Limit Theorems for Excursion Sets of Stationary Random Fields

Evgeny Spodarev

Abstract We give an overview of the recent asymptotic results on the geometry of excursion sets of stationary random fields. Namely, we cover a number of limit theorems of central type for the volume of excursions of stationary (quasi-, positively or negatively) associated random fields with stochastically continuous realizations for a fixed excursion level. This class includes in particular Gaussian, Poisson shot noise, certain infinitely divisible, α-stable, and max-stable random fields satisfying some extra dependence conditions. Functional limit theorems (with the excursion level being an argument of the limiting Gaussian process) are reviewed as well. For stationary isotropic C^1-smooth Gaussian random fields similar results are available also for the surface area of the excursion set. Statistical tests of Gaussianity of a random field which are of importance to real data analysis as well as results for an increasing excursion level round up the paper.

1 Introduction

Geometric characteristics such as Minkowski functionals (or intrinsic volumes, curvature measures, etc.) of excursions of random fields are widely used for data analysis purposes in medicine (brain fMRI analysis; see, e.g., [5, 55, 60, 62]), physics and cosmology (microwave background radiation analysis; see, e.g., [41] and references therein), and materials science (quantification of porous media; see, e.g., [42, 61]), to name just a few. Minkowski functionals include the volume, the surface area, and the Euler–Poincaré characteristic (reflecting porosity) of a set with a sufficiently regular boundary.

E. Spodarev (✉)
Ulm University, Institute of Stochastics, 89069 Ulm, Germany
e-mail: evgeny.spodarev@uni-ulm.de

V. Korolyuk et al. (eds.), *Modern Stochastics and Applications*, Springer Optimization and Its Applications 90, DOI 10.1007/978-3-319-03512-3_13,

Among the possible abundance of random field models, Gaussian random fields are best studied due to their analytic tractability. A number of results starting with explicit calculation of the moments of Minkowski functionals are available for them since the mid-1970s. We briefly review these results in Sect. 4. However, our attention is focused on the asymptotic arguments for (mainly non-Gaussian) stationary random fields. There has been a recent breakthrough in this domain starting with the paper [15] where a central limit theorem (CLT) for the volume of excursions of a large class of quasi-associated random fields was proved. We also cover a number of hard-to-find results from recent preprints and PhD theses.

The paper is organized as follows: After introducing some basic facts on excursions and dependence structure of stationary random fields in Sect. 2, we briefly review the limit theorems for excursions of stationary Gaussian processes ($d = 1$) in the next section. However, our focus is on the recent results in the multidimensional case $d > 1$ which is considered in Sects. 5 and 6. Thus, Sect. 5 gives (uni- and multivariate as well as functional) central limit theorems for the volume of excursion sets of stationary (in general, non-Gaussian) random fields over fixed, variable, or increasing excursion levels. In Sect. 6, a similar scope of results is covered for the surface area of the boundary of excursion sets of stationary (but possibly anisotropic) Gaussian random fields in different functional spaces. The paper concludes with a number of open problems.

2 Preliminaries

Fix a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $X = \{X(t,\omega),\ t \in \mathbb{R}^d,\ \omega \in \Omega\}$ be a stationary (in the strict sense) real-valued measurable (in $(t,\omega) \in \mathbb{R}^d \times \Omega$) random field. Later on we suppress ω in the notation. For integrable X we assume X to be centered (i.e., $\mathsf{E}X(o) = 0$ where $o \in \mathbb{R}^d$ is the origin point). If the second moment of $X(o)$ exists, then we denote by $C(t) = \mathsf{E}\,(X(o)X(t))$, $t \in \mathbb{R}^d$ the covariance function of X.

Let $\|\cdot\|_2$ be the Euclidean norm in $\mathbb{R}^d$ and dist_2 the Euclidean distance: for two sets $A, B \subset \mathbb{R}^d$, we put $\mathsf{dist}_2(A,B) = \inf\{\|x-y\|_2 : x \in A, y \in B\}$. Denote by $\|\cdot\|_\infty$ the supremum norm in $\mathbb{R}^d$ and by dist_∞ the corresponding distance function.

Let $\xrightarrow{d}$ mean convergence in distribution. Denote by A^c the complement and by $\mathrm{int}(A)$ the interior of a set A in the corresponding ambient space which will be clear from the context. Let $\mathrm{card}(A)$ be the cardinality of a finite set A. Denote by $B_r(x)$ the closed Euclidean ball with center in $x \in \mathbb{R}^d$ and radius $r > 0$. Let $\mathcal{H}^k(\cdot)$ be the k-dimensional Hausdorff measure in $\mathbb{R}^d$, $0 \le k \le d$. In the sequel, we use the notation $\kappa_j = \mathcal{H}^j(B_1(o))$, $j = 0, \dots, d$.

To state limit theorems, one has to specify the way of expansion of windows $W_n \subset T$, where the random field $X = \{X(t),\ t \in T\}$ is observed, to the whole index space $T = \mathbb{R}^d$ or $\mathbb{Z}^d$. A sequence of compact Borel sets $(W_n)_{n\in\mathbb{N}}$ is called a *Van Hove sequence* (*VH*) if $W_n \uparrow \mathbb{R}^d$ with

$$\lim_{n\to\infty} V_d\left(W_n\right) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{V_d\left(\partial W_n \oplus B_r(o)\right)}{V_d\left(W_n\right)} = 0, \quad r > 0.$$

A sequence of finite subsets $U_n \subset \mathbb{Z}^d$, $n \in \mathbb{N}$ is called *regular growing* if

$$\text{card}(U_n) \to \infty \quad \text{and} \quad \text{card}(\delta U_n)/\text{card}(U_n) \to 0 \text{ as } n \to \infty$$

where $\delta U_n = \{j \in \mathbb{Z}^d \setminus U_n : \ \mathsf{dist}_\infty(j, U_n) = 1\}$ is the discrete boundary of U_n in $\mathbb{Z}^d$.

2.1 Excursion Sets and Their Intrinsic Volumes

The *excursion set* of X at level $u \in \mathbb{R}$ in the compact observation window $W \subset \mathbb{R}^d$ is given by $A_u(X, W) = \{t \in W : X(t) \geq u\}$. The *sojourn set* under the level u is $S_u(X, W) = \{t \in W : X(t) \leq u\}$, respectively.

Due to measurability of X, $A_u(X, W)$ and $S_u(X, W)$ are random Borel sets. If X is a.s. upper (lower) semicontinuous, then $A_u(X, W)$ ($S_u(X, W)$, respectively) is a random closed set (cf. [45, Sect. 5.2.1]).

A popular way to describe the geometry of excursion sets is via their *intrinsic volumes* V_j, $j = 0, \dots, d$. They can be introduced for various families of sets such as convex and polyconvex sets [54, Chap. 4], sets of positive reach, and their finite unions [22], unions of basic complexes [4, Chap. 6]. One possibility to define $V_j(K)$, $j = 0, \dots, d$ for a set K belonging to the corresponding family is given by the *Steiner formula* (see, e.g., [53, Sect. 13.3]) as the coefficients in the polynomial expansion of the volume of the tubular neighborhood $K_r = \{x \in \mathbb{R}^d : \mathsf{dist}_2(x, K) \leq r\}$ of K with respect to the radius $r > 0$ of this neighborhood:

$$\mathcal{H}^d\left(K_r\right) = \sum_{j=0}^{d} \kappa_{d-j} V_j(K) r^{d-j}$$

for admissible $r > 0$ (for convex K, these are all positive r). The geometric interpretation of intrinsic volumes $V_j(K)$, $j = 1, \dots, d-2$ can be given in terms of integrals of elementary symmetric polynomials of principal curvatures for convex sets K with C^2-smooth boundary, cf. [53, Sects. 13.5–6]. Without going into details here, let us discuss the meaning of some of $V_j\left(A_u(X, W)\right)$, $j = 0, \dots, d$ in several dimensions.

For $d = 1$, $V_1\left(A_u(X, W)\right)$ is the length of excursion intervals and $V_0\left(A_u(X, W)\right)$ is the number of upcrossings of level u by the random process X within W.

For dimensions $d \geq 2$, $V_d\left(A_u(X, W)\right)$ is always the volume (i.e., the Lebesgue measure) of $A_u(X, W)$ and $V_{d-1}\left(A_u(X, W)\right)$ is half the surface area, i.e., $1/2 \cdot \mathcal{H}^{d-1}\left(\partial A_u(X, W)\right)$. The *Euler characteristic* $V_0\left(A_u(X, W)\right)$ is a topological measure of "porosity" of excursion set $A_u(X, W)$. For "basic" sets A (e.g., nonempty convex sets or sets of positive reach), we set $V_0(A) = 1$. Then V_0 is defined for unions of basic sets by additivity. One can show that for $d = 2$, it holds

$$V_0(A) = \text{card}\{\text{connected components of } A\} - \text{card}\{\text{holes of } A\}.$$

The existence of $V_j\,(A_u(X, W))$, $j = d, d-1$, is clear since $A_u(X, W)$ is a Borel set whose Lebesgue and Hausdorff measures are well defined. Intrinsic volumes V_j of lower orders $j = 0, \dots, d-2$ are well defined, e.g., for excursion sets of sufficiently smooth (at least C^2) deterministic functions (cf. [4, Theorem 6.2.2]) and Gaussian random fields (cf. [4, Theorem 11.3.3]) satisfying some additional conditions.

2.2 Dependence Concepts for Random Fields

To prove limit theorems for a random field X, some conditions have to be imposed on the structure of the dependence of X. Mixing conditions that are usually required (cf., e.g., [13, 20]) are however rather difficult to check for a particular random field under consideration. For this practical reason, we follow the books [16], [58, Chap. 10] and introduce *association* as well as related dependence concepts.

A random field $X = \{X(t),\ t \in \mathbb{R}^d\}$ is called *associated (**A**)* if

$$\text{cov}\,(f\,(X_I)\,, g\,(X_I)) \geq 0$$

for any finite subset $I \subset \mathbb{R}^d$, and for any bounded coordinatewise non-decreasing functions $f : \mathbb{R}^{\text{card}(I)} \to \mathbb{R}$, $g : \mathbb{R}^{\text{card}(I)} \to \mathbb{R}$ where $X_I = \{X(t),\ t \in I\}$.

A random field $X = \{X(t),\ t \in \mathbb{R}^d\}$ is called *positively (**PA**)* or *negatively (**NA**) associated* if

$$\text{cov}\,(f\,(X_I)\,, g\,(X_J)) \geq 0 \quad (\leq 0,\ \text{resp.})$$

for all finite disjoint subsets $I, J \subset \mathbb{R}^d$, and for any bounded coordinatewise non-decreasing functions $f : \mathbb{R}^{\text{card}(I)} \to \mathbb{R}$, $g : \mathbb{R}^{\text{card}(J)} \to \mathbb{R}$. It is clear that if $X \in \mathbf{A}$, then $X \in \mathbf{PA}$.

Subclasses of **A** (**PA**, **NA**)-fields are certain infinitely divisible (e.g., max-stable and α-stable) random fields. In particular, a Gaussian random field with nonnegative covariance function is associated.

A random field $X = \{X(t), t \in \mathbb{R}^d\}$ with finite second moments is called *quasi-associated* (**QA**) if

$$|\text{cov}\,(f\,(X_I)\,, g\,(X_J))| \leq \sum_{i \in I} \sum_{j \in J} \text{Lip}_i\,(f)\,\text{Lip}_j\,(g)\,|\text{cov}\,(X\,(i)\,, X\,(j))|$$

for all finite disjoint subsets $I, J \subset \mathbb{R}^d$, and for any Lipschitz functions $f : \mathbb{R}^{\mathrm{card}(I)} \to \mathbb{R}$, $g : \mathbb{R}^{\mathrm{card}(J)} \to \mathbb{R}$ where $\mathrm{Lip}_i(f)$ is the Lipschitz constant of function f for coordinate i. It is known that if square-integrable $X \in \mathbf{A}(\mathbf{PA}, \mathbf{NA})$, then $X \in \mathbf{QA}$, cf. [16, Theorem 5.3].

A real-valued random field $X = \{X(t), t \in \mathbb{R}^d\}$ is called (BL, θ)-*dependent* if there exists a nonincreasing sequence $\theta = \{\theta_r\}_{r \in \mathbb{R}_0^+}$, $\theta_r \downarrow 0$ as $r \to \infty$ such that for any finite disjoint sets $I,\ J \subset \mathbb{R}^d$ with $\mathrm{dist}_\infty(I, J) = r \in \mathbb{R}_0^+$ and any bounded Lipschitz functions $f : \mathbb{R}^{\mathrm{card}(I)} \to \mathbb{R}$, $g : \mathbb{R}^{\mathrm{card}(J)} \to \mathbb{R}$, one has

$$|\mathrm{cov}(f(X_I), g(X_J))| \leq \sum_{i \in I} \sum_{j \in J} \mathrm{Lip}_i(f)\,\mathrm{Lip}_j(g)\,|\mathrm{cov}(X(i), X(j))|\,\theta_r.$$

It is often possible to choose θ as the *Cox–Grimmett coefficient*

$$\theta_r = \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus B_r^\infty(y)} |\mathrm{cov}(X(y), X(t))|\, dt$$

where $B_r^\infty(y) = \{x \in \mathbb{R}^d : \|x - y\|_\infty \leq r\}$. It can be easily seen that if $X \in \mathbf{QA}$ and its covariance function is absolutely integrable on $\mathbb{R}^d$, then X is (BL, θ) dependent.

3 Excursions of Stationary Gaussian Processes

Excursions of stochastic processes is a popular research topic in probability theory since many years; see, e.g., [10] and references in [27]. The vast literature on this subject for different classes of processes such as Lévy, diffusion, stable, and Gaussian ones can be hardly covered by one review. For this reason, we concentrate on the excursions of (mainly stationary) Gaussian processes here.

Let $X = \{X(t), t \geq 0\}$ be a centered real-valued Gaussian process. If X is a polynomial of degree n with iid $N(0, 1)$-distributed coefficients, then the mean number of real roots of the equation $X(t) = 0$ was first obtained by M. Kac [28]. It initiated a substantial amount of papers on the roots of random algebraic polynomials; see [12] for a review. For C^1-smooth stationary Gaussian processes X, expectation of the number of upcrossings of a level u by X in time interval $[0, 1]$ has been studied in [14, 50, 51], etc. Higher-order factorial moments are considered in [17]; see also references therein and [7, 8]. For reviews (also including results on non-Gaussian stationary processes), see [33, Sects. 7.2 and 7.3] and [6, Chap. 3]. In [1] and [2], the notion of the number of upcrossings of level u for random processes has been generalized to the Euler–Poincaré characteristic of excursion sets of random fields.

The first proof of a central limit theorem for the number of zeros of a stationary Gaussian process within an increasing time interval was given in [40]. Cuzick [18] refined the assumptions given in [40] and proved a central limit theorem for the number of zeros $N_X(T) = 2V_0(A_0(X; [0, T]))$ of a centered separable stationary Gaussian process $X = \{X(t), t \geq 0\}$ in the time interval $[0, T]$ as well as analogous results for integrals $\int_0^T g(X(t))\,dt$ as $T \to \infty$. He used approximations by m-dependent random processes with spectral representation as a method borrowed from [40]. In more detail, let $C(t)$ be twice differentiable with $C(0) = 1$, $C''(0) = -\lambda_2$, and variogram γ of X' be given by $\gamma(h) = C''(h) - C''(0) = 1/2\mathsf{E}(X'(h) - X'(0))$, $h \geq 0$.

Theorem 1 ([18]). *If C, C'' are square integrable on $\mathbb{R}_+$, $\int_0^\varepsilon \gamma(t)/t\,dt < \infty$ for some $\varepsilon > 0$ and*

$$\mathsf{Var} N_X(T)/T \to \sigma_2 > 0 \ \ as \ \ T \to +\infty \tag{1}$$

then

$$T^{-1/2}\left(N_X(T) - \mathsf{E}\, N_X(T)\right) \xrightarrow{d} N(0, \sigma^2) \ \ as \ \ T \to +\infty$$

where

$$\sigma^2 = \pi^{-1}\left(\lambda_2^{1/2} + \int_0^\infty \left(\frac{\mathsf{E}\ (|X'(0)X'(t)||X(0) = X(t) = 0)}{\sqrt{1 - C^2(t)}} - (\mathsf{E}\,|X'(0)|)^2\right) dt\right).$$

Condition (1) is difficult to check and is substituted in [18, Lemma 5] by a more tractable sufficient condition involving C and λ_2. Piterbarg [48] managed to prove the above theorem by substituting condition (1) with

$$\int_0^\infty t\left(|C(t)| + |C'(t)| + |C''(t)|\right)\,dt < \infty.$$

He approximates the point process of upcrossings of X of level u by a strongly mixing point process.

Theorem 2 ([18]). *Let X be a stationary Gaussian process with covariance function C being integrable on $\mathbb{R}_+$. For any measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $\mathsf{E}\, g^2(X(0)) < \infty$ and $g(x) - g(0)$ is not odd, it holds*

$$T^{-1/2}\left(\int_0^T g\big(X(t)\big)\,dt - T\,\mathsf{E}\, g\big(X(0)\big)\right) \xrightarrow{d} N(0, \sigma^2) \ \ as \ \ T \to +\infty \tag{2}$$

where $\sigma^2 > 0$.

It is clear that the choice $g(x) = \mathbf{1}\{x \in \mathbb{R} : x \geq u\}$ for any $u \in \mathbb{R}$ leads to the central limit theorem for the length $V_1(A_u(X; [0, T]))$ of excursion intervals of X in $[0, T]$.

Elizarov [21] first proved a functional central limit theorem for the sojourn times of the stationary Gaussian process under the level u, in our terms, for $V_1(S_u(X;[0,T]))$ if excursion level u is allowed to vary within $\mathbb{R}$. Additionally, an analogous result for local times

$$\lim_{\varepsilon\to+0}\frac{1}{2\varepsilon}\left(V_1(S_{u+\varepsilon}(X;[0,T]))-V_1(S_{u-\varepsilon}(X;[0,T]))\right)$$

was given. Both results were proved in the functional space $C[0,1]$ after the substitution $u\mapsto f(x)$, $x\in[0,1]$ where $f\in C[0,1]$ is a monotonously increasing function with $f(0)=-\infty$, $f(1)=\infty$.

Belyaev and Nosko [9] proved limit theorems for $V_1(A_u(X;[0,T]))$, $T\to\infty$ as $u\to\infty$ for stationary ergodic processes X satisfying a number of additional (quite technical) assumptions. In particular, these assumptions are satisfied if X is an ergodic Gaussian stationary process with twice continuously differentiable covariance function such that

$$\left|C''(t)-C''(0)\right|\le a/|\log|t||^{1+\varepsilon},\quad t\downarrow 0$$

for some constants $a,\varepsilon>0$.

Slud [57] gave a multiple Wiener- Itô representation for the number of crossings of a C^1-function ψ by X. In [31], methods of [40] and [18] are generalized to the case of functionals of X, X', and X''. CLTs for the number of crossings of a smooth curve ψ by a Gaussian process X as well as for the number of specular points of X (if X is a Gaussian process in time and space) are given in [32]. For a review of results on moments and limit theorems for different characteristics of stationary Gaussian processes, see [30]. In [27], CLTs for the multivariate nonlinear weighted functionals (similar to those in (2)) of Gaussian stationary processes with multiple singularities in their spectra, having a covariance function belonging to a certain parametric family, are proved.

4 Moments of $V_j\,(A_u(X,W))$ for Gaussian Random Fields

We briefly review the state of the art for $\mathsf{E}\,V_j\,(A_u(X,W))$ of Gaussian random fields X. For recent extended surveys, see the books [4] and [6]. For stationary (isotropic) Gaussian fields X, stratified C^2-smooth compact manifolds $W\subset\mathbb{R}^d$, and any $u\in\mathbb{R}$, formulae for $\mathsf{E}\,V_j\,(A_u(X,W))$, $j=0,\dots,d$ are given in [4, Theorems 13.2.1 and 13.4.1].

Apart from obtaining exact (or asymptotic as $u\to\infty$) formulae for $\mathsf{E}\,V_j\,(A_u(X,W))$, $j=0,\dots,d$, the possibility of an estimate

$$\left|\mathsf{P}\left(\sup_{t\in W}X(t)>u\right)-\mathsf{E}\,V_0\,(A_u(X,W))\right|\le g(u)\qquad(3)$$

(the so-called *Euler–Poincaré heuristic*) with $g(u) = o(1)$ as $u \to \infty$ is of special interest. It has been proved in [4, Theorem 14.3.3] with $g(u) = c_0 \exp\{-u^2(1+\alpha)/2\}$ for some positive constants c_0 and α if X is a (non)stationary Gaussian random field with constant variance on a stratified manifold W as $u \to \infty$. Lower and upper bounds for the density of supremum of stationary Gaussian random fields X (which imply relation (3)) for any $u \in \mathbb{R}$ are given in [6, Theorem 8.4]. Similar bounds are proven in [6, Theorem 8.10] for nonstationary Gaussian random fields X with a unique point of maximum of variance in $\mathrm{int}(W)$ as $u \to \infty$.

In [59], asymptotic behavior of $\mathsf{E}\, V_j\left(A_u(X,[a,b]^d)\right)$, $j = 0, d-1, d$ of nonstationary sufficiently smooth Gaussian random fields is studied as the excursion level $u \to \infty$. The variance of these fields is assumed to attain a global maximum at a vertex of $[a,b]^d$. It is shown that the heuristic (3) still holds true.

An interesting rather general formula for the mean surface area of Gaussian excursions is proven in [24]. Let W be a compact subset of $\mathbb{R}^d$ with a nonempty interior and a finite Hausdorff measure of the boundary. Let $X = \{X(t),\ t \in W\}$ be a Gaussian random field with mean $\mu(t) = \mathsf{E}X(t)$ and variance $\sigma^2(t) = \mathsf{Var}X(t)$. For an arbitrary (but fixed) excursion level $u \in \mathbb{R}$, introduce the zero set $\nabla_X^{-1}(0)$ of the gradient of the normalized field $(X-u)/\sigma$ by $\nabla_X^{-1}(0) = \{t \in W : \nabla\left((X(t)-u)/\sigma(t)\right) = 0\}$.

Theorem 3 ([24]). *Assume that* $X \in C^1(W)$ *a.s.,* $\mathsf{E}V_{d-1}\left(\nabla_X^{-1}(0)\right) < \infty$ *and* $\sigma(t) > 0$ *for all* $t \in W$. *Then*

$$\mathsf{E}V_{d-1}\left(\partial A_u(X,W)\right) = \frac{1}{2\sqrt{2\pi}} \int_W \exp\left[-\frac{(\mu(t)-u)^2}{2\sigma^2(t)}\right] \mathsf{E}\left\|\nabla\left((X(t)-u)/\sigma(t)\right)\right\|_2 dt.$$

Asymptotic formulae for $\mathsf{E}\, V_j\left(A_u(X,W)\right)$, $j = 0,\dots,d$ as $u \to \infty$ of three subclasses of stable random fields (subgaussian, harmonizable, concatenated-harmonizable ones) are given in [3].

5 Volume of Excursion Sets of Stationary Random Fields

The first limit theorems of central type for the volume of excursion sets (over a fixed level u) of stationary isotropic Gaussian random fields were proved in [26, Chap. 2]. There, the case of short- and long-range dependence (Theorem 2.2.4 and Example 2.2.1, Theorem 2.4.6) was considered. The CLT followed from a general Berry–Esséen-type bound for the distribution function of properly normed integral functionals

$$\int_{B_r(o)} G\left(X(t)\right) dt \tag{4}$$

as $r \to \infty$ where $G : \mathbb{R} \to \mathbb{R}$ is a function such that $\mathsf{E}\, G^2(X(o)) < \infty$ satisfying some additional assumptions, cf. also [36]. To get the volume $V_d\,(A_u\,(X, B_r(o)))$ out of (4), set $G(x) = \mathbf{1}(x \geq u)$. The isotropy of X was essential as one used expansions with respect to the basis of Chebyshev–Hermite polynomials in the proofs. The cases of

$$G(x) = \mathbf{1}(|x| \geq u),\ \max\{0, x\},\ |x|$$

as well as of G depending on a parameter and of weighted integrals in (4) are considered as well.

In a remark [26, p. 81], it was noticed that similar CLTs can be expected for non-Gaussian mixing random fields. The aim of this section is to review the recent advances in proving such CLTs for various classes of stationary random fields that include also the (not necessarily isotropic) Gaussian case.

For instance, random fields with singularities of their spectral densities are considered in [37]. In Sect. 3.2 of that book, noncentral limit theorems for the volume of excursions of stationary isotropic Gamma-correlated and χ^2-random fields over a radial surface (i.e., the level u is not constant anymore, but a function of $\|t\|_2$, where $t \in \mathbb{R}^d$ is the integration variable in (4)) are proved. (Non)central limit theorems for functionals (4) of stationary isotropic vector-valued Gaussian random fields are given in the recent preprint [34]. There, the case of long- and short-range dependence is considered as well as applications to F- and t-distributed random fields.

The asymptotic behavior of tail probabilities

$$\mathsf{P}\left(\int_W e^{X(t)}\, dt > x\right), \quad x \to \infty$$

for a homogeneous smooth Gaussian random field X on a compact $W \subset \mathbb{R}^d$ is considered in [38]; see [39] for further extensions.

5.1 Limit Theorems for a Fixed Excursion Level

The main result (which we call a *methatheorem*) can be formulated as follows:

Theorem 4 (Methatheorem). *Let X be a strictly stationary random field satisfying* some additional conditions *and $u \in \mathbb{R}$ fixed. Then, for any sequence of VH-growing sets $W_n \subset \mathbb{R}^d$, one has*

$$\frac{V_d\,(A_u\,(X, W_n)) - \mathsf{P}(X(o) \geq u) \cdot V_d\,(W_n)}{\sqrt{V_d\,(W_n)}} \xrightarrow{\mathsf{d}} \mathcal{N}\left(0, \sigma^2(u)\right) \tag{5}$$

as $n \to \infty$. Here

$$\sigma^2(u) = \int_{\mathbb{R}^d} \mathsf{cov}\left(\mathbf{1}\{X(o) \geq u\}, \mathbf{1}\{X(t) \geq u\}\right) dt. \tag{6}$$

Depending on the class of random fields, these additional conditions will vary. First we consider the family of square-integrable random fields.

5.1.1 Quasi-Associated Random Fields

Theorem 5 ([15]). *Let $X = \{X(t), t \in \mathbb{R}^d\} \in$ **QA** be a stationary square-integrable random field with a continuous covariance function C such that $|C(t)| = \mathcal{O}\left(\|t\|_2^{-\alpha}\right)$ for some $\alpha > 3d$ as $\|t\|_2 \to \infty$. Let $X(o)$ have a bounded density. Then $\sigma^2(u) \in (0, \infty)$ and Theorem 4 hold true.*

Let us give an idea of the proof. Introduce the random field $Z = \{Z(j),\ j \in \mathbb{Z}^d\}$ by

$$Z(j) = \int_{j+[0,1]^d} \mathbf{1}\big\{X(t) \geq u\big\}\, dt - \Psi(u), \quad j \in \mathbb{Z}^d. \tag{7}$$

Here $\Psi(u) = \mathsf{P}\big(X(o) > u\big)$ is the tail distribution function of $X(o)$. It is clear that the sum of $Z(j)$ over indices $j \in W_n \cap \mathbb{Z}^d$ approximates the numerator in (5). One has to show that Z can be approximated by a sequence of (BL, θ)-dependent stationary centered square-integrable random fields Z_γ, $\gamma \downarrow 0$, on $\mathbb{Z}^d$. The proof finishes by applying the following CLT to Z_γ for each $\gamma > 0$.

Theorem 6 ([16], Theorem 3.1.12). *Let $Z = \{Z(j), j \in \mathbb{Z}^d\}$ be a (BL, θ)-dependent strictly stationary centered square-integrable random field. Then, for any sequence of regularly growing sets $U_n \subset \mathbb{Z}^d$, one has*

$$S(U_n) / \sqrt{card(U_n)} \xrightarrow{\mathsf{d}} \mathcal{N}\left(0, \sigma^2\right)$$

as $n \to \infty$, with

$$\sigma^2 = \sum_{j \in \mathbb{Z}^d} \mathsf{cov}\left(Z(o), Z(j)\right).$$

We give two examples of random fields satisfying Theorem 5.

Example 1 ([15]). Let $X = \{X(t),\ t \in \mathbb{R}^d\}$ be a stationary *shot noise random field* given by $X(t) = \sum_{i \in \mathbb{N}} \xi_i \varphi(t - x_i)$ where $\Pi_\lambda = \{x_i\}$ is a homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda \in (0, \infty)$ and $\{\xi_i\}$ is a family of i.i.d. nonnegative random variables with $\mathsf{E}\,\xi_i^2 < \infty$ and characteristic function φ_ξ. Assume that Π_λ and $\{\xi_i\}$ are independent. Moreover, let $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ be a bounded and uniformly continuous Borel function with $\varphi(t) \leq g_0(\|t\|_2) = \mathcal{O}\left(\|t\|_2^{-\alpha}\right)$ as $\|t\|_2 \to \infty$ for a function $g_0 : \mathbb{R}_+ \to \mathbb{R}_+$, $\alpha > 3d$, and

$$\int\limits_{\mathbb{R}^d} \left| \exp\left\{ \lambda \int_{\mathbb{R}^d} \left(\varphi_\xi(s\varphi(t)) - 1 \right) \, dt \right\} \right| \, ds < \infty.$$

Then Theorem 5 holds true.

Example 2 ([15]). Consider a stationary Gaussian random field $X = \{X(t), t \in \mathbb{R}^d\}$ with a continuous covariance function $C(\cdot)$ such that $|C(t)| = \mathcal{O}\left(\|t\|_2^{-\alpha}\right)$ for some $\alpha > d$ as $\|t\|_2 \to \infty$. Let $X(o) \sim \mathcal{N}\left(a, \tau^2\right)$. Then, Theorem 5 holds true with

$$\sigma^2(u) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_0^{\rho(t)} \frac{1}{\sqrt{1-s^2}} \, e^{-\frac{(u-a)^2}{\tau^2(1+s)}} \, ds \, dt,$$

where $\rho(t) = \mathsf{corr}(X(o), X(t))$. In particular, for $u = a$ one has

$$\sigma^2(a) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \arcsin\left(\rho(t)\right) \, dt.$$

5.1.2 PA- or NA-Random Fields

What happens if the field X does not have the finite second moment? In this case, another set of conditions for our methatheorem to hold was proven in [29, Theorem 3.59].

Theorem 7. *Let $X = \{X(t),\ t \in \mathbb{R}^d\} \in$ **PA**(**NA**) be stochastically continuous satisfying the following properties:*

1. *The asymptotic variance $\sigma^2(u) \in (0, \infty)$ (cf. its definition in (6)),*
2. $\mathsf{P}\left(X(o) = u\right) = 0$ *for the chosen level $u \in \mathbb{R}$.*

Then Theorem 4 holds.

The idea of the proof is first to show that the random field $Z = \{Z(j),\ j \in \mathbb{Z}^d\}$ defined in (7) is **PA** (**NA**). Second, use [16, Theorem 1.5.17] to prove that Z is (BL, θ)–dependent. Then apply Theorem 6 to Z.

A number of important classes of random fields satisfy Theorem 7. For instance, stationary infinitely divisible random fields $X = \{X(t),\ t \in \mathbb{R}^d\}$ with spectral representation

$$X(t) = \int_E f_t(x) \, \Lambda(dx), \quad t \in \mathbb{R}^d,$$

where Λ is a centered independently scattered infinitely divisible random measure on space E and $f_t : E \to \mathbb{R}_+$ are Λ-integrable kernels, are associated, and hence **PA** by [16, Chap. 1, Theorem 3.27]. The finite susceptibility condition

$\sigma^2(u) \in (0, \infty)$ can be verified by [29, Lemma 3.71]. Further examples of random fields satisfying Theorem 7 are *stable* random fields which we consider in more detail following [29, Sect. 3.5.3].

Max-Stable Random Fields

Let $X = \{X(t),\ t \in \mathbb{R}^d\}$ be a stationary max-stable random field with spectral representation

$$X(t) = \max_{i \in \mathbb{N}} \xi_i f_t(y_i), \quad t \in \mathbb{R}^d,$$

where $f_t : E \to \mathbb{R}_+$ is a measurable function defined on the measurable space (E, μ) for all $t \in \mathbb{R}^d$ with

$$\int_E f_t(y)\, \mu(dy) = 1, \quad t \in \mathbb{R}^d,$$

and $\{(\xi_i, y_i)\}_{i \in \mathbb{N}}$ is the Poisson point process on $(0, \infty) \times E$ with intensity measure $\xi^{-2} d\xi \times \mu(dy)$. It is known that all max-stable distributions are associated and hence **PA** by [49, Proposition 5.5.29]. The field X is stochastically continuous if $\|f_s - f_t\|_{L^1} \to 0$ as $s \to t$ (cf. [23, Lemma 2]). Condition $\sigma^2(u) \in (0, \infty)$ is satisfied if

$$\int_{\mathbb{R}^d} \int_E \min\{f_0(y), f_t(y)\}\, \mu(dy)\, dt < \infty.$$

α-Stable Random Fields

Let $X = \{X(t),\ t \in \mathbb{R}^d\}$ be a stationary α-stable random field ($\alpha \in (0, 2)$, for simplicity $\alpha \neq 1$) with spectral representation

$$X(t) = \int_E f_t(x)\, \Lambda(dx), \quad t \in \mathbb{R}^d,$$

where Λ is a centered independently scattered α-stable random measure on space E with control measure m and skewness intensity $\beta : E \to [-1, 1]$, $f_t : E \to \mathbb{R}_+$ is a measurable function on (E, m) for all $t \in \mathbb{R}^d$. By [52, Proposition 3.5.1], X is stochastically continuous if $\int_E |f_s(x) - f_t(x)|^\alpha\, m(dx) \to 0$ as $s \to t$ for any $t \in \mathbb{R}^d$. Condition $\sigma^2(u) \in (0, \infty)$ is satisfied if

$$\int_{\mathbb{R}^d} \left(\int_E \min\{|f_0(x)|^\alpha, |f_t(x)|^\alpha\}\, m(dx) \right)^{1/(1+\alpha)} dt < \infty.$$

5.2 A Multivariate Central Limit Theorem

If a finite number of excursion levels $u_k \in \mathbb{R}$, $k = 1, \ldots, r$ is considered simultaneously, a multivariate analogue of Theorem 4 can be proven. Introduce the notation

$$S_{\mathbf{u}}(W_n) = \left(V_d\left(A_{u_1}(X, W_n)\right), \ldots, V_d\left(A_{u_r}(X, W_n)\right)\right)^\top, \quad \Psi(\mathbf{u}) = (\Psi(u_1), \ldots, \Psi(u_r))^\top.$$

Theorem 8 ([15,29]). *Let X be the above random field satisfying Theorem 4. Then, for any sequence of VH-growing sets $W_n \subset \mathbb{R}^d$, one has*

$$V_d(W_n)^{-1/2}\left(S_{\mathbf{u}}(W_n) - \Psi(\mathbf{u})\, V_d(W_n)\right) \xrightarrow{d} \mathcal{N}(0, \Sigma(\mathbf{u}))$$

as $n \to \infty$. Here, $\Sigma(\mathbf{u}) = (\sigma_{lm}(\mathbf{u}))_{l,m=1}^r$ with

$$\sigma_{lm}(\mathbf{u}) = \int_{\mathbb{R}^d} \operatorname{cov}\left(\mathbf{1}\{X(0) \geq u_l\}, \mathbf{1}\{X(t) \geq u_m\}\right) dt.$$

If X is Gaussian as in Example 2, we have

$$\sigma_{lm}(\mathbf{u}) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_0^{\rho(t)} \frac{1}{\sqrt{1-s^2}} \exp\left\{-\frac{(u_l-a)^2 - 2r(u_l-a)(u_m-a) + (u_m-a)^2}{2\tau^2(1-s^2)}\right\} ds\, dt.$$

However, the explicit computation of the elements of matrix Σ for the majority of fields X (except for Gaussianity) seems to be a very complex task. In order to overcome this difficulty in statistical applications of the methatheorem to testing, the matrix Σ can be (weakly) consistently estimated from one observation of a stationary random field X; see [47], [58, Sect. 9.8.3] and references therein.

Statistical Version of the CLT and Tests

Let X be a random field satisfying Theorem 4, $u_k \in \mathbb{R}$, $k = 1, \ldots, r$, and $(W_n)_{n\in\mathbb{N}}$ be a sequence of VH-growing sets. Let $\hat{C}_n = (\hat{c}_{nlm})_{l,m=1}^r$ be a weakly consistent estimator for the nondegenerate asymptotic covariance matrix $\Sigma(\mathbf{u})$, i.e., for any $l, m = 1, \ldots, r$

$$\hat{c}_{nlm} \xrightarrow{P} \sigma_{lm}(\mathbf{u}) \quad \text{as } n \to \infty.$$

Then

$$\hat{C}_n^{-1/2} V_d\,(W_n)^{-1/2}\,(S_{\mathbf{u}}(W_n) - \Psi(\mathbf{u})\,V_d\,(W_n)) \xrightarrow{d} \mathcal{N}(0, I). \tag{8}$$

Based on the latter relation, an asymptotic test for the following hypotheses can be constructed:
H_0 : *X is a random field satisfying Theorem 4 with tail distribution function* $\Psi(\cdot)$ vs. H_1 : *negation of* H_0. As a test statistic, we use

$$T_n = V_d\,(W_n)^{-1}\,(S_{\mathbf{u}}(W_n) - \Psi(\mathbf{u})\,V_d\,(W_n))^{\top}\,\hat{C}_n^{-1}\,(S_{\mathbf{u}}(W_n) - \Psi(\mathbf{u})\,V_d\,(W_n))$$

which is asymptotically χ_r^2 distributed by continuous mapping theorem and relation (8): $T_n \xrightarrow{\text{d}} \chi_r^2$ as $n \to \infty$. Hence, reject the null hypothesis at a confidence level $1-\nu$ if $T_n > \chi^2_{r,1-\nu}$ where $\chi^2_{r,1-\nu}$ is the $(1-\nu)$-quantile of χ_r^2-law.

5.3 Functional Limit Theorems

A natural generalization of multivariate CLTs is a functional CLT where the excursion level $u \in \mathbb{R}$ is treated as a variable, which also appears as a ("time") index in the limiting Gaussian process. In order to state the main results, introduce the *Skorokhod space* $D(\mathbb{R})$ of càdlàg functions on $\mathbb{R}$ endowed with the usual Skorokhod topology, cf. [11, Sect. 12]. Denote by $\Rightarrow$ the weak convergence in $D(\mathbb{R})$.

Define the stochastic processes $Y_n = \{Y_n(u),\ u \in \mathbb{R}\}$ by

$$Y_n(u) = \frac{1}{n^{d/2}}\left(V_d\left(A_u(X, [0,n]^d)\right) - n^d \Psi(u)\right), \quad u \in \mathbb{R}. \tag{9}$$

Introduce the following condition:

($\star$) For any subset $T = \{t_1, \ldots, t_k\} \subset \mathbb{R}^d$ and its partition $T = T_1 \cup T_2$, there exist some constants $c(T), \gamma > 0$ such that

$$\operatorname{cov}\left(\prod_{t_i \in T_1} \phi_{a,b}\left(X(t_i)\right), \prod_{t_j \in T_2} \phi_{a,b}\left(X(t_j)\right)\right) \le c(T)\,(1+\operatorname{dist}_\infty(T_1, T_2))^{-(3d+\gamma)},$$

where $\phi_{a,b}(x) = \mathbf{1}(a < x \le b) - \mathsf{P}(a < X(o) \le b)$ for any real numbers $a < b$.

The following functional CLT is proven in [43, Theorem 1 and Lemma 1].

Theorem 9. *Let $X = \{X(t),\ t \in \mathbb{R}^d\}$ be a real-valued stationary random field with a.s. continuous sample paths and a bounded density of the distribution of $X(o)$. Let condition ($\star$) and Theorem 4 be satisfied. Then $Y_n \Rightarrow Y$ as $n \to \infty$ where $Y = \{Y(u),\ u \in \mathbb{R}\}$ is a centered Gaussian stochastic process with covariance function*

$$C_Y(u,v) = \int_{\mathbb{R}^d} \mathsf{cov}\left(\mathbf{1}\{X(0) \geq u\}, \mathbf{1}\{X(t) \geq v\}\right) dt, \quad u, v \in \mathbb{R}.$$

In particular, condition ($\star$) is satisfied if $X \in \mathbf{A}$ is square integrable with covariance function C that admits a bound

$$|C(t)| \leq \zeta\,(1 + \|t\|_\infty)^{-\lambda}$$

for all $t \in \mathbb{R}^d$ and some $\zeta > 0$, $\lambda > 9d$. The proofs are quite technical involving a Móricz bound for the moment of a supremum of (absolute values of) partial sums of random fields on $\mathbb{Z}^d$, cf. [46, Theorem 2].

For max-stable random fields introduced in Sect. 5.1.2, condition ($\star$) is satisfied if for any $T = \{t_1, \ldots, t_k\} \subset \mathbb{R}^d$ and its partition $T = T_1 \cup T_2$, there exist some constants $c(T), \gamma > 0$ such that

$$\int_E \min\left\{\max_{t_i \in T_1} f_{t_i}(y), \max_{t_j \in T_2} f_{t_j}(y)\right\} \mu(dy) \leq c(T)\,(1 + \mathsf{dist}_\infty(T_1, T_2))^{-(3d+\gamma)}. \tag{10}$$

For α-stable moving averages, i.e., α-stable random fields from Sect. 5.1.2 with $f_t(\cdot) = f(t - \cdot)$ for any $t \in \mathbb{R}^d$, condition (10) should be replaced by

$$\left(\int_{\mathbb{R}^d} \min\left\{\max_{t_i \in T_1} f(t_i - y), \max_{t_j \in T_2} f(t_j - y)\right\}^\alpha m(dy)\right)^{1/(1+\alpha)}$$
$$\leq c(T)\,(1 + \mathsf{dist}_\infty(T_1, T_2))^{-(3d+\gamma)}.$$

These results are proven (under slightly more general assumptions) in [29, Sect. 3.5.5] together with analogous conditions for infinitely divisible random fields (that are too lengthy to give them in a review paper) as well as examples of random fields satisfying them.

Theorem 9 together with the continuous mapping theorem can be used to test hypotheses of Sect. 5.2 with test statistic

$$T_n = \frac{\sup_{u \in \mathbb{R}} Y_n(u)}{\sqrt{\mathsf{E}\,Y_n^2(0)}}$$

if a large deviation result for the limiting Gaussian process Y is available, cf. [43, Corollary 1].

5.4 Limit Theorem for an Increasing Excursion Level

If the level $u \to \infty$, one may also expect that a CLT for the volume of the corresponding excursion set holds, provided that a particular rate of convergence of r to infinity is chosen in accordance with the expansion rate of the observation window.

First, results of this type were proven in [26, Theorems 2.7.1, 2.7.2, 2.8.1] for stationary isotropic Gaussian random fields with short- or long-range dependence. A generalization to the case of stationary **PA**-random fields is given in a recent preprint [19]:

Theorem 10. *Let* $X = \{X(t),\ t \in \mathbb{R}^d\} \in \mathbf{PA}$ *be a stationary random field with a continuous covariance function* C *such that* $|C(t)| = \mathcal{O}\left(\|t\|_2^{-\alpha}\right)$ *for some* $\alpha > 3d$ *as* $\|t\|_2 \to \infty$. *Let* $X(o)$ *have a bounded density* $p_{X(o)}$. *Assume that the variance of* $V_d\left(A_{u_n}\left(X, [0, n]^d\right)\right)$ *being equal to*

$$\sigma_n^2 = \int\limits_{[0,n]^d} \int\limits_{[-x,n-x]^d} \mathsf{cov}\left(\mathbf{1}\{X(o) \ge u_n\}, \mathbf{1}\{X(t) \ge u_n\}\right)\, dt\, dx$$

satisfies

$$\sigma_n^2 \to \infty, \quad n \to \infty. \tag{11}$$

Introduce $\gamma(x) = \sup_{y \ge x} p_{X(o)}(y)$, $x \in \mathbb{R}$. *Choose a sequence of excursion levels* $u_n \to \infty$ *such that*

$$\frac{n^d \gamma^{2/3}(u_n)}{\sigma_n^{2(\alpha+3)/3}} \to 0, \quad n \to \infty. \tag{12}$$

Then it holds

$$\frac{V_d\left(A_{u_n}\left(X, [0, n]^d\right)\right) - n^d \mathsf{P}s(X(o) \ge u_n)}{\sigma_n} \xrightarrow{\mathsf{d}} \mathcal{N}(0, 1) \tag{13}$$

as $n \to \infty$.

Conditions (11), (12) are checked in [19] explicitly for stationary (non-isotropic) Gaussian as well as shot noise random fields leading to quite tractable simple expressions. For instance, it suffices to choose $u_n = O(\sqrt{\log n})$, $n \to \infty$ in the Gaussian case.

Student and Fisher–Snedecor random fields are considered in the recent preprint [34, Sect. 7]. CLTs for spherical measures of excess

$$\int_{\partial B_r(o)} \mathbf{1}\{X(t) > u(r)\}\, \mathcal{H}^{d-1}(dt)$$

of a stationary Gaussian isotropic random field X over the moving level $u(r) \to \infty$, $r \to \infty$ are proved in [37, Sect. 3.3]. For yet another type of geometric measures of excess over a moving level, see [35].

6 Surface Area of Excursion Sets of Stationary Gaussian Random Fields

Limit theorems for $V_{d-1}\left(A_u(X, W_n)\right)$ have been first proven for one fixed level u and a stationary isotropic Gaussian random field X in [31] in dimension $d = 2$. There, the expansion of $V_{d-1}\left(A_u(X, W_n)\right)$ in Hermite polynomials is used. In higher dimensions, a multivariate analogue of this result can be proven along the same guidelines; see [56, Proof of Theorem 1] for a shorter proof. A CLT for the integral of a continuous function along a level curve $\partial A_u(X, W)$ for an a.s. C^1-smooth centered mixing stationary random field $X = \{X(t),\ t \in \mathbb{R}^2\}$ in a rectangle W is proved in [25].

6.1 Functional Limit Theorems

Let us focus on functional LTs for $V_{d-1}\left(\partial A_u(X, W_n)\right)$ proven in [44] for the phase space $L^2(\mathbb{R}, \nu)$ (where ν is a standard Gaussian measure in $\mathbb{R}$) and in [56] for the phase space $C(\mathbb{R})$.

Let $X = \{X(t),\ t \in \mathbb{R}^d\}$, $d > 1$, be a centered stationary and isotropic Gaussian random field with a.s. C^1-smooth paths and covariance function $C \in C^2(\mathbb{R}^d)$ satisfying $C(o) = 1$ as well as

$$|C(t)| + \frac{1}{1 - C(t)} \sum_{i=1}^{d} \left| \frac{\partial C(t)}{\partial t_i} \right| + \sum_{i,j=1}^{d} \left| \frac{\partial^2 C(t)}{\partial t_i \partial t_j} \right| < g(t) \tag{14}$$

for large $\|t\|_2$ (where $t = (t_1, \ldots, t_d)^\top$) and a bounded continuous function $g : \mathbb{R}^d \to \mathbb{R}_+$ such that $\lim_{\|t\|_2 \to \infty} g(t) = 0$ and

$$\int_{\mathbb{R}^d} \sqrt{g(t)}\, dt < \infty.$$

Denote by $\nabla X(t)$ the gradient of $X(t)$. Assume that the $(2d + 2)$-dimensional random vector $(X(o), X(t), \nabla X(o), \nabla X(t))^\top$ is nondegenerate for all $t \in \mathbb{R}^d \setminus \{o\}$. Let $\lambda^2 = -\partial^2 C(o)/\partial t_1^2$.

Introduce the sequence of random processes $\{Y_n\}$, $n \in \mathbb{N}$ by

$$Y_n(u) = \frac{2\lambda^{d/2-1}}{n^{d/2}} \left(V_{d-1}\left(\partial A_u(X, [0,n]^d)\right) - \mathsf{E}\, V_{d-1}\left(\partial A_u(X, [0,n]^d)\right)\right) \tag{15}$$

where $u \in \mathbb{R}$. They will be interpreted as random elements in $L^2(\mathbb{R}, \nu)$. Let $\rightharpoonup$ denote the weak convergence of random elements in $L^2(\mathbb{R}, \nu)$. Let

$$\kappa(t) = f\left(X(t)\right) \exp\{-X^2(t)/2\} \|\nabla X(t)\|_2, \quad t \in \mathbb{R}^d.$$

Theorem 11 ([44]). *Under the above assumptions on X and C, it holds $Y_n \rightharpoonup Y$ as $n \to \infty$ where Y is a centered Gaussian random element in $L^2(\mathbb{R}, \nu)$ with covariance operator*

$$\mathsf{Var}\langle Y, f\rangle_{L^2(\mathbb{R},\nu)} = \frac{1}{2\pi}\int_{\mathbb{R}^d} \mathsf{cov}\left(\kappa(o), \kappa(t)\right)\, dt, \quad f \in L^2(\mathbb{R}, \nu).$$

For $d \geq 3$, processes Y_n have a continuous modification $\tilde{Y}_n$ if conditions on X starting from (14) are replaced by the following ones:

1. Covariance function C as well as all its first- and second- order derivatives belong to $L^1(\mathbb{R})$
2. There exist $\tau \in (0,1)$ and $\beta > 0$ such that for all $h \in [-\tau, \tau]$ and $e_h = (h, 0, 0 \ldots, 0)^\top \in \mathbb{R}^d$, the determinant of the covariance matrix of the vector

$$\left(X(o), X(e_h), \frac{\partial X(o)}{\partial t_1}, \frac{\partial X(e_h)}{\partial t_1}\right)^\top$$

   is not less than $|h|^\beta$.

Let $\rightharpoondown$ denote the weak convergence of random elements in $C(\mathbb{R})$. Denote by $p_{X(t)}$ ($p_{X(o),X(t)}$) the density of $X(t)$ ($(X(o), X(t))^\top$), $t \in \mathbb{R}^d$, respectively. Set

$$H_t(u,v) = \mathsf{E}\left(\|\nabla X(o)\|_2 \|\nabla X(t)\|_2 \,|X(o) = u,\; X(t) = v\right), \quad u, v \in \mathbb{R}, \quad t \in \mathbb{R}^d.$$

In definition (15), assume $\lambda = 1$.

Theorem 12 ([56]). *Under the above assumptions on X and C, it holds $\tilde{Y}_n \rightharpoondown Y$ as $n \to \infty$ for $d \geq 3$ where Y is a centered Gaussian random process with covariance function*

$$\mathsf{cov}\left(Y(u), Y(v)\right) = \int_{\mathbb{R}^d} \left(H_t(u,v) p_{X(o),X(t)}(u,v) - (\mathsf{E}\|\nabla X(o)\|_2)^2\, p_{X(o)}(u) p_{X(t)}(v)\right) dt$$

for $u, v \in \mathbb{R}$.

The case $d = 2$ is still open.

7 Open Problems

It is a challenging problem to prove the whole spectrum of limit theorems for $V_j\,(A_u(X, W_n))$ of lower orders $j = 0, \ldots, d-2$ for isotropic C^2-smooth stationary Gaussian random fields. Functional limit theorems and the case of increasing level $u \to \infty$ are therein of special interest. Further perspective of research is the generalization of these (still hypothetic) results to non-Gaussian random fields.

Another open problem is to prove limit theorems for a large class of functionals of non-Gaussian stationary random fields that includes the volume of excursion sets. It is quite straightforward to do this for

$$\int_{W_n} g(X(t))\, dt$$

for a measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $\mathsf{E}\, g^2(X(o)) < \infty$. For more general classes of functionals of the field X and the observation window W_n, it is still *terra incognita*.

References

1. Adler, R.: On generalizing the notion of upcrossings to random fields. Adv. Appl. Probab. **8**(4), 789–805 (1976)
2. Adler, R.: The Geometry of Random Fields. J. Wiley & Sons, Chichester (1981)
3. Adler, R., Samorodnitsky, G., Taylor, J.: Excursion sets of three classes of stable random fields. Adv. Appl. Probab. **42**, 293–318 (2010)
4. Adler, R., Taylor, J.: Random Fields and Geometry. Springer, New York (2007)
5. Adler, R.J., Taylor, J.E., Worsley, K.J.: Applications of random fields and geometry: Foundations and case studies. Springer Series in Statistics Springer, New York (2009) (In preparation). Http://webee.technion.ac.il/people/adler/hrf.pdf
6. Azais, J.M., Wschebor, M.: Level Sets and Extrema of Random Processes and Fields. Wiley, New York (2009)
7. Belyaev, Y.K.: On the number of intersections of a level by a Gaussian stochastic process, I. Theor. Probab. Appl. **11**(1), 106–113 (1966)
8. Belyaev, Y.K.: On the number of intersections of a level by a Gaussian stochastic process, II. Theor. Probab. Appl. **12**(3), 392–404 (1967)
9. Belyaev, Y.K., Nosko, V.P.: Characteristics of excursions above a high level for a Gaussian process and its envelope. Theor. Probab. Appl. **14**(2), 296–309 (1969)
10. Berman, S.M.: Sojourns and extremes of stochastic processes. Wadsworth & Brooks, CA (1992)
11. Billingsley, P.: Convergence of probability measures, 2nd edn. Wiley Series in Probability and Statistics: Probability and Statistics. Wiley, New York (1999)
12. Bordenave, C., Chafaï, D.: Around the circular law. Probab. Surv. **9**, 1–89 (2012)
13. Bradley, R.C.: Introduction to Strong Mixing Conditions. Vol. 1,2,3. Kendrick Press, Heber City, UT (2007)
14. Bulinskaya, E.V.: On the mean number of crossings of a level by a stationary Gaussian process. Theor. Probab. Appl. **6**, 435–438 (1961)
15. Bulinski, A., Spodarev, E., Timmermann, F.: Central limit theorems for the excursion sets volumes of weakly dependent random fields. Bernoulli **18**, 100–118 (2012)
16. Bulinski, A.V., Shashkin, A.P.: Limit theorems for associated random fields and related systems. World Scientific, Singapore (2007)
17. Cramér, H., Leadbetter, M.: The moments of the number of crossings of a level by a stationary normal process. Ann. Math. Statist. **36**(6), 1656–1663 (1965)
18. Cuzick, J.: A central limit theorem for the number of zeros of a stationary Gaussian process. Ann. Probab. **4**(4), 547–556 (1976)
19. Demichev, V., Schmidt, J.: A central limit theorem for the volume of high excursions of stationary associated random fields. Preprint (2013). Ulm University

20. Doukhan, P.: Mixing: properties and examples. Lecture Notes in Statistics, vol. 85, Springer, Berlin (1994)
21. Elizarov, A.: Central limit theorem for the sojourn time and local time of a stationary process. Theory Probab. Appl. **33**(1), 161–164 (1988)
22. Federer, H.: Curvature measures. Trans. Amer. Math. Soc. **93**, 418–491 (1959)
23. de Haan, L.: A spectral representation for max–stable processes. Ann. Probab. **12**(4), 1194–1204 (1984)
24. Ibragimov, I., Zaporozhets, D.: On the area of a random surface. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **312**, 154–175 (2010) arXiv:1102.3509
25. Iribarren, I.: Asymptotic behaviour of the integral of a function on the level set of a mixing random field. Probab. Math. Stat. **10**(1), 45–56 (1989)
26. Ivanov, A.V., Leonenko, N.N.: Statistical Analysis of Random Fields. Kluwer, Dordrecht (1989)
27. Ivanov, A.V., Leonenko, N.N., Ruiz-Medina, M.D., Savich, I.N.: Limit theorems for weighted nonlinear transformations of Gaussian stationary processes with singular spectra. Ann. Probab. **41**(2) (2013)
28. Kac, M.: On the average number of real roots of a random algebraic equation. Bull. Amer. Math. Soc. **49** 314-320 (1943)
29. Karcher, W.: On infinitely divisible random fields with an application in insurance. Ph.D. thesis, Ulm University, Ulm (2012)
30. Kratz, M.: Level crossings and other level functionals of stationary Gaussian processes. Probab. Surv. **3**, 230–288 (2006)
31. Kratz, M., Léon, J.: Central limit theorems for level functionals of stationary Gaussian processes and fields. J. Theor. Probab. **14**, 639–672 (2001)
32. Kratz, M., Léon, J.: Level curves crossings and applications for Gaussian models. Extremes **13**(3), 315–351 (2010)
33. Leadbetter, M.R., Lindgren, G., Rootzen, H.: Extremes and related properties of random sequences and processes, 1st edn. Springer Series in Statistics. Springer, New York (1983)
34. Leonenko, N., Olenko, A.: Sojourn measures of Student and Fischer–Snedecor random fields. Bernoulli (2013) (in press)
35. Leonenko, N.N.: Limit distributions of characteristics of exceeding a level by a Gaussian field. Math. notes of the Acad. Sci. of the USSR **41**(4), 339–345 (1987)
36. Leonenko, N.N.: Sharpness of the normal approximation of functionals of strongly correlated Gaussian random fields. Math. notes Acad. Sci. USSR **43**(2), 161–171 (1988)
37. Leonenko, N.N.: Limit theorems for random fields with singular spectrum. Kluwer, Dordrecht (1999)
38. Liu, J.: Tail approximations of integrals of Gaussian random fields. Ann. Probab. **40**(3), 1069–1104 (2012)
39. Liu, J., Xu, G.: Some asymptotic results of Gaussian random fields with varying mean functions and the associated processes. Ann. Statist. **40**(1), 262–293 (2012)
40. Malevich, T.L.: Asymptotic normality of the number crossings of level zero by a Gaussian process. Theor. Probab. Appl. **14**, 287–295 (1969)
41. Marinucci, D., Peccati, G.: Random Fields on the Sphere. Representation, Limit Theorems and Cosmological Applications, Lecture Notes of the London Mathematical Society, vol. 389. Cambridge University Press, Cambridge (2011)
42. Mecke, K., Stoyan, D. (eds.): Morphology of condensed matter. Physics and Geometry of Spatially Complex Systems, Lecture Notes in Physics, vol. 600. Springer, Berlin (2002)
43. Meschenmoser, D., Shashkin, A.: Functional central limit theorem for the volume of excursion sets generated by associated random fields. Statist. Probab. Lett. **81**(6), 642–646 (2011)
44. Meschenmoser, D., Shashkin, A.: Functional central limit theorem for the measures of level surfaces of the Gaussian random field. Theor. Probab. Appl. **57**(1), 162–172 (2013)
45. Molchanov, I.: Theory of Random Sets. Springer, London (2005)
46. Móricz, F.: A general moment inequality for the maximum of the rectangular partial sums of multiple series. Acta Math. Hung. **41**, 337–346 (1983)

47. Pantle, U., Schmidt, V., Spodarev, E.: On the estimation of integrated covariance functions of stationary random fields. Scand. J. Statist. **37**, 47–66 (2010)
48. Piterbarg, V.I.: The central limit theorem for the number of level crossings of a stationary Gaussian process. Theor. Probab. Appl. **23**(1), 178–182 (1978)
49. Resnik, S.I.: Extreme values, regular variation and point processes. Springer, Berlin (2008)
50. Rice, S.: Mathematical analysis of random noise. Bell Syst. Tech. J. **23**, 282–332 (1944)
51. Rice, S.: Mathematical analysis of random noise. Bell Syst. Tech. J. **24**, 46–156 (1945)
52. Samorodnitsky, G., Taqqu, M.: Stable non-Gaussian random processes. Chapman & Hall/CRC, London (1994)
53. Santaló, L.: Integral Geometry and Geometric Probability. Addison-Welsey, Reading, Mass (1976)
54. Schneider, R.: Convex Bodies. The Brunn–Minkowski Theory. Cambridge University Press, Cambridge (1993)
55. Schwartzman, A., Dougherty, R.F., Taylor, J.E.: False discovery rate analysis of brain diffusion direction maps. Ann. Appl. Stat. **2**(1), 153–175 (2008)
56. Shashkin, A.: A functional central limit theorem for the level measure of a Gaussian random field. Statist. Probab. Lett. **83**(2), 637–643 (2013)
57. Slud, E.: MWI representation of the number of curve–crossings by a differentiable Gaussian process, with applications. Ann. Probab. **22**(3), 1355–1380 (1994)
58. Spodarev, E. (ed.): Stochastic Geometry, Spatial Statistics and Random Fields. Asymptotics Methods, Lecture Notes in Mathematics, vol. 2068. Springer, Berlin (2013)
59. Spodarev, E., Zaporozhets, D.: Asymptotics of the mean Minkowski functionals of Gaussian excursions (2012). Preprint
60. Taylor, J.E., Worsley, K.J.: Detecting sparse signal in random fields, with an application to brain mapping. J. Am. Stat. Assoc. **102**(479), 913–928 (2007)
61. Torquato, S.: Random Heterogeneous Materials: Microstructure and Macroscopic Properties. Springer, New York (2002)
62. Worsley, K.J., Taylor, J.: Detecting fMRI activation allowing for unknown latency of the hemodynamic response. Neuroimage **29**, 649–654 (2006)