Chapter 3
Multiplex Networks

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Abstract Typical complex system operates through multiple types of interactions between its constituents. The collective function of these multiple interactions, or multiple network layers, is often non-additive, resulting in nontrivial effects on the network structure and dynamics. To better model such situations, the concept of multiplex network, the network with explicit multiple types of links, has recently been applied. In this contribution, we survey recent studies on this subject, focused on the notion of correlated multiplexity. Empirical multiplex network analysis as well as analytical results on the random graph models of correlated multiplex networks are presented, followed by a brief summary of dynamical processes on multiplex networks. It is illustrated that a multiplex complex system can indeed exhibit structural and dynamical properties that cannot be represented by its individual layer's properties alone, establishing the network multiplexity as an essential ingredient in the new physics of "network of networks."

3.1 Introduction

In the last decade, network science has successfully established itself as a unified framework for studying complex systems [1, 2]. Along with its impressive success, the framework has continuously been evolving. One of the most current evolution of complex network theory is the study of multiplex networks, the networks with more than one type of links [3]. Indeed, most studies until quite recently have focused on isolated, single networks, ignoring the existence of multiple types of interactions. In most, if not all, real-world complex systems, however, nodes in the system can engage in more than one type of interactions, and such multiple interactions can make a non-additive effect on network structure and the dynamics on it. For example, as illustrated

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Fig. 3.1 A cartoon of multi-
plex social network as a triplex
network consisting of friend-
ship, family, and work-related
acquaintanceship layers
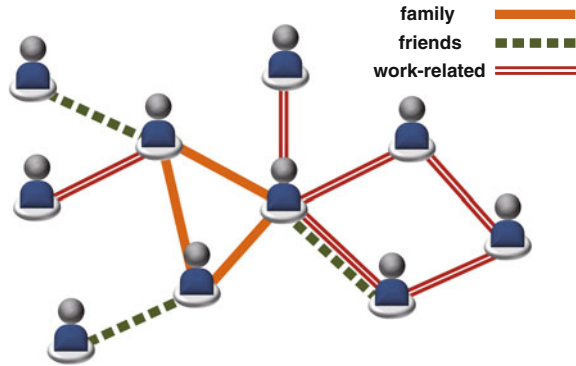
in Fig. 3.1, people in a society interact via their friendship, family relationship, and/or
more formal work-related acquaintanceship, *etc.*, which are collectively responsible
for complex emergent social phenomena [4, 5]. Countries in the global economic
system also interact via various international relations ranging from commodity trade
to political alliance [6]. Even proteins in a cell participate in multiple layers of
interactions and regulations, from transcriptional regulations and metabolic synthesis
to signaling [7]. Obviously, in dealing with such problems the multiplex network
representation would be a more appropriate description than the single network, or
simplex, one.

In this contribution, we will survey recent works on the topic of multiplex net-
works. We begin with an analysis of real-world multiplex coauthorship network data
to introduce the notion of correlated multiplexity in Sect. 3.2. Then the random graph
model of correlated multiplex network is introduced in Sect. 3.3. In Sect. 3.4, ana-
lytical formalism based on the joint degree distribution for analyzing the structural
properties of multiplex random graph models is developed. The cases of duplex ran-
dom graphs and duplex scale-free networks are studied in detail in Sects. 3.5 and 3.6,
respectively. Topics of network robustness and network dynamics are briefly dis-
cussed in Sects. 3.7 and 3.8, respectively. Finally, we will conclude our contribution
with a summary and outlook.

3.2 Correlated Multiplexity

In most previous studies of coupled networks—in context of layered, interacting,
interdependent networks [8–10]—network layers were coupled randomly. In real-
world complex systems, however, nonrandom structure in network multiplexity can
be prominent. For example, a person with many links in the friendship layer is likely
to also have many links in another social network layer, being a friendly person. We
termed the correlated multiplexity [3] to refer such a nonrandom pattern of network

multiplexity. Examples of correlated multiplexity are widespread. Some of examples reported in the literature are:

- Social networks: online-game network [11], coauthorship network [12].
- Organizational networks [13].
- Transportation networks [8, 14, 15].
- Cellular network: Interaction network and perturbation network [16].
- Economic networks: Trade networks in different industrial sectors [17].

The most frequent pattern of correlated multiplexity is the positively correlated multiplexity, such that a node with large degree in one layer likely has more links in the other layer as well. For example, in the online game social network data [11], it was shown that different positive social relations such as friendship and trade are highly correlated as well as overlap.

In Fig. 3.2, we present our own analysis of a multiplex coauthorship network [12]. The network consists of a set of researchers who are connected with one another by three types of collaboration links, first being due to publications in the field of fractal surface growth (denoted KPZ, representing Kardar-Parisi-Zhang equation), second in the field of self-organized criticality (denoted SOC, representing Self-Organized Criticality), and third in the field of complex network theory (denoted CNR, representing Complex Network Research), resulting in a triplex network (for more details on the data collection, see [12]). Despite the separation of timescales of three research topics, degree distributions of the three network layers, and that of the superposed network, are indistinguishable (Fig. 3.2a, inset). Within the individual layer, analysis of degree distributions of restricted set of nodes that participate in more than one layers reveals that there indeed exists a positively correlated multiplexity pattern: the more layer a node participates to, the more likely would they have larger degrees (Fig. 3.2a). The analysis of joint degree distributions (Fig. 3.2c,d) confirms this finding. There is a systematic enrichment of joint degree distribution near the diagonal of the plots, revealing strong correlation between degrees of a node in two network layers. In addition, it was found that a pair of nodes which are closer in one layer tend to be also closer in another layer (Fig. 3.2b). This result extends the classical concept of multiplexity that accounts only for direct link overlap [4] and demonstrates the effect of network multiplexity at all scales.

3.3 Random Graph Model of Correlated Multiplexity

For a systematic mathematical understanding of correlated multiplexity, one needs a graph model. There exist a few random graph models with multiple link-types (or colored edges) [3, 18, 19]. Here we present a way to build correlated multiplex networks, following [3].

Given two network layers with equal number of nodes, we define three particular couplings: (i) uncorrelated, (ii) maximally-positive (MP), and (iii) maximally-negative (MN) correlated couplings (Fig. 3.3). In the uncorrelated coupling, we
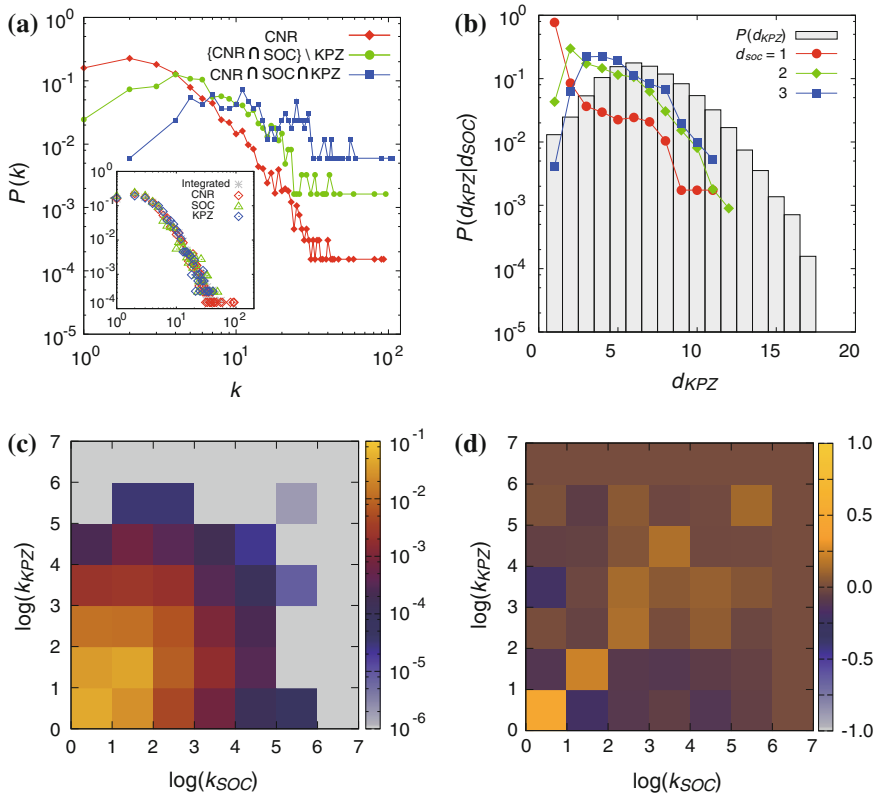
Fig. 3.2 Patterns of correlated multiplexity in multiplex coauthorship network. **a** Degree distribution of nodes participating in a single (*diamond*), double (*circle*), and triple layers (*square*). **b** Conditional distance distribution $P(d_{KPZ}|d_{SOC})$ in KPZ-layer of pairs of nodes of distance d_{SOC} in SOC-layer. **c** Joint degree distribution $P(k_{SOC}, k_{KPZ})$, and **d** Significance plot based on Z-score with respect to randomly coupled counterpart. Z-score is obtained as $Z = (P_{real} - \langle P_{random} \rangle)/\sigma_{P_{random}}$, where the average and standard deviation for P_{random} are evaluated over 10^4 independent randomizations

couple the two layers randomly, that is, we use a random matching between a node in one layer to a node in the other layer. In the MP correlated coupling, a node's degrees in different layers are maximally correlated in their degree order; the node that is hub in one layer is also the hub in the other layer, and the node that has the smallest degree in one layer also has the smallest degree in other layer. Likewise, in the MN correlated coupling, a node's degrees in different layers are maximally anti-correlated in their degree order.

These three particular couplings are useful in their mathematical simplicity and tractability, thus highlighting the effect of correlated multiplexity. Yet in real-world multiplex systems the correlated multiplexity would hardly be maximal. The cases of partially correlated multiplexity can be constructed by maximally correlating a
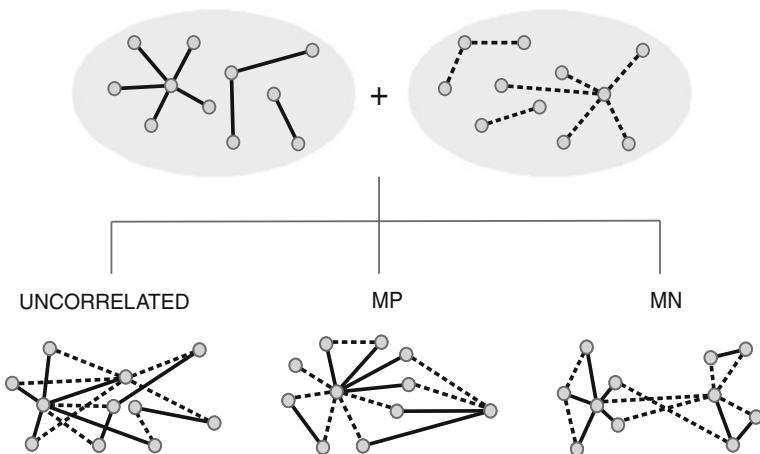
Fig. 3.3 Schematic illustration of constructing the correlated multiplex networks discussed in the text. MP (MN) stands for maximally-positive (maximally-negative) correlated multiplexity

fraction q of nodes in the network while randomly coupling the rest fraction $1 - q$. Using this method one can interpolate between MP, through uncorrelated, and MN couplings, modulating the strength of correlated multiplexity.

3.4 Analytical Formalisms

3.4.1 Degree Distributions

The information of degree distribution of a multiplex network with ℓ layers (ℓ-plex network) can be encoded in the joint degree distribution $P(\{k_\alpha\}) \equiv P(k_1, k_2, \cdots, k_\ell)$. (Throughout this work, we will use Greek subscript to denote the layer index). The degree distribution within a layer α, denoted as $\pi_\alpha(k_\alpha)$, can be obtained as the marginal distribution, $\pi_\alpha(k_\alpha) = \sum_{\{k_{\beta \neq \alpha}\}} P(k_1, k_2, \cdots, k_\ell)$. The total degree of a node in the multiplex network is given by $k = \sum_\alpha k_\alpha$, which can differ from the number of distinct connected nodes when there are link overlaps between network layers. Such link overlaps can be neglected for large, sparse random graphs, but can be significant in real-world multiplex networks as in multiplex social network data [11, 12]. One can obtain the total degree distribution $P(k)$ from the joint degree distribution as $P(k) = \sum_{\{k_\mu\}} P(\{k_\mu\}) \delta_{k, \sum_\nu k_\nu}$, where δ denotes Kronecker delta symbol.

3.4.2 Emergence of the Giant Component

Having established a way to construct the total degree distribution $P(k)$, it is tempting to use it to calculate the connected components properties via standard generating function technique [20]. It turns out that, however, this simplified procedure works only when the degree distributions of all layers are identical, as we will see shortly.

Now we develop a theory which exploits the full joint degree distribution $P(\{k_\alpha\})$, applicable when every layer is uncorrelated and locally tree-like, as in random graph models. Let us define u_α to be probability that a node reached by a randomly chosen link in layer α does not belong to the giant component (which is connected via *any* types of links). Following a similar reasoning as the standard generating function technique, one can construct the self-consistency equations for u_α's as

$$u_\alpha = \sum_{\{k_\mu\}} \frac{k_\alpha P(\{k_\mu\}) \prod_\nu u_\nu^{k_\nu}}{z_\alpha} \frac{1}{u_\alpha} \qquad (\alpha = 1, \cdots, \ell) , \tag{3.1}$$

where z_α is the mean degree of layer α. Then the probability that a randomly chosen node belongs to the giant component (that is, the giant component size), denoted S, can be obtained as

$$S = 1 - \sum_{\{k_\mu\}} P(\{k_\mu\}) \prod_\nu u_\nu^{k_\nu} , \tag{3.2}$$

with u_ν's being the solution of Eq. (3.1). Therefore, the giant component exists (that is, $S > 0$) if Eq. (3.1) has a nontrivial solution other than $(u_1, \cdots, u_\ell) = (1, \cdots, 1)$. This condition can be extracted from the Jacobian of Eq. (3.1), which reads in the case of duplex network

$$\frac{1}{4} \left[\left(\frac{\kappa_1}{z_1} + \frac{\kappa_2}{z_2} \right) + \sqrt{ \left(\frac{\kappa_1}{z_1} - \frac{\kappa_2}{z_2} \right)^2 + \frac{4\kappa_{12}^2}{z_1 z_2} } \right] > 1 , \tag{3.3}$$

where $\kappa_1 = \langle k_1^2 \rangle$, $\kappa_2 = \langle k_2^2 \rangle$, and $\kappa_{12} = \langle k_1 k_2 \rangle$ are second-order moments of joint degree distribution.

When the degree distributions of all layers are identical, one has the solution of Eq. (3.1) satisfying $u_1 = u_2 = \cdots = u_\ell$, which reduces Eqs. (3.1–3.3) to those of standard generating function technique [20]. For example, Eq. (3.3) reduces to the well-known Molloy-Reed criterion for the total degree distribution, $\langle k^2 \rangle - 2\langle k \rangle > 0$, with $k = k_1 + k_2$ [21]. This shows that in such a case, one can use the reduced total degree distribution $P(k)$ to study the component structure, but in general Eqs. (3.1–3.3) should be used to have the correct results. Note that similar generating function-type techniques for clustered [22], multi-type [23], and interdependent networks [24] have also been developed recently, which slightly differ from the current formalism.

3.4.3 Degree-Degree Correlations

The fact that one cannot use the reduced total degree distribution $P(k)$ for component structure of correlated multiplex network suggests that the superposed network possesses degree correlations even when uncorrelated random networks are coupled. To show this explicitly, let us consider the assortativity coefficient r defined as [25]

$$r = \frac{\langle kk' \rangle_l - \langle k \rangle_l^2}{\langle k^2 \rangle_l - \langle k \rangle_l^2}, \tag{3.4}$$

where k and k' are the *total* degrees of nodes at two ends of an edge and $\langle \cdots \rangle_l$ denotes the average over all edges in the superposed network. Nonzero value of r dictates the presence of degree-degree correlations between connected nodes. Following the steps developed in [22], one can show that the numerator of Eq. (3.4) can be expressed, after some manipulations, as

$$\sum_{k,k'} kk'Q(k,k') - \left(\sum_{k,k'} kQ(k,k') \right)^2 = \sum_{\mu} c_{\mu} X_{\mu}^2 - \left(\sum_{\mu} c_{\mu} X_{\mu} \right)^2$$

$$= \frac{1}{2} \sum_{\mu,\nu} c_{\mu} c_{\nu} (X_{\mu} - X_{\nu})^2 \geq 0, \tag{3.5}$$

where $Q(k,k')$ denotes the probability that a randomly chosen link (of any kind) connects two nodes with total degree k and k' at each end, c_{α} is the fraction of links of type α, such that $\sum_{\alpha} c_{\alpha} = 1$, and X_{α} is the expected total degree of a node that is reached by following a randomly chosen link of type α, which is related to the joint degree distribution as

$$X_{\alpha} = \sum_{k} k \sum_{\{k_{\mu}\}} k_{\alpha} P(\{k_{\mu}\}) \delta(k - \sum_{\nu} k_{\nu}) / z_{\alpha} . \tag{3.6}$$

Therefore, a multiplex network can become assortative ($r > 0$), even when uncorrelated layers are coupled, unless the degree distributions of all layers are identical, so that all X_{α}'s are equal. (Another exception is the uncorrelated multiplex ER graphs, see Sect. 3.5.1.) It also allows one to calculate the assortativity coefficient r, once the joint degree distribution is given.

3.5 Duplex ER Graphs

To illustrate basic effects of multiplex couplings, in this section we apply the formalism to duplex Erdős-Rényi (ER) graphs [26] in which two ER graph layers are multiplex coupled, summarizing the results reported in [3].

3.5.1 Uncorrelated Duplex ER Graphs

In the absence of correlation between network layers, the joint degree distribution factorizes, $P_{uncorr}(k_1, k_2) = \pi_1(k_1)\pi_2(k_2)$. The total degree distribution is then given by the convolution of $\pi_\alpha(k_\alpha)$, $P_{uncorr}(k) = \sum_{k_1=0}^{k} \pi_1(k_1)\pi_2(k - k_1)$. It is easy to see that the resulting superposed network is nothing but an ER graph with the total mean degree $z_1 + z_2$, so that

$$P_{uncorr}(k) = \frac{e^{-z}z^k}{k!} \tag{3.7}$$

with $z = z_1 + z_2$. The connectivity and component properties follow the conventional behaviors [20, 26].

3.5.2 Duplex ER Networks with Equal Link Densities

The case of duplex ER networks with layers of equal link densities is particularly simple, as one can use standard generating function technique with the total degree distribution. Furthermore it is amenable for a number of explicit exact results.

MP coupling.—In this case, degrees of a node in the two layers would become almost equal in the thermodynamic limit (more precisely, relative dispersion of the two degrees would decay with N and vanish as $N \to \infty$), so that the total degree distribution of the duplex network can be approximated as

$$P_{MP}(k) = \begin{cases} e^{-z_1}z_1^{k/2}/(k/2)! & (k \text{ even}), \\ 0 & (k \text{ odd}), \end{cases} \tag{3.8}$$

where z_1 is the mean degree of the layer 1. Therefore, the Molloy-Reed criterion is fulfilled for all nonzero z_1, as $\langle k^2 \rangle - 2\langle k \rangle = 4(z_1 + z_1^2) - 2(2z_1) = 4z_1^2 > 0$ for $z_1 \neq 0$, which can also follow from the condition Eq. (3.3). This means that surprisingly the giant component exists for any nonzero link density, that is, the critical single-layer mean degree z_c above which the giant component exists vanishes,

$$z_c^{MP} = 0 \,. \tag{3.9}$$

One can further obtain the giant component size S and the average size of finite components $\langle s \rangle$ from the standard generating function technique [20], which are given explicitly by:

$$S = 1 - P(0) = 1 - e^{-z_1} \,, \tag{3.10}$$

and

$$\langle s \rangle = 1 \,. \tag{3.11}$$

This shows that the giant component grows linearly in the vicinity of z_c^{MP}, and that only the isolated nodes are outside the giant component and all the linked nodes form a single giant component. All these predictions are fully supported by numerical simulations (Fig. 3.4).

MN coupling.—In this case, there exist distinct regimes of z_1, three of which among them are of relevance for the giant component properties (in $N \to \infty$ limit).

(i) $0 \leq z_1 \leq \ln 2$.

In this regime, more than half of nodes are of degree zero in each layer so every linked node in one layer is coupled with a degree-0 node in the other layer under MN coupling. After some inspection one obtains the total degree distribution $P(k)$ as

$$P_{MN}(k) = \begin{cases} 2\pi(0) - 1 & (k = 0), \\ 2\pi(k) & (k \geq 1). \end{cases} \tag{3.12}$$

In this regime there is no giant component.

(ii) $\ln 2 \leq z_1 \leq z^*$.

Following similar steps, $P(k)$ in this regime is obtained as

$$P_{MN}(k) = \begin{cases} 0 & (k = 0), \\ 2[2\pi(0) + \pi(1) - 1] & (k = 1), \\ 2\pi(2) - 2\pi(0) + 1 & (k = 2), \\ 2\pi(k) & (k \geq 3). \end{cases} \tag{3.13}$$

In this regime, $\langle k^2 \rangle - 2\langle k \rangle = 2(z_1^2 - z_1 - 2e^{-z_1} + 1)$, which becomes positive for $z_1 > z_c^{MN}$ where

$$z_c^{MN} = 0.838587497... \tag{3.14}$$

Therefore the giant component emerges at a much higher link density. Being delayed in its birth, however, the giant component grows more abruptly once formed (Fig. 3.4c). This regime is terminated at $z_1 = z^*$, determined by the condition $2\pi(0) + \pi(1) = 1$, from which we have $z^* = 1.14619322...$

(iii) $z_1 \geq z^*$.
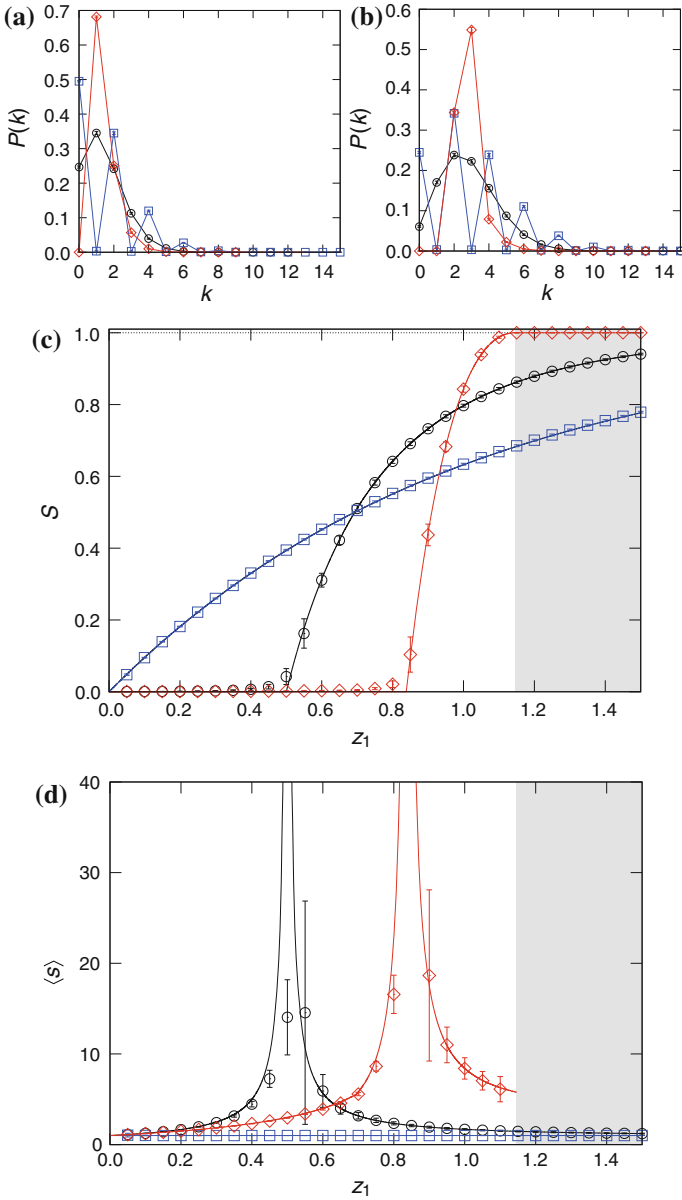
Fig. 3.4 a, **b** Total degree distribution $P(k)$ of duplex ER graphs with $z_1 = z_2 = 0.7$ (**a**) and $z_1 = z_2 = 1.4$ (**b**). Different symbols denotes MP (*square*), uncorrelated (*circle*), and MN (*diamond*) couplings. **c**, **d** The giant component size S (**c**) and the average size of finite components $\langle s \rangle$ (**d**) as a function of z_1 of duplex ER graphs with $z_1 = z_2$. Same symbols as (a, b) are used. Gray shade denotes the region in which $S = 1$ for the MN case ($z_1 > z^*$). Lines represent the theoretical curves and symbols the numerical simulation results. Errorbars denote standard deviations. Adapted from [3]
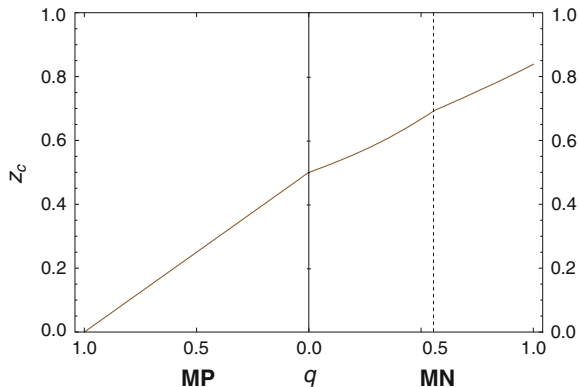
Fig. 3.5 Plot of Eq. (3.15a) for the critical mean degree z_c as a function of q, the fraction of correlated multipex nodes. The cases $q = 1$ denote maximally correlated multiplexity, $0 < q < 1$ partially correlated multiplexity, and $q = 0$ uncorrelated multiplextiy. The vertical dotted line is drawn at $q = 2 - 1/\ln 2$ across which z_c takes different formulae in Eq. (3.15b). Adapted from [3]

In this regime we have $P(0) = P(1) = 0$ and thereby $S = 1$. This means that the entire network becomes connected into a single component at this finite link density z^*, which can never be achieved for ordinary ER networks.

All these theoretical results are confirmed numerically (Fig. 3.4). Meanwhile, it is noteworthy that despite these abnormal behaviors and apparently more rapid growth of S near z_c, the critical behavior in the MN case is found to be consistent with that of standard mean-field [3].

Imperfect correlated multiplexity.—So far we have seen that maximally correlated or anti-correlated multiplexity crucially affects the onset of emergence of giant component in multiplex ER networks. For a partially correlated duplex ER network (with equal link densities) in which a fraction q of nodes are maximally correlated coupled while the rest fraction $1 - q$ are randomly coupled, the total degree distribution can be obtained as $P_{partial}(k) = qP_{maximal}(k) + (1 - q)P_{uncorr}(k)$, where *maximal* is either *MP* or *MN*. Using Eqs. (3.8, 3.12, 3.13) and following similar steps as in the previous section we obtain the critical link density as a function of q as

$$z_c = (1 - q)/2 \tag{3.15a}$$

for positively correlated case and

$$z_c = \begin{cases} 1/(2 - q) & (q < 2 - 1/\ln 2), \\ z_1(q) & (q > 2 - 1/\ln 2) \end{cases} \tag{3.15b}$$

for negatively correlated case, where $z_1(q)$ is the solution of $(2 - q)z_1^2 - z_1 - 2qe^{-z_1} + q = 0$. This result shows that z_c depends continuously on q (Fig. 3.5), illustrating that the effect of correlated multiplexity is present for general q.
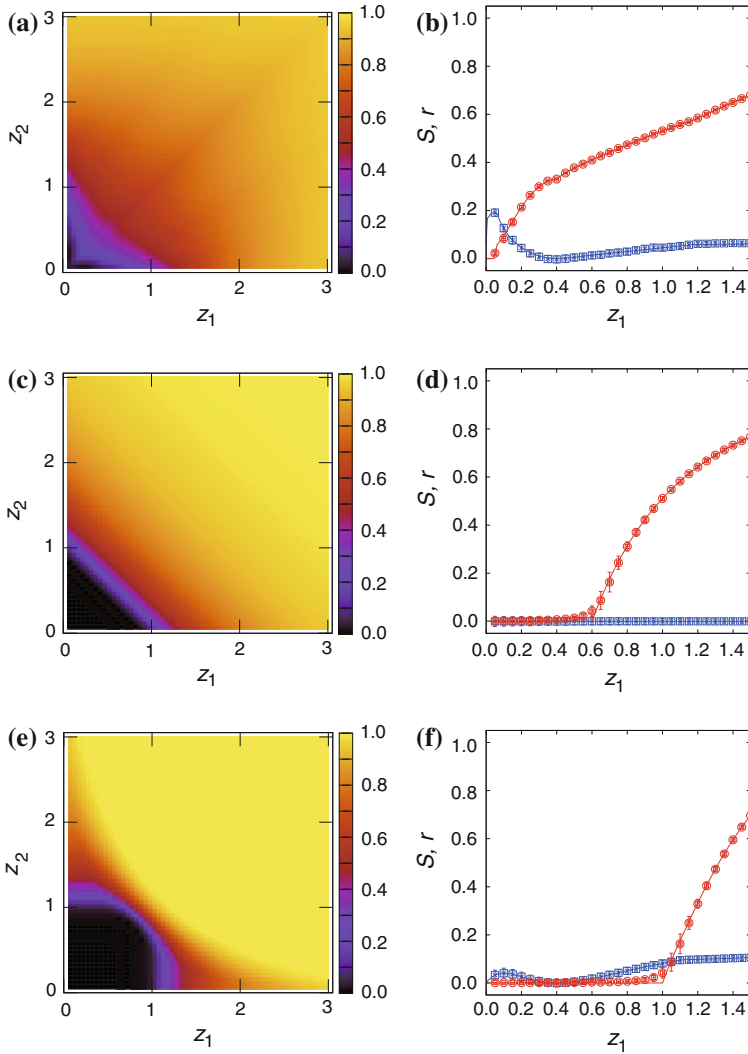
Fig. 3.6 **a**, **c**, **e** Numerical simulation results of the size of giant component of duplex ER networks of size $N = 10^4$ with **a** MP, **c** uncorrelated, and **e** MN couplings. **b**, **d**, **f** The giant component size S (*red*) is plotted for $z_2 = 0.4$, along with the assortativity coefficient r (*blue*) for the MP (**b**), uncorrelated (**d**), and MN (**f**) cases. Theoretical predictions based on the joint degree distribution in Sect. 3.4 are shown in lines, demonstrating excellent agreements with simulations. Errorbars denote standard deviations from 10^4 independent runs. Adapted partly from [3]

### 3.5.3 Duplex ER Networks with General Link Densities

In this section we consider general duplex ER networks with $z_1 \neq z_2$. Qualitative picture of behavior of giant component size is similar to the equal link density case: the giant component emerges at lower link densities for the MP case but grows more slowly than the uncorrelated case, whereas it emerges at higher link densities for the MN case but grows more abruptly and connects all the nodes in the network at finite link density (Fig. 3.6).

It should be emphasized, however, that one should use the formalism in Sect. 3.4, which fully exploits the joint degree distribution, in order to obtain correct theoretical results for $z_1 \neq z_2$ (Fig. 3.4b, d, f). Indeed, the assortativity coefficient r calculated both analytically by Eqs. (3.4–3.6) and numerically shows that it is assortative in MP and MN cases, except for $z_1 = z_2$. This clearly shows that the correlated multiplexity can not only modulate the total degree distribution $P(k)$ of the superposed network but also introduce higher-order correlations in its network structure.

3.6 Duplex SF Networks

Now we consider a duplex scale-free (SF) network, in which two SF networks constructed by the static model [27] are multiplex-coupled. The static model network is constructed as follows. Each node i ($i = 1, \cdots, N$) is assigned a weight $w_i = i^{-a}$, where a is a constant greater than 1. By successively connecting two nodes each chosen with probability proportional to its weight until desired number of links are made, one obtains a network with asymptotic power-law degree distribution $\pi(k) \sim k^{-\gamma}$, with γ (called the degree exponent) given by $\gamma = 1 + 1/a$ [27]. Thus one can tune both the degree exponent and the mean degree of the network.

An important property of SF networks is the vanishing percolation threshold for $\gamma \leq 3$ [28], fundamentally different from the case with $\gamma > 3$. The case of $\gamma = 2.5$ is examined first (Fig. 3.7a). In this case the giant component exists for any $z_1 > 0$ even in the single layer, so $z_c = 0$ in all three cases. For small z_1, MP has the largest giant component size as in the ER case. Peculiar behavior is observed for the MN coupling, in which the giant component size increases slowly until it makes a jump around $z_{jump} \approx 1.05$, almost doubling its size. This unusual behavior is rooted in the fact that with MN coupling each layer's hub supports giant component of its own and the two giant components are totally disjoint until the link density reaches the threshold z_{jump}. Beyond this threshold, the two equally-large giant components cannot but overlap and merge, thereby making a jump. This picture is supported by the observations that sizes of the largest and second largest component are almost equal, and the position of jump coincides with the point at which all nodes in the network acquire at least one link (Fig. 3.7a, inset). The case of $\gamma = 5.0$ is examined next (Fig. 3.7b), yielding overall similar qualitative behaviors as the duplex ER networks, without any discontinuous jump.
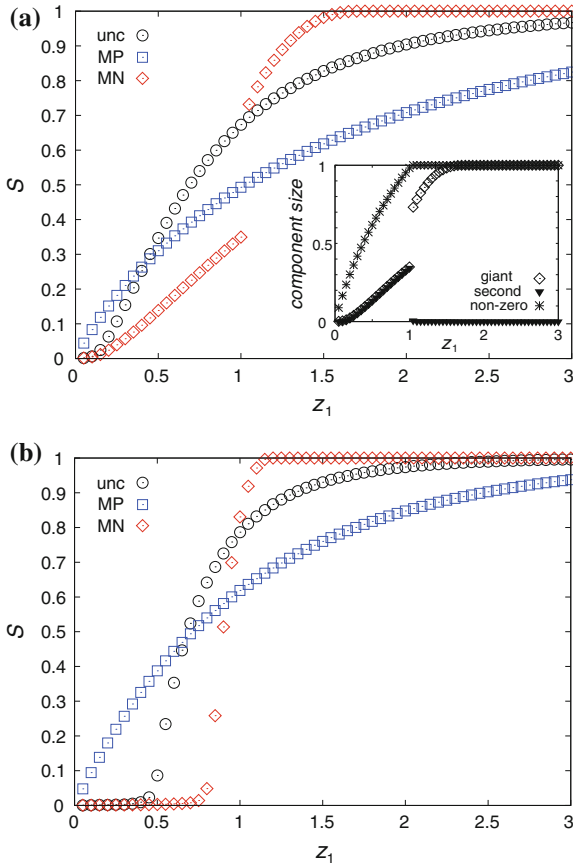
Fig. 3.7 Giant component size of duplex SF networks of equal link densities with $\gamma = 2.5$ (**a**) and $\gamma = 5.0$ (**b**). Symbols stand for uncorrelated (○), MP (□), and MN (◇) couplings. (Inset) Size of largest (◇) and second largest (▽) components, together with the fraction of nonzero-degree nodes (∗), for MN coupling

3.6.1 Betweenness and Load

Betweenness centrality [29] or load [27] is a widely-used centrality measure which characterizes the potential burden or traffic over a node in a network due to simple shortest path-based transport protocols. It has been shown that the load distribution of SF network also follows a power law, with the exponent ≈ 2.2 for non-tree SF networks with $2 < \gamma \le 3$ [27]. Here we examine how the betweenness and its distribution are affected by the multiplex coupling of SF networks. From the scaling perspective, neither the degree exponent nor the power-law exponent for betweenness distribution is found to be affected by the multiplex coupling (Fig. 3.8a, b). Looking at the individual node level, it is found that the betweenness changes most when the
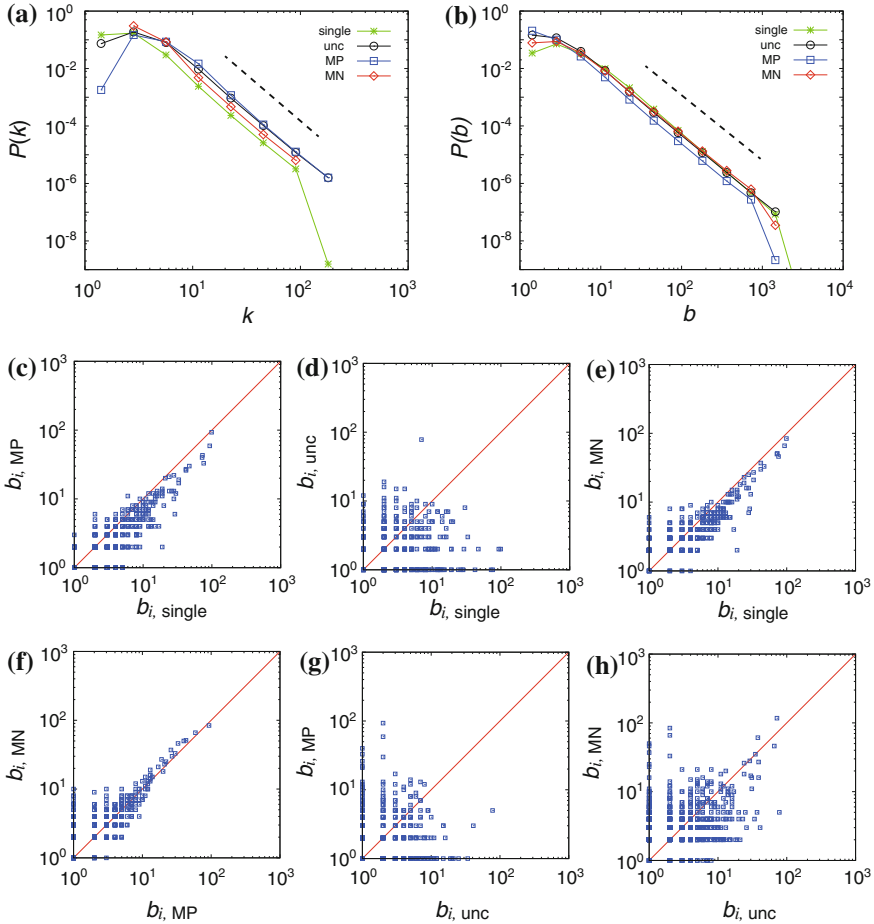
Fig. 3.8 **a** Total degree distribution and **b** betweenness distributions of duplex SF networks. Dashed line has the slope -3.0 (**a**) and -2.2 (**b**), drawn as a guide to the eye. **c–h** Scatterplots of betweenness centralities of a node in the two layers for different multiplex couplings. Diagonal lines are drawn as a guide to the eye

two networks are coupled randomly, rather than in a MP or MN way (Fig. 3.8c–h). This suggests that in MP or MN coupling the pathway structure is weakly affected and topological centralities of hub nodes are largely preserved. Concepts of betweenness and load are intimately related with the definition of shortest path. One interesting issue in this regard is the concept of optimal path in multiplex networks with the context and interplay between layers fully taken into account, which deserves further study.
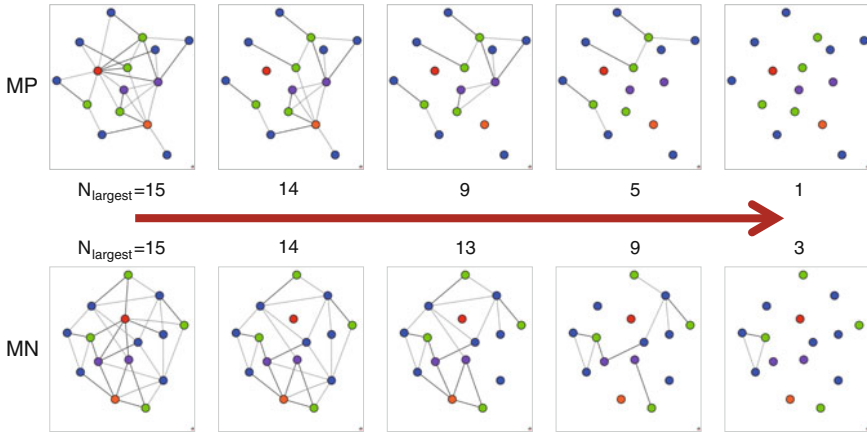
Fig. 3.9 Schematic diagram of intentional attack on MP and MN type multiplex networks. From left to right, nodes are removed in descending order of the total degree to simulate an intentional attack, and the size of largest connected component in the remaining superposed network is monitored

3.7 Robustness of Multiplex Networks

Having established that correlated multiplexity can significantly affect the overall connectivity of multiplex networks, the next question we might have is its impact on network robustness against random failures or intentional attack [2]. For example, as the cartoon diagram in Fig. 3.9 shows, the way how the network layers are multiplex-coupled can alter the resilience of the superposed network against attack. It has also been shown that robustness of interdependent networks to cascade of failures can be affected by the correlated coupling [14, 30].

As a preliminary case study, here we use the multiplex coauthorship network introduced in Sect. 3.2 and examine the topological robustness under various failure and attack scenarios. We construct the SOC-KPZ coauthorship network, consisting of the nodes participating in both layers and the links among them. Then we simulate virtual random node or link failures and degree-based intentional node attacks, and measure the fraction of nodes in the initial largest component that still form largest component in the remaining superposed network, denoted S/S_0, as a function of the fraction of removed nodes or links f. We also compare the results against those obtained from three shuffled networks, in which the two layers are MP, uncorrelated, and MN-coupled (obtained by shuffling the node names in each layer according to the coupling rule, while controlling the link density of the superposed network to be equal) (Fig. 3.10). It is noteworthy that even though the coauthorship networks show positive correlated multiplexity (Fig. 3.2), the topological robustness properties do not always correspond to those of MP-correlated networks. For example, the real coauthorship network is more vulnerable, albeit slightly, to random link removals than its uncorrelated versions, in contrast to the higher robustness of MP-correlated
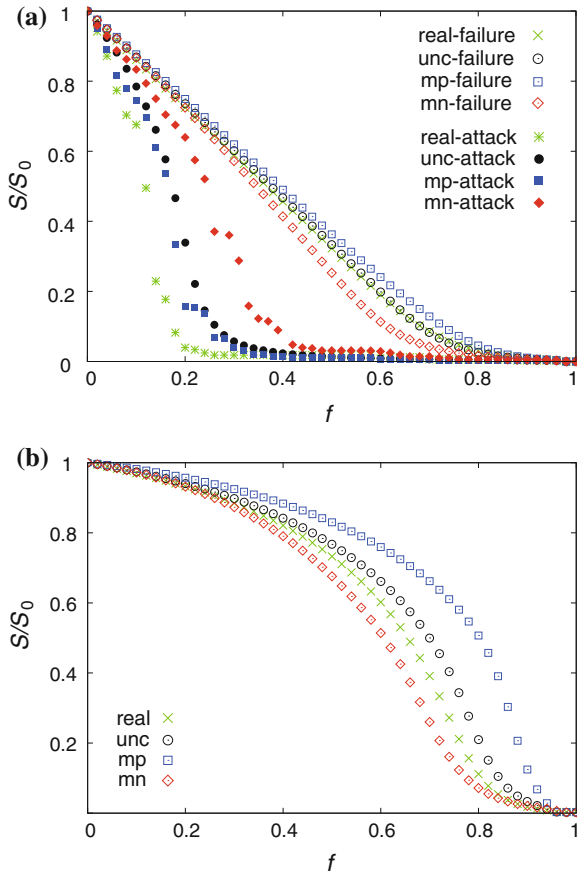
Fig. 3.10 Topological robustness of correlated multiplex networks. **a** Relative size of largest component S/S_0 of the superposed network under random (failure) and intentional (attack) removal of nodes of fraction f. The intentional attack was simulated by removing nodes in descending order of total degree. Shown are results for the multiplex coauthorship network (SOC-KPZ) ($\times$, $*$) and its three shuffled versions, MP (*square*), uncorrelated (*circle*), and MN-coupled networks (*textitdiamond*). **b** Same plots for random link removals. Data are averaged over 10^4 independent simulation runs

networks than the uncorrelated ones (Fig. 3.10b). Such discrepancy indicates the presence of higher-order correlations in the coupling structure of real multiplex networks, beyond the degree correlation. More systematic investigation on this topic using model networks is currently underway (B. Min et al., arXiv:1307.1253).

3.8 Dynamics on Multiplex Networks

Multiplexity can also have impact on network dynamics [31]; in fact it is one of the ultimate goals of the study of multiplex networks to understand what the generic effect of correlated multiplexity on various dynamic processes occurring on top of real-world multiplex complex systems. This may have implications on many profound real-world complex systems problems, such as understanding, predicting, and controlling systemic risk and collective social movement. Dynamics with multiplexity in general, poses the question of how the interplay of different network layers can bring about emergent dynamic consequences, and in many cases calls for development of new theoretical tools, similarly to what we did in Sect. 3.4 for structural analysis, which raises theoretical challenge as well.

Study of dynamical processes on multiplex networks is still in its infancy, yet is rapidly growing over the years [32–39]. Surveying all these recent effort would already require a separate contribution; here we could merely compile them with a brief summary of key findings. Given the obvious relevance of multiplex-network framework for many real-world problems, such as social cascades in social networks [5] or dynamics of systemic risk [40], this list is expected to expand quickly so is by no means meant to be exhaustive.

One of the first studies on multiplex dynamics was the study of sandpile dynamics [32], where it is found that the scaling behavior of avalanche does not change by the multiplex coupling, despite alterations in the detailed cascade dynamics. Generalized models of behavioral cascades in multiplex social networks [33, 34] showed that the multiplexity can facilitate global cascades compared to null models of simplex networks. In the study of random Boolean network on multiplex networks [35], the multiplex coupling is shown to support stabilization of the system even when each single layer is in the unstable chaotic state. In studies of evolutionary dynamics on multiplex networks, it was shown that the cooperative behavior is enhanced when individuals interact through multiple network layers [36, 37]. In the study of diffusion dynamics on multiplex networks [38], the existence of multiple channels of diffusive motion is shown to speed up the diffusion process. These studies collectively highlight how the dynamical properties on multiplex networks can differ from those of a single or simplex network.

3.9 Summary and Outlook

In summary, we have surveyed recent studies on multiplex networks, the networks with explicit multiple types of links, which is a better representation of real-world complex systems. Particularly emphasized are the notion of correlated multiplexity and its effect on the structural properties of multiplex network system. We have introduced the random graph models of correlated multiplex networks and developed analytical formalism to study its structural properties. Applications to multiplex

ER and SF networks demonstrated that the correlated multiplexity can dramatically change the properties of the giant component. This shows that a multiplex complex system can exhibit structural properties that cannot be represented by its individual network layer's properties alone. Such nontrivial, emerging multiplex structure should entail significant impact on dynamical processes occurring on it, opening a vast avenue of future studies on the impact of correlated multiplexity on network dynamics and function [14, 30].

The concepts and tools for the multiplex network should also be useful in the study of related subjects of recent interest such as layered [8], multi-type [23], interacting [9, 41], and interdependent networks [10, 24, 42], which share similar theoretical framework and mathematical techniques. Notable areas for further investigation would be, to name but a few, the multiplex network evolution [43] and the role of negative or antagonistic interactions between layers [11, 44]. Altogether, these studies will cooperatively help establish unified framework for the emerging paradigm of "network of networks," and the concept of network multiplexity will play an essential role in this collective endeavor.

References

1. M. E. J. Newman, *Networks: An introduction* (Oxford University Press, Oxford, 2010)
2. S. Havlin and R. Cohen, *Complex neworks: Structure, robustness, and function* (Cambridge Unviersity Press, Cambridge, 2010).
3. K.-M. Lee, J. Y. Kim, W.-K. Cho, K.-I. Goh, and I.-M. Kim, New J. Phys. **14**, 033027 (2012).
4. S. Wasserman and K. Faust, *Social network analysis* (Cambridge University Press, Cambridge, 1994).
5. M. O. Jackson, *Social and economic networks* (Princeton University Press, Princeton, NJ, 2008).
6. Z. Maoz, *Networks of nations: The Evolution, structure, and impact of international networks, 1816-2001* (Cambridge University Press, Cambridge, 2010).
7. M. Buchanan, G. Caldarelli, P. De Los Rios, F. Rao, and M. Vendruscolo, *Networks in cell biology* (Cambridge University Press, Cambridge, 2010).
8. M. Kurant and P. Thiran, Phys. Rev. Lett. **96**, 138701 (2006).
9. E. A. Leicht and R. M. D'Souza, e-print (2009) arXiv:0907.0894.
10. S. V. Buldyrev, R. Parshani, G. Paul, H. E. Stanley, and S. Havlin, Nature **464**, 1025 (2010).
11. M. Szell, R. Lambiotte, and S. Thurner, Proc. Natl. Acad. Sci. U.S.A. **107**, 13636 (2010).
12. D. Lee, K.-I. Goh, B. Kahng, D. Kim, Phys. Rev. E **82**, 026112 (2010).
13. A. Lomi, P. Pattison, Org. Sci. **17**, 313 (2006).
14. R. Parshani, C. Rozenblat, D. Ietri, C. Ducruet, and S. Havlin, EPL **92**, 68002 (2010).
15. R. G. Morris and M. Barthelemy, Phys. Rev. Lett. **109**, 128703 (2012).
16. H. W. Han, et al. Nucl. Acids Res. **41**, 9209 (2013).
17. M. Barigozzi, G. Fagiolo, and D. Garlaschelli, Phys. Rev. E **81**, 046104 (2010).
18. B. Soderberg, Phys. Rev. E **68**, 015102 (2003).
19. D.-H. Kim, B. Kahng, and D. Kim, Eur. Phys. J. B **38**, 305 (2004).

20. M. E. J. Newman, S. H. Strogatz, and D. J. Watts, Phys. Rev. E **64**, 026118 (2001).
21. M. Molloy and B. Reed, Random Struct Algorithms **6**, 161 (1995).
22. J. P. Gleeson, S. Melnik, and A. Hackett, Phys. Rev. E **81**, 066114 (2010).
23. A. Allard, P.-A. Noël, L. J. Dubé, and B. Pourbohloul, Phys. Rev. E **79**, 036113 (2009).
24. S.-W. Son, G. Bizhani, C. Christensen, P. Grassberger, and M. Paczuski, EPL **97**, 16006 (2012).
25. M. E. J. Newman, Phys. Rev. Lett. **89**, 208701 (2002).
26. P. Erdős and A. Rényi, Publ. Math. Inst. Hung. Acad. Sci. **5**, 17 (1960).
27. K.-I. Goh, B. Kahng, and D. Kim, Phys. Rev. Lett. **87**, 278701 (2001).
28. R. Cohen, K. Erez, D. ben-Avraham, and S. Havlin, Phys. Rev. Lett. **85**, 4626 (2000).
29. L. C. Freeman, Sociometry **40**, 35 (1977).
30. S. V. Buldyrev, N. Shere, and G. A. Cwilich, Phys. Rev. E **83**, 016112 (2011).
31. A. Barrat, M. Barthélemy, and A. Vespignani, *Dynamic processes on complex networks* (Cambridge University Press, Cambridge, 2008).
32. K.-M. Lee, K.-I. Goh, and I.-M. Kim, J. Korean Phys. Soc. **60**, 641 (2012).
33. C. D. Brummitt, K.-M. Lee, and K.-I. Goh, Phys. Rev. E **85**, 045102(R) (2012).
34. O. Yağan and V. Gligor, Phys. Rev. E **86**, 036103 (2012).
35. E. Cozzo, A. Arenas, and Y. Moreno, Phys. Rev. E **86**, 036115 (2012).
36. J. Gómez-Gardeñes, I. Reinares, A. Arenas, and L. M. Floria, Sci. Rep. **2**, 620 (2012).
37. J. Gómez-Gardeñes, C. Gracia-Lázaro, L. M. Floría, and Y. Moreno, Phys. Rev. E **86**, 056113 (2012).
38. S. Gómez, A. Díaz-Guilera, J. Gómez-Gardeñes, C. J. Pérez-Vicente, Y. Moreno, and A. Arenas, Phys. Rev. Lett. **110**, 028701 (2013).
39. S. Shai and S. Dobson, Phys. Rev. E **86**, 066120 (2012).
40. K.-M. Lee, J.-S. Yang, G. Kim, J. Lee, K.-I. Goh, and I.-M. Kim, PLoS ONE **6**, e18443 (2011).
41. J. F. Donges, H. C. H. Schultz, N. Marwan, Y. Zou, and J. Kurths, Eur. Phys. J. B **84**, 635 (2011).
42. D. Zhou, H. E. Stanley, G. D'Agostino, and A. Scala, Phys. Rev. E **86**, 066103 (2012).
43. J. Y. Kim and K.-I. Goh, Phys. Rev. Lett. **111**, 058702 (2013).
44. K. Zhao and G. Bianconi, J. Stat. Mech. P05005 (2013).