calculation of the field content, i.e., the BRST cohomology, which is illustrated with some examples. In order to formulate actions, a measure is needed, which is developed in Sect. 4.4, based on the “non-minimal” variables of Berkovits. Section 4.5 gives the field–antifield machinery needed in order to formulate consistent interactions. The following sections deal with gauge fixing, necessary for quantum calculations, and with an application: to find higher-derivative terms. Finally, in Sect. 4.8 some (hopefully) interesting open questions and possible developments are mentioned.

4.2 Pure Spinors from Superspace

We denote bosonic and fermionic indices in coordinate basis (“curved indices”) by $M, N \dots = (m, n, \dots; \mu, \nu, \dots)$ and in Lorentz basis (“flat indices”) by $A, B, \dots = (a, b, \dots; \alpha, \beta, \dots)$. Wess–Zumino superspace has a torsion

$$T_{\alpha\beta}{}^a = 2\gamma_{\alpha\beta}^a \quad (4.1)$$

(there might be slight formal variations on this expression, e.g. when there is some R-symmetry in case of extended supersymmetry, but with a liberal interpretation (4.1) is always true). Note that we always express components in Lorentz indices, since fermionic directions otherwise can not be seen as spinors. This is typically the only non-vanishing torsion component at dimension zero (in on-shell theories), dimension here being defined so that a bosonic derivative has dimension 1 and a fermionic $\frac{1}{2}$. In flat superspace, this statement amounts to the anticommutator between fermionic covariant derivatives being

$$\{D_\alpha, D_\beta\} = -T_{\alpha\beta}{}^a \partial_a = -2\gamma_{\alpha\beta}^a \partial_a . \quad (4.2)$$

In flat space, these are the ordinary derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - (\gamma^a\theta)_\alpha \partial_a , \quad (4.3)$$

which anticommute with the global supersymmetry generators (superspace Killing vectors)

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} + (\gamma^a\theta)_\alpha \partial_a . \quad (4.4)$$

Some special possible rôle of pure spinors can be seen already here. Suppose that λ is pure (in the sense mentioned in the introduction), i.e., that

$$(\lambda\gamma^a\lambda) = 0 . \quad (4.5)$$

If one forms the scalar fermionic operator

$$Q = \lambda^\alpha D_\alpha , \quad (4.6)$$

it becomes immediately clear from (4.1) and (4.5) that

$$Q^2 = 0 . \quad (4.7)$$

It is possible to think of Q as a BRST operator, and examine its cohomology. This cohomology will be non-trivial due to the pure spinor constraint. This will actually be

the BRST operator used in the (minimal) pure spinor formalism, and its cohomology will consist of the physical states.

In order to understand why this happens, and why it indeed is little more than a reformulation of the traditional superspace formalism, it is suitable to reexamine the canonical example, $D = 10$ SYM (the procedure describes equally well its dimensional reductions) [31, 50]. For simplicity, we will use an abelian field.

Note that we aim at going directly to the field theory, without passing via a first-quantised superparticle model. The BRST operator (4.6) is not obtained as the BRST operator for some local symmetry on the world-line of a superparticle, but postulated more or less *ad hoc*. It will soon be motivated from superspace arguments, though. Some work has been done on showing the equivalence of the first-quantised superparticle or string with the formulation based on Q [103–105]. We take a more pragmatic point of view—if the correct field theories are produced we are happy with that.

4.2.1 SYM

We work in $D = 10$, where a chiral spinor has 16 components. The theory starts from a gauge theory on superspace [31, 50]. This means that the connection 1-form *a priori* is completely general,

$$A = E^A A_A = E^a A_a(x, \theta) + E^\alpha A_\alpha(x, \theta) \quad (4.8)$$

(where $E^A = dZ^M E_M^A$ is the superspace vielbein). In order to reduce the very large number of component fields, some constraints must be imposed. One such constraint, which goes under the name of conventional constraint, completely expresses the superfield A_a in terms of A_α . This is desirable, since there is another component 1-form at level θ in A_α , and only one in the physical theory. The conventional constraint is formulated in terms of the field strength, in order not to destroy gauge symmetry, and reads (in the abelian case)

$$\gamma_a^{\alpha\beta} F_{\alpha\beta} = 0 . \quad (4.9)$$

Since this part of F is expressed as

$$F_{\alpha\beta} = 2D_{(\alpha} A_{\beta)} + T_{\alpha\beta}{}^a A_a , \quad (4.10)$$

the conventional constraint does exactly what it is supposed to. Then, one is left with A_α , the lowest-dimensional superfield.

In order to take the fields on-shell the remaining part of $F_{\alpha\beta}$ is also set to zero. This is a selfdual 5-form. We will not exhibit the detailed calculation here, but contend ourselves with the well known statement the setting the dimension-0 field strength to zero gives the equations of motion for the component fields. These sit in the

superfield at order θ (the gauge connection) and θ^2 (the fermion) (and of course also at higher orders if they contain non-zero modes). Traditionally, to keep gauge invariance manifest, the superfield A_a is not actually eliminated. Instead one uses the Bianchi identities for the superspace field strength F , which will give the equations of motions once $F_{\alpha\beta} = 0$. This is not the path taken here. Instead we leave A_a completely aside and focus on A_α .

We can then observe that the conditions imposed are exactly those implied by demanding that a field $\Psi = \lambda^\alpha A_\alpha(x, \theta)$ is annihilated by the BRST operator $Q = \lambda^\alpha D_\alpha$. The fermionic covariant derivative acts on the superfield A_β , and the bilinear in λ contains only the 5-form part, due to the pure spinor condition. In addition, gauge invariance is implemented as $\delta_\Lambda \Psi = Q\Lambda$ (that this is true for the bosonic connection at level θ of course requires a small calculation), which makes clear that the cohomology of Q describes precisely the on-shell physical fields. The cohomology will be examined to greater generality in the following section.

Expanding out the λ -dependence of the field Ψ , we thus have an infinite set of superfields,

$$\Psi(x, \theta, \lambda) = \sum_{n=0}^{\infty} \lambda^{\alpha_1} \dots \lambda^{\alpha_n} A_{\alpha_1 \dots \alpha_n}(x, \theta). \quad (4.11)$$

In order for $Q = \lambda D$ to behave as a BRST operator, it is natural to assign a ghost number 1 to λ . We have already mentioned that the cohomology of Q at order λ reproduces the gauge connection and the fermion, subject to their linearised equations of motion (the remaining cohomology will be left for Sect. 4.3). The field Ψ then also carries ghost number 1, so that the physical fields have ghost number 0.

Already at this point we see that relaxing the equations of motion is equivalent to relaxing the condition $Q\Psi = 0$. If a suitable integration measure is found, a true off-shell formulation could be provided by an action of the type $S \sim \int \Psi Q\Psi + \dots$, which will be the objective of Sects. 4.4 and 4.5.

4.2.2 SG

What will be said in this section will apply to $D = 11$ supergravity, and its dimensional reductions.

A spinor in $D = 11$ has 32 components. The symmetric spinor bilinears are a 1-form, a 2-form and a 5-form. In addition to the metric field, $D = 11$ SG also contains a 3-form potential C with 4-form field strength $H = dC$ and a gravitino. The component action for the bosonic fields,

$$S = \frac{1}{2\kappa^2} \left(\int d^{11}x \left(R - \frac{1}{48} H^2 \right) + \frac{1}{6} \int C \wedge H \wedge H \right), \quad (4.12)$$

contains a Chern–Simons term for C .

There are two ways of approaching the superspace construction of the supergravity. The first one is via the actual supergeometry, examined in [41, 42, 60–62, 106]. Here one starts with the vielbein on superspace E_M^A together with a Lorentz algebra-valued connection Ω_M . Just like in the case of gauge theory, all the superfields except the one of lowest dimension, E_μ^a , are effectively eliminated as independent degrees of freedom via conventional constraints [62, 107, 108]. This is slightly more involved than in the SYM case, and we refer to [62] for a complete treatment. Essentially, by formulating constraints on the superspace torsion,

$$T^A = dE^A + E^B \wedge \Omega_B^A, \quad (4.13)$$

all connection superfields and all of the vielbein become expressible in E_μ^a . The conventional constraints reduce the possible dimension-0 torsion $T_{\alpha\beta}^a$ (apart from the standard part $2\gamma_{\alpha\beta}^a$) to the irreducible modules

$$\begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|c|c|} \hline \square & \square & \square & \square \\ \hline \end{array}, \quad (4.14)$$

where the 2 or 5 antisymmetrised indices come from the contraction of the two spinor indices with γ^{ab} or γ^{abcde} .

Like in SYM, the standard procedure for deriving the full equations of motion is not to actually solve for the vielbein and spin connection superfields, but to use torsion Bianchi identities [39],

$$DT^A = E^B \wedge R_B^A, \quad (4.15)$$

to obtain the equations of motion without giving up any manifest gauge invariance.

Suppose we now want to interpret this, at the linearised level, in terms of pure spinors. Then we again leave all the superfield except the lowest-dimensional one out. After converting the form index on E_μ^a to a flat spinor index, we have a field ϕ_α^a . It is actually only its γ -traceless part that is not eliminated by conventional constraints. Note that the spinor bilinears appearing above in the torsion $T_{\alpha\beta}^a$ after conventional constraints have been used, the 2-form and 5-form, are exactly those which are non-vanishing for a pure spinor. It looks reasonable to think of the linearised superfield ϕ_α^a as appearing at order λ in a pure spinor superfield $\Phi^a(x, \theta, \lambda)$. The linearised equations of motion then come from $Q\Phi^a = 0$. There is only a small ingredient missing here, namely that ϕ_α^a is γ -traceless, as are the two torsion modules. This is achieved by declaring an equivalence relation

$$\Phi^a \approx \phi^a + (\lambda\gamma^a)_\rho. \quad (4.16)$$

We call this type of equivalence relation a “shift symmetry” [69–72, 109], and we will come back to its rôle in the following sections.

The other way of obtaining the linearised equations of motion is from the 3-form C , which extends to a 3-form on superspace. This method has not traditionally been used alone as a formulation of supergravity, since the geometry (via the torsion) will enter its Bianchi identities. Nevertheless, at the linearised level this produces all the supergravity fields, without involving superspace geometry; this will be made clear in Sect. 4.3. Without going into details about conventional constraints, it is again the lowest-dimensional superfield that is relevant. This is $C_{\alpha\beta\gamma}$, of dimension $-\frac{3}{2}$, and actually only the irreducible modules consisting of γ -traceless 2-form- and 5-form-spinors. These modules fit perfectly in the expansion of a scalar pure spinor superfield $\Psi(x, \theta, \lambda)$ to third order in λ ,

$$\Psi = \dots + \frac{1}{6} \lambda^\alpha \lambda^\beta \lambda^\gamma C_{\alpha\beta\gamma} + \dots \quad (4.17)$$

The linearised supergravity equations of motion come from demanding that

$$H_{\alpha\beta\gamma\delta} \Big|_{\begin{array}{c} \boxplus \\ \oplus \\ \boxplus \\ \oplus \\ \boxplus \end{array}} = 0, \quad (4.18)$$

which is equivalent to the condition

$$Q\Psi = 0, \quad (4.19)$$

since these three irreducible modules are precisely the ones occurring in a quadrilinear of a pure spinor.

4.2.3 Summary

We have seen, in the two main examples of $D = 10$ SYM and $D = 11$ SG, that the linearised equations of motion (and gauge symmetries) are reproduced precisely by considering the physical fields as part of a pure spinor superfield with appropriate properties annihilated by the pure spinor BRST operator $Q = \lambda D$. The price paid for this is that interactions are (for the moment) ignored, and that only some lowest-dimensional superfield is considered. This also means that gauge symmetry (including diffeomorphisms and local supersymmetry in the SG case) are not kept “manifest” or “geometrical”. We will comment more on this issue when interactions are introduced, in Sect. 4.5.2.

4.3 Cohomology

In this section, we will take a closer look at the cohomology of the BRST operator in the two examples of Sect. 4.2 and some other models. The statements about it reproducing the fields of the models in question will be made more precise, and

some interesting structure pointing forward to a field–antifield formalism will be pointed out.

Notice that if λ had been unconstrained (and there was no shift symmetry, for the case of non-scalar fields), the cohomology had been trivial. It is the pure spinor property of λ that gives room for some interesting cohomology. Consider, for example, a scalar pure spinor superfield $\Psi(x, \theta, \lambda)$, and let us for the moment forget about the x -dependence. A field $\Psi = (\lambda\gamma^a\theta)A_a$ represents cohomology: acting with Q gives

$$Q \cdot (\lambda\gamma^a\theta)A_a = \lambda \frac{\partial}{\partial\theta} \cdot (\lambda\gamma^a\theta)A_a = (\lambda\gamma^a\lambda)A_a = 0, \quad (4.20)$$

and it is also obvious that such a field can not be written as a Q -exact expression. In the SYM case, this cohomology is precisely the zero mode of the gauge connection. Obviously, Ψ should be taken to be fermionic.

It is clear that the algebraic properties of the pure spinor λ play a decisive rôle for determining the cohomology. Indeed, as we will see in the following sections, a partition function for the pure spinor contains essentially all information needed to determine the full cohomology.

We have seen one example above of an element of the cohomology of a scalar superfield, the zero mode of the gauge connection. We also argued in Sect. 4.2.1 that the cohomology at order λ precisely reproduces the fields of $D = 10$ SYM, subject to the linearised equations of motion. What is the general cohomology? One more example is the constant field, $\Psi = c$. This is a cohomology of ghost number 1 (given the ghost number assignment of Sect. 4.2.1), and given the gauge transformation of Ψ it is natural to identify it as the ghost for the gauge symmetry.

Both these examples concern zero mode cohomology, i.e., elements of cohomology independent of the coordinates x . It turns out to be very instructive to first consider general zero mode cohomology. Not only is it much easier to calculate, since it is a purely algebraic problem (the operator Q reduces to $\lambda^\alpha \frac{\partial}{\partial\theta^\alpha}$), it will also give all essential information concerning the full cohomology. Namely, consider a zero mode cohomology of Ψ at order $\lambda^p\theta^q$. Such a cohomology will have ghost number $\text{gh}\#(\Psi) - p$ and dimension $\dim(\Psi) + \frac{1}{2}(p + q)$. If then x -dependence is introduced, how will the corresponding cohomology behave? The only possibility is to have some field in the same module as the zero mode, but subject to some differential equation, an equation of motion. This equation of motion must in turn have support in the zero mode cohomology. This means that the zero mode cohomology can be used to read off the possible full cohomology. If there is also a zero mode cohomology at $\lambda^{p+1}\theta^{q+2n-1}$ (i.e., at ghost number $\text{gh}\#(\Psi) - p - 1$ and dimension $\dim(\Psi) + \frac{1}{2}(p + q) + n$), a field $\phi(x)$ in some module determined by the zero mode cohomology at $\lambda^p\theta^q$ can be subject to a (linearised) equation of motion of the form $\partial^n\phi = 0$, given that the modules of the two zero mode cohomologies match. The corresponding x -dependent cohomology will of course take the generic form

$$\Psi \sim \lambda^p(\theta^q\phi + \theta^{q+2}\partial\phi + \theta^{q+4}\partial^2\phi + \dots). \quad (4.21)$$

4.3.1 SYM

As mentioned, the algebraic problem of calculating the zero mode cohomology can be used to gain information about the full cohomology [54, 57, 110]. The problem can be solved by computer methods [57] or algebraically [111]. For the field Ψ of ghost number 1 and dimension 0, the result may be summarised in Table 4.1, where the horizontal direction is the expansion in λ (i.e., decreasing ghost number of the component fields) and the vertical is the expansion in θ (i.e., increasing dimension within each superfield). The expansion of the superfields in θ has been shifted, so that components on the same horizontal level have the same dimension. The modules have been labeled by the Dynkin labels of the Lorentz group $Spin(1, 9)$. As already discussed we see the gauge ghost at λ^0 and the physical fields (gauge connection A_a and spinor χ^α) at λ^1 . In addition there are cohomologies at λ^2 and λ^3 . The ones at λ^2 indicate, according to the discussion above, that the physical fields are subject to equations of motion. Their interpretation as components of the field Ψ is as *antifields* A^{*a} and χ_α^* , fields of ghost number -1 with the same dimensions as the equations of motion. The singlet at $\lambda^3\theta^5$ is the ghost antifield c^* . Its presence in cohomology in turn implies the divergencelessness of the on-shell antifield, corresponding to conservation of the gauge current. This is then strong evidence that using a pure spinor to go off shell implies introducing a Batalin–Vilkovisky field–antifield structure. This will be formalised in detail in Sect. 4.5.

As argued in the beginning of the present section, there is a more direct way of deducing the zero mode cohomology (and thereby the full cohomology) from the partition function for a pure spinor. Consider the expansion of a function $f(\lambda)$ in a power series expansion in λ , just as we have done for the pure spinor superfield. The pure spinor λ itself is in the module (00001), and the pure spinor constraint ensures that only the module (0000 n) occurs at λ^n . Therefore, the component fields in the expansion will come in the conjugate module $R_n = (000n0)$. A formal partition function [111–113] containing all information about the expansion is

$$\mathcal{P}(t) = \bigoplus_{n=0}^{\infty} R_n t^n = \bigoplus_{n=0}^{\infty} (000n0) t^n . \quad (4.22)$$

A less refined partition function is one that only counts the dimensions of the modules, i.e.,

$$\begin{aligned} P(t) &= \sum_{n=0}^{\infty} \dim(000n0) t^n = \sum_{n=0}^{\infty} \frac{1}{10} \binom{n+7}{7} \binom{n+5}{3} t^n \\ &= (1-t)^{-11} (1+t)(1+4t+t^2) \\ &= (1-t)^{-16} (1-10t^2+16t^3-16t^5+10t^6-t^8) . \end{aligned} \quad (4.23)$$

Various information can be collected here. The next to last line indicates that the number of degrees of a pure spinor in $D = 10$ is 11 (more on this in Sect. 4.4). The last line (where the factor $(1-t)^{-16}$ represents the partition function of an unconstrained spinor) is where the zero mode cohomology can be read off: note

Table 4.1 The zero mode cohomology in Ψ for $D = 10$ super-Yang–Mills theory

dim	gh#				
	1	0	-1	-2	-3
0	(00000)				
$\frac{1}{2}$	•	•			
1	•	(10000)	•		
$\frac{3}{2}$	•	(00001)	•	•	
2	•	•	•	•	•
$\frac{5}{2}$	•	•	(00010)	•	•
3	•	•	(10000)	•	•
$\frac{7}{2}$	•	•	•	•	•
4	•	•	•	(00000)	•
$\frac{9}{2}$	•	•	•	•	•

The horizontal direction represents the expansion of the superfield in terms of λ whereas the corresponding for the vertical (in each row) is θ (downward). The irreducible representations of the component fields are listed at the positions which describe their ghost numbers and dimensions

the agreement between the numbers in the polynomial $1 - 10t^2 + 16t^3 - 16t^5 + 10t^6 - t^8$ and the dimensions of the modules in Table 4.1. In addition, the signs of the monomials indicate the bosonic (plus) or fermionic (minus) character of the cohomologies (remember that Ψ is fermionic, so all signs change). This property is of course expressible also in the more refined partition \mathcal{P} , which can be shown to be

$$\begin{aligned}
 \mathcal{P}(t) = & \left(\bigoplus_{k=0}^{\infty} \vee^k (00010)t^k \right) \\
 & \otimes \left((00000) \oplus (10000)(-t^2) \oplus (00001)t^3 \right. \\
 & \left. \oplus (00010)(-t^5) \oplus (10000)t^6 \oplus (00000)(-t^8) \right), \quad (4.24)
 \end{aligned}$$

where \vee denotes the symmetric product, and the first line is the refined partition function for an unconstrained spinor. This unconstrained factor can formally be written as $(1 - t)^{-(00010)}$, see [114], where the pure spinor partition function is related to a certain Borchers algebra.

4.3.2 Supergravity

The analogous procedure can be performed for $D = 11$ supergravity, and the resulting zero mode cohomologies [57] are listed in Table 4.2. This list is based on the cohomologies in a *scalar* superfield of ghost number 3 and dimension -3 , i.e., the field Ψ of Sect. 4.2.2, based on the superspace 3-form. This field must indeed be taken as the basic field of $D = 11$ supergravity, since the “geometric field” Φ^a does not exhibit the gauge invariance of the C -field—only the field strength H appears in the torsion—so one can not hope to reproduce the Chern–Simons term of the action of (4.12) from Φ^a alone (although the equations of motion are reproducible, one of them being the Bianchi identity for H). We will not bother to write down the detailed partition function for the $D = 11$ pure spinor [111]; the relation to the cohomology is completely analogous to the case of SYM.

The reason for Ψ having ghost number -3 is now obvious; the lowest cohomology represents the ghost for ghost for ghost of the the twice reducible gauge transformations of the 3-form field. Consequently, the “highest” cohomology, the corresponding antifield, is a scalar at $\lambda^7\theta^9$. The content of Table 4.2 verifies that indeed all degrees of freedom of the supergravity are present at λ^3 , also the gravitational ones (and even some without local degrees of freedom, related to the Weyl invariance of [106]). We also note the presence of ghosts for diffeomorphisms and local supersymmetry, appearing alongside the ghost for tensor gauge transformations at λ^2 . As in the SYM case, the zero mode cohomology (and the partition function) is completely symmetric with respect to exchange of fields and antifields.

4.3.3 Other Models

The method may be extended to other models. Specifically, it has been used [71–73] for superconformal models in $D = 3$: the $N = 8$ Bagger–Lambert–Gustavsson (BLG) [115–117] and $N = 6$ Aharony–Bergman–Jafferis–Maldacena (ABJM) [118] models. Here the Chern–Simons connection comes in one (scalar) pure spinor superfield, and the matter multiplets in another, which, in the absence of ghosts, comes in the same module as the scalar fields, subject to a shift symmetry. We refer to the papers [71–73] for details.

We can also note that models containing selfdual fields follow part of the pattern. Take for example the $N = (2, 0)$ tensor multiplet in $D = 6$. Without exhibiting the details [57] here, we note that the correct cohomologies for fields and ghosts are produced. When it comes to “antifields”, however, the pattern is broken. The equation of motion for the tensor field is the selfduality of its field strength, and there is no symmetry between fields and antifields in the cohomology. Therefore, equations of motion $Q\Psi = 0$ are meaningful, but the construction of an action along the lines of Sect. 4.5 becomes obstructed.

Table 4.3 The zero mode cohomology in Ψ for $D = 6$ $N = (1, 0)$ super-Yang–Mills theory

dim	gh#		
	1	0	-1
0	(000)(0)		
$\frac{1}{2}$	•	•	
1	•	(100)(0)	•
$\frac{3}{2}$	•	(001)(1)	•
2	•	(000)(2)	•
$\frac{5}{2}$	•	•	•

4.3.4 Less than Maximal Supersymmetry

The procedure sketched here is not unique for maximally supersymmetric models, although it is there that it seems to have its highest potential. What happens if the method is attempted for a theory with less than maximal supersymmetry? If the pure spinors are appropriately chosen, the traditional superspace formulation should be reproduced also here. This is indeed the case. If such a superspace formulation results in an *off-shell* supermultiplet including auxiliary fields, this also happens in the pure spinor formulation. The result, then, will be a cohomology without the antifields, since we have argued that the presence of antifield cohomology is what puts the physical fields on shell.

This can be illustrated by $N = (1, 0)$ SYM in $D = 6$ [119]. There is an $SU(2)$ R-symmetry, and with standard assignment of Dynkin labels for $Spin(1, 5) \times SU(2)$ we let λ^α transform in the module (001)(1). With the pure spinor constraint $(\lambda\gamma^\alpha\lambda) = 0$, the only remaining spinor bilinear is the $SU(2)$ triplet selfdual 3-form (002)(2). Note that such a pure spinor is non-trivially constrained, unlike a Cartan pure spinor in $D = 6$, which has no R-symmetry. The superfields in the λ expansion of a scalar pure spinor superfield Ψ are fields $A_{\alpha_1 \dots \alpha_n}$ in (00n)(n). A direct calculation of the zero mode cohomology, or equivalently, of the pure spinor partition function, gives at hand that cohomology only occurs at λ^0 (the ghost) and λ^1 (the physical fields). No higher cohomologies exist, and there is no room for equations of motion for the physical fields. The cohomology is listed in Table 4.3, where it is clear that in addition to the gauge connection and fermion field, the triplet of auxiliary fields also appears.

Since all equations of motion follow from setting the auxiliary fields to zero, it is natural that the antifields should occur as cohomology of a separate pure spinor superfield of dimension 2 and ghost number -1 transforming as a triplet. This is indeed the case. The antifields (or, the current multiplet) is described by a pure

Table 4.4 The zero mode cohomology in Ψ^{*I} for the antifields of $D = 6$ $N = (1, 0)$ super-Yang–Mills theory

dim	gh#		
	-1	-2	-3
2	(000)(2)		
$\frac{5}{2}$	(010)(1)	•	
3	(100)(0)	•	•
$\frac{7}{2}$	•	•	•
4	•	(000)(0)	•
$\frac{9}{2}$	•	•	•

spinor superfield Ψ^{*I} , which has a shift symmetry of the form

$$\Psi^{*I} \approx \Psi^{*I} + (\lambda \sigma^I \rho) . \tag{4.25}$$

The cohomology in Ψ^* is the mirror of the one in Ψ , and listed in Table 4.4.

The condition for Ψ being on-shell must be separately formulated as another condition $s^I \Psi = 0$, where s^I is an operator with ghost number -1 and dimension 2, such that $s^I \Psi$ effectively starts out with the auxiliary field [119].

Similar considerations could be applied to other non-maximally supersymmetric models. It has been used to check the multiplet structure of $D = 3$, $N = 8$ supergravity [120]. The cohomology (Cederwall, unpublished) of $D = 10$, $N = 1$ SG has also been verified to agree with known results [122, 123].

4.4 Pure Spinor Space and Integration

As noted in Sect. 4.2.1, if a reasonable (non-degenerate) integration measure $[dZ]$ (Z denoting the ordinary superspace coordinates together with the pure spinor variables) can be found, an action of the form

$$S = \frac{1}{2} \int [dZ] \Psi Q \Psi + \text{interactions} \tag{4.26}$$

will provide an off-shell formulation of the model in question, and a solution to the problem of finding an action for maximally supersymmetric models. In view of the discussion on cohomology of the previous section, such an action would be a classical Batalin–Vilkovisky (field–antifield) action (see Sect. 4.5).

A measure on the pure spinor space has to fulfil a number of requirements. First, as already noted, it has to be non-degenerate in order that the variation of the action actually implies the equations of motion $Q\Psi = 0$. In addition, and depending on the model at hand, there are restrictions on the dimension and ghost number of the integration.

For the case of $D = 10$ SYM, Ψ has ghost number 1 and dimension 0. Therefore $\int[dZ]$ needs to have ghost number -3 , and since $\frac{1}{g^2} \int d^{10}x d^{16}\theta$ has dimension $-4 + \frac{1}{2} \times 16 = 4$, “ $\int[d\lambda]$ ” must have dimension 4. Correspondingly, in $D = 11$ SG, the pure spinor integration measure must contribute ghost number -7 and, since the dimension of $\frac{1}{k^2} \int d^{11}x d^{32}\theta$ is $-2 + \frac{1}{2} \times 32 = 14$ and that of Ψ is -3 , it also must give dimension -8 . In addition the measures should have the property that $\int[dZ]Q\Lambda = 0$, so that BRST-trivial states have zero integral and partial integration with respect to Q is possible.

The second thing to note is there are natural operations with precisely these quantum numbers. If we check the highest ghost antifield cohomology, they come at $\lambda^3\theta^5$ and $\lambda^7\theta^9$, respectively. So, an “integration” that picks out the corresponding term in the expansion of a pure spinor superfield would have $(\text{gh}\#, \text{dim}) = (-3, 4)$ and $(-7, 8)$ respectively, as desired. This is correct in spirit, but is still a degenerate measure, since the expansion in λ only contains positive powers. Some adjustment is needed.

The solution to this problem was provided, for $D = 10$ pure spinors, by Berkovits [74] with the introduction of so called non-minimal variables. By the introduction of another set of pure spinors called $\bar{\lambda}_\alpha$ and a spinor of fermionic variables r_α which is pure relative $\bar{\lambda}$, i.e., fulfilling $(\bar{\lambda}\gamma^\alpha r) = 0$, the measure could be made non-degenerate. Non-minimal sets of variables are quite standard when it comes to field-antifield quantisation, but the present ones are even more natural, even from a purely geometric point of view. Namely, although solutions to the pure spinor constraints are complex (unless one is in split signature), we have so far assumed that the fields depend on λ and not on $\bar{\lambda}$. Unless we have some kind of residue measure, it seems more natural to integrate over the full complex variable $(\lambda, \bar{\lambda})$. The interpretation of the fermion r_α is as the differential $d\bar{\lambda}_\alpha$ (with the fermionic statistics coming from the wedge product), which obviously satisfies $(\bar{\lambda}\gamma^\alpha d\bar{\lambda}) = 0$ [80]. When more variables are introduced, the BRST operator must be changed accordingly in order to keep the cohomology intact. This is done by adding a term to Q :

$$Q = (\lambda D) + \left(r \frac{\partial}{\partial \bar{\lambda}} \right) = Q_0 + \left(d\bar{\lambda} \frac{\partial}{\partial \bar{\lambda}} \right) = Q_0 + \bar{\partial} , \quad (4.27)$$

where $\bar{\partial}$ is the antiholomorphic exterior derivative, the Dolbeault operator. The cohomology is unchanged, and any cohomology will have a representative that is independent of $\bar{\lambda}$ and $d\bar{\lambda}$.

A field $\Psi(x, \theta; \lambda, \bar{\lambda}, d\bar{\lambda})$ is then seen as an antiholomorphic form on pure spinor space (meaning, it can depend on both λ and $\bar{\lambda}$, but has only antiholomorphic indices, seen as a tensor). A suitable assignment of quantum numbers for $\bar{\lambda}$ and $d\bar{\lambda}$ is that

$\bar{\lambda}$ has ghost number -1 and dimension $\frac{1}{2}$ (the opposite to λ), while $d\bar{\lambda}$ has ghost number 0 and dimension $\frac{1}{2}$ (there is some irrelevant arbitrariness in the assignment, as long as it comes out right for the BRST operator).

Suppose that the integration can be written as an integral of a form over the pure spinor space. Since no fields contain $d\lambda$, the integration measure needs to contain a top form Ω with the maximum number of holomorphic indices. In $D = 10$, this number is 11 (see below). In order for partial integration of $\bar{\partial}$ to be allowed, this form should in addition depend on λ only, so that $\bar{\partial}\Omega = 0$. We now try an expression for the full integral over the non-minimal pure spinor variables,

$$\int [d\lambda]X = \int \Omega \wedge X . \quad (4.28)$$

Again counting quantum numbers (for the $D = 10$ case), the λ and $\bar{\lambda}$ integrals cancel, while the r integration (“removal of $d^{11}\bar{\lambda}$ ”) provides ghost number 0 and dimension $-\frac{11}{2}$. In order to land at the desired quantum numbers for the integration, ghost number -3 and dimension -4 , the components of Ω must have ghost number -3 and dimension $\frac{3}{2}$, which is accomplished by precisely three negative powers of λ ,

$$\Omega \sim \lambda^{-3}d^{11}\lambda \quad (4.29)$$

(we leave it as a trivial exercise to show that the same applies to any assignment of quantum numbers to $\bar{\lambda}$ and $d\bar{\lambda}$ that respects the ones of Q , and that the assignments for $d\lambda$ are irrelevant).

The requirement that the holomorphic top form with $\bar{\partial}\Omega$ exists is equivalent to the existence of a Calabi–Yau structure on the pure spinor space, defined by Ω . There is indeed a unique $Spin(10)$ -invariant Calabi–Yau metric (up to a scale) on the pure spinor space, following from the Kähler potential [124]

$$K(\lambda, \bar{\lambda}) = (\lambda\bar{\lambda})^{8/11} . \quad (4.30)$$

The pure spinor constraint may be solved in a basis where manifest $Spin(10)$ is broken to $SU(5) \times U(1)$. Then, $\mathbf{16} \rightarrow \mathbf{1}_{-5/2} \oplus \mathbf{10}_{-1/2} \oplus \bar{\mathbf{5}}_{3/2}$, and a spinor is represented by a 0-form ℓ , a 2-form Λ and a 4-form M . The pure spinor constraint reads $\ell M - \frac{1}{2}\Lambda \wedge \Lambda = 0$, so the 11 coordinates can be taken as ℓ and Λ in a patch where $\ell \neq 0$. It is obvious that

$$\Omega = \ell^{-3}d\ell d^{10}\Lambda \quad (4.31)$$

has vanishing $U(1)$ charge, and it can be checked that it is fully $Spin(10)$ -invariant. In [124], it was checked by explicit calculation that this is the Calabi–Yau top form corresponding to the Kähler potential (4.30). It can of course also be given a covariant form. The expression

$$\Omega \sim (\lambda\bar{\lambda})^{-3}\bar{\lambda}_{\alpha_1}\bar{\lambda}_{\alpha_2}\bar{\lambda}_{\alpha_3}\star T^{\alpha_1\alpha_2\alpha_3}_{\beta_1\dots\beta_{11}}d\lambda^{\beta_1} \wedge \dots \wedge d\lambda^{\beta_{11}}$$

is indeed independent of $\bar{\lambda}$ [125] (which thus can be replaced by any constant spinor), where the the tensor T is precisely what, after dualisation of the 11 antisymmetric lower indices to 5 upper ones, defines the ghost antifield cohomology,

$$\begin{aligned} \Psi &\sim T_{\alpha_1\alpha_2\alpha_3,\beta_1\beta_2\beta_3\beta_4\beta_5}\lambda^{\alpha_1}\lambda^{\alpha_2}\lambda^{\alpha_3}\theta^{\beta_1}\theta^{\beta_2}\theta^{\beta_3}\theta^{\beta_4}\theta^{\beta_5} \\ &\sim (\lambda\gamma^a\theta)(\lambda\gamma^b\theta)(\lambda\gamma^c\theta)(\theta\gamma_{abc}\theta) . \end{aligned} \quad (4.32)$$

This whole procedure may be repeated for the $D = 11$ pure spinors. The introduction of non-minimal variables is completely analogous, as is the formulation of the integration in terms of a Calabi–Yau top form. The dimension of the pure spinor space is 23, which can be deduces from an explicit solution similar to the one for $D = 10$. When $Spin(11) \rightarrow SU(5) \times U(1)$,

$$\mathbf{32} \rightarrow \mathbf{1}_{-5/2} \oplus \mathbf{5}_{-3/2} \oplus \mathbf{10}_{-1/2} \oplus \bar{\mathbf{10}}_{1/2} \oplus \bar{\mathbf{5}}_{3/2} \oplus \mathbf{1}_{5/2} . \quad (4.33)$$

A spinor is thus parametrised by an arbitrary form. If we write it as

$$\lambda = \ell \oplus \bigoplus_{p=1}^5 \Lambda_p \quad (4.34)$$

(ℓ being the 0-form, and the subscript p denoting form degree), the solution to the pure spinor constraint is

$$\begin{aligned} \Lambda_3 &= \ell^{-1} \Lambda_1 \wedge \Lambda_2 + \Sigma, \\ \Lambda_4 &= \ell^{-1} (-\Lambda_1 \wedge \Lambda_3 + \frac{1}{2} \Lambda_2 \wedge \Lambda_2), \\ \Lambda_5 &= \ell^{-2} \Lambda_2 \wedge \Lambda_3 - \frac{1}{2} \Lambda_1 \wedge \Lambda_2 \wedge \Lambda_2, \end{aligned} \quad (4.35)$$

where Σ is a 3-from satisfying

$$t_v \Sigma \wedge \Sigma = 0 \quad (4.36)$$

for all vectors v , i.e., $\epsilon^{ijklmn} \Sigma_{ijk} \Sigma_{lmn} = 0$ [111, 126].

An important difference compared to the $D = 10$ pure spinors is that there is a singular locus away from the origin, where the 3-form Σ vanishes. It is straightforward to see that then $(\lambda\gamma^{ab}\lambda) = 0$. This is the space of $D = 12$ Cartan pure spinors, a 16-dimensional space. The degrees of freedom contained in Σ consists, modulo a scale, of the Grassmannian $Gr(2, 5) = \frac{SU(5)}{S(U(3)\times U(2))}$ of 2-planes in 5-dimensions. So the appearance of Σ provides 14 more real, or 7 complex dimensions, to make a total of 23. A similar parametrisation of the solution of the constraint on Σ in terms of modules of $su(3) \oplus su(2) \oplus u(1)$, with s being the singlet, gives at hand that the the measure, i.e., the holomorphic top form carries the factor $\ell^{-5}s^{-2}$ [126], and here is the ghost number -7 as announced. Again, the measure can be cast in a Lorentz-covariant form, but we will not go into the details (see [69, 100, 127]). The above reflects the fact that the top cohomology at $\lambda^7\theta^9$ contains 2 powers of

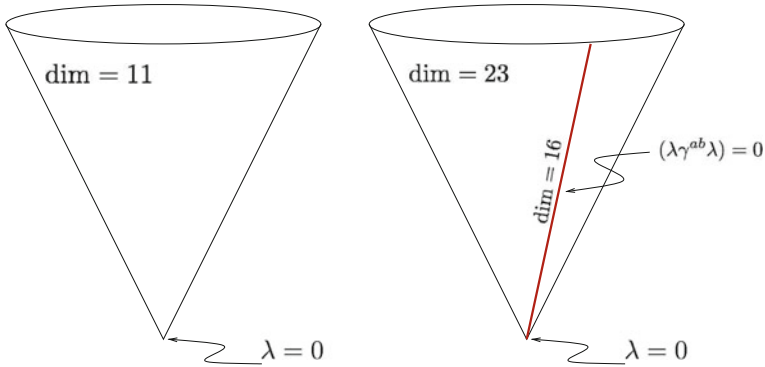


Fig. 4.1 A rough sketch of the $D = 10$ and $D = 11$ pure spinor spaces, with their respective singular subspaces marked

$(\lambda\gamma^{(2)}\lambda)$. The corresponding Kähler potential and metric have not been explicitly constructed, but this should be straightforward.

We finally want to say a few words about integration and regularisation [74]. It was mentioned that the cohomology, also after the introduction of $(\bar{\lambda}, d\bar{\lambda})$, has representatives that are independent of these variables. In other words, they are holomorphic functions (0-forms). How can integrals of (products of) such functions give a non-vanishing result? One will always obtain 0, due the undersaturation of the form degree (the fermionic variables). On the other hand, the polynomial behaviour of the cohomologies at infinity gives ∞ , if radial integration is performed first. The integrals are ill-defined, of the form $0 \times \infty$. This can be remedied in two (equivalent) ways. Either we note that the representatives in the minimal variables are a bad choice, and change them into some BRST-equivalent representatives that give well-defined integrals, or we use a BRST-invariant regularisation of the measure. The same type of regulator, an expression of the form $e^{-t\langle Q, \chi \rangle}$, works in both cases. A standard choice for χ is $\chi = \theta^\alpha \bar{\lambda}_\alpha$, giving a regulator

$$e^{-t((\lambda\bar{\lambda})+(\theta d\bar{\lambda}))} \tag{4.37}$$

If such a regulated measure (with $t > 0$) is used with the minimal representatives, we see that it regulates the bosonic integrals at infinity. At the same time 11 ($D = 10$) or 23 ($D = 11$) $d\bar{\lambda}$'s are needed to saturate the form degree (fermionic integral), and the corresponding term in the expansion of the exponential carries 11 (23) θ 's. In order to saturate the θ integration, another 5 (9) are needed, and we see that this agrees with picking out the top cohomology, as was the first, too naïve, candidate for integration. It is thus no coincidence that the number of θ 's in the top cohomology agrees with the number of independent constraints on a pure spinor (Fig. 4.1).

The regulated integrals will of course be independent of the parameter t . This looks much like localisation—taking t to be very big localises the integral close to

the origin. The dependence on the pure spinor variables is indeed “topological”, in the sense that they do not provide new functional dependence, only a finite spectrum. We have not seen any good way of making use of localisation. The origin is not a regular point in pure spinor space, rather a boundary [128].

4.5 Batalin–Vilkovisky Formalism and Actions

We have seen in Sect. 4.3 that the content of the pure spinor superfields is not only the physical fields, but also a full set of ghosts and antifields (at least for maximal supersymmetry). This indicates that the proper framework for introducing interactions (so far, everything has been at a linearised level) is the Batalin–Vilkovisky formalism [129–131].

4.5.1 Field-Antifield Structure

The Batalin–Vilkovisky (BV) formalism can be thought of in several ways. It seems to have originated as an attempt to find something similar to a Hamiltonian formalism, without breaking manifest Lorentz symmetry, in that sense uniting the advantages of the Lagrange and Hamilton methods. Another way of viewing it is that it naturally lifts the BRST method to possibly include nonlinear terms and transformations, i.e., interactions. It should be noted that some textbooks (e.g. [132]) introduce the BV formalism in connection with gauge fixing, which tends to somewhat obscure the simplicity. What we will do here is classical BV field theory, although we will discuss gauge fixing in Sect. 4.7.

In the BV framework, a ghost field is introduced for each gauge symmetry (and reducibility) and each of the fields ϕ^I (by which is meant physical fields as well as ghosts) is supplemented by its antifield ϕ_I^* with opposite statistics and a ghost number assignment fulfilling $\text{gh}\#(\phi) + \text{gh}\#(\phi^*) = -1$. A fermionic bracket, the so called antibracket, between functions of fields and antifields is introduced as

$$(A, B) = \int d^D x \left(A \frac{\overleftarrow{\delta}}{\delta \phi^I(x)} \frac{\overrightarrow{\delta}}{\delta \phi_I^*(x)} B - A \frac{\overleftarrow{\delta}}{\delta \phi_I^*(x)} \frac{\overrightarrow{\delta}}{\delta \phi^I(x)} B \right). \quad (4.38)$$

The (classical) BV action is defined as a solution to the master equation

$$(S, S) = 0, \quad (4.39)$$

which reduces to the action for the physical fields when ghosts and antifields are removed. The action itself generates gauge transformations via the antibracket (in a generalised sense, where e.g. antifields are transformed by the equations of motion

for the physical fields), so the master equation (4.39) can be seen as the invariance of the action itself.

In the situation at hand, with the pure spinor superfields for maximally supersymmetric theories, we have seen that the cohomology describes both fields and antifields, so a split in the two sets looks problematic. In addition, it is of course necessary to define the antibracket off shell, so that also components outside cohomology takes part. The field–antifield symmetry of the cohomology makes it natural to think of a field Ψ as self-conjugate with respect to the antibracket, and define it as [69]

$$(A, B) = \int A \overleftarrow{\frac{\delta}{\delta\Psi(Z)}} [dZ] \overrightarrow{\frac{\delta}{\delta\Psi(Z)}} B . \quad (4.40)$$

It is straightforward to show that this antibracket (in all cases we have considered) carries the correct quantum numbers, and that a free action of the form

$$S_2 = \frac{1}{2} \int [dZ] \Psi Q \Psi \quad (4.41)$$

indeed generates gauge transformations. At this non-interacting level, the master equation is equivalent to the nilpotency of the BRST operator. Actions of this form thus describes both SYM and SG at linearised order.

4.5.2 Interactions from the Master Equation

We now have at our disposal all ingredients necessary to introduce interactions in a consistent way. The guiding principle is the master equation (4.39).

4.5.2.1 SYM

The SYM case is easy. The linearised action has the form of an abelian Chern–Simons action, and since Ψ and Q carry the same quantum numbers a Ψ^3 term can be added, turning the full action into Chern–Simons form,

$$S = \int [dZ] \text{tr} \left(\frac{1}{2} \Psi Q \Psi + \frac{1}{3} \Psi^3 \right) . \quad (4.42)$$

This leads to equations of motion

$$Q\Psi + \Psi^2 = 0 , \quad (4.43)$$

which could of course equally well be directly deduced from the superspace formalism, where its restriction to the ghost number zero fields reads $\lambda^\alpha \lambda^\beta F_{\alpha\beta} = 0$.

A notable feature is that although the component action contains 4-point couplings, such terms are not present in the manifestly supersymmetric pure spinor superfield action. Instead they are reproduced when the equations of motion are solved sequentially in the θ expansion of the superfields A_α . Such simplifications are typical. We mentioned them in passing for the 3-dimensional conformal models of Sect. 4.3.3, and similar simplifications turn out to happen also for supergravity.

4.5.2.2 SG

The interactions of $D = 11$ supergravity [69, 70] are more subtle. Remember that Q has ghost number 1 and dimension 0, while Ψ has ghost number 3 and dimension -3 . The first step will be to construct a 3-point coupling. How can it be formed, given that the integrand in the action must have ghost number 7 and dimension -6 ?

Here, the geometric field Φ^a comes into play. We remind that it has ghost number 1 and dimension -1 . It contains the field strength H but not the potential C . Guided by the form of the Chern–Simons term $C \wedge H \wedge H$, is it possible that something like $\Psi \Phi^2$ may work? Such a combination has ghost number 5 and dimension -5 . If it is supplemented by two powers of λ , the quantum numbers are the correct ones. A hypothetical 3-point coupling is then

$$S_3 \sim \int [dZ] (\lambda \gamma_{ab} \lambda) \Psi \Phi^a \Phi^b . \quad (4.44)$$

Apart from the matching of quantum numbers, the factor $(\lambda \gamma_{ab} \lambda)$ has two other rôles: the antisymmetry in $[ab]$ makes it possible to contract the indices on the (fermionic) Φ fields; and it ensures the invariance under the shift symmetry of (4.16), thanks to the Fierz identity $(\gamma^b \lambda)_\alpha (\lambda \gamma_{ab} \lambda) = 0$, satisfied by a pure spinor λ (but not by an unconstrained one).

This is of course not the final answer for the 3-point coupling. We have argued that Ψ is the fundamental field, but (4.44) is meaningless until we declare how Φ^a is formed from Ψ . Let us assume that there is some operator R^a of ghost number -2 and dimension 2 (defined modulo shift symmetry) such that

$$\Phi^a = R^a \Psi . \quad (4.45)$$

Then the master equation, stating the consistency of the tentative 3-point coupling, demands that $[Q, R^a] = 0$ (again modulo shift symmetry). Such an operator was constructed in [69], and it takes the form

$$R^a = \eta^{-1} \left(\bar{\lambda} \gamma^{ab} \bar{\lambda} \right) \partial_b + \dots , \quad (4.46)$$

where $\eta = (\lambda\gamma^{ab}\lambda)(\bar{\lambda}\gamma_{ab}\bar{\lambda})$ is the scalar invariant vanishing on the the codimension-7 subspace of 12-dimensional pure spinors, and where the ellipsis denotes terms with $d\bar{\lambda}$ and $d\bar{\lambda}^2$.

This means that we have a consistent 3-point interaction. It is clearly also non-trivial, and since already the 3-point coupling for gravity is cohomologically unique [133], it must be the full 3-point coupling of $D = 11$ SG in Minkowski space. A concrete check on component field couplings would nevertheless be encouraging. In [69, 70], it has been verified that the Chern–Simons term is correctly reproduced, and that the ghost couplings corresponding to the diffeomorphism algebra are the right ones.

Surprisingly, the 3-point interactions provide almost the full answer. When checking the master equation to higher order in Ψ , a very simple 4-point coupling arises, containing a simple nilpotent operator T . The properties of this operator ensures that the master equation is satisfied to all orders, and the full action for $D = 11$ SG is

$$S = \int [dZ] \left[\frac{1}{2} \Psi Q \Psi + \frac{1}{6} (\lambda\gamma_{ab}\lambda) \left(1 - \frac{3}{2} T \Psi \right) \Psi R^a \Psi R^b \Psi \right]. \quad (4.47)$$

We refer to [70] for the details.

Strikingly enough, the full action for $D = 11$ supergravity becomes polynomial. The 4-point coupling may even be removed by a field redefinition (at the price of having a redefined field which is not canonical with respect to the antibracket, and has a less standard kinetic term). However, it should be said that geometry is somewhat obscured. By basing the formulation on the lowest-dimensional part of the superspace fields, and treating the fields as deformation of the flat background, geometry is not manifest. Still, the appearance of all ghosts, including the ones for diffeomorphisms and local supersymmetry, in the cohomology, together with the master equation, ensures full gauge invariance, although in a form that is not easily recognisable as geometric. Therefore it may be interesting to try to “rebuild” a geometric picture based on the present formalism. We do not have any concrete ideas about how this may be done, but it might involve further variables, reintroducing the superfields that were discarded (the higher-dimensional parts of the super-vielbein). Formally, an analogue statement is true for the SYM action, but the simple Chern–Simons form there makes gauge invariance (almost) manifest. In close connection with this, it is not clear how to best find solutions to the equations of motion. It is not known even how to embed simple, purely gravitational, solutions like the Schwarzschild geometry into the superfield Ψ . For perturbation theory around flat space, on the other hand, the formulation is ideal, both for keeping control over the symmetries and for having a very limited number of couplings, and it has been used for amplitude calculations [126, 134].

4.5.2.3 Other Models

Actions, along the lines drawn up here, can also be constructed for the BLG and ABJM models described briefly in Sect. 4.3.3. Since the fields describing the scalar multiplets are non-scalar, their kinetic terms contain extra λ 's ensuring shift symmetry. The interactions consist essentially of a minimal coupling to the Chern–Simons field, replacing and reproducing the higher order interactions among the component fields (e.g. a sixth order potential in the scalars). We again refer to [71–73] for details.

In principle, actions could be formed also for models with less supersymmetry. Then we know from the discussion in Sect. 4.3.4 that separate pure spinor superfields must be introduced for the fields and the antifields. The full formalism for lower supersymmetry has not been developed. In [119] minimal $D = 6$ SYM was treated, but only at the level of equations of motion, and in a minimal pure spinor formalism. Especially issues concerning gauge fixing may turn out to be easier in such models (see Sect. 4.7). In particular, $D = 10$, $N = 1$ supergravity and its dimensional reductions may be interesting, e.g. concerning the investigation of possible counterterms.

4.6 Higher Derivative Terms and Born–Infeld Theory

As an example of an application of our formalism, we will briefly describe the construction of a higher-derivative term. Even though the example is specific—the F^4 deformation of $D = 10$, $N = 1$ SYM, it may be applied to any supersymmetric deformation of a maximally supersymmetric model with a pure spinor action. As we will see, the drastic simplifications of interaction terms persist also here, and although an F^4 deformation in component language will come together with an infinite number of terms of arbitrarily high order in derivatives, a single quartic term turns out to contain the full deformation in the pure spinor superfield language for the abelian model. We conjecture that it describes Born–Infeld theory.

The question addressed here was actually one starting point for the development of the present formalism [55–59]. The work described in this section is based on [109].

Precisely as for any interaction term, the guide to consistent deformation is the master equation. What is needed is some Ansatz for the form of the interactions. In [55, 58], it was observed that the 5-form part of $F_{\alpha\beta} = 0$ must be changed in order to deform the theory. It was also noted that the appropriate $\alpha'^2 F^4$ terms for SYM were generated by

$$F_{\alpha\beta}^A \sim \alpha'^2 t^A{}_{BCD} (\gamma^a \chi^B)_\alpha (\gamma^b \chi^C)_\beta F_{ab}^D, \quad (4.48)$$

where t is a symmetric invariant tensor, and χ and F denote the superfields with the corresponding component fields as lowest components. We will from now on drop

the explicit factor α'^2 . This was then used in [58] in order to derive for the first time the complete deformation at this order, including all fermion couplings.

We need some systematics for lifting expressions like (4.48) to full pure spinor superfield expressions, containing not only fields of definite ghost number. The method introduced in [109] was to form “physical operators”, solving this problem. Take for example the physical fermion. We would like to find an operator $\hat{\chi}^\alpha$ that, roughly speaking, strips the pure spinor superfield Ψ of one power of λ and two powers of θ and forms a pure spinor superfield that “starts” with χ^α , and similarly for other component fields. These operators were systematically constructed in the non-minimal formalism. For example, the operator $\hat{\chi}^\alpha$ takes the form

$$\hat{\chi}^\alpha = \frac{1}{2}(\lambda\bar{\lambda})^{-1}(\gamma^a\bar{\lambda})^\alpha\partial_a + \dots, \quad (4.49)$$

with the ellipsis denoting terms with more singular behaviour in $(\lambda\bar{\lambda})$ and with one or two powers of $d\bar{\lambda}$. The physical operators turn out to satisfy a number of interesting algebraic and differential relations (among them, a somewhat surprising relation to the b operator of Sect. 4.7).

We found that a quartic term in the action

$$S_4 = \frac{1}{4} \int [dZ] \Psi(\lambda\gamma^a\hat{\chi})\Psi(\lambda\gamma^b\hat{\chi})\Psi\hat{F}_{ab}\Psi \quad (4.50)$$

solves the master equation in the Maxwell case, not only to this order but to all orders, and conjectured that it describe supersymmetric Born–Infeld theory. In the non-abelian case, the same term, dressed up with a four-index tensor, describes the full totally symmetric part of the interaction to all orders. We found various ways of rewriting this 4-point coupling in more symmetric ways, and refer to [109] for the details.

The generalisation to supergravity has not been performed, but should not present any other difficulties than purely technical, and may be useful in the search for supersymmetric counterterms. Note that, while in a component language one must make separate Ansätze for the deformed action and the deformed supersymmetry, here everything is uniformly encoded in the master equation.

4.7 Gauge Fixing

We will finally briefly mention gauge fixing, which is an important issue when it comes to quantum calculations and path integrals.

There is a well developed theory of gauge fixing in the BV framework. One must of course eliminate the antifields as independent propagating degrees of freedom, and this is achieved by the introduction of a gauge fermion χ . One then demands that

$$\phi_I^* = \frac{\delta\chi}{\delta\phi^I} . \quad (4.51)$$

This makes physical quantities independent of gauge choice. Normally, in a gauge theory, this procedure involves extra non-minimal fields, the “antighost” and Nakanishi-Lautrup fields.

In the pure spinor superfield framework (for maximally supersymmetric models), we have fields Ψ which effectively contain both fields and antifields and are self-conjugate under the antibracket. We can not form a condition like (4.51) without a contrived and unnatural splitting of the field Ψ . Therefore it is necessary to fix the gauge in some other way.

A standard way to fix gauge in string theory is Siegel gauge [135]. The gauge fixing condition is

$$b\Psi = 0 , \quad (4.52)$$

where b is a ghost field corresponding to the Virasoro constraint. However, in the pure spinor formalism, no world-sheet or world-line reparametrisation is *a priori* present—as we have seen the equations of motion of the massless fields is an “indirect” consequence of cohomology, and do not follow from “ $p^2 = 0$ ” of some particle model with reparametrisation symmetry. Such a b operator has to be constructed as a composite operator if it exists. This was done for string theory in [74]. The field theory version of this b operator, relevant for SYM, is

$$\begin{aligned} b = & -\frac{1}{2}(\lambda\bar{\lambda})^{-1}(\bar{\lambda}\gamma^a D)\partial_a + \frac{1}{16}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma^{abc}d\bar{\lambda})\left(N_{ab}\partial_c + \frac{1}{24}(D\gamma_{abc}D)\right) \\ & -\frac{1}{64}(\lambda\bar{\lambda})^{-3}(d\bar{\lambda}\gamma^{abc}d\bar{\lambda})(\bar{\lambda}\gamma_a D)N_{bc} \\ & -\frac{1}{1024}(\lambda\bar{\lambda})^{-4}(\bar{\lambda}\gamma^{abe}d\bar{\lambda})(d\bar{\lambda}\gamma^{cd}{}_e d\bar{\lambda})N_{ab}N_{cd}, \end{aligned} \quad (4.53)$$

where $N_{ab} = (\lambda\gamma_{ab}\frac{\partial}{\partial\lambda})$. The defining property of the b operator is

$$\{Q, b\} = \square . \quad (4.54)$$

The whole purpose of gauge fixing is of course to make the kinetic operator (in this case Q) invertible. With this gauge choice, the propagator G (“ Q^{-1} ”) is formally

$$G = \frac{b}{\square} . \quad (4.55)$$

So, even if b is a complicated operator, it does precisely what is needed for gauge fixing: it eliminates almost all the antifields and implies Lorenz gauge for the gauge connection. By “almost all” we mean that there is a small remainder of the antifield $A^{*\alpha}$, connected to its on-shell divergencelessness, that gives place for the antighost,

which otherwise is normally introduced by hand. That this happens follows from the deliberations in [87].

The consistency of the gauge fixing also relies on the property $b^2 = 0$. This identity is quite cumbersome to show—in string theory so much so that the full calculation was performed only recently [136, 137].

In $D = 11$ the b operator is quite complicated,

$$b = \frac{1}{2}\eta^{-1}(\bar{\lambda}\gamma_{ab}\bar{\lambda})(\lambda\gamma^{ab}\gamma^c D)\partial_c + \dots \quad (4.56)$$

We will not display it in full detail here, and refer to [126].

The fact that the b operators, and also other operators carrying negative ghost number such as the R^a operator of the supergravity and the physical operators of Sect. 4.6, have quite complicated expression has been the source of some activity searching for simpler versions. See e.g. [78, 81, 126, 138].

Once gauge fixing has thus been performed, it is possible to use the pure spinor superfield formalism for calculation of amplitudes. There will be further (resolvable) questions about regularisation that we will completely forgo here, see [80, 126, 139, 140]. In [126], amplitudes derived from the supergravity action were shown to be finite up to six loops, in agreement with [139, 140] (see the talk presented by Anna Karlsson [134]).

It might be expected that gauge fixing in models with less than maximal supersymmetry can be performed in a way which is more along the standard lines of the BV formalism, i.e., with a gauge fixing fermion, since then fields and antifields are naturally separated in different pure spinor superfields. This remains to be investigated.

4.8 Discussion

We have given a brief overview of the pure spinor superfield formalism, and how it leads to off-shell superfield actions for maximally supersymmetric models. The main focus has been on $D = 10$ SYM and $D = 11$ SG, but also other models have been mentioned. Some of the more technically intricate parts of the formalism have been left out, but we hope that the general message is clear: this is a solution to the problem of going off-shell with maximal supersymmetry.

We have repeatedly pointed out the simplicity of the resulting actions. Indeed, the many terms in a supersymmetric component action generically reduce to some quite simple expression, which is of lower order in fields than the component interactions. In a couple of cases, we even get polynomial expressions where the component ones are non-polynomial. This is of course an advantage when it comes to quantum calculations: the number of vertices is very limited. The other advantage for amplitude calculations is that the presence of an action (as opposed to a first-quantised formalism) directly yields the form of the vertices consistent with all symmetries.

The formulation of supergravity has some drawbacks, though. Since only part of the supervielbein is used, the geometric structure of the theory is obscured. Background invariance is not manifest, since some background is needed in order even to define the BRST operator. In this sense, the behaviour is similar to closed string field theory [135]. It is not clear whether geometry, or some aspects of it can be regained without losing the obvious advantages of the pure spinor formalism. This means also that solutions beyond the linearised level around some background are difficult to find, as is e.g. the dynamics of extended objects and their coupling to supergravity.

We believe that there is something to learn from the application of pure spinor techniques to theories with less supersymmetry. This is however a largely unexplored subject.

Finally, we would be very interested in extending the formalism to other structure groups. The type of models we primarily have in mind are models with “manifest U-duality”, formulated as gauge theories within the framework of generalised geometry. Some supermultiplets are already known in connection with U-duality [141–144], and it would be very interesting to continue to a superfield formalism and maybe a (generalisation of the) pure spinor version. A manifest control over both supersymmetry and U-duality would be the ideal situation for examining the ultraviolet properties of maximal supergravity.

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