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Modern Stochastics and Applications

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Aims and Scope
Optimization has been expanding in all directions at an astonishing rate during the last few decades. New algorithmic and theoretical techniques have been developed, the diffusion into other disciplines has proceeded at a rapid pace, and our knowledge of all aspects of the field has grown even more profound. At the same time, one of the most striking trends in optimization is the constantly increasing emphasis on the interdisciplinary nature of the field. Optimization has been a basic tool in all areas of applied mathematics, engineering, medicine, economics, and other sciences.

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Modern Stochastics and Applications

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Dedicated to the memory of our teachers,
Boris Gnedenko and Mykhaylo Yadrenko

Preface

This special volume contains contributed papers of selected speakers of the International Conference "Modern Stochastics: Theory and Applications III." This conference was held on September 10–14, 2012, at Taras Shevchenko National University of Kyiv, Ukraine. It was dedicated to anniversaries of prominent Ukrainian scientists of international recognition: 100th anniversary of B.V. Gnedenko and 80th anniversary of M.I. Yadrenko.

This conference is third in the "Modern Stochastics: Theory and Applications" series, the first two having taken place in 2006 and 2010. It was a major scientific event, providing an excellent opportunity for exchanging ideas and discussing recent results and new trends in probability, statistics, and their applications. The conference covered all research areas in probability theory and its applications: stochastic analysis, stochastic processes and fields, random matrices, optimization methods in probability, stochastic models of evolution systems, financial mathematics, risk processes and actuarial mathematics, statistics, information security, etc. Over 250 scientists from 29 countries took part in the conference, including both top-level specialists as well as young researchers.

The editors pursued two goals in collecting chapters for this volume: to present the most deep and bleeding-edge results and to make this volume accessible to as wide audience as possible. This resulted in extensive overview of some modern trends in probability and stochastic analysis and its applications. Scientific researchers will find in the volume a variety of new tools, ideas, and optimization methods, while practitioners will find a rigorous mathematical background for their studies.

The volume consists of five parts.

The first part is devoted to properties of probability distributions and their applications. The chapter by F. Hirsch and M. Yor describes the relation between the inequalities for the integrands of stochastic integrals and convex order of the integrals. This result is of particular interest in financial mathematics for pricing of contingent claims in continuous financial market models. Yu.V. Kozachenko and R.E. Yamnenko show how sub-Gaussian random processes can be applied to queuing theory. New probabilistic tools for mathematical physics are given in the chapter by E. Orsingher, where pseudoprocesses governed by higher-order heat-type equations are considered and a probabilistic representation of their densities is given. The chapter by S. Rœlly is concerned with reciprocal processes (whose concept can be traced back to E. Schrödinger), discussing an approach to characterize different types of these processes via duality formulae on path spaces.

The second part discusses stochastic ordinary and partial differential equations. The chapter by Ya.I. Belopolskaya shows how theory of backward stochastic differential equations can be extended in order to construct a viscosity solution to the Cauchy problem for a system of quasilinear parabolic equations. Stochastic partial differential equations with fractional noise are studied in the chapter by M. Dozzi, E.T. Kolkovska, and J.A. López-Mimbela, which focuses mainly on the cases where solution of equation does not exist globally, but blows up in a finite time. The authors make substantial contribution by giving both lower and upper bounds for the blowup time of solutions of stochastic partial differential equations, thus providing tools for sharp estimation of risks and reliability. S. Fang shows how variational principles for the Navier–Stokes equation can be established by analysis of ordinary and stochastic differential equations with Sobolev coefficients, thus opening new ways to investigate this famous equation from hydrodynamics. Stochastic partial differential equations driven by fractional noises with long memory are also studied in the chapter by M. Hinz, E. Issoglio, and M. Zähle, who survey recent results on solvability and regularity of such equations. Stochastic integrals with respect to general stochastic measures are studied in the chapter by V. Radchenko. The uniqueness of the approach is that no special assumptions, such as martingale property or integrability, are imposed, which allows to apply these results in a vast variety of situations. It allows to establish the solvability of parabolic stochastic partial differential equations under mildest assumptions on the random driver.

The third part of this collection is about limit theorems for stochastic processes and fields. S.V. Anulova and A.Yu. Veretennikov make a considerable progress in the investigation of Langevin–Smoluchowski-type system, proving existence of unique solution with a strong Markov property and establishing exponential stability of the system. V.P. Knopova and A.M. Kulik study fractional Lévy motion both in long-memory and in short-memory cases and provide examples showing the difference of asymptotic behavior of the distribution density of the process in these cases. V.S. Korolyuk and I.V. Samoilenko discuss the problem of large deviations for random evolutions in the scheme of asymptotically small diffusion. The chapter by E. Spodarev is devoted to limit behavior of geometric characteristics of random fields, which are widely used for data analysis purposes in medicine (FMR image processing), physics and cosmology (e.g., microwave background radiation analysis), and materials science (quantification of porous media).

Particular attention in the volume is given to financial applications of stochastic analysis, which is the subject of *the fourth part* of the book. J.M. Corcuera, G. Farkas, and A. Valdivia review ambit processes, which share their mathematical structure with the solutions of random evolution equations, allowing them great flexibility for modelling. They discuss applications of ambit processes to finance. A.A. Gushchin, R.V. Khasanov, and I.S. Morozov give a review of new tools in the utility maximization problem. They propose new ideas on how to embed the original maximization problem in an appropriate functional space and then to deduce the dual variational problem. The techniques developed by the authors allow to get rid of singular functionals in the dual problem. O. Ragulina considers the classical risk model where an insurance company has the opportunity to adjust franchise amount continuously. She solves the problem of optimal control by franchise amount.

The last part of this book is devoted to statistics. The chapter by Yu. Mishura, K. Ral' chenko, O. Seleznev, and G. Shevchenko discusses parameter estimation for stochastic differential equations driven by fractional Brownian motion. The chapter by L. Sakhno deals with the parameter estimation of stationary fields in the spectral domain based on minimum contrast principle. The chapter by S. Shklyar presents different methods of estimation for generalized linear models with measurement errors (Poissonian regression, Gamma regression, exponential regression).

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Part I
Probability Distributions in Applications

Comparing Brownian Stochastic Integrals for the Convex Order

Francis Hirsch and Marc Yor

Abstract We show that, in general, inequalities between integrands with respect to Brownian motion do not lead to majorization in the convex order for the corresponding stochastic integrals. Particular examples and counterexamples are discussed.

1 Introduction

In this chapter, we are interested in the following general question. Let X and Y be square integrable Brownian centered random variables given by their predictable representations:

$$X = \int_0^\infty H_t \, \mathrm{d}B_t \qquad Y = \int_0^\infty K_t \, \mathrm{d}B_t$$

with

$$\int_0^\infty \mathbb{E}[H_t^2] \, \mathrm{d}t < \infty \ \text{ and } \int_0^\infty \mathbb{E}[K_t^2] \, \mathrm{d}t < \infty. \tag{1}$$

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Is it possible to give conditions on H and K ensuring that $X \le Y$ in the convex order?

We recall that two integrable random variables X and Y are said to satisfy $X \le Y$ in the convex order, which will be denoted in the sequel by $X \overset{(c)}{\le} Y$, if, for every convex $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$,

$$-\infty < \mathbb{E}[\varphi(X)] \le \mathbb{E}[\varphi(Y)] \le +\infty.$$

There is an obvious necessary condition:

$$X \overset{(c)}{\le} Y \Longrightarrow \int_0^\infty \mathbb{E}[H_t^2]\,\mathrm{d}t \le \int_0^\infty \mathbb{E}[K_t^2]\,\mathrm{d}t. \tag{2}$$

This condition is far from being sufficient. In Sect. 2 we present an easy counterexample.

Then, in Sect. 3, we consider the elementary case where (H_t) or (K_t) is deterministic. In this case, there is a simple sufficient condition.

In the three following sections, we study particular families $(X^f(a); a \ge 0)$ defined by

$$X^f(a) = \int_0^\infty f(a, s, B_s)\,\mathrm{d}B_s,$$

where f denotes a nonnegative Borel function on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}$.

In Sect. 4, $f(a, s, x) = 1_{(0,1)}(s)\,1_{(a,+\infty)}(x)$. Then f is decreasing with respect to a, and the map: $a \longrightarrow X^f(a)$ also is decreasing in the convex order.

In Sect. 5, $f(a, s, x) = 1_{(0,1)}(s)\,1_{(-\infty,a)}(x)$. Then f is increasing with respect to a, and the map: $a \longrightarrow X^f(a)$ also is increasing in the convex order.

In Sect. 6, $f(a, s, x) = a\,1_{(0,1)}(s) + 1_{(1,2)}(s)\,1_{(-\infty,0)}(x)$. Then f is increasing with respect to a, and the map: $a \longrightarrow X^f(a)$ is not monotone in the convex order. More precisely, $a \longrightarrow \mathbb{E}[(X^f(a))^2]$ is obviously increasing, but $a \longrightarrow \mathbb{E}[\exp(X^f(a))]$ is strictly decreasing on $[0, a_0]$ for some $a_0 > 0$.

2 A Simple Example

In this section, we show that, in general, the converse of (2) does not hold, even if H is deterministic.

Proposition 1. *We set, for $\lambda \ge 0$,*

$$X_\lambda = \int_0^1 \exp(\lambda\, B_s)\,\mathrm{d}B_s.$$

Obviously, for any $\lambda \geq 0$,

$$\int_0^1 \mathbb{E}[\exp(2\lambda\, B_s)]\, \mathrm{d}s = \int_0^1 \exp(2\lambda^2 s)\, \mathrm{d}s \geq 1.$$

However, there exists $\lambda > 0$ *such that* $B_1 \overset{(c)}{\leq} X_\lambda$ *does not hold.*

Proof. For every C^1-function φ with bounded first derivative, we have

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \mathbb{E}[\varphi(X_\lambda)]\bigg|_{\lambda=0} = \frac{1}{2}\, \mathbb{E}[\varphi'(B_1)\,(B_1^2 - 1)].$$

Suppose that for every $\lambda > 0$, one has $B_1 \overset{(c)}{\leq} X_\lambda$. Then, for every convex C^1-function φ with bounded first derivative, $\mathbb{E}[\varphi'(B_1)\,(B_1^2 - 1)] \geq 0$. In particular, we obtain for the convex function: $\varphi(x) = (x+1)^2\, 1_{(-1,0)}(x) + (1+2x)\, 1_{(0,+\infty)}(x)$,

$$2\,\mathbb{E}[((B_1 + 1)^+ \wedge 1)\,(B_1^2 - 1)] \geq 0.$$

Now,

$$\mathbb{E}[((B_1 + 1)^+ \wedge 1)\,(B_1^2 - 1)] = \frac{1}{\sqrt{2\pi}}\,(\mathrm{e}^{-1/2} - 1) < 0,$$

which yields a contradiction. □

3 Case Where *H* Or *K* Is Deterministic

The following proposition partially extends a result of Pagès [8, Proposition 2.4] with a different method.

Proposition 2. *Let* (H_t) *be an adapted process and* k *be a deterministic Borel function such that*

$$\int_0^\infty \mathbb{E}[H_t^2]\, \mathrm{d}t < \infty \quad \text{and} \quad \int_0^\infty k^2(t)\, \mathrm{d}t < \infty.$$

We set

$$X = \int_0^\infty H_t\, \mathrm{d}B_t \qquad Y = \int_0^\infty k(t)\, \mathrm{d}B_t.$$

One has

1. *if* $\int_0^\infty H_t^2 \, \mathrm{d}t \leq \int_0^\infty k(t)^2 \, \mathrm{d}t$ *a.s., then* $X \overset{(c)}{\leq} Y$*;*
2. *if* $\int_0^\infty k(t)^2 \, \mathrm{d}t \leq \int_0^\infty H_t^2 \, \mathrm{d}t$ *a.s., then* $Y \overset{(c)}{\leq} X$.

Proof. By the Dubins–Schwarz theorem (see, e.g., Revuz-Yor [9, Chap. V]), there exist a filtration $(\mathcal{G}_u)$ and a $\mathcal{G}$-Brownian motion (β_u) such that: $X = \beta_T$, with $T := \int_0^\infty H_t^2 \, \mathrm{d}t$ a $(\mathcal{G}_u)$-stopping time. By hypothesis, T is integrable. Denote by s the deterministic time $\int_0^\infty k(t)^2 \, \mathrm{d}t$. Let also $\tilde{Y} = \beta_s$. Since k is deterministic, we have: $Y \overset{\text{(law)}}{=} \tilde{Y}$. Now, if $T \leq s$, then $X = \mathbb{E}[\tilde{Y} \mid \mathcal{G}_T]$, and if $s \leq T$, then $\tilde{Y} = \mathbb{E}[X \mid \mathcal{G}_s]$. The desired result then follows from Jensen's inequality. □

Remark 1. We shall show in Sect. 6 that in the above proposition, the hypothesis that k is deterministic cannot be deleted. Likewise, the hypothesis: $\int_0^\infty \mathbb{E}[H_t^2] \, \mathrm{d}t < \infty$ cannot be deleted as shown by the following example. Suppose that $k(t) = 1_{(0,1)}(t)$ and $H_t = 1_{(0,d_1)}(t)$ with $d_1 = \inf\{t \geq 1; B_t = 0\}$. Then, since $1 \leq d_1$, one has $0 \leq k(t) \leq H_t$. But, $X = B_{d_1} = 0$, $Y = B_1$ and, obviously, $B_1 \overset{(c)}{\leq} 0$ does not hold. On the contrary, we have $0 \overset{(c)}{\leq} B_1$, which shows that, in general, the implication (2) does not hold if the condition (1) is not fulfilled.

4 A Decreasing Family

We set, for $t \geq 0$ and $a \geq 0$,

$$X_t(a) = \int_0^t 1_{(B_s > a)} \, \mathrm{d}B_s$$

and we denote $X_1(a)$ simply by $X(a)$. We also denote as usual by T_a the entrance time of (B_t) in $[a, +\infty[$.

Proposition 3. *The map:* $a \geq 0 \longrightarrow X(a)$ *is decreasing in the convex order.*

Proof. Denote by $B^{(a)}$ the Brownian motion defined by $B_t^{(a)} = B_{t+T_a} - a$. It is independent of $\mathcal{F}_{T_a}^B$ (where $(\mathcal{F}_t^B)$ denotes the natural filtration of B) and, in particular, it is independent of T_a. We set, for $t \geq 0$,

$$X_t^{(a)} = \int_0^t 1_{(B_s^{(a)} > 0)} \, \mathrm{d}B_s^{(a)}.$$

We clearly have

$$X(a) = X_{(1-T_a)^+}^{(a)}. \tag{3}$$

Let $\tilde{B}$ be an independent copy of B. We deduce from (3) that

$$X(a) \overset{\text{(law)}}{=} \tilde{X}_{(1-T_a)^+}(0), \tag{4}$$

where $\tilde{X}_t(a)$ is defined as $X_t(a)$ from $\tilde{B}$ in place of B. We set, for $a \geq 0$,

$$\hat{\mathfrak{F}}_a = \sigma\{B_s, s \geq 0; \tilde{B}_{s\wedge(1-T_a)^+}, s \geq 0\}.$$

Then, the above family of σ-algebras is decreasing with respect to a. We have, if $a \leq b$,

$$\tilde{X}_{(1-T_b)^+}(0) = \mathbb{E}[\tilde{X}_{(1-T_a)^+}(0) \,|\, \hat{\mathfrak{F}}_b].$$

The desired result follows from (4) and Jensen's inequality. □

Remark 2. Proposition 3 says, in the terminology of Hirsch et al. [3], that $(X(a); a \geq 0)$ is an *inverse peacock*. By the general theorem of Kellerer (see [2, 5] and also [3, Exercise 1.6]), there exists an inverse martingale which is *associated* to $(X(a); a \geq 0)$, which means that both processes have the same 1-marginals (we also say that $(X(a); a \geq 0)$ is a *1-inverse martingale*). In the previous proof, we showed that we may take as associated inverse martingale: $(\tilde{X}_{(1-T_a)^+}(0), a \geq 0)$.

In the sequel of this section, we shall give another proof, more analytic, of Proposition 3, from a computation of the law of $X(a)$.

Proposition 4. *The law of $X(a)$ is*

$$\sqrt{\frac{2}{\pi}}\left(\left(\int_0^a e^{-u^2/2}\,du\right)\delta_0 + h_a(z)\,dz\right),$$

where δ_0 denotes the Dirac measure at 0, and

$$h_a(z) = \frac{4}{3}\exp\left(-\frac{(a-2z)^2}{2}\right)1_{(-\infty,0)}(z) + \frac{1}{3}\exp\left(-\frac{(a+z)^2}{2}\right)1_{(0,+\infty)}(z).$$

Proof. Denote by μ_a the law of T_a. One has

$$\mu_a(dy) = \frac{a}{\sqrt{2\pi}}\,1_{(0,+\infty)}(y)\,y^{-3/2}\exp\left(-\frac{a^2}{2y}\right)\,dy. \tag{5}$$

By (4), we have for every nonnegative φ,

$$\mathbb{E}[\varphi(X(a))] = \varphi(0)\int_1^{+\infty}\mu_a(dy) + \int_0^1 \mathbb{E}[\varphi(X_{1-y}(0))]\,\mu_a(dy). \tag{6}$$

By (5) and the change of variable $y = a^2 u^{-2}$, we obtain

$$\int_1^{+\infty} \mu_a(\mathrm{d}y) = \sqrt{\frac{2}{\pi}} \int_0^a \mathrm{e}^{-u^2/2}\, \mathrm{d}u. \tag{7}$$

By Tanaka's formula, $X_t(0) = B_t^+ - \frac{1}{2} L_t$ where L_t denotes the local time at 0 of the Brownian B. According to [9, Exercise 2.18], the joint law of (B_t, L_t) has a density given by

$$\frac{1}{\sqrt{2\pi t^3}} \left[(a+b) \exp\left(-\frac{(a+b)^2}{2t}\right)\right] \quad \text{for } a, b \geq 0 \text{ and}$$
$$\frac{1}{\sqrt{2\pi t^3}} \left[(-a+b) \exp\left(-\frac{(-a+b)^2}{2t}\right)\right] \quad \text{for } a \leq 0, b \geq 0.$$

Consequently, the density of the law of $X_t(0)$ is

$$\sqrt{\frac{2}{\pi\, t}} \left[\frac{4}{3} \exp\left(-\frac{2u^2}{t}\right) 1_{(u<0)} + \frac{1}{3} \exp\left(-\frac{u^2}{2t}\right) 1_{(u>0)}\right]. \tag{8}$$

Thus, by (5) and (8),

$$\int_0^1 \mathbb{E}[\varphi(X_{1-y}(0))]\, \mu_a(\mathrm{d}y) = \frac{a}{3\pi} \int_0^\infty I_a(z)\, [2\varphi(-z/2) + \varphi(z)]\, \mathrm{d}z \tag{9}$$

with

$$I_a(z) = \int_0^1 y^{-3/2}(1-y)^{-1/2} \exp\left(-\frac{a^2}{2y}\right) \exp\left(-\frac{z^2}{2(1-y)}\right)\, \mathrm{d}y.$$

The change of variable: $y = a^2 u (1 + a^2 u)^{-1}$ yields

$$\begin{aligned} I_a(z) &= a^{-1} \exp\left(-\frac{(a^2+z^2)}{2}\right) \int_0^\infty u^{-3/2} \exp\left(-\frac{1}{2u}\right) \exp\left(-\frac{a^2 z^2 u}{2}\right)\, \mathrm{d}u \\ &= \sqrt{2\pi}\, a^{-1} \exp\left(-\frac{(a^2+z^2)}{2}\right) \mathbb{E}\left[\exp\left(-\frac{(az)^2 T_1}{2}\right)\right] \end{aligned}$$

and hence

$$I_a(z) = \sqrt{2\pi}\, a^{-1} \exp\left(-\frac{(a+z)^2}{2}\right). \tag{10}$$

Finally, gathering (6), (7), (9), and (10), we obtain the announced result, after an obvious change of variable. □

The next corollary follows easily from Proposition 4.

Corollary 1. *Let φ be a suitably integrable function. Then, for any $a \geq 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[\varphi(X(a))]$$
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty (a+z) \exp\left(-\frac{(a+z)^2}{2}\right) \left(\varphi(0) - \frac{2}{3}\varphi(-z/2) - \frac{1}{3}\varphi(z)\right) \mathrm{d}z.$$

In particular, if moreover φ is convex, then

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[\varphi(X(a))] \leq 0$$

and the inequality is strict if and only if φ is not an affine function.

Clearly, the above corollary entails Proposition 3.

By Remark 2, $(X(a); a \geq 0)$ is a 1-inverse martingale. One may wonder whether it also is a *2-inverse martingale*, that is, whether it has the same 2-marginals as an inverse martingale. We answer this question in the next proposition.

Proposition 5. *For every $a > 0$, $\mathbb{E}[X(0)\, X(a)^2] > \mathbb{E}[X(a)^3]$.*
Consequently, $(X(a); a \geq 0)$ is not a 2-inverse martingale.

Proof. Set:

$$E(a) = \mathbb{E}[X(0)\, X(a)^2] - \mathbb{E}[X(a)^3].$$

By Itô's formula:

$$E(a) = \int_0^1 \mathbb{E}\left[\left(\int_0^t 1_{(0<B_s<a)}\, \mathrm{d}B_s\right) 1_{(B_t>a)}\right] \mathrm{d}t.$$

We set, for $t > 0$,

$$U(t) = \mathbb{E}\left[\left(\int_0^t 1_{(0<B_s<1)}\, \mathrm{d}B_s\right) 1_{(B_t>1)}\right] \tag{11}$$

By scaling,

$$\mathbb{E}\left[\left(\int_0^t 1_{(0<B_s<a)}\, \mathrm{d}B_s\right) 1_{(B_t>a)}\right] = a\, U(a^{-2}t).$$

Hence,

$$E(a) = a \int_0^1 U(a^{-2}t)\, \mathrm{d}t = a^3 \int_0^{a^{-2}} U(t)\, \mathrm{d}t. \tag{12}$$

The result will then follow from the next lemma.

Lemma 1. *For every $t > 0$,*

$$U(t) = \frac{1}{2\sqrt{2\pi}} \int_{t^{-1}}^{\infty} e^{-u/2} u^{-1/2} \, du,$$

where $U(t)$ is defined in (11).

Proof. By Tanaka's formula,

$$U(t) = \mathbb{E}\left[\left(1 + \frac{1}{2} L_t^1 - \frac{1}{2} L_t^0\right) 1_{(B_t > 1)}\right],$$

where L_t^1 (resp. L_t^0) denotes the local time of the Brownian motion at time t in 1 (resp. 0). Using the classical property

$$L_t^x = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t 1_{(x < B_s < x+\varepsilon)} \, ds$$

we obtain

$$\mathbb{E}[L_t^1 \, 1_{(B_t > 1)}] = \frac{1}{2\sqrt{2\pi}} \int_0^t e^{-1/2s} s^{-1/2} \, ds, \tag{13}$$

$$\mathbb{E}[L_t^0 \, 1_{(B_t > 1)}] = \frac{1}{2\pi} \int_0^t e^{-1/2s} (t-s)^{1/2} s^{-3/2} \, ds. \tag{14}$$

By (14),

$$\frac{d}{dt} \mathbb{E}[L_t^0 \, 1_{(B_t > 1)}] = \frac{1}{4\pi} \int_0^t e^{-1/2s} (t-s)^{-1/2} s^{-3/2} \, ds$$

and the change of variable $s = t\,(2vt + 1)^{-1}$ yields

$$\frac{d}{dt} \mathbb{E}[L_t^0 \, 1_{(B_t > 1)}] = \frac{1}{2\sqrt{2\pi}} e^{-1/2t} t^{-1/2},$$

which is equal to $\frac{d}{dt} \mathbb{E}[L_t^1 \, 1_{(B_t > 1)}]$ by (13). Thus, $\mathbb{E}[L_t^0 \, 1_{(B_t > 1)}] = \mathbb{E}[L_t^1 \, 1_{(B_t > 1)}]$, and consequently $U(t) = \mathbb{P}[B_t > 1]$, which is the announced result. □

The proposition follows then from the above lemma and (12). □

Remark 3. By Lemma 1, $\lim_{t \to \infty} U(t) = \frac{1}{2}$ and $U(t)$ is equivalent to $\frac{1}{\sqrt{2\pi}} e^{-1/2t} t^{1/2}$ when t tends to 0. Therefore, by (12), $E(a)$ is equivalent to $\frac{a}{2}$ when a tends to 0 and is equivalent to $\sqrt{\frac{2}{\pi}} e^{-a^2/2} a^{-2}$ when a tends to ∞.

5 An Increasing Family

We set, for $t \geq 0$ and $a \geq 0$,

$$Y_t(a) = \int_0^t 1_{(B_s<a)} \, \mathrm{d}B_s$$

and we denote $Y_1(a)$ simply by $Y(a)$.

Proposition 6. *The law of $Y(a)$ is*

$$\sqrt{\frac{2}{\pi}} \left(A_a(z)\, 1_{(-\infty,a)}(z) + D_a(z)\, 1_{(a,+\infty)}(z)\right) \, \mathrm{d}z$$

with

$$A_a(z) = \left[\frac{1}{2}\exp\left(-\frac{z^2}{2}\right) - \frac{1}{6}\exp\left(-\frac{(2a-z)^2}{2}\right)\right]$$

$$\text{and} \qquad D_a(z) = \frac{4}{3}\exp\left(-\frac{(2z-a)^2}{2}\right).$$

Proof. We shall follow the same lines as in the proof of Proposition 4, of which we keep the notation.
We set, for $t \geq 0$,

$$Y_t^{(a)} = \int_0^t 1_{(B_s^{(a)}<0)} \, \mathrm{d}B_s^{(a)}.$$

We clearly have

$$Y(a) = 1_{(T_a>1)}\, B_1 + 1_{(T_a<1)} \left(a + Y_{(1-T_a)}^{(a)}\right) \tag{15}$$

and

$$1_{(T_a<1)}\, B_1 = 1_{(T_a<1)} \left(a + B_{(1-T_a)}^{(a)}\right). \tag{16}$$

Since $Y^{(a)}$ and $B^{(a)}$ are independent of T_a, by (15) and (16), we have for every nonnegative φ,

$$\mathbb{E}[\varphi(Y(a))] = \mathbb{E}[\varphi(B_1)] + \int_0^1 \left(\mathbb{E}[\varphi(a + Y_{1-y}(0))] - \mathbb{E}[\varphi(a + B_{1-y})]\right) \, \mu_a(\mathrm{d}y). \tag{17}$$

Clearly, $Y_t(0) \overset{\text{(law)}}{=} -X_t(0)$. Consequently, by (8), the density of the law of $Y_t(0)$ is

$$\sqrt{\frac{2}{\pi\, t}} \left[\frac{1}{3}\exp\left(-\frac{u^2}{2t}\right) 1_{(u<0)} + \frac{4}{3}\exp\left(-\frac{2u^2}{t}\right) 1_{(u>0)}\right]. \tag{18}$$

By (18) and (5),

$$\int_0^1 \mathbb{E}[\varphi(a+Y_{1-y}(0))]\ \mu_a(\mathrm{d}y) = \frac{a}{3\pi}\int_0^\infty I_a(z)\,[2\varphi(a+z/2)+\varphi(a-z)]\ \mathrm{d}z, \quad (19)$$

where $I_a(z)$ is given in (10). Likewise,

$$\int_0^1 \mathbb{E}[\varphi(a+B_{1-y})]\ \mu_a(\mathrm{d}y) = \frac{a}{2\pi}\int_0^\infty I_a(z)\,[\varphi(a+z)+\varphi(a-z)]\ \mathrm{d}z. \quad (20)$$

Finally, gathering (17), (19), (20), and (10), we obtain the announced result, after obvious changes of variable. □

The next corollary follows easily from Proposition 6.

Corollary 2. *Let φ be a suitably integrable function. Then, for any $a \geq 0$,*

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[\varphi(Y(a))]$$
$$= \sqrt{\frac{2}{\pi}}\int_0^\infty (a+z)\exp\left(-\frac{(a+z)^2}{2}\right)\left(\frac{2}{3}\varphi(a+z/2)+\frac{1}{3}\varphi(a-z)-\varphi(a)\right)\mathrm{d}z.$$

In particular, if moreover φ is convex, then

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[\varphi(Y(a))] \geq 0$$

and the inequality is strict if and only if φ is not an affine function.

Clearly, the above corollary entails the following proposition.

Proposition 7. *The map: $a \geq 0 \longrightarrow Y(a)$ is increasing in the convex order.*

Proposition 7 says, in the terminology of Hirsch et al. [3], that $(Y(a); a \geq 0)$ is a *peacock*. By the general aforementioned theorem of Kellerer, there exists a martingale which is associated to $(Y(a); a \geq 0)$ (in the sense that both processes have the same 1-marginals). We shall exhibit such a martingale, using *the stochastic differential equation method* (see [3, Chap. 6] and [2]).

We first introduce some further notation. We set, for $(a,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$p(a,x) = \frac{1}{\sqrt{2\pi}}\left[\exp\left(-\frac{x^2}{2}\right) - \frac{1}{3}\exp\left(-\frac{(2a-x)^2}{2}\right)\right] \ \text{ if } x < a$$

$$\text{and} \ \ p(a,x) = \frac{1}{\sqrt{2\pi}}\,\frac{8}{3}\exp\left(-\frac{(2x-a)^2}{2}\right) \ \text{ if } x \geq a.$$

Thus, by Proposition 6, the law of $Y(a)$ is $p(a,x)\,\mathrm{d}x$. We then define the *call function* C by

$$\forall (a,x) \in \mathbb{R}_+ \times \mathbb{R}, \quad C(a,x) = \mathbb{E}[(Y(a)-x)^+].$$

We also set, for $x \in \mathbb{R}$,

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp\left(-\frac{u^2}{2}\right)\, \mathrm{d}u.$$

Lemma 2.

1. For every $(a,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$p(a,x) \geq \frac{2}{3\sqrt{2\pi}} \exp\left(-\frac{[(2x-a)\vee(2a-x)]^2}{2}\right) \quad \textit{and}$$

$$p(a,x) \leq \frac{8}{3\sqrt{2\pi}} \exp\left(-\frac{[(2x-a)\vee x]^2}{2}\right).$$

2. For $(a,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\frac{\partial}{\partial a} C(a,x) = \frac{2}{3}\,\mathcal{N}((2x-a)\vee(2a-x)).$$

3. For $(a,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\frac{\partial}{\partial a} C(a,x) \leq \frac{\sqrt{2\pi}}{2}\; p(a,x).$$

Proof. The first point follows from the above definition of p.
The second point is a direct consequence of Corollary 2.
For the third point, we remark that $(2x-a)\vee(2a-x) \geq a \geq 0$ and

$$\sup\{\exp(u^2/2)\,\mathcal{N}(u); u \geq 0\} = \mathcal{N}(0) = 1/2.$$ □

We now set, for $(a,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\sigma(a,x) = \left(2\,\frac{\frac{\partial}{\partial a} C(a,x)}{p(a,x)}\right)^{1/2}.$$

This definition of σ comes from Dupire [1].

Proposition 8. *The stochastic differential equation:*

$$M_t = M_0 + \int_0^t \sigma(s, M_s)\, \mathrm{d}B_s, \quad M_0 \overset{\text{(law)}}{=} Y(0) \tag{21}$$

admits a weak solution which is unique in law. Such a solution is a continuous, strong Markov martingale, which is associated to the peacock $(Y(a); a \geq 0)$.

Proof. We shall first prove the existence of a weak solution to (21). We remark that, by Lemma 2, one has $0 < \sigma \leq (2\pi)^{1/4}$. However, σ is not continuous on $\mathbb{R}_+ \times \mathbb{R}$, but only continuous on the complement of $\{(a,a); a \geq 0\}$. So, we need to approximate σ. We set, for $\varepsilon > 0$ and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$p_\varepsilon(t,x) = \int_0^1 p(t, x + \varepsilon\, u)\, \mathrm{d}u.$$

Thus, $p_\varepsilon(t,x)\,\mathrm{d}x$ is the law of $Y_t^\varepsilon := Y(t) - \varepsilon\, U$, where U denotes a uniform variable on [0, 1], independent of $Y(t)$. Clearly, p_ε is continuous and > 0 on $\mathbb{R}_+ \times \mathbb{R}$. We set:

$$C_\varepsilon(t,x) = \mathbb{E}[(Y_t^\varepsilon - x)^+] = \int_0^1 C(t, x + \varepsilon\, u)\, \mathrm{d}u.$$

Consequently, by Lemma 2, for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$0 < \frac{\partial}{\partial t} C_\varepsilon(t,x) = \int_0^1 \frac{\partial}{\partial t} C(t, x + \varepsilon\, u)\, \mathrm{d}u \leq \frac{\sqrt{2\pi}}{2}\, p_\varepsilon(t,x).$$

We then set, for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\sigma_\varepsilon(t,x) = \left(2\, \frac{\frac{\partial}{\partial t} C_\varepsilon(t,x)}{p_\varepsilon(t,x)} \right)^{1/2}.$$

Thus, σ_ε is continuous and, for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$0 < \sigma_\varepsilon(t,x) \leq (2\pi)^{1/4}. \tag{22}$$

Therefore, the stochastic differential equation

$$M_t = M_0 + \int_0^t \sigma_\varepsilon(s, M_s)\, \mathrm{d}B_s, \quad M_0 \overset{\text{(law)}}{=} Y_0^\varepsilon \tag{23}$$

admits a weak solution, and, using M. Pierre's uniqueness theorem ([3, Theorem 6.1]), one sees as in the proof of Theorem 6.2 in [3] that such a solution M^ε is unique in law and M^ε is a continuous martingale, which is associated to the peacock $(Y_t^\varepsilon; t \geq 0)$. Besides, by (22) and BDG inequalities, for every $\gamma > 0$, there exists $c_\gamma > 0$ such that

$$\forall \varepsilon > 0,\ \forall s, t \geq 0, \quad \mathbb{E}[|M_t^\varepsilon - M_s^\varepsilon|^\gamma] \leq c_\gamma |t-s|^{\gamma/2}. \tag{24}$$

We denote by $\mathbb{P}^\varepsilon$ the law of M^ε on $C(\mathbb{R}_+;\mathbb{R})$. We deduce from (24) and Kolmogorov's criterion (see, e.g., [9, Theorem 1.8, Chap. XIII]) that the family of laws $\{\mathbb{P}^\varepsilon; \varepsilon > 0\}$ is weakly relatively compact. Therefore, there exists a sequence (ε_n) tending to 0 and a probability $\mathbb{P}$ on $C(\mathbb{R}_+;\mathbb{R})$ such that $\mathbb{P}^{\varepsilon_n}$ weakly converges to $\mathbb{P}$ when n tends to infinity. We denote by M a continuous process with law $\mathbb{P}$. Obviously, the law of M_t is $p(t,x)\,\mathrm{d}x$. To show that M is a weak solution to (21), we shall prove that $\mathbb{P}$ is a solution to the corresponding martingale problem. Let f be a C^2-function with compact support and let $0 \leq s \leq t$ and g be a bounded continuous function on $C([0,s];\mathbb{R})$. By (23), for every n,

$$\mathbb{E}^{\varepsilon_n}\left[\left(f(y_t) - f(y_s) - \frac{1}{2}\int_s^t \sigma_{\varepsilon_n}^2(u, y_u)\, f''(y_u)\,\mathrm{d}u\right) g(y|_{[0,s]})\right] = 0.$$

We set

$$R_n = \mathbb{E}^{\varepsilon_n}\left[\int_s^t |\sigma_{\varepsilon_n}^2(u, y_u) - \sigma^2(u, y_u)|\,\mathrm{d}u\right].$$

Then,

$$R_n = \int_{\mathbb{R}} \mathrm{d}x \int_0^1 \mathrm{d}v \int_s^t \mathrm{d}u\, |\sigma_{\varepsilon_n}^2(u,x) - \sigma^2(u,x)|\, p(u, x+\varepsilon_n v).$$

Since, for every $u \geq 0$, $x \longrightarrow p(u,x)$ is right-continuous, we obtain by dominated convergence (see point 1 in Lemma 2): $\lim_{n\to\infty} R_n = 0$. We define the bounded function H on $C(\mathbb{R}_+;\mathbb{R})$ by

$$H(y) = \left(f(y_t) - f(y_s) - \frac{1}{2}\int_s^t \sigma^2(u, y_u)\, f''(y_u)\,\mathrm{d}u\right) g(y|_{[0,s]}).$$

We obtain by what precedes: $\lim_{n\to\infty} \mathbb{E}^{\varepsilon_n}[H] = 0$. On the other hand, H is continuous at any y such that $u \neq y_u$ $\mathrm{d}u$-a.e. Now, since the law of y_u under $\mathbb{P}$ admits a density, namely, $p(u,x)$, one has

$$\int_0^\infty \int 1_{\{u=y_u\}}\,\mathbb{P}(\mathrm{d}y)\,\mathrm{d}u = 0.$$

Therefore, $\mathbb{P}$-a.s., $u \neq y_u$ $\mathrm{d}u$-a.e. Thus, H is continuous at every point of the complement of a $\mathbb{P}$-negligible set. By a classical result, this entails $\lim_{n\to\infty} \mathbb{E}^{\varepsilon_n}[H] = \mathbb{E}[H]$, and therefore, $\mathbb{E}[H] = 0$ which means

$$\mathbb{E}\left[\left(f(y_t) - f(y_s) - \frac{1}{2}\int_s^t \sigma^2(u, y_u)\, f''(y_u)\,\mathrm{d}u\right) g(y|_{[0,s]})\right] = 0.$$

So, $\mathbb{P}$ is a solution to the martingale problem corresponding to $(Y(0), \sigma^2)$, which is equivalent to say that M is a weak solution of (21).

Now, by Lemma 2, σ satisfies the conditions allowing to apply M. Pierre's uniqueness theorem [3, Theorem 6.1] with $a = \frac{1}{2}\sigma^2$. Consequently, we may show, as in the proof of Theorem 6.2 in [3], the uniqueness in law of the weak solution of (21), and the strong Markov property follows. □

Remark 4. By the results of Lowther [6], the martingale M, weak solution of (21), is the only (in law) continuous, strong Markov martingale associated to $(Y(a); a \geq 0)$. It also follows from Lowther [6, 7] (see also [4, Theorem 4.4]) that there exists a continuous inverse martingale associated to $(X(a); a \geq 0)$. This inverse martingale is therefore different, in law, of the one proposed in Remark 2, which is not continuous. Note that the above method does not apply to $(X(a); a \geq 0)$, since, as seen in Proposition 4, the law of $X(a)$ is not absolutely continuous.

Here again, one may wonder whether $(Y(a); a \geq 0)$ is a *2-martingale*, that is, whether it has the same 2-marginals as a martingale. We answer this question in the next proposition.

Proposition 9. *For every* $a > 0$, $\mathbb{E}[Y(0)^2\, Y(a)] < \mathbb{E}[Y(0)^3]$.
Consequently, $(Y(a); a \geq 0)$ *is not a 2-martingale.*

Proof. We follow the same lines as in the proof of Proposition 5. Set:

$$F(a) = \mathbb{E}[Y(0)^2\, Y(a)] - \mathbb{E}[Y(0)^3].$$

By Itô's formula:

$$F(a) = \int_0^1 \mathbb{E}\left[\left(\int_0^t 1_{(0<B_s<a)}\, \mathrm{d}B_s\right) 1_{(B_t<0)}\right] \mathrm{d}t.$$

We set, for $t > 0$,

$$V(t) = \mathbb{E}\left[\left(\int_0^t 1_{(0<B_s<1)}\, \mathrm{d}B_s\right) 1_{(B_t<0)}\right] \tag{25}$$

By scaling,

$$\mathbb{E}\left[\left(\int_0^t 1_{(0<B_s<a)}\, \mathrm{d}B_s\right) 1_{(B_t<0)}\right] = a\, V(a^{-2}t).$$

Hence,

$$F(a) = a \int_0^1 V(a^{-2}t)\, \mathrm{d}t = a^3 \int_0^{a^{-2}} V(t)\, \mathrm{d}t. \tag{26}$$

The result will then follow from the next lemma.

Lemma 3. *For every $t > 0$,*

$$V(t) = \frac{-1}{4\sqrt{2\pi}} \int_{t^{-1}}^{\infty} (1 - \mathrm{e}^{-2u})\, u^{-3/2}\, \mathrm{d}u,$$

where $V(t)$ is defined in (25).

Proof. By Tanaka's formula,

$$V(t) = \frac{1}{2}\,\mathbb{E}\left[\left(L_t^1 - L_t^0\right) 1_{(B_t<0)}\right],$$

where L_t^1 (resp. L_t^0) still denotes the local time of the Brownian motion at time t in 1 (resp. 0). Using again

$$L_t^x = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_0^t 1_{(x<B_s<x+\varepsilon)}\, \mathrm{d}s$$

we obtain

$$\mathbb{E}[L_t^0\, 1_{(B_t<0)}] = \frac{t^{1/2}}{\sqrt{2\pi}}$$

$$\mathbb{E}[L_t^1\, 1_{(B_t<0)}] = \frac{1}{4\pi} \int_0^t \mathrm{e}^{-1/2s}\, s^{-1/2} \int_0^{t-s} \mathrm{e}^{-1/2u}\, u^{-3/2}\, \mathrm{d}u\, \mathrm{d}s. \tag{27}$$

By (27),

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[L_t^1\, 1_{(B_t<0)}] = \frac{1}{4\pi} \int_0^t \exp\left(-\frac{t}{2}\,\frac{1}{s\,(t-s)}\right)(t-s)^{-3/2}\, s^{-1/2}\, \mathrm{d}s$$

and the change of variable $s = t\,(v+1)^{-1}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[L_t^1\, 1_{(B_t<0)}] = \frac{1}{2\sqrt{2\pi}}\, \mathrm{e}^{-2/t}\, t^{-1/2}.$$

Thus, we obtain

$$V(t) = \frac{-1}{4\sqrt{2\pi}} \int_0^t (1 - \mathrm{e}^{-2/s})\, s^{-1/2}\, \mathrm{d}s$$

which yields the desired result. □

The proposition follows then from the above lemma and (26). □

Remark 5. By Lemma 3, $\lim_{t\to\infty} V(t) = -\frac{1}{2}$ and $V(t)$ is equivalent to $-\frac{1}{2\sqrt{2\pi}} t^{1/2}$ when t tends to 0. Therefore, by (26), $F(a)$ is equivalent to $-\frac{a}{2}$ when a tends to 0, and $\lim_{a\to\infty} F(a) = -\frac{1}{3\sqrt{2\pi}}$.

6 An Example of Non-monotony

The following example (which is inspired from Pagès [8, Example 2.4.3]) shows that there exist functions f and g, satisfying

$$\forall s \geq 0, \forall x \in \mathbb{R}, \quad 0 \leq f(s,x) \leq g(s,x) \leq 1$$

and such that $\int f(s,B_s)\mathrm{d}B_s$ and $\int g(s,B_s)\mathrm{d}B_s$ are not comparable in the convex order. In particular, Proposition 2 does not hold in general, if neither H nor K is deterministic.

Proposition 10. *Set, for $a \geq 0$,*

$$Z(a) = a\,B_1 + \int_1^2 1_{(B_s<0)}\,\mathrm{d}B_s.$$

Then, there exists $a_0 > 0$ such that $a \longrightarrow \mathbb{E}[\exp(Z(a))]$ is strictly decreasing on $[0,a_0]$. In particular, if $0 \leq a_1 < a_2 \leq a_0$, then $Z(a_1)$ and $Z(a_2)$ are not comparable in the convex order.

Proof. We begin with the following lemma.

Lemma 4. *We set, for $x \in \mathbb{R}$,*

$$\mathcal{E}(x) = \mathbb{E}\left[\exp\left(\int_0^1 1_{(B_s<x)}\,\mathrm{d}B_s\right)\right].$$

Then, for every $x \in \mathbb{R}$, $\mathcal{E}'(x) > 0$, and hence $\mathcal{E}$ is strictly increasing on $\mathbb{R}$.

Proof. For $x \geq 0$, $\mathcal{E}(x) = \mathbb{E}[\exp(Y(x))]$, and, for $x \leq 0$, $\mathcal{E}(x) = \mathbb{E}[\exp(-X(-x))]$. The result follows then from Corollaries 1 and 2. □

We have:

$$Z(a) = a\,B_1 + \int_0^1 1_{(\tilde{B}_s<-B_1)}\,\mathrm{d}\tilde{B}_s$$

with $\tilde{B}_s = B_{1+s} - B_1$. Since $(\tilde{B}_s)_{0\leq s\leq 1}$ is independent of B_1, one obtains:

$$\mathbb{E}[\exp(Z(a))] = \mathbb{E}[\exp(aB_1)\,\mathcal{E}(-B_1)].$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[\exp(Z(a))]\bigg|_{a=0} = \mathbb{E}[B_1\,\mathcal{E}(-B_1)].$$

Since $x \longrightarrow \mathcal{E}(-x)$ is strictly decreasing by Lemma 4 and $x \longrightarrow x$ is strictly increasing,

$$\frac{\mathrm{d}}{\mathrm{d}a}\mathbb{E}[\exp(Z(a))]\bigg|_{a=0} < \mathbb{E}[B_1]\,\mathbb{E}[\mathcal{E}(-B_1)] = 0.$$

This entails the desired result. □

References

1. Dupire, B.: Pricing with a smile. Risk Mag. **7**, 18–20 (1994)
2. Hirsch, F., Roynette, B.: A new proof of Kellerer's theorem. ESAIM: PS **16**, 48–60 (2012)
3. Hirsch, F., Profeta, C., Roynette, B., Yor, M.: Peacocks and Associated Martingales, with Explicit Constructions. Bocconi and Springer Series, vol. 3. Springer, New York (2011)
4. Hirsch, F., Roynette, B., Yor, M.: Kellerer's theorem revisited, Prépublication Université dÉvry, vol. 361 (2012)
5. Kellerer, H.G.: Markov-Komposition und eine Anwendung auf Martingale. Math. Ann. **198**, 99–122 (1972)
6. Lowther, G.: Fitting martingales to given marginals. http://arxiv.org/abs/0808.2319v1 (2008)
7. Lowther, G.: Limits of one-dimensional diffusions. Ann. Probab. **37–1**, 78–106 (2009)
8. Pagès, G.: Convex order for path-dependent American options using the Euler scheme of martingale jump diffusion process. http://hal.archives-ouvertes.fr/hal-00767885 (2012)
9. Revuz, D., Yor, M.: Continuous Martingales and Brownian Motion, 3rd edn. Springer, New York (1999)

Application of φ-Sub-Gaussian Random Processes in Queueing Theory

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Abstract The chapter is devoted to investigation of the class $V(\varphi, \psi)$ of φ-sub-Gaussian random processes with application to queueing theory. This class of stochastic processes is more general than the Gaussian one; therefore, all results obtained in general case are valid for Gaussian processes by selection of certain Orlicz N-functions φ and ψ. We consider different queues filled by an aggregate of such independent sources and obtain estimates for the tail distribution of some extremal functionals of incoming random processes and their increments which describe behavior of the queue. We obtain the upper bound for the buffer overflow probability for the corresponding storage process and apply obtained result to the aggregate of sub-Gaussian generalized fractional Brownian motion processes.

1 Introduction

Consider a single server queue that is filled by the aggregate of **N** independent (uncorrelated) random sources $X_i = \{X_i(t), t \in T\}$, where T is some parametric set, e.g., an interval $[a, b]$. Our main interest is focused on studying the distribution of the following functionals that depend on the incoming aggregate and characterize the behavior of the queue:

$$\sup_{t \in T} \left(\sum_{i=1}^{\mathbf{N}} X_i(t) - f(t) \right),$$

$$\sup_{s \le t,\ s,t \in T} \left(\sum_{i=1}^{\mathbf{N}} X_i(t) - f(t) - \left(\sum_{i=1}^{\mathbf{N}} X_i(s) - f(s) \right) \right),$$

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and

$$\sup_{s,t \in T} \sup_{s \leq t} \left(\sum_{i=1}^{\mathbf{N}} X_i(t) - f(t) - \left(\sum_{i=1}^{\mathbf{N}} X_i(s) - f(s) \right) \right),$$

where $f(t)$ is a continuous function which describes intensity of the queue serving.

Here we summarize recent studies [6–8, 13–18] for a general class of incoming processes X_i. We assume that incoming streams belong to the class $V(\varphi, \psi)$ and study the buffer overflow probability for the storage process $Q(t)$. Our results are illustrated by sub-Gaussian generalized fractional Brownian motion (GFBM). Recall that the normalized FBM process with Hurst parameter $H \in (0.5, 1)$ is the Gaussian centered process with stationary increments, continuous paths, and covariance function of the form

$$R_H(t,s) = \left(t^{2H} + s^{2H} - |s-t|^{2H}\right)/2. \tag{1}$$

Its long-range dependence and self-similarity properties make the FBM process a natural choice for modeling traffic through telecommunication networks (see more results on FBM storage models in [1, 10–12]).The paper is organized as follows. Section 2 is devoted to the general theory of φ-sub-Gaussian random processes and is based on the works [2–6, 8, 9, 13, 18]. In Sect. 3 we consider the storage process from the class $V(\varphi, \psi)$ with application to GFBM processes.

2 Random Variables from Spaces $\mathrm{Sub}_\varphi(\Omega)$, $\mathrm{SSub}_\varphi(\Omega)$, and Class $V(\varphi, \psi)$

This section contains some basic notions, definitions and properties of random variables, and processes from the spaces $\mathrm{Sub}_\varphi(\Omega)$, $\mathrm{SSub}_\varphi(\Omega)$, and the class $V(\varphi, \psi)$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a standard probability space and let (T, ρ) be a pseudometric (metric) compact space equipped by pseudometric (metric) ρ.

Definition 1 ([2]). Metric entropy with regard to pseudometric (metric) ρ or just metric entropy is a function

$$H_{(T,\rho)}(u) = H_T(u) = H(u) = \begin{cases} \log N_{(T,\rho)}(u), & \text{if } N_{(T,\rho)}(u) < +\infty \\ +\infty, & \text{if } N_{(T,\rho)}(u) = +\infty \end{cases},$$

where $N_{(T,\rho)}(u) = N_T(u) = N(u)$ denotes the least number of closed ρ-balls with radius u covering space (T, ρ).

Example 1. If T is an interval $[a, b]$ and ρ is the Euclidean distance, then

$$\ln\left(\max\left\{\frac{b-a}{2u}, 1\right\}\right) \le H(u) \le \ln\left(\frac{b-a}{2u}+1\right).$$

Definition 2 ([2]). A continuous even convex function φ is said to be an Orlicz N-function if it is strictly increasing for $x > 0$, $\varphi(0) = 0$ and

$$\frac{\varphi(x)}{x} \to 0 \text{ as } x \to 0 \quad \text{and} \quad \frac{\varphi(x)}{x} \to \infty \text{ as } x \to \infty.$$

Example 2. The following functions are N-functions.

- $\varphi(x) = \alpha|x|^\beta, \ \alpha > 0, \ \beta > 1;$
- $\varphi(x) = \exp\{|x|\} - |x| - 1;$
- $\varphi(x) = \exp\{\alpha|x|^\beta\} - 1, \ \alpha > 0, \ \beta > 1;$
- $\varphi(x) = \begin{cases} \left(e\alpha/2\right)^{2/\alpha} x^2, & |x| \le \left(2/\alpha\right)^{1/\alpha}; \\ \exp\{|x|^\alpha\}, & |x| > \left(2/\alpha\right)^{1/\alpha}, \ 0 < \alpha < 1. \end{cases}$

Condition Q ([3]). We say that an N-function φ satisfies Condition Q if

$$\liminf_{x\to 0} \frac{\varphi(x)}{x^2} = \alpha > 0. \tag{2}$$

It is permitted α to be infinite.

Example 3. Condition Q is fulfilled for N-function $\varphi(x) = c|x|^\beta, \ c > 0$, when $1 < \beta \le 2$ and is not fulfilled when $\beta > 2$.

Definition 3 ([2]). Let φ be an Orlicz N-function satisfying Condition Q. The random variable ξ belongs to the space $\mathrm{SUB}_\varphi(\Omega)$ (a space of φ-sub-Gaussian random variables), if it is centered, i.e., $\mathbf{E}\xi = 0$, the moment generating function $\mathbf{E}\exp\{\lambda\xi\}$ exists for all $\lambda \in R$ and there exists a positive constant a such that the following inequality

$$\mathbf{E}\exp(\lambda\xi) \le \exp(\varphi(a\lambda)) \tag{3}$$

holds for all $\lambda \in R$.

Theorem 1 ([2]). *The space* $\mathrm{SUB}_\varphi(\Omega)$ *is a Banach space with respect to the norm* $\tau_\varphi(\xi) = \inf\{a \ge 0 \colon \mathbf{E}\exp(\lambda\xi) \le \exp(\varphi(a\lambda)), \lambda \in R\}$ *and the inequality*

$$\mathbf{E}\exp(\lambda\xi) \le \exp\big(\varphi\big(\lambda\tau_\varphi(\xi)\big)\big), \tag{4}$$

holds for all $\lambda \in R$. *Moreover, for all* $r > 0$ *there exists constant* $c_r > 0$ *such that*

$$\left(\mathbf{E}\xi^r\right)^{1/r} \le c_r \tau_\varphi(\xi). \tag{5}$$

When $\varphi(x) = x^2/2$, the space $\mathrm{SUB}_\varphi(\Omega)$ is called the space of *sub-Gaussian* random variables and is denoted by $\mathrm{SUB}(\Omega)$. The simplest examples of sub-Gaussian random variables are the following:

- Centered Gaussian random variables $\xi = N(0, \sigma^2)$ belong to the space $\mathrm{SUB}(\Omega)$ and $\tau(\xi) = (\mathbf{E}\xi^2)^{1/2}$.
- Let ξ be a centered bounded random variable, i.e., $\mathbf{E}\xi = 0$, and there exists number $c > 0$ that $|\xi| \leq c$ almost surely. Then $\xi \in \mathrm{SUB}(\Omega)$ and $\tau(\xi) \leq c$.

Theorem 2 ([8]). *Let $\xi \in \mathrm{SUB}_\varphi(\Omega)$. Then for all $\varepsilon > 0$ the following inequality holds:*

$$\mathbf{P}\{|\xi| > \varepsilon\} \leq 2\exp\left\{-\varphi\left(\frac{\varepsilon}{\tau_\varphi(\xi)}\right)\right\}.$$

Definition 4 ([2]). Random process $X = \{X(t), t \in T\}$ is called a *φ-sub-Gaussian process* if for all $t \in T$ $X(t) \in \mathrm{SUB}_\varphi(\Omega)$.

Condition Σ. Suppose there exists such a continuous monotonically increasing function $\sigma = \{\sigma(h), h > 0\}$ that $\sigma(h) \to 0$, as $h \to 0$, and the following inequality for increments of the process is true:

$$\sup_{\rho(t,s)\leq h} \tau_\varphi(Y(t) - Y(s)) \leq \sigma(h). \tag{6}$$

If a process $X(t)$ is continuous in norm $\tau_\varphi(\cdot)$, then the function

$$\sigma(h) = \sup_{\rho(t,s)\leq h} \tau_\varphi(Y(t) - Y(s))$$

satisfies Condition Σ.

Theorem 3 ([2]). *Let φ be an Orlicz N-function satisfying Condition Q and the function $\varphi(\sqrt{\cdot})$ be convex. Suppose that $\xi_1, \xi_2, \ldots, \xi_n$ are independent random variables from the space $\mathrm{SUB}_\varphi(\Omega)$. Then*

$$\tau_\varphi^2\left(\sum_{i=1}^n \xi_i\right) \leq \sum_{i=1}^n \tau_\varphi^2(\xi_i). \tag{7}$$

Definition 5 ([4]). A family of random variables Δ from the space $\mathrm{SUB}_\varphi(\Omega)$ is called *strictly* $\mathrm{SUB}_\varphi(\Omega)$, if there exists a constant $C_\Delta > 0$ such that for arbitrary finite set $I : \xi_i \in \Delta, i \in I$ and for any $\lambda_i \in R$, the following inequality takes place:

$$\tau_\varphi\left(\sum_{i\in I} \lambda_i \xi_i\right) \leq C_\Delta\left(\mathbf{E}\left(\sum_{i\in I} \lambda_i \xi_i\right)^2\right)^{1/2}. \tag{8}$$

If Δ is a family of strictly $\mathrm{SUB}_{\varphi}(\Omega)$ random variables, then linear closure $\overline{\Delta}$ of the family Δ in the space $L_2(\Omega)$ is also strictly $\mathrm{SUB}_{\varphi}(\Omega)$ family of random variables. Linearly closed families of strictly $\mathrm{SUB}_{\varphi}(\Omega)$ random variables form a space of *strictly φ-sub-Gaussian* random variables. This space is denoted by $\mathrm{SSUB}_{\varphi}(\Omega)$.

When $\varphi(x) = x^2/2$, the space $\mathrm{SSUB}_{\varphi}(\Omega)$ is called the space of *strictly sub-Gaussian* random variables and is denoted as $\mathrm{SSUB}(\Omega)$. The space of jointly Gaussian random variables belongs to the space $\mathrm{SSUB}(\Omega)$ and $\tau^2(\xi) = \mathbf{E}\xi^2$, i.e., $C_{\Delta} = 1$.

Definition 6 ([2]). A random process $X = \{X(t), t \in T\}$ is a *strictly φ-sub-Gaussian process* if the corresponding family of random variables belongs to the space $\mathrm{SSUB}_{\varphi}(\Omega)$.

Example 4 ([4]). Let φ be such an Orlicz N-function that the function $\varphi(\sqrt{\cdot})$ is convex and

$$X(t) = \sum_{k=1}^{\infty} \xi_k \phi_k(t),$$

where series $\sum\limits_{k=1}^{\infty} \xi_k \phi_k(t)$ converges in mean square sense for all $t \in T$ and family $\{\xi_k, k \geq 1\}$ belongs to the space $\mathrm{SSUB}_{\varphi}(\Omega)$, for instance, $\{\xi_k, k \geq 1\}$ are independent random variables from $\mathrm{SSUB}_{\varphi}(\Omega)$. Then $X(t)$ is a strictly φ-sub-Gaussian random process.

Example 5 ([7]). We call the process $Z^H = (Z^H(t), t \in T)$ *strictly φ-sub-Gaussian GFBM* with Hurst index $H \in (0, 1)$, if Z^H is a strictly φ-sub-Gaussian process with stationary increments and covariance function as defined by (1).

In order to give an example of such a process, let's consider a sequence of independent strictly φ-sub-Gaussian random variables $\{\eta_n, n = 1, 2, \ldots\}$ for which $\mathbf{E}\eta_n = 0$, $\mathbf{E}\eta_n^2 = 1$, and φ is such an N-function that function $\varphi(\sqrt{\cdot})$ is convex and $\tau_{\varphi}(\eta_n) \leq \tau < +\infty$. Then the process $Z^H(t) = \sum\limits_{n=1}^{\infty} \lambda_n \eta_n \psi_n(t)$ is a centered strictly φ-sub-Gaussian random process with covariance function R_H from (1), if λ_n are eigenvalues and ψ_n are corresponding eigenfunctions of the following integral equation:

$$\psi(s) = \lambda^{-2} \int_0^T R_H(t, s) \psi(t) \, \mathrm{d}t.$$

Definition 7 ([6]). N-function φ is subordinated by an Orlicz N-function ψ ($\varphi \prec \psi$) if there are exist such numbers $x_0 > 0$ and $k > 0$ that $\varphi(x) < \psi(kx)$ for $x > x_0$.

Theorem 4 ([13]). *Let φ_1 and φ_2 be such N-functions that $\varphi_1 \prec \varphi_2$. If $\xi \in \mathrm{Sub}_{\varphi_1}(\Omega)$, then $\xi \in \mathrm{Sub}_{\varphi_2}(\Omega)$, and there exists such a constant c_{φ_1,φ_2} that $\tau_{\varphi_2}(\xi) \le c_{\varphi_1,\varphi_2}\tau_{\varphi_1}(\xi)$.*

Definition 8 ([6]). Let $\varphi \prec \psi$ are two Orlicz N-functions. Random process $X = \{X(t), t \in T\}$ belongs to class $V(\varphi,\psi)$ if for all $t \in T$ the random variable $X(t)$ is from $\mathrm{Sub}_{\psi}(\Omega)$ and, for all $s,t \in T$ increments $(X(t) - X(s))$ belong to the family $\mathrm{Sub}_{\varphi}(\Omega)$.

Example 6. Sub-Gaussian random processes belong to the class $V(\varphi,\varphi)$ with $\varphi(x) = x^2/2$.

Example 7. Let

$$X(t) = \xi_0 + \sum_{k=1}^{\infty} \xi_k f_k(t),$$

where φ is such an Orlicz N-function that $\varphi(\sqrt{\cdot})$ is a convex function. Let ξ_0 be a ψ-sub-Gaussian random variable and $\{\xi_k, k = 1,2,\dots\}$ be a sequence of φ-sub-Gaussian random variables such that $\sum\limits_{k=1}^{\infty} \tau_\varphi(\xi_k)|f_k(t)| < \infty$. Then the process $X(t)$ belongs to the class $V(\varphi,\psi)$.

Condition *F*. A continuous function $f = \{f(t), t \in T\}$ satisfies Condition *F* if

$$|f(u) - f(v)| \le \delta(\rho(u,v)),$$

where $\delta = \{\delta(s), s > 0\}$ is some monotonically increasing nonnegative function.

Condition *R*. A continuous function $r = \{r(u), u \ge 1\}$ satisfies Condition *R* if $r(u) > 0$ when $u > 1$ and function $s(t) = r(\exp\{t\}), t \ge 0$, is convex.

Let $B \subset T$ be a compact set. In what follows we use the following notation:

- $\gamma(u) = \tau_\psi(X(u)) < \infty$;
- $\beta > 0$ is such a number that $\beta \le \sigma\left(\inf\limits_{s\in B}\sup\limits_{t\in B}\rho(t,s)\right)$;
- $B_t = \{u \in B : u \le t\}$;
- $L(u) = \big(N(u)^2 + N(u)\big)/2$, where function $N(u)$ is denoted in Definition 1.

Theorem 5 ([18]). *Let $X = \{X(t), t \in B\}$ be a separable random process from the class $V(\varphi,\psi)$ which satisfies Condition Σ. Let functions $f = \{f(t), t \in B\}$ and $r = \{r(u): u \ge 1\}$ satisfy Conditions F and R, respectively. If*

$$\int_0^{\beta} r\big(N(\sigma^{(-1)}(u))\big)\,\mathrm{d}u < \infty,$$

then for all $p \in (0,1)$ *and* $x > 0$ *the following inequality holds*

$$\mathbf{P}\left\{\sup_{t\in T}(X(t)-f(t)) > x\right\} \le \inf_{\lambda>0} Z_r(\lambda, p, \beta),$$

where

$$Z_r(\lambda, p, \beta)$$
$$= \exp\left\{\theta_\psi(\lambda, p) + p\varphi\left(\frac{\lambda\beta}{1-p}\right) + \lambda\left(\sum_{k=2}^{\infty}\delta(\sigma^{(-1)}(\beta p^{k-1})) - x\right)\right\}$$
$$\times r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p} r(\mathsf{N}(\sigma^{(-1)}(u)))\,\mathrm{d}u\right), \tag{9}$$

$$\theta_\psi(\lambda, p) = \sup_{u\in T}\left((1-p)\psi\left(\frac{\lambda\gamma(u)}{1-p}\right) - \lambda f(u)\right). \tag{10}$$

Example 8 ([18]). Let $Z^H = \{Z^H(t), t \in [a,b]\}$, $0 \le a < b < \infty$ be a GFBM process from the class $V(\varphi, \psi)$ with Hurst index $H \in (0,1)$ and let $c > 0$ be a constant service rate. Then for all $p \in (0,1)$, $\beta \in \left(0, (b-a/2)^H\right]$, and $\lambda > 0$, the following inequality holds:

$$\mathsf{P}\left\{\sup_{a\le t\le b}\left(Z^H(t) - ct\right) > x\right\} \le (b-a)\left(\frac{e}{\beta p}\right)^{1/H}$$
$$\times\exp\left\{\frac{\lambda c(\beta p)^{1/H}}{C_\Delta(1-p^{1/H})} + p\varphi\left(\frac{\lambda\beta}{1-p}\right) + (1-p)\theta_\psi(\lambda, p) - \frac{\lambda x}{C_\Delta}\right\}, \tag{11}$$

where $\theta_\psi(\lambda, p) = \sup\limits_{a\le u\le b}\left(\psi\left(\frac{\lambda u^H}{1-p}\right) - \frac{\lambda c u}{C_\Delta(1-p)}\right)$.

More details on the GFBM process can be found in papers [7, 13–15].

Condition ΣN. We say that independent separable random processes $X_i = \{X_i(t), t \in B\}$ from classes $V(\varphi_i, \psi_i)$ defined on a compact set $B \subset T$ satisfy Condition ΣN if there exist such continuous monotone increasing functions $\{\sigma_i(h), h \ge 0\}$ that $\sigma_i(h) \to 0$ when $h \to 0$ and

$$\sup_{\rho(t,s)\le h} \tau_\varphi(X_i(t) - X_i(s)) \le \sigma_i(h), \tag{12}$$

$$\gamma_i(u) = \tau_\psi(X_i(u)) < \infty, \tag{13}$$

$$\sigma(h) = \sup_{1\le i\le \mathbf{N}} \sigma_i(h) < \infty, \quad i = \overline{1, \mathbf{N}}. \tag{14}$$

Theorem 6. *Let random process* $X(t) = \sum_{i=1}^{\mathbf{N}} X_i(t)$ *satisfy the assumptions of Theorem 5, where functions* $\gamma_i(u)$ *and* $\sigma(h)$ *are given in (13) and (14). Then for all* $p \in (0,1)$ *and* $x > 0$*, following inequalities hold:*

$$\mathbf{P}\left\{\sup_{t\in B}\left(\sum_{i=1}^{\mathbf{N}} X_i(t) - f(t)\right) > x\right\} \le Z_r(p,\beta,x),$$

$$\mathbf{P}\left\{\inf_{t\in B}\left(\sum_{i=1}^{\mathbf{N}} X_i(t) - f(t)\right) < -x\right\} \le Z_r(p,\beta,x),$$

$$\mathbf{P}\left\{\sup_{t\in B}\left|\sum_{i=1}^{\mathbf{N}} X_i(t) - f(t)\right| > x\right\} \le 2Z_r(p,\beta,x),$$

where

$$Z_r(p,\beta,x) = r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p} r\left(N_B(\sigma^{(-1)}(u))\right) \mathrm{d}u\right)$$
$$\times \inf_{\lambda>0}\exp\left\{\theta_\varphi(\lambda,p) + p\sum_{i=1}^{\mathbf{N}}\varphi_i\left(\frac{\lambda\beta}{1-p}\right) + \lambda\left(\sum_{k=2}^{\infty}\delta\left(\sigma^{(-1)}(\beta p^{k-1})\right) - x\right)\right\},$$

$$\theta_\psi(\lambda,p) = \sup_{u\in B}\left((1-p)\sum_{i=1}^{\mathbf{N}}\psi_i\left(\frac{\lambda\gamma_i(u)}{1-p}\right) - \lambda f(u)\right).$$

Proof. Let V_{ε_k} denote a set of the centers of closed balls with radii $\varepsilon_k = \sigma^{(-1)}(\beta p^k)$, $p \in (0,1)$, $k = 0,1,2,\ldots$, which forms minimal covering of the space (B,ρ). Number of elements in the set V_{ε_k} is equal to $N_{(B,\rho)}(\varepsilon_k) = N_B(\varepsilon_k)$.

It follows from Theorem 2 and Condition Σ that for any $\varepsilon > 0$

$$\mathbf{P}\{|X_i(t) - X_i(s)| > \varepsilon\}$$
$$\le 2\exp\left\{-\varphi_i\left(\frac{\varepsilon}{\tau_{\varphi_i}(X_i(t)-X_i(s))}\right)\right\} \le 2\exp\left\{-\varphi_i\left(\frac{\varepsilon}{\sigma(\rho(t,s))}\right)\right\}.$$

Therefore the processes $X_i(t)$ are continuous in probability, and the process $X(t) = \sum_{i=1}^{\mathbf{N}} X_i(t) - f(t)$ is continuous in probability as well. Hence the set $V = \bigcup_{k=1}^{\infty} V_{\varepsilon_k}$ is a set of separability of the process X and with probability one

$$\sup_{t\in T} X(t) = \sup_{t\in V} X(t). \tag{15}$$

Consider a mapping $\alpha_n = \{\alpha_n(t), n = 0, 1 \ldots\}$ of the set $V = \bigcup V_{\varepsilon_k}$ into the set V_{ε_n}, where $\alpha_n(t)$ is such a point from the set V_{ε_n} that $\rho(t, \alpha_n(t)) < \varepsilon_n$. If $t \in V_{\varepsilon_n}$, then $\alpha_n(t) = t$. If there exist several such points from the set V_{ε_n} that $\rho(t, \alpha_n(t)) < \varepsilon_n$, then we choose one of them and denote it $\alpha_n(t)$.

It follows from Chebyshev's inequality, Theorem 1, and Condition Σ that

$$\mathbf{P}\left\{|X_i(t) - X_i(\alpha_n(t))| > p^{\frac{n}{2}}\right\}$$
$$\leq \frac{\mathbf{E}(X_i(t) - X_i(\alpha_n(t)))^2}{p^n} \leq \frac{c_2^2 \tau_\varphi^2(Y(t) - Y(\alpha_n(t)))}{p^n} \leq \frac{c_2^2 \sigma^2(\varepsilon_n)}{p^n} = c_2^2 \beta^2 p^n,$$

where c_2 is the constant from (5). This inequality implies that

$$\sum_{n=1}^{\infty} \mathbf{P}\left\{|X_i(t) - X_i(\alpha_n(t))| > p^{\frac{n}{2}}\right\} < \infty.$$

It follows from the Borel-Kantelli lemma that $X_i(t) - X_i(\alpha_n(t)) \to 0$ as $n \to \infty$ with probability one. Since the function f is continuous, then $X(t) - X(\alpha_n(t)) \to 0$ as $n \to \infty$ with probability one as well. Since the set V is countable, then $X(t) - X(\alpha_n(t)) \to 0$ as $n \to \infty$ for all t simultaneously.

Let t be an arbitrary point from the set V. Denote by $t_m = \alpha_m(t)$, $t_{m-1} = \alpha_{m-1}(t_m), \ldots, t_1 = \alpha_1(t_2)$ for any $m \geq 1$. Since for all $m \geq 2$

$$X(t) = X(t_1) + \sum_{k=2}^{m} (X(t_k) - X(t_{k-1})) + X(t) - X(\alpha_m(t))$$
$$\leq \max_{u \in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u \in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)) + X(t) - X(\alpha_m(t))$$

we have

$$X(t) \leq \lim_{m\to\infty} \inf \left(\max_{u \in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u \in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)) + X(t) - X(\alpha_m(t))) \right)$$
$$= \lim_{m\to\infty} \inf \left(\max_{u \in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u \in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u))) \right). \quad (16)$$

It follows from (15) and (16) that with probability one

$$\sup_{t \in T} X(t) \leq \lim_{m\to\infty} \inf \left(\max_{u \in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u \in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u))) \right). \quad (17)$$

Let $\{q_k, k = 1, 2, \ldots\}$ be such a sequence that $q_k > 1$ and $\sum_{k=1}^{\infty} q_k^{-1} \leq 1$. It follows from the Hölder's inequality, the Fatou's lemma, and (17) that for all $\lambda > 0$

$$\begin{aligned}
&\mathbf{E}\exp\left\{\lambda \sup_{t\in T} X(t)\right\} \\
&\leq \mathbf{E} \lim_{m\to\infty} \inf \exp\left\{\lambda\left(\max_{u\in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\right)\right\} \\
&\leq \lim_{m\to\infty} \inf \mathbf{E}\exp\left\{\lambda\left(\max_{u\in V_{\varepsilon_1}} X(u) + \sum_{k=2}^{m} \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\right)\right\} \\
&\leq \lim_{m\to\infty} \inf \left(\left(\mathbf{E}\exp\left\{q_1\lambda \max_{u\in V_{\varepsilon_1}} X(u)\right\}\right)^{1/q_1}\right. \\
&\quad \left.\times \prod_{k=2}^{m}\left(\mathbf{E}\exp\left\{q_k\lambda \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\right\}\right)^{1/q_k}\right) \\
&\leq \left(\mathbf{E}\exp\left\{q_1\lambda \max_{u\in V_{\varepsilon_1}} X(u)\right\}\right)^{1/q_1} \\
&\quad \times \prod_{k=2}^{\infty}\left(\mathbf{E}\exp\left\{q_k\lambda \max_{u\in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u)))\right\}\right)^{1/q_k}. \qquad (18)
\end{aligned}$$

Consider each of the factors in the right-hand side of (18) separately. It follows from (4) that for all $1 \leq i \leq \mathbf{N}$

$$\mathbf{E}\exp\{q_1\lambda X_i(u)\} \leq \exp\{\psi_i(q_1\lambda\gamma_i(u))\}$$

and

$$\mathbf{E}\exp\{q_k\lambda(X_i(u) - X_i(\alpha_{k-1}(u)))\} \leq \exp\{\varphi_i(q_k\lambda\sigma_i(\varepsilon_{k-1}))\}.$$

Therefore,

$$\begin{aligned}
&\left(\mathbf{E}\exp\left\{q_1\lambda \max_{u\in V_{\varepsilon_1}} X(u)\right\}\right)^{1/q_1} \\
&\leq \left(\sum_{u\in V_{\varepsilon_1}} \mathbf{E}\exp\left\{q_1\lambda \sum_{i=1}^{\mathbf{N}} X_i(u)\right\} \exp\left\{-q_1\lambda f(u)\right\}\right)^{1/q_1}
\end{aligned}$$

$$\leq \left(\sum_{u \in V_{\varepsilon_1}} \prod_{i=1}^{\mathbf{N}} \mathbf{E} \exp\left\{ q_1 \lambda X_i(u) \right\} \exp\left\{ -q_1 \lambda f(u) \right\} \right)^{1/q_1}$$

$$\leq \left(N_B(\varepsilon_1) \right)^{1/q_1} \exp\left\{ \frac{1}{q_1} \sup_{u \in B} \left(\sum_{i=1}^{\mathbf{N}} \psi_i(q_1 \lambda \gamma_i(u)) - q_1 \lambda f(u) \right) \right\} .$$

Using the assumption $|f(u) - f(v)| \leq \delta(\rho(u, v))$, we obtain that

$$\left(\mathbf{E} \exp\left\{ q_k \lambda \max_{u \in V_{\varepsilon_k}} (X(u) - X(\alpha_{k-1}(u))) \right\} \right)^{1/q_k}$$

$$\leq \left(N_B(\varepsilon_k) \max_{u \in V_{\varepsilon_k}} \mathbf{E} \exp\left\{ q_k \lambda \sum_{i=1}^{\mathbf{N}} \left(X_i(u) - X_i(\alpha_{k-1}(u)) \right) \right\} \right.$$

$$\left. \times \exp\left\{ -q_k \lambda \left(f(u) - f(\alpha_{k-1}(u)) \right) \right\} \right)^{1/q_k} \leq \left(N_B(\varepsilon_k) \right)^{1/q_k}$$

$$\times \left(\max_{u \in V_{\varepsilon_k}} \exp\left\{ \sum_{i=1}^{\mathbf{N}} \varphi_i \left(q_k \lambda \sigma(\varepsilon_{k-1}) \right) + q_k \lambda \delta(\rho(u, \alpha_{k-1}(u))) \right\} \right)^{1/q_k}$$

$$\leq \left(N_B(\varepsilon_k) \right)^{1/q_k} \exp\left\{ q_k^{-1} \sum_{i=1}^{\mathbf{N}} \varphi_i \left(q_k \lambda \beta p^{k-1} \right) + \lambda \delta \left(\sigma^{(-1)}(\beta p^{k-1}) \right) \right\} .$$

From inequality (18) after substitution of $q_k = p^{1-k}/(1-p)$, $k \geq 1$, we have

$$\mathbf{E} \exp\left\{ \lambda \sup_{t \in B} X(t) \right\}$$

$$\leq \exp\left\{ \sum_{k=2}^{\infty} (1-p) p^{k-1} \sum_{i=1}^{\mathbf{N}} \varphi_i \left(\frac{\lambda \beta}{1-p} \right) + \lambda \sum_{k=2}^{\infty} \delta \left(\sigma^{(-1)}(\beta p^{k-1}) \right) \right\}$$

$$\times \exp\left\{ \theta_\psi(\lambda, p) + \sum_{k=1}^{\infty} (1-p) p^{k-1} H_B \left(\sigma^{(-1)}(\beta p^k) \right) \right\} . \tag{19}$$

As in Theorem 5 Condition R implies the next inequality for the function $r(t)$

$$\exp\left\{ \sum_{k=1}^{\infty} (1-p) p^{k-1} H_B \left(\sigma^{(-1)}(\beta p^k) \right) \right\} \leq r^{(-1)} \left(\frac{1}{\beta p} \int_0^{\beta p} r \left(N_B \left(\sigma^{(-1)}(u) \right) \right) \mathrm{d}u \right). \tag{20}$$

So, we obtain the assertion of Theorem 6 from (19), (20), and Chebyshev's inequality.

3 Storage Processes

This section is devoted to the study of a storage process with mixed input from class $V(\varphi, \psi)$.

Definition 9. We call a random process $Q(t) = \{Q(t), t \in T\}$ storage process of the queue with input process $X(t) = \{X(t), t \in T\}$ if

$$Q(t) = \sup_{s \le t} \big(X(t) - X(s) - (f(t) - f(s))\big), \quad s, t \in T, \tag{21}$$

where function $f(t)$ denotes intensity of queue serving.

If more work arrives than can be processed by the server, then the surplus is stored in a buffer of size $x \ge 0$. Obviously that part of the input received after the moment when buffer overflow is lost. Therefore estimation of the buffer overflow probability is an important task in the queueing theory. Such problem can also be reformulated in terms of the risk theory as estimation of the probability of bankruptcy for the corresponding risk process, e.g., [1, 15].

The following theorem gives an upper estimate for the tail distribution of a storage process with aggregate input

$$Q(t) = \sup_{s \le t} \left(\sum_{i=1}^{\mathbf{N}} (X_i(t) - X_i(s)) - (f(t) - f(s)) \right), \quad s, t \in B. \tag{22}$$

Theorem 7. *Let $X_i = \{X_i(t), t \in B\}$ be independent separable random processes from classes $V(\varphi_i, \psi_i)$ defined on a compact set $B \subset T$ and satisfying Condition $\Sigma\mathbf{N}$. Let $f = \{f(t), t \in B\}$ be a continuous function satisfying Condition F, and let $r = \{r(u), u \ge 1\}$ be a continuous function satisfying Condition R. If, in addition,*

$$R_\beta(t) = \int_0^\beta r(N_{B_t}(\sigma^{(-1)}(u)))\, \mathrm{d}u < \infty, \tag{23}$$

then for all $p \in (0, 1)$ and $x > 0$ the following inequality holds for the storage process $Q(t)$ defined in (22)

$$\mathbf{P}\{Q(t) > x\} \le Z_r(p, t, x), \tag{24}$$

where

$$Z_r(p,t,x) = r^{(-1)}\big(R_\beta(t)/(\beta p)\big)$$
$$\times \inf_{\lambda > 0} W(\lambda, p, t) \exp\left\{ p \sum_{i=1}^{\mathbf{N}} \varphi_i\left(\frac{\lambda\beta}{1-p}\right) + \lambda \left(\sum_{k=2}^{\infty} \delta\left(\sigma^{(-1)}(\beta p^{k-1})\right) - x \right) \right\},$$

$$W(\lambda, t, p) = \min\{U_1(\lambda, t, p);\ \ U_2(\lambda, t, p)\},$$

$$(U_1(\lambda, t, p))^{1/(1-p)}$$

$$= \int_1^{N_{B_t}\left(\sigma^{(-1)}(\beta p)\right)+1} \exp\left\{ \sum_{i=1}^{\mathbf{N}} \varphi\left(\frac{\lambda\sigma\left(2x\sigma^{(-1)}(\beta p)\right)}{1-p}\right) + \frac{\lambda\delta(2x\sigma^{(-1)}(\beta p))}{1-p}\right\} \mathrm{d}x,$$

$$U_2(\lambda, t, p) = \left(N_{B_t}(\sigma^{(-1)}(\beta p))\right)^{1-p} \inf_{v>1} \exp\left\{(1-p)\sum_{i=1}^{\mathbf{N}} \psi_i\left(\frac{v\lambda\gamma(t)}{1-p}\right)/v\right.$$

$$\left. -\lambda f(t) + \max_{u\in B_t}\left((v-1)(1-p)\sum_{i=1}^{\mathbf{N}}\psi_i\left(\frac{v\lambda\gamma_i(t)}{(v-1)(1-p)}\right)/v + \lambda f(u)\right)\right\}.$$

The assertion of Theorem 7 follows from Theorem 6 and comes as a natural generalization of the results in [17] for a sum of random processes from classes $V(\varphi_i, \psi_i)$.

Theorem 8 also follows from Theorem 6 and can be easily obtained through the generalization of the results of paper [16] for an aggregate of random processes from classes $V(\varphi_i, \psi_i)$.

Theorem 8. *Let $X_i = \{X_i(t), t \in B\}$ be independent separable random processes from classes $V(\varphi_i, \psi_i)$ defined on a compact set $B \subset T$ and satisfying Condition $\Sigma\mathbf{N}$. Let $f = \{f(t), t \in B\}$ be a continuous function satisfying Condition F, and let $r = \{r(u), u \geq 1\}$ be a continuous function satisfying Condition R. If, in addition, the following condition holds*

$$\int_0^{\beta} r\left(L\left(\sigma^{(-1)}(u)\right)\right) \mathrm{d}u < \infty, \tag{25}$$

then for all $p \in (0,1)$ and $x > 0$ following estimates hold for the storage process $Q(t)$ defined in (22)

$$\mathbf{P}\left\{\sup_{s\leq t;\ s,t\in B} Q(t) > x\right\} \leq Z_r(p, x), \tag{26}$$

$$\mathbf{P}\left\{\inf_{s\leq t;\ s,t\in B} Q(t) < -x\right\} \leq Z_r(p, x), \tag{27}$$

$$\mathbf{P}\left\{\sup_{s\leq t;\ s,t\in B} |Q(t)| > x\right\} \leq 2Z_r(p, x), \tag{28}$$

where

$$Z_r(p,t,x) = r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p^2} r\left(\mathsf{L}\left(\sigma^{(-1)}(u)\right)\right)\,\mathrm{d}u\right)$$

$$\times \inf_{\lambda>0} W(\lambda,p)\exp\left\{p\sum_{i=1}^{\mathbf{N}}\varphi_i\left(\frac{2\lambda\beta}{1-p}\right)+\lambda\left(2\sum_{k=1}^{\infty}\delta\left(\sigma^{(-1)}(\beta p^k)\right)-x\right)\right\},$$

$$W(\lambda,p) = \left(\sum_{l=0}^{\mathsf{N}(\sigma^{(-1)}(\beta p))-1}\left(\mathsf{N}\left(\sigma^{(-1)}(\beta p)\right)-l\right)\right.$$

$$\left.\times\exp\left\{\sum_{i=1}^{\mathbf{N}}\varphi_i\left(\frac{\lambda\sigma(2l\sigma^{(-1)}(\beta p))}{1-p}\right)+\frac{\lambda\delta(2l\sigma^{(-1)}(\beta p))}{1-p}\right\}\right)^{1-p}.$$

Example 9. Consider independent centered normalized GFBM processes $X_i(t) = \{X_i(t),\ t\in[a,b]\}$ from classes $V(\psi_i,\varphi_i)$ and N-functions $\varphi_i(x) = x^2/x_i^2$, where $x_i > 0$ are some constants, $i = \overline{1,\mathbf{N}}$. Let $f(t)$ be a continuous service function with the following property:

$$|f(t)-f(s)| \le c|t-s|^n, \quad t,s\in[a,b], \tag{29}$$

where $c > 0$ and $0 < n \le 1$ are some constants. It is easy to see that Condition F holds for the function f.

Let Condition ΣN be fulfilled for processes X_i. Then the following estimates follow from Theorem 8.

Theorem 9. *Let $X_i(t) = \{X_i(t),\ t\in[a,b]\}$ be independent GFBM processes with Hurst indexes $H_i \in (0,1)$ from classes $V(\psi_i,\varphi_i)$ defined by Orlicz N-functions $\varphi(x) = x^2/x_i^2$, $i = \overline{1,\mathbf{N}}$, and let $f = \{f(t), t\in[a,b]\}$ be a continuous function which satisfies (29). Then for all*

$$p \in \left(0, \min\left\{(2/3)^{H_{\max}}; 1/\beta\right\}\right] \quad \textit{and} \quad x \ge 0 \tag{30}$$

the following estimates hold

$$\mathbf{P}\left\{\sup_{s\le t;s,t\in B}\left(\sum_{i=1}^{\mathbf{N}}(X_i(t)-X_i(s))-(f(t)-f(s))\right) > x\right\} \le Z(p,x),$$

$$\mathbf{P}\left\{\inf_{s\le t;s,t\in B}\left(\sum_{i=1}^{\mathbf{N}}(X_i(t)-X_i(s))-(f(t)-f(s))\right) < -x\right\} \le Z(p,x),$$

$$\mathbf{P}\left\{\sup_{s\le t;s,t\in B}\left|\sum_{i=1}^{\mathbf{N}}(X_i(t)-X_i(s))-(f(t)-f(s))\right| > x\right\} \le 2Z(p,x),$$

where $H_{\min} = \min_{i=\overline{1,\mathbf{N}}} H_i$, $H_{\max} = \max_{i=\overline{1,\mathbf{N}}} H_i$,

$$Z(p,x) = \frac{(b-a)^2}{2}(\beta p e)^{2/H_{\max}} \left(\sum_{l=0}^{\frac{b-a}{2(\beta p)^{1/H_{\max}}}} \left(\frac{b-a}{2(\beta p)^{1/H_{\max}}} + 1 - l \right) \right.$$
$$\left. \times \exp\left\{ - \frac{\left(x - c(2l)^n (\beta p)^{n/H_{\max}} - 2c(\beta p^2)^{n/H_{\max}}/(1-p^{n/H_{\max}})\right)^2}{4p\beta^2 \sum_{i=1}^{\mathbf{N}} x_i^{-2}\,(4+p(2l)^{2H_i})} \right\} \right)^{1-p}.$$

Proof. From (8), (12), and (14), we obtain the following bounds for sub-Gaussian increments of the aggregate $\sum X_i(t)$:

$$\sigma(u) = \sup_{i=\overline{1,\mathbf{N}}} \{\sigma_i(u)\} = \sup_{i=\overline{1,\mathbf{N}}} u^{H_i} = \begin{cases} u^{H_{\min}}, \ 0 \le u \le 1, \\ u^{H_{\max}}, \ u > 1, \end{cases} \tag{31}$$

and

$$\sigma^{(-1)}(u) = \begin{cases} u^{1/H_{\max}}, \ 0 \le u \le 1, \\ u^{1/H_{\min}}, \ u > 1. \end{cases} \tag{32}$$

Let $r(u) = u^\alpha$, $0 < \alpha < \frac{H}{2}$. If $p \le \min\left\{(2/3)^{H_{\max}}; 1/\beta\right\}$, then $\frac{b-a}{2u^{1/H_{\max}}} > \frac{3}{2}$, since $u \le (2/3)^{H_{\max}}(b-a/2)^{H_{\max}} \le p\beta \le 1$. Therefore, we obtain

$$r^{(-1)}\left(\frac{1}{\beta p}\int_0^{\beta p} r\left(L\left(\sigma^{(-1)}(u)\right)\right)\,\mathrm{d}u\right)$$
$$\le \left(\frac{1}{\beta p}\int_0^{\beta p}\left(\left(\frac{b-a}{2u^{1/H_{\max}}}+1\right)^2 + \frac{b-a}{2u^{1/H_{\max}}}+1\right)^\alpha /2^\alpha\,\mathrm{d}u\right)^{1/\alpha}$$
$$\le \frac{1}{2}\left(\frac{1}{\beta p}\int_0^{\beta p}\left(\frac{b-a}{2u^{1/H_{\max}}}+\frac{3}{2}\right)^{2\alpha}\mathrm{d}u\right)^{1/\alpha} < \frac{1}{2}\left(\frac{1}{\beta p}\int_0^{\beta p}\left(\frac{b-a}{u^{1/H_{\max}}}\right)^{2\alpha}\mathrm{d}u\right)^{1/\alpha}$$
$$= \frac{(b-a)^2}{2}(\beta p)^{2/H_{\max}}\left(1-\frac{2\alpha}{H_{\max}}\right)^{-1/\alpha} \to \frac{(b-a)^2}{2}(\beta p e)^{2/H_{\max}},\ \alpha \to 0. \tag{33}$$

Also

$$\sum_{k=1}^{\infty} \delta\left(\sigma^{(-1)}(\beta p^k)\right) = \sum_{k=1}^{\infty} c(\beta p^k)^{n/H_{\max}} = \frac{c\beta^{n/H_{\max}} p^{n/H_{\max}}}{1-p^{n/H_{\max}}}. \tag{34}$$

Applying (33), (34), and the following chain of transforms to Theorem 5, we obtain the assertion of Theorem 9 (Fig. 1).

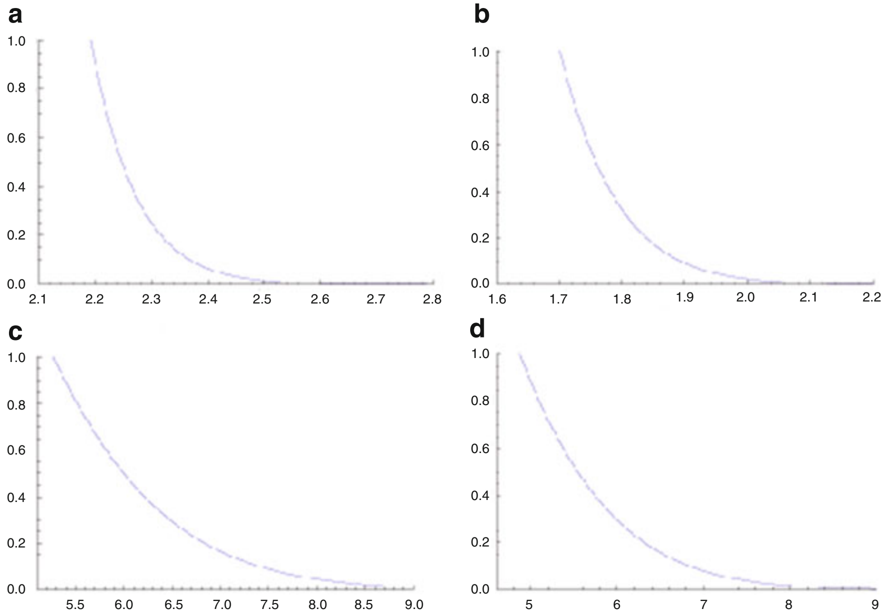

Fig. 1 Here is depicted upper estimating function $Z(p,x)$ from Theorem 9 for various sets of sub-Gaussian GFBM incoming processes considered on interval $[a,b]=[0,1]$ with the following values: $c=1$, $n=1$ (i.e., $f(t)=t$), $x_i=\sqrt{2}$, $\beta=\left(\frac{b-a}{2}\right)^{H_{\max}}=(0.5)^{H_{\max}}$, $p=0.25$ (**a**) $\mathbf{N}=1$, $H=0.5$ (**b**) $\mathbf{N}=1$, $H=0.75$ (**c**) $\mathbf{N}=5$, $H_i=0.9$ (**d**) $\mathbf{N}=5$, $H_i\in\{0.5,0.6,0.7,0.8,0.9\}$

$$\inf_{\lambda>0}\exp\left\{p\sum_{i=1}^{\mathbf{N}}\varphi_i\left(\frac{2\lambda\beta}{1-p}\right)+\lambda\left(2\sum_{k=1}^{\infty}\delta\left(\sigma^{(-1)}(\beta p^k)\right)-x\right)\right\}W(\lambda,p)$$

$$=\inf_{\lambda>0}\exp\left\{\sum_{i=1}^{\mathbf{N}}\frac{4p\lambda^2\beta^2}{x_i^2(1-p)^2}+\lambda\left(\frac{2c\beta^{n/H_{\max}}p^{n/H_{\max}}}{1-p^{n/H_{\max}}}-x\right)\right\}$$

$$\times\left(\sum_{l=0}^{\frac{b-a}{2(\beta p)^{1/H_{\max}}}}\left(\frac{b-a}{2(\beta p)^{\frac{1}{H_{\max}}}}+1-l\right)\exp\left\{\sum_{i=1}^{\mathbf{N}}\frac{\lambda^2(2l)^{2H_i}(\beta p)^2}{x_i^2(1-p)^2}\right.\right.$$

$$\left.\left.+\frac{\lambda(2l)^n c(\beta p)^{\frac{n}{H_{\max}}}}{1-p}\right\}\right)^{1-p}\leq\left(\sum_{l=0}^{\frac{b-a}{2(\beta p)^{1/H_{\max}}}}\left(\frac{b-a}{2(\beta p)^{\frac{1}{H\max}}}+1-l\right)\right.$$

$$\times\inf_{\lambda>0}\exp\left\{\lambda^2\sum_{i=1}^{\mathbf{N}}\left(\frac{4p\beta^2}{x_i^2(1-p)^2}+\frac{(2l)^{2H_i}(\beta p)^2}{x_i^2(1-p)^2}\right)\right.$$

$$-\lambda\left(\frac{x}{1-p}-\frac{2c\beta^{n/H_{\max}}p^{n/H_{\max}}}{(1-p)(1-p^{n/H_{\max}})}-\frac{(2l)^n c(\beta p)^{n/H_{\max}}}{1-p}\right)\Bigg\}\Bigg)^{1-p}$$

$$\leq\left(\sum_{l=0}^{\frac{b-a}{2(\beta p)^{1/H_{\max}}}}\left(\frac{b-a}{2(\beta p)^{1/H_{\max}}}+1-l\right)\right.$$

$$\left.\times\exp\left\{-\frac{\left(x-c(2l)^n(\beta p)^{n/H_{\max}}-2c(\beta p^2)^{n/H_{\max}}/(1-p^{n/H_{\max}})\right)^2}{4p\beta^2\sum_{i=1}^{\mathbf{N}}x_i^{-2}\left(4+p(2l)^{2H_i}\right)}\right\}\right)^{1-p}.$$

4 Conclusions

The paper summarizes some recent studies that have been made for a general class $V(\phi,\psi)$ of incoming processes X_i. As an example, we consider sub-Gaussian GFBM storage process with aggregated input formed by independent sources. We show that obtained estimate for the tail distribution of such storage process depends on the buffer size x as $o\left(\exp\left\{-\alpha x^2\right\}\right)$. Also we provide several illustrations of the buffer overflow probability for different values of Hurst parameter of the incoming processes.

References

1. Boulongne, P., Pierre-Loti-Viaud, D., Piterbarg, V.: On average losses in the ruin problem with fractional Brownian motion as input. Extremes **12**, 77–91 (2009)
2. Buldygin, V.V., Kozachenko, Yu.V.: Metric Characterization of Random Variables and Random Processes. AMS, Providence, RI (2000)
3. Giuliano-Antonini, R., Kozachenko, Yu., Nikitina, T.: Spaces of ϕ-subgaussian random variables. Rendiconti, Academia Nazionale delle Scienze detta dei XL. Memorie di Matematica e Applicazioni, 121^o, XXVII, fasc.1, 95–124 (2003)
4. Kozachenko, Yu.V., Kovalchuk, Yu.A.: Boundary value problems with random initial conditions, and functional series from $\mathrm{Sub}_\varphi(\Omega)$. I. Ukr. Math. J. **50**(4), 504–515 (1998)
5. Kozachenko, Yu.V., Ostrovskij, E.I.: Banach spaces of random variables of sub-Gaussian type. Theory Probab. Math. Stat. **32**, 45–56 (1986)
6. Kozachenko, Yu., Vasylyk, O.: Random processes from classes $V(\varphi,\psi)$. Theory Probab. Math. Stat. **63**, 100–111 (2000)
7. Kozachenko, Yu., Sottinen, T., Vasilik, O.: Weakly self-similar stationary increment processes from the space $SSub_\varphi(\Omega)$. Theory Probab. Math. Stat. **65**, 77–88 (2002)
8. Kozachenko, Yu., Vasylyk, O., Yamnenko, R.: Upper estimate of overrunning by $\mathrm{Sub}_\varphi(\Omega)$ random process the level specified by continuous function. Random Oper. Stoch. Equ. **13**(2), 111–128 (2005)
9. Krasnoselskii, M.A., Rutitskii, Ya.B.: Convex Functions in the Orlicz spaces. Noordhoff, Gröningen (1961)

10. Massoulie, L., Simonian, A.: Large buffer asymptotics for the queue with fractional Brownian input. J. Appl. Prob. **36**, 894–906 (1999)
11. Norros, I.: A storage model with self-similar input. Queueing Syst. **16**(3–4), 387–396 (1994)
12. Norros, I.: On the use of fractional Brownian motions in the theory of connectionless networks. IEEE J. Sel. Areas Commun. **13**(6), 953–962 (1995)
13. Vasylyk, O.I., Kozachenko, Yu.V., Yamnenko R.E.: φ-Subgaussovi vypadkovi protsesy: monographia (in Ukrainian). VPC "Kyiv University", Kyiv (2008)
14. Yakovenko, T., Yamnenko, R.: Generalized fractional Brownian motion in Orlicz spaces. Theory Stoch. Process. **14**(3–4), 174–188 (2008)
15. Yamnenko, R.: Ruin probability for generalized φ-sub-Gaussian fractional Brownian motion. Theory Stoch. Process. **12**(28), part no. 3–4, 261–275 (2006)
16. Yamnenko, R.E.: Bounds for the distribution of some functionals of processes with φ-sub-Gaussian increments. Theory Probab. Math. Stat. **85**, 181–197 (2012)
17. Yamnenko, R.E., Shramko, O.S.: On the distribution of storage processes from the class $V(\varphi, \psi)$. Theory Probab. Math. Stat. **83**, 191–206 (2011)
18. Yamnenko, R., Vasylyk, O.: Random process from the class $V(\varphi, \psi)$: exceeding a curve. Theory Stoch. Process. **13**(29), part no. 4, 219–232 (2007)

A Review on Time-Changed Pseudoprocesses and Related Distributions

Enzo Orsingher

Abstract Pseudoprocesses governed by higher-order heat-type equations are considered, and a probabilistic representation of their densities is presented. The composition of pseudoprocesses $X_n(t)$, $t > 0$, with stable processes $T_{\frac{1}{n}}(t)$, $t > 0$, produces Cauchy processes which for odd values of n are asymmetric. Cauchy-type processes satisfying higher-order Laplace equations are considered and some of their properties are discussed. Pseudoprocesses on a unit-radius circle with densities obtained by wrapping up their counterparts on the line are examined. Various forms of Poisson kernels related to the composition of circular pseudoprocesses with stable processes are considered. The Fourier representation of the signed densities of circular pseudoprocesses is given and represents an extension of the classical circular Brownian motion.

1 Pseudoprocesses and Generalized Cauchy Processes

The study of the structure of solutions of higher-order heat-type equations of the form

$$\begin{cases} \frac{\partial}{\partial t} u_m(x,t) = \kappa_m \frac{\partial^m}{\partial x^m} u_m(x,t), & x \in \mathbb{R},\ t > 0,\ m \geq 2, \\ u_m(x,0) = \delta(x), \end{cases} \tag{1}$$

where

$$\kappa_m = \begin{cases} (-1)^{\frac{m}{2}+1}, & \text{if } m \text{ is even} \\ \pm 1, & \text{if } m \text{ is odd}, \end{cases} \tag{2}$$

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has started with the papers by [2, 4, 13, 21]. These mathematicians have shown that the solutions to (1) have an infinite number of zeros so that they differ from the fundamental solution of the heat equation because they are sign-varying.

The analysis of the asymptotic behavior of the solutions

$$u_m(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\beta x+\kappa_m(-i\beta)^m t}\,d\beta \tag{3}$$

in the even-order case has been performed by [14]. This analysis for the case $m = 4$ was done by [7] and for $m = 3$ by [1].

A probabilistic representation of the solutions to (1) was recently published by [18] and can be stated as follows. The structure of the solutions of (1) has the form

$$u_{2n}(x,t) = \frac{1}{\pi x}\mathbb{E}\left\{\sin\left(xG^{2n}\left(\frac{1}{t}\right)\right)\right\}, \quad x\in\mathbb{R},\ n\geq 1, \tag{4}$$

$$u_{2n+1}(x,t) = \frac{1}{\pi x}\mathbb{E}\left\{e^{-b_n xG^{2n+1}\left(\frac{1}{t}\right)}\sin a_n xG^{2n+1}\left(\frac{1}{t}\right)\right\}, \quad x\in\mathbb{R}, n\geq 1, \tag{5}$$

where $G^{\gamma}(t)$ is the generalized gamma r.v. with density

$$g^{\gamma}(x,t) = \gamma\frac{x^{\gamma-1}}{t}e^{-\frac{x^{\gamma}}{t}}, \qquad x>0, \gamma>0, t>0. \tag{6}$$

The numbers a_n and b_n are defined as

$$a_n = \cos\frac{\pi}{2(2n+1)}, \qquad b_n = \sin\frac{\pi}{2(2n+1)}, \tag{7}$$

and are related to the asymmetric structure of the solutions to (1) for odd values of m.

The representation (5) shows that the exponential factor is responsible for the asymmetric structure of the fundamental solutions to the heat-type equations (1). Formula (7) shows that the asymmetry decreases as the order of equations increases. The third-order case is therefore the most asymmetric of all solutions to (1) and takes the form

$$\begin{aligned} u_3(x,t) &= \frac{1}{\sqrt[3]{3t}}\,\mathrm{Ai}\left(\frac{x}{\sqrt[3]{3t}}\right) \\ &= \frac{3t}{\pi x}\int_0^{\infty} e^{-\frac{xy}{2}}\sin\left(\frac{\sqrt{3}}{2}xy\right)y^2 e^{-ty^3}\,dy \\ &= -\frac{1}{\pi x}\sum_{k=1}^{\infty}\frac{1}{k!}\sin\left(\frac{\pi k}{3}\right)\Gamma\left(1+\frac{k}{3}\right)\left(-\frac{x}{\sqrt[3]{t}}\right)^k. \end{aligned} \tag{8}$$

The fourth-order case has been analyzed by several authors and can be represented in different forms as

$$\begin{aligned} u_4(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{y^2 t}{2^2}} \cos xy dy \\ &= \frac{1}{2\pi\sqrt{2}t^{\frac{1}{2}}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}\Gamma\left(\frac{k}{2}+\frac{1}{4}\right)\left(\frac{-\sqrt{2}|x|}{t^{\frac{1}{4}}}\right)^{2k} \\ &= 2\int_0^{\infty}\frac{1}{\sqrt{2\pi s}}\cos\left(\frac{x^2}{2s}-\frac{\pi}{4}\right)\frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}}\, ds. \end{aligned} \tag{9}$$

Pseudoprocesses related to the solutions of the mth-order heat equations (1) have been constructed by several authors. The functions $x : t \to x(t)$ which can be regarded as the sample paths of the pseudoprocess X are first introduced. The cylinders C defined as

$$C = \{x : a_j \le x(t_j) \le b_j,\ j = 1, \ldots, n\} \tag{10}$$

with signed measure

$$\mu(C) = \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} \prod \left(p(t_j - t_{j-1}; x_j - x_{j-1})\right) dx_j, \tag{11}$$

where $p(x,t)$ is the fundamental solution of equation (1) having Fourier representation (3) are then considered. The measure (11) is then extended to the field generated by the cylinder sets. For this construction, see, for example, [7–9, 15, 16]). Its properties were analyzed in the fourth-order case by [7, 15]. Various types of functionals related to pseudoprocesses have been investigated recently in a systematic and general way in a series of papers by [5, 6, 9–11]. See also [12] for a recent review on pseudoprocesses and related functionals.

Some functionals related to pseudoprocesses have the surprising property of being genuine r.v.'s. This happens, for example, for the sojourn time

$$\Gamma_t = \int_0^t \mathbb{I}_{[0,\infty]}(X(s))\, ds \tag{12}$$

for which we have that

$$\frac{\Pr\{\Gamma_t \in ds\}}{ds} = \frac{1}{\pi}\frac{1}{\sqrt{s(t-s)}}, \qquad 0 < s < t, \tag{13}$$

for even-order pseudoprocesses, while for odd-order pseudoprocesses the sojourn time Γ_t possesses a Beta distribution (see [8, 9]).

We have investigated compositions of pseudoprocesses and positively skewed stable processes and obtained the following result. If $X_{2n+1}(t)$, $t > 0$, is a pseudoprocess of order $2n + 1$ (i.e., its law is related to (1) for $m = 2n + 1$), and $T_{\frac{1}{2n+1}}(t)$, $t > 0$, is a positively skewed stable process of order $\frac{1}{2n+1}$ (i.e., a stable subordinator of order $\frac{1}{2n+1}$), then $X_{2n+1}\left(T_{\frac{1}{2n+1}}(t)\right)$, $t > 0$, is a r.v. with the following Cauchy distribution for $x \in \mathbb{R}$ (see [18]):

$$\Pr\left\{X_{2n+1}\left(T_{\frac{1}{2n+1}}(t)\right) \in dx\right\} = dx \frac{t \cos\frac{\pi}{2(2n+1)}}{\pi\left[\left(x + t\sin\frac{\pi}{2(2n+1)}\right)^2 + t^2\cos^2\frac{\pi}{2(2n+1)}\right]}. \tag{14}$$

The Cauchy r.v. with distribution (14) has therefore the location parameter $-t\sin\frac{\pi}{2(2n+1)}$ and the scale parameter equal to $t\cos\frac{\pi}{2(2n+1)}$. The asymmetry of (14) decreases with n increasing and is maximal for the third-order case. The composition of $X_{2n}\left(T_{\frac{1}{2n}}(t)\right)$ yields, for all n, a symmetric Cauchy, and thus we have that

$$\Pr\left\{X_{2n}\left(T_{\frac{1}{2n}}(t)\right) \in dx\right\} = dx \frac{t}{\pi\,(t^2 + x^2)}. \tag{15}$$

In view of (14) we have the following fine relationship for the Airy functions:

$$\begin{aligned}\frac{\Pr\left\{X_3\left(T_{\frac{1}{3}}(t)\right) \in dx\right\}}{dx} &= \int_0^\infty \frac{1}{\sqrt[3]{3s}}\mathrm{Ai}\left(\frac{x}{\sqrt[3]{3s}}\right)\frac{t}{s}\frac{1}{\sqrt[3]{3s}}\mathrm{Ai}\left(\frac{t}{\sqrt[3]{3s}}\right) \\ &= \frac{\sqrt{3}}{2\pi}\frac{t}{x^2 + xt + t^2}. \end{aligned} \tag{16}$$

The r-times iterated pseudoprocess

$$Z_r(t) = X_{2n+1}\left(T^1_{\frac{1}{2n+1}}\left(T^2_{\frac{1}{2n+1}}\left(\cdots\left(T^r_{\frac{1}{2n+1}}(t)\right)\cdots\right)\right)\right) \tag{17}$$

is for any $t > 0$ a stable r.v. of order $\frac{1}{(2n+1)^r}$ and possesses the characteristic function

$$\mathbb{E}e^{i\beta Z_r(t)} = e^{-t|\beta|^{\frac{1}{(2n+1)^{r-1}}}\left(\cos\frac{\pi}{2(2n+1)^r} + i\sin\frac{\pi}{2(2n+1)^r}\right)}. \tag{18}$$

For $r = 1$ we retrieve result (14) because

$$\mathbb{E}e^{i\beta Z_1(t)} = e^{-t|\beta|\left(\cos\frac{\pi}{2(2n+1)} + i\sin\frac{\pi}{2(2n+1)}\right)} \tag{19}$$

which is the characteristic function of the Cauchy r.v. with density (14).

For pseudoprocesses related to

$$\frac{\partial}{\partial t}u = -\frac{\partial^{2k+1}}{\partial x^{2k+1}}u \tag{20}$$

the composition $X_{2n+1}\left(T_{\frac{1}{2n+1}}(t)\right)$ has distribution:

$$\Pr\left\{X_{2k+1}\left(T_{\frac{1}{2k+1}}(t)\right)\in dx\right\} = \frac{dx\, t\cos\frac{\pi}{2(2k+1)}}{\pi\left[\left(x+(-1)^{k+1}t\sin\frac{\pi}{2(2k+1)}\right)^2+t^2\cos^2\frac{\pi}{2(2k+1)}\right]}. \tag{21}$$

The density function (21) is a solution to the higher-order Laplace equation

$$\left(\frac{\partial^{2k+1}}{\partial t^{2k+1}}+\frac{\partial^{2k+1}}{\partial x^{2k+1}}\right)u = 0. \tag{22}$$

The characteristic function of (21) has the form

$$\begin{aligned} U(\beta,t) &= \int_0^\infty e^{i\beta x}\Pr\left\{X_{2k+1}\left(T_{\frac{1}{2k+1}}(t)\right)\in dx\right\} \\ &= e^{-t|\beta|\cos\frac{\pi}{2(2k+1)}-i(-1)^{k+1}t\beta\sin\frac{\pi}{2(2k+1)}} \\ &= e^{-t|\beta| e^{-\frac{i\pi}{2(2k+1)}\frac{\beta}{|\beta|}(-1)^{k+1}}}. \end{aligned} \tag{23}$$

The Fourier transform of (22) becomes

$$\left(\frac{\partial^{2k+1}}{\partial t^{2k+1}}+(-i\beta)^{2k+1}\right)U(\beta,t) = 0, \tag{24}$$

and the derivative of (23) is therefore

$$\begin{aligned} \frac{\partial^{2k+1}}{\partial t^{2k+1}}U(\beta,t) &= \left(-|\beta| e^{-\frac{i\pi}{2(2k+1)}\frac{\beta}{|\beta|}(-1)^{k+1}}\right)^{2k+1}U(\beta,t) \\ &= (-|\beta|)^{2k+1}\, e^{-\frac{i\pi}{2}(-1)^{(k+1)(2k+1)}\left(\frac{\beta}{|\beta|}\right)^{2k+1}}U(\beta,t) \\ &= -\left(-i|\beta|\frac{\beta}{|\beta|}\right)^{2k+1}U(\beta,t) \\ &= -(-i\beta)^{2k+1}U(\beta,t). \end{aligned} \tag{25}$$

We note also that the asymmetric Cauchy densities of the form

$$f(x,t;m) = \frac{1}{\pi}\frac{t\cos\frac{\pi}{2m}}{\left(x+t\sin\frac{\pi}{2m}\right)^2 + t^2\cos^2\frac{\pi}{2m}} \tag{26}$$

are solutions of the second-order equation

$$\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial x^2}\right) f = 2\sin\frac{\pi}{2m}\frac{\partial^2}{\partial x\partial t} f \tag{27}$$

as proved in [19].

A different class of higher-order Cauchy distributions is obtained by considering Laplace equations of the form (see [19])

$$\left(\frac{\partial^{2^n}}{\partial t^{2^n}}+\frac{\partial^{2^n}}{\partial x^{2^n}}\right) u = 0, \qquad n \geq 1. \tag{28}$$

In this case we arrive at densities with the following structure:

$$p_{2^n}(x,t) = \frac{t(x^2+t^2)}{2^{n-2}\pi\,(x^{2^n}+t^{2^n})} g(x,t) \tag{29}$$

where $g(x,t)$ is a polynomial of order $2^n - 2^2$. For $n = 2$ we have

$$p_4(x,t) = \frac{t(x^2+t^2)}{\sqrt{2}\pi(x^4+t^4)} \tag{30}$$

which is the distribution of $F(T_t)$ of the Fresnel pseudoprocess F with the first passage time $T_t = \inf\{s : B(s) = t\}$ of a Brownian motion B through level t (on this point consult [17]). The density (30) as well as (29) have a bimodal structure. The explicit form of (29) can be given in several different forms as

$$\begin{aligned} p_{2^n}(x,t) &= \frac{1}{2^{n-1}\pi}\sum_{\substack{k=-(2^{n-1}-1)\\ k \text{ odd}}}^{2^{n-1}-1} \frac{te^{\frac{i\pi k}{2^n}}}{x^2+\left(te^{\frac{i\pi k}{2^n}}\right)^2} \\ &= \frac{t(x^2+t^2)}{2^{n-2}\pi}\sum_{\substack{k=1\\ k \text{ odd}}}^{2^{n-1}-1} \frac{\cos\frac{k\pi}{2^n}}{x^4+t^4+2x^2t^2\cos\frac{k\pi}{2^{n-1}}}. \end{aligned} \tag{31}$$

Each component of (31) is obtained by folding and symmetrizing the density of the r.v.

$$V(t) = C\left(t\cos\frac{k\pi}{2^n}\right) - t\sin\frac{k\pi}{2^n}, \tag{32}$$

where $C(t)$, $t > 0$, is the symmetric Cauchy process.

2 Pseudoprocesses on Circles

In this section we consider pseudoprocesses $\Theta_n(t)$, $t > 0$, on a unit-radius ring $\mathcal{R}$. They can be obtained by wrapping up around $\mathcal{R}$ the sample paths of pseudoprocesses $X_n(t)$, $t > 0$, on the line (see [20]). Increasing sample paths of X_n correspond to trajectories running counterclockwise on $\mathcal{R}$. The distributions of $\Theta_n(t)$, $t > 0$, can be obtained by wrapping up those of $X_n(t)$ in the same way as in the case of circular Brownian motion. If we denote by $v_n(\theta, t)$ the density of the signed measure related to $\Theta_n(t)$, we have that it solves the initial-value problem

$$\begin{cases} \frac{\partial}{\partial t} v_n(\theta, t) = c_n \frac{\partial^n}{\partial \theta^n} v_n(\theta, t), & \theta \in [0, 2\pi], t > 0, \\ v_n(\theta, 0) = \delta(\theta). \end{cases} \tag{33}$$

The constants c_n coincide with those of the pseudoprocesses on the line defined in (2). The Fourier expansion of the solutions to (33) has the following form:

$$\begin{cases} v_{2n}(\theta, t) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} e^{-k^{2n} t} \cos k\theta, \\ v_{2n+1}(\theta, t) = \frac{1}{2\pi} + \frac{1}{\pi} \sum_{k=1}^{\infty} \cos \left(k^{2n+1} t + k\theta \right). \end{cases} \tag{34}$$

We observe that for $n = 1$ we extract from (34) the Fourier expansion of the circular Brownian motion (with infinitesimal variance equal to 2). For $n = 1$ the second formula represents a sort of discretized Airy function. In both cases they represent the Fourier expansion of the wrapped up laws (5). In other words (34) is obtained by expanding

$$\begin{cases} v_{2n}(\theta, t) = \sum_{m=-\infty}^{\infty} \frac{1}{\pi(\theta + 2m\pi)} \mathbb{E} \left\{ \sin \left(a_n (\theta + 2m\pi) G^{2n} \left(\frac{1}{t} \right) \right) \right\} \\ v_{2n+1}(\theta, t) = \sum_{m=-\infty}^{\infty} \frac{1}{\pi(\theta + 2m\pi)} \mathbb{E} \left\{ e^{-b_n \left((\theta + 2m\pi) G^{2n+1} \left(\frac{1}{t} \right) \right)} \sin \left(a_n (\theta + 2m\pi) G^{2n+1} \left(\frac{1}{t} \right) \right) \right\}, \end{cases} \tag{35}$$

where $G^{2n+1}\left(\frac{1}{t}\right)$, a_n, b_n, are defined by (6) and (7), respectively. Circular even-order pseudoprocesses after a small initial time interval (when their density measure is sign-varying) become genuine r.v.'s. For $t \to \infty$ their distributions become uniform on the ring $\mathcal{R}$ (see Fig. 1).

The process obtained as the composition of circular Brownian motion $\mathfrak{B}$ with the inverse of a stable subordinator $L^\nu(t)$, $t > 0$, $0 < \nu \leq 1$, defined as

$$L^\nu(t) = \inf \{ s > 0 : H^\nu(s) \geq t \}, \tag{36}$$

where $H^\nu(t)$ is a positively skewed stable process, has a distribution governed by a higher-order fractional equation (see [20]). In other words

$$v_{2n}^\nu(\theta, t)\, d\theta = \Pr \{ \mathfrak{B} \left(L^\nu(t) \right) \in d\theta \} \tag{37}$$

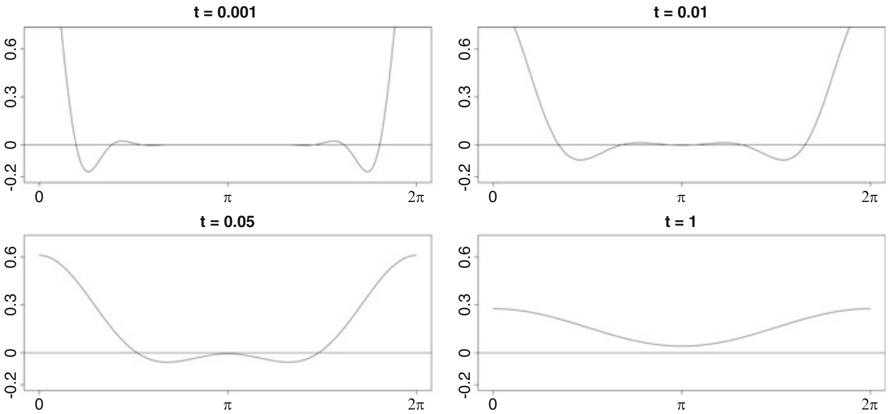

Fig. 1 The distributions of the fourth-order circular pseudoprocess for different values of t

solves

$$\begin{cases} \frac{\partial^\nu}{\partial t^\nu} v_{2n}^\nu(\theta, t) = -\left(-\frac{\partial^2}{\partial \theta^2}\right)^n v_{2n}^\nu(\theta, t) \\ v_{2n}^\nu(\theta, 0) = \delta(\theta). \end{cases} \tag{38}$$

The fractional derivative in (38) must be understood in the Dzerbayshan–Caputo sense, that is as

$$\frac{\partial^\nu}{\partial t^\nu} v_{2n}^\nu(\theta, t) = \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{\frac{\partial}{\partial s} v_{2n}^\nu(\theta, s)}{(t-s)^\nu} \, ds. \tag{39}$$

The explicit form of the solution to (38) is

$$v_{2n}^\nu(\theta, t) = \frac{1}{2\pi} \left(1 + 2 \sum_{k=1}^{\infty} E_{\nu,1}\left(-k^{2n} t^\nu\right) \cos k\theta \right), \tag{40}$$

where

$$E_{\nu,1}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\nu k + 1)}, \qquad x \in \mathbb{R}, t > 0, \tag{41}$$

is the Mittag-Leffler function. For $\nu = 1$, we retrieve from (40) the solutions (34) of higher-order circular heat equations.

Wrapped up stable symmetric process can be represented as $\mathfrak{B}\left(H^\beta(t)\right)$, $t > 0$, $\mathfrak{B}$ is a circular Brownian motion stopped at the positively skewed stable process $H^\beta(t)$, $t > 0$. Furthermore their probability distributions

$$p^{\beta}(\theta,t) = \frac{1}{2\pi}\left(1+2\sum_{k=1}^{\infty} e^{-\left(\frac{k^2}{2}\right)^{\beta} t}\cos k\theta\right) \tag{42}$$

are solutions to the space-fractional equation

$$\begin{cases} \frac{\partial}{\partial t} p^{\beta}(\theta,t) = -\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^{\beta} p^{\beta}(\theta,t) \\ p^{\beta}(\theta,0) = \delta(\theta). \end{cases} \tag{43}$$

The differential operator appearing in (43) has the following integral representation (see [3]):

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^{\beta} = -\frac{\sin\pi\beta}{2}\int_0^{\infty}\left(\lambda+\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)\right)^{-1}\lambda^{\beta}\,d\lambda. \tag{44}$$

Fractionality in time and space leads to the equation

$$\begin{cases} \frac{\partial^{\nu}}{\partial t^{\nu}} p^{\nu,\beta}(\theta,t) = -\left(-\frac{1}{2}\frac{\partial^2}{\partial\theta^2}\right)^{\beta} p^{\nu,\beta}(\theta,t) \\ p^{\nu,\beta}(\theta,0) = \delta(\theta), \end{cases} \tag{45}$$

whose solutions have the following form:

$$p^{\nu,\beta}(\theta,t) = \frac{1}{2\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty} E_{\nu,1}\left(-\left(\frac{k^2}{2}\right)^{\beta} t^{\nu}\right)\cos k\theta. \tag{46}$$

The distribution (46) coincides with the law of

$$\mathcal{F}^{\nu,\beta}(t) = \mathfrak{B}\left(H^{\beta}\left(L^{\nu}(t)\right)\right) \tag{47}$$

as the following check shows:

$$\begin{aligned} &\Pr\left\{\mathcal{F}^{\nu,\beta}(t)\in d\theta\right\} \\ &= d\theta\int_0^{\infty}\Pr\{\mathfrak{B}(s)\in d\theta\}\int_0^{\infty}\Pr\left\{H^{\beta}(w)\in ds\right\}\Pr\{L^{\nu}(t)\in dw\} \\ &= \frac{d\theta}{2\pi}+\frac{d\theta}{\pi}\int_0^{\infty}\sum_{k=1}^{\infty} e^{-\frac{k^2}{2}s}\cos k\theta\int_0^{\infty}\Pr\left\{H^{\beta}(w)\in ds\right\}\Pr\{L^{\nu}(t)\in dw\} \\ &= \frac{d\theta}{2\pi}+\frac{d\theta}{\pi}\int_0^{\infty}\sum_{k=1}^{\infty} e^{-\left(\frac{k^2}{2}\right)^{\beta} w}\cos k\theta\Pr\{\nu(t)\in dw\} \\ &= \frac{d\theta}{2\pi}+\frac{d\theta}{\pi}\sum_{k=1}^{\infty}\cos k\theta E_{\nu,1}\left(-\left(\frac{k^2}{2}\right)^{\beta} t^{\nu}\right). \end{aligned} \tag{48}$$

3 Poisson Kernels

The composition of pseudoprocesses on the ring $\mathcal{R}$ with positively skewed random variables leads to Poisson kernels of various forms (see [20]). We recall that the general form of the Poisson kernel is

$$\mathcal{P}(\theta, r; \psi, R) = \frac{R^2 - r^2}{\pi \left[R^2 + r^2 - 2rR\cos(\psi - \theta)\right]}, \qquad r < R;\ \psi, \theta \in [0, 2\pi]. \tag{49}$$

Here we consider the circular counterpart of the composition of pseudoprocesses on the line which leads to Cauchy processes (see [19]). In both cases pseudoprocesses stopped at suitably chosen stable processes give rise to genuine random variables. The circular pseudoprocesses Θ_{2n} stopped at times represented by the subordinator $H^{\frac{1}{2n}}(t)$, $t > 0$, have the probability density

$$\Pr\left\{\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right) \in d\theta\right\} = \frac{d\theta}{2\pi}\frac{1 - e^{-2t}}{1 + e^{-2t} - 2e^{-t}\cos\theta}, \qquad 0 < \theta < 2\pi \tag{50}$$

with distribution function

$$\Pr\left\{\Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right) < \theta\right\} = \begin{cases} \frac{1}{\pi}\arctan\frac{1+e^{-t}}{1-e^{-t}}\tan\frac{\theta}{2}, & 0 < \theta < \pi \\ 1 + \frac{1}{\pi}\arctan\frac{1+e^{-t}}{1-e^{-t}}\tan\frac{\theta}{2}, & \pi < \theta < 2\pi. \end{cases} \tag{51}$$

The Poisson kernel can be interpreted as the distribution of the hitting point on $\mathcal{R}$ of a planar Brownian motion starting from $(0, e^{-t})$. If $\tau_{\mathcal{R}} = \inf\{t > 0 : \boldsymbol{B}(t) \in \mathcal{R}\}$, we have therefore the following equality in distribution:

$$\boldsymbol{B}\left(\tau_{\mathcal{R}}\right) \stackrel{\text{law}}{=} \Theta_{2n}\left(H^{\frac{1}{2n}}(t)\right). \tag{52}$$

In the odd-order case the composition $\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right)$ has the probability density

$$\Pr\left\{\Theta_{2n+1}\left(H^{\frac{1}{2n+1}}(t)\right) \in d\theta\right\} = \frac{d\theta}{2\pi}\frac{1 - e^{-2a_n t}}{1 + e^{-2a_n t} - 2e^{-a_n t}\cos(\theta + b_n t)}, \tag{53}$$

where

$$a_n = \cos\frac{\pi}{2(2n+1)}, \qquad b_n = \sin\frac{\pi}{2(2n+1)}. \tag{54}$$

While the distribution (50) is independent from n, the distribution (53) changes with n and its asymmetry decreases with increasing values of n. For $n \to \infty$ the density (53) takes the form (50). The distribution (53) corresponds to the law of the hitting point of a planar Brownian motion starting from the point with polar coordinates $(e^{-a_n t}, -b_n t)$. The densities (50) and (53) converge in distribution to the uniform law as $t \to \infty$.

4 Circular Fresnel Pseudoprocesses

The equation of vibrations of rods

$$\frac{\partial^2}{\partial t^2}u = -\frac{1}{2^2}\frac{\partial^4}{\partial x^4}u, \qquad x \in \mathbb{R}, t > 0, \tag{55}$$

has fundamental solution

$$u(x,t) = \frac{1}{\sqrt{2\pi t}}\cos\left(\frac{x^2}{2t} - \frac{\pi}{4}\right). \tag{56}$$

On the base of (64), it is possible to construct a non-Markovian pseudoprocess with measure of cylinders given by

$$\mu(C) = \int_{a_1}^{b_1}\cdots\int_{a_n}^{b_n} \frac{(2\pi)^{-\frac{n}{2}}}{\prod_{j=1}^n \sqrt{t_j - t_{j-1}}}\cos\left(\sum_{j=1}^n \frac{\left(x_j - x_{j-1}\right)^2}{2(t_j - t_{j-1})} - n\frac{\pi}{4}\right)\prod_{j=1}^n dx_j. \tag{57}$$

The wrapped up density

$$f(\theta,t) = \frac{1}{\sqrt{2\pi t}}\sum_{m=-\infty}^{\infty}\cos\left(\frac{(\theta+2m\pi)^2}{2t} - \frac{\pi}{4}\right) \tag{58}$$

has Fourier expansion

$$f(\theta,t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty}\cos\frac{k^2 t}{2}\cos k\theta \tag{59}$$

and solves

$$\begin{cases} \frac{\partial^2}{\partial t^2} f = -\frac{1}{2^2}\frac{\partial^4}{\partial \theta^4} f, & \theta \in [0, 2\pi], \\ f(\theta,0) = \delta(\theta). \end{cases} \tag{60}$$

The law (59) can be regarded as the superposition of two circular Brownian motions running on $\mathcal{R}$ and taken formally at imaginary times $\pm it$. This construction is similar to that of Fresnel pseudoprocess on the line developed in [17]. The composition of the Fresnel pseudoprocess F with the positively skewed stable processes $H^\beta(t)$, $t > 0$, has density

$$f^\beta(\theta,t) = \frac{1}{2\pi} + \frac{1}{2\pi}\sum_{k=1}^{\infty}\cos k\theta e^{-\frac{k^{2\beta}}{2}t\cos\frac{\beta\pi}{2}}\cos\left(\frac{k^{2\beta}t}{2^\beta}\right)\sin\frac{\beta\pi}{2}. \tag{61}$$

The composition $F(L^\nu(t))$ where $L^\nu(t)$ is the inverse of H^ν has density

$$f^\nu(\theta, t) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{\infty} \cos k\theta E_{2\nu,1}\left(-\frac{t^{2\nu}k^4}{2''}\right) \tag{62}$$

and solves the time fractional equation

$$\frac{\partial^{2\nu}}{\partial t^{2\nu}} f^\nu(\theta, t) = -\frac{1}{2^2}\frac{\partial^4}{\partial\theta^4} f^\nu(\theta, t) \tag{63}$$

which includes the equation of vibrations of rods when $\nu = 1$.

For $\nu = \frac{1}{2}$ we extract from (62) the Fourier expansion of the fourth-order heat equation on the ring $\mathcal{R}$. For $\nu = 1$, since $E_{2,1}(-x) = \cos\sqrt{x}$ we retrieve (59).

The superposition of Poisson kernels is obtained by considering Brownian motions starting from (r, Ψ) where Ψ is a r.v. which takes values $\pm\psi$ with equal probability $\frac{1}{2}$. The Poisson kernel becomes

$$\begin{aligned} \mathbb{E}\mathcal{P}(r, \Psi, R, \Phi) &= \frac{1}{2\pi}\left[\frac{1}{2}\mathcal{P}(r, +\psi, R, \phi) + \frac{1}{2}\mathcal{P}(r, -\psi, R, \phi)\right] \\ &= \frac{1}{2\pi}\frac{\left(R^1 - r^2\right)\left(R^2 + r^2 - 2rR\cos\psi\right)}{\left(R^2 + r^2 - 2rR\cos\psi\cos\phi\right)^2 - \left(2rR\sin\psi\sin\phi\right)^2}, \end{aligned} \tag{64}$$

where

$$\mathcal{P}(r, \pm\psi, R, \phi) = \frac{R^2 - r^2}{2\pi\left(R^2 + r^2 + 2rR\cos(\phi \pm \psi)\right)}. \tag{65}$$

The expression (64) considerably simplifies for $\Psi = \pm\frac{\pi}{2}$ and becomes

$$\mathbb{E}\mathcal{P}\left(r, \Psi = \pm\frac{\pi}{2}, R, \phi\right) = \frac{1}{2\pi}\frac{R^4 - r^4}{R^4 + r^4 + 2r^r R^2\cos 2\phi}. \tag{66}$$

In the case of the Fresnel pseudoprocesses one must take $F(\tau_t)$, $\tau_t = \inf\{s : B(s) = t\}$, and the Poisson kernel becomes

$$f^{\tau_t}(\theta, t) = \frac{1}{2^2\pi}\left[\frac{1 - r_1^2}{1 + r_1^2 - 2r_1\cos\theta} + \frac{1 - r_2^2}{1 + r_2^2 - 2r_1\cos\theta}\right] \tag{67}$$

with

$$r_1 = e^{-\frac{t}{\sqrt{2}}(1-i)}, \qquad r_2 = e^{-\frac{t}{\sqrt{2}}(1+i)}. \tag{68}$$

References

1. Accetta, G., Orsingher, E.: Asymptotic expansion of fundamental solutions of higher order heat equations. Random Oper. Stoch. Equ. **5**(3), 217–226 (1997)
2. Bernstein, F.: Über das Fourierintegral $\int_0^\infty e^{-x*} \cos txdx$. Math. Ann. **79**(3), 265–268 (1918)
3. Bochner, S.: Diffusion equation and stochastic processes. Proc. Nat. Acad. Sci. U. S. A. **35**, 368–370 (1949)
4. Burwell, W.R.: Asymptotic expansions of generalized hyper-geometric functions. Proc. London Math. Soc. **S2-22**(1), 57
5. Cammarota, V., Lachal, A.: Joint distribution of the process and its sojourn time on the positive half-line for pseudo-processes governed by high-order heat equation. Electron. J. Probab. **15**, 895–931 (2010)
6. Cammarota, V., Lachal, A.: Joint distribution of the process and its sojourn time in a half-line $[a, +\infty)$ for pseudo-processes driven by a high-order heat-type equation. Stoch. Process. Appl. **122**(1), 217–249 (2012)
7. Hochberg, K.J.: A signed measure on path space related to Wiener measure. Ann. Probab. **6**(3), 433–458 (1978)
8. Krylov, V.J.: Some properties of the distribution corresponding to the equation $\partial u/\partial t = (-1)^{q+1}\partial^{2q}u/\partial x^{2q}$. Soviet Math. Dokl. **1**, 760–763 (1960)
9. Lachal, A.: Distributions of sojourn time, maximum and minimum for pseudo-processes governed by higher-order heat-type equations. Electron. J. Probab. **8**(20), 53 (2003) (electronic)
10. Lachal, A.: First hitting time and place, monopoles and multipoles for pseudo-processes driven by the equation $\frac{\partial}{\partial t} = \pm\frac{\partial^N}{\partial x^N}$. Electron. J. Probab. **12**(29), 300–353 (2007)
11. Lachal, A.: First hitting time and place for pseudo-processes driven by the equation $\frac{\partial}{\partial t} = \pm\frac{\partial^N}{\partial x^N}$ subject to a linear drift. Stoch. Process. Appl. **118**(1), 1–27 (2008)
12. Lachal, A.: A survey on the pseudo-process driven by the high-order heat-type equation $\partial/\partial t = \pm\partial^N/\partial x^N$ concerning the hitting and sojourn times. Methodol. Comput. Appl. Probab. **14**(3), 549–566 (2012)
13. Lévy, P.: Sur une application de la dérivée d'ordre non entier au calcul des probabilités. C. R. **176**, 1118–1120 (1923)
14. Li, X., Wong, R.: Asymptotic behaviour of the fundamental solution to $\partial u/\partial t = -(-\Delta)^m u$. Proc. Roy. Soc. London Ser. A **441**(1912), 423–432 (1993)
15. Nishioka, K.: The first hitting time and place of a half-line by a biharmonic pseudo process. Japan. J. Math. (N.S.) **23**(2), 235–280 (1997)
16. Orsingher, E.: Processes governed by signed measures connected with third-order "heat-type" equations. Litovsk. Mat. Sb. **31**(2), 323–336 (1991)
17. Orsingher, E., D'Ovidio, M.: Vibrations and fractional vibrations of rods, plates and Fresnel pseudo-processes. J. Stat. Phys. **145**(1), 143–174 (2011)
18. Orsingher, E., D'Ovidio, M.: Probabilistic representation of fundamental solutions to $\frac{\partial u}{\partial t} = \kappa_m \frac{\partial^m u}{\partial x^m}$. Electron. Commun. Probab. **17**(1885), 12 (2012)
19. Orsingher, E., D'Ovidio, M.: Higher-order Laplace equations and hyper-Cauchy distributions. J. Theor. Probab. (2013, published online on February 28th 2013)
20. Orsingher, E., Toaldo, B.: Even-order pseudoprocesses on a circle and related Poisson kernels (2012)
21. Pólya, G.: On the zeros of an integral function represented by Fourier's integral. Messenger **52**, 185–188 (1923)

Reciprocal Processes: A Stochastic Analysis Approach

Sylvie Rœlly

Abstract Reciprocal processes, whose concept can be traced back to E. Schrödinger, form a class of stochastic processes constructed as mixture of bridges. They are Markov fields indexed by a time interval. We discuss here a new unifying approach to characterize several types of reciprocal processes via duality formulae on path spaces: The case of reciprocal processes with continuous paths associated to Brownian diffusions and the case of pure jump reciprocal processes associated to counting processes are treated. This chapter is based on joint works with M. Thieullen, R. Murr, and C. Léonard.

1 Introduction and Historical Remarks

The theory of reciprocal processes evolved from an idea by Schrödinger. In [25], he described the motion of a Brownian particle under constraints at initial and final times as a stochastic variational problem and proposed that its solutions are stochastic processes that have the same bridges as the Brownian motion. Bernstein called them *réciproques* and pointed out that they are Markov fields indexed by time, which allows to state probabilistic models based on a symmetric notion of past and future: *ces grandeurs deviennent stochastiquement parfaites*! See [1].

Various aspects of reciprocal processes have been examined by several authors. Many fundamental reciprocal properties were given by Jamison in a series of articles [12–14], first in the context of Gaussian processes. Contributions to a physical interpretation and to the development of a stochastic calculus adjusted to

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reciprocal diffusions have been made by Zambrini and various coauthors in their interest of creating a *Euclidean version* of quantum mechanics (see [8, 27] and the monograph [6]). Krener in [17] and then Clark in [7] exhibited reciprocal invariants associated with classes of reciprocal diffusions.

This chapter reviews and unifies for the first time current results on characterizing various types of reciprocal processes by duality formulae.

A first duality formula appeared under the Wiener measure as an analytical tool in Malliavin calculus; see [2]. It is an integration by parts on the set of continuous paths, which reflects the duality between a stochastic derivative operator and a stochastic integral operator. In [24], the authors indeed characterize the Brownian motion as the unique continuous process for which the Malliavin derivative and the Skorohod integral are dual operators.

In the framework of jump processes, a characterization of the Poisson process as the unique process for which a difference operator and a compensated stochastic integral are in duality was first given by Slivnjak [26] and extended to Poisson measures by Mecke [19].

We present here duality formulae as unifying tool to characterize classes of reciprocal processes in following contexts:

- In the framework of Brownian diffusions, reviewing results of [22, 23]
- In the framework of pure jump processes, namely, counting processes, following the recent studies of Murr [20]

2 Reciprocal Processes and Reciprocal Classes

We mainly work on the canonical càdlàg path space $\Omega = \mathbb{D}([0,1],\mathbb{R})$ or some subset of it. It is endowed with the canonical σ-algebra $\mathcal{A}$, induced by the *canonical process* $X = (X_t)_{t\in[0,1]}$.

For a time interval $[s,u] \subset [0,1]$ one defines:

- $X_{[s,u]} := (X_t)_{t\in[s,u]}$
- $\mathcal{A}_{[s,u]} := \sigma(X_{[s,u]})$, internal story of the process between time s and time u

$\mathcal{P}(\Omega)$ denotes the space of probability measures on Ω.

For a probability measure $P \in \mathcal{P}(\Omega)$,

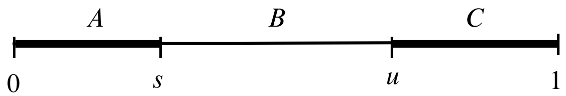
$$P_{01} := P \circ (X_0, X_1)^{-1} \in \mathcal{P}(\mathbb{R}^2)$$

denotes its endpoint marginal law.

2.1 Definition and First Properties

Definition 1. The probability measure $P \in \mathcal{P}(\Omega)$ is reciprocal, or the law of a *reciprocal process*, if for any $s \le u$ in $[0,1]$ and any event $A \in \mathcal{A}_{[0,s]}, B \in \mathcal{A}_{[s,u]}, C \in \mathcal{A}_{[u,1]}$,

$$P(A \cap B \cap C \mid X_s, X_u) = P(A \cap C \mid X_s, X_u) P(B \mid X_s, X_u) P\text{-a.e.} \tag{1}$$



This property—which is time symmetric—makes explicit the conditional independence under P of the future of u and the past of s with the events happened between s and u, given the σ-algebras at boundary times s and u.

The reciprocality can be expressed in several equivalent ways.

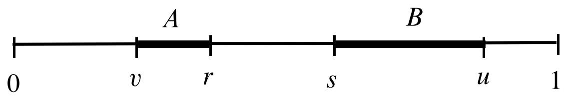
Theorem 1. *Let $P \in \mathcal{P}(\Omega)$. Following assertions are equivalent:*

(1) *The probability measure P is reciprocal.*
(1)* *The reversed probability measure $P^* := P \circ (X_{1-\cdot})^{-1}$ is reciprocal.*
(2) *For any $0 \le s \le u \le 1$ and $B \in \mathcal{A}_{[s,u]}$*

$$P(B \mid X_{[0,s]}, X_{[u,1]}) = P(B \mid X_s, X_u). \tag{2}$$

(3) *For any $0 \le v \le r \le s \le u \le 1$, and $A \in \mathcal{A}_{[v,r]}$, $B \in \mathcal{A}_{[s,u]}$,*

$$P(A \cap B \mid X_{[0,v]}, X_{[r,s]}, X_{[u,1]}) = P(A \mid X_v, X_r) P(B \mid X_s, X_u).$$



Proof. See, e.g., Theorem 2.3 in [18]. □

The identity (2) points out that any reciprocal process is a Markov field parametrized by the time interval [0,1]: To condition events between s and u, knowing the future of u and past of s is equivalent to condition them knowing only the σ-algebras at both times s and u. This property is sometimes called *two-side Markov property*. Therefore

Proposition 1. *Any Markov process is reciprocal but the inverse is false.*

Proof. The first assertion was first done in [12] in a Gaussian framework.

Take P the law of a Markov process, $0 \le s \le u \le 1$ and $A \in \mathcal{A}_{[0,s]}$, $B \in \mathcal{A}_{[s,u]}$, and $C \in \mathcal{A}_{[u,1]}$. The following holds:

$$\begin{aligned}
P(A \cap B \cap C) &= E[P(A \cap B \cap C \mid X_{[s,u]})] \\
&\overset{*}{=} E[P(A \mid X_s)\mathbf{1}_B P(C \mid X_u)] \\
&= E[P(A \mid X_s)P(B \mid X_s, X_u)P(C \mid X_u)] \\
&\overset{*}{=} E[P(A \mid X_s)P(B \mid X_s, X_u)P(C \mid X_{[0,u]})] \\
&= E[P(A \mid X_s)P(B \mid X_s, X_u)\mathbf{1}_C] \\
&\overset{*}{=} E[P(A \mid X_{[s,1]})P(B \mid X_s, X_u)\mathbf{1}_C] \\
&= E[\mathbf{1}_A P(B \mid X_s, X_u)\mathbf{1}_C],
\end{aligned}$$

where the Markov property was used to prove equalities with *. Therefore, (2) holds and P is reciprocal.

As a counterexample, take, e.g., the periodic process constructed in Sect. 3.1.4.

□

Indeed a canonical method to construct reciprocal processes is to mix Markovian bridges. Take $P \in \mathcal{P}(\Omega)$ the law of a Markov process whose bridges $(P^{xy})_{x,y\in\mathbb{R}}$ can be constructed *for all* $x, y \in \mathbb{R}$ as a regular version of the family of conditional laws $P(\cdot \mid X_0 = x, X_1 = y), x, y \in \mathbb{R}$. (It is a difficult challenge in a general non-Markov setting, but it is already done if P is a Lévy process, see [15, 21] Proposition 3.1, or if P is a right process [11] or a Feller process, see the recent paper [4].) One can now associate with P a class of reciprocal processes as follows.

Definition 2. The set of probability measures on Ω obtained as mixture of bridges of $P \in \mathcal{P}(\Omega)$,

$$\mathfrak{R}_c(P) := \{Q \in \mathcal{P}(\Omega) : Q(\cdot) = \int_{\mathbb{R}\times\mathbb{R}} P^{xy}(\cdot)\, Q_{01}(dxdy)\}, \tag{3}$$

is the so-called *reciprocal class* associated with P.

This concept was introduced by Jamison in [13] in the case of a Markov reference process P whose transition kernels admit densities.

Note that, in spite of its name, a reciprocal *class* is not an *equivalence class* because the relation is often not symmetric: The periodic process P^{per} constructed in Sect. 3.1.4 belongs to $\mathfrak{R}_c(P)$ but $P \notin \mathfrak{R}_c(P^{\text{per}})$ if P is not periodic.

Proposition 2. *Any process in the reciprocal class $\mathfrak{R}_c(P)$ is reciprocal and its bridges coincide a.s. with those of P.*

Proof. Let $Q \in \mathfrak{R}_c(P)$ as in (3). Let us show that Q satisfies (2). Let $0 \leq s \leq t \leq 1$, $A \in \mathcal{A}_{[0,s]}$, $B \in \mathcal{A}_{[s,u]}$, and $C \in \mathcal{A}_{[u,1]}$. Then

$$\begin{aligned} E_Q[\mathbf{1}_A Q(B \mid X_{[0,s]}, X_{[u,1]})\mathbf{1}_C] &= Q(A \cap B \cap C) = \int_{\mathbb{R}\times\mathbb{R}} P^{xy}(A \cap B \cap C)\,\pi(dxdy) \\ &\overset{\checkmark}{=} \int_{\mathbb{R}\times\mathbb{R}} E_{P^{xy}}[\mathbf{1}_A P(B \mid X_s, X_t)\mathbf{1}_C]\,\pi(dxdy) \\ &= E_Q[\mathbf{1}_A P(B \mid X_s, X_t)\mathbf{1}_C], \end{aligned}$$

where the reciprocality of P was used at the marked equality. Thus $Q(B \mid X_{[0,s]}, X_{[t,1]})$ only depends on (X_s, X_t) and $Q(B \mid X_{[0,s]}, X_{[t,1]}) = P(B \mid X_s, X_t)$, Q-a.e. which completes the proof. □

2.2 *Reciprocal Characteristics*

Let us now introduce, in two important frameworks, functionals of the reference process which are invariant on its reciprocal class. They indeed characterize the reciprocal class, as we will see in Theorems 2 and 3.

2.2.1 Case of Brownian Diffusions

In this paragraph the path space is restricted to the set of continuous paths $\Omega_c := \mathbf{C}([0,1];\mathbb{R})$. Consider as reference probability measure $\mathbb{P}_b \in \mathcal{P}(\Omega_c)$ a Brownian diffusion with regular drift b, that is, the law of the SDE

$$dX_t = dB_t + b(t, X_t)\,dt,$$

where B is a Brownian motion and $b(t,x) \in \mathbf{C}^{1,2}([0;1] \times \mathbb{R};\mathbb{R})$.

The family of its bridges $(\mathbb{P}_b^{xy})_{x,y\in\mathbb{R}}$ can be constructed for all $x, y \in \mathbb{R}$ as mentioned in the preceding section. Since we are only interested in its reciprocal class, the marginal at time 0 of $\mathbb{P}_b$ does not play any role, and, therefore, we do not mention it.

Clark proved a conjecture of Krener, stating that the reciprocal class of $\mathbb{P}_b$ is, in some sense, characterized by the time–space function

$$F_b(t,x) := \partial_t b(t,x) + \frac{1}{2}\partial_x(b^2 + \partial_x b)(t,x),$$

thus called *reciprocal characteristics* associated with $\mathbb{P}_b$.

Theorem 2. *Let $\mathbb{P}_b$ and $\mathbb{P}_{\tilde{b}}$ be two Brownian diffusions with smooth drifts b and $\tilde{b}$.*

$$\mathfrak{R}_c(\mathbb{P}_b) = \mathfrak{R}_c(\mathbb{P}_{\tilde{b}}) \Leftrightarrow F_b \equiv F_{\tilde{b}}.$$

Proof. See [7] Theorem 1. □

Example 1. 1. The reciprocal characteristics of a Wiener measure $\mathbb{P}_0$, law of a Brownian motion with any initial condition, vanishes since $b \equiv 0 \Rightarrow F_b \equiv 0$.
2. The reciprocal characteristics of the Ornstein–Uhlenbeck process with linear time-independent drift $b(x) = -\lambda x$ is the linear function $x \mapsto \lambda^2 x$.
3. It is known that if the Brownian diffusion $\mathbb{P}_b$ admits a smooth transition density p_b, then its bridge $\mathbb{P}_b^{xy}$ between x and y can be constructed as a Brownian diffusion with drift b^{xy} given by

$$b^{xy}(t,z) = b(t,z) + \partial_z \log p_b(t,z;1,y), \quad t < 1.$$

Let us compute $F_{b^{xy}}$:

$$\begin{aligned}
&F_{b^{xy}}(t,z) - F_b(t,z)\\
&\quad = \partial_t \partial_z \log p_b(t,z;1,y) + \partial_z b(t,z)\,\partial_z \log p_b(t,z;1,y) + b(t,z)\,\partial_z^2 \log p_b(t,z;1,y)\\
&\qquad + \big(\partial_z \log p_b\, \partial_z^2 \log p_b\big)(t,z;1,y) + \frac{1}{2}\partial_z^3 \log p_b(t,z;1,y)\\
&\quad = 0,
\end{aligned}$$

where we used the identity

$$\partial_t p_b(t,z;1,y) + \partial_z^2 p_b(t,z;1,y) + b(t,z)\partial_z p_b(t,z;1,y) = 0.$$

It confirms the fact that $\mathbb{P}_b \in \mathfrak{R}_c(\mathbb{P}_b^{xy})$.

Remark 1. In the multidimensional case, when the path space is $\mathbf{C}([0,1];\mathbb{R}^d)$, $d > 1$, one needs one more function to characterize the reciprocal class $\mathfrak{R}_c(\mathbb{P}_b)$. It is denoted by G_b and defined as an $\mathbb{R}^{d\otimes d}$-valued function $G_b(t,x) = (G_b^{i,j}(t,x))_{i,j}$ as follows $G_b^{i,j} := \partial_j b^i - \partial_i b^j$; see [7].

2.2.2 Case of Counting Processes

In this paragraph, let us now restrict the path space to the set of càdlàg step functions with unit jumps on [0,1]. It can be described as follows:

$$\Omega_j := \left\{ \omega = x\,\delta_0 + \sum_{i=1}^{n} \delta_{t_i}, 0 < t_1 < \cdots < t_n < 1, x \in \mathbb{R}, n \in \mathbb{N} \right\},$$

Consider, as reference Markov probability measure $\mathbf{P}_\ell \in \mathcal{P}(\Omega_j)$, the law of a counting process with a regular uniformly bounded Markovian jump intensity ℓ, satisfying for all $x \in \mathbb{R}, \ell(\cdot, x) \in \mathbf{C}^1([0;1];\mathbb{R})$ and $0 < \inf_{t,x} \ell(t,x) \leq \sup_{t,x} \ell(t,x) < +\infty$.

Note that the definition of $\mathfrak{R}_c(\mathbf{P}_\ell)$ makes sense: On one side the family of bridges $\mathbf{P}_\ell^{xy}$ can be constructed for all x, y such that $y - x \in \mathbb{N}$; on the other side, for any $Q \in \mathcal{P}(\Omega_j)$, its endpoint marginal law Q_{01} is concentrated on such configurations. Murr identified a time–space functional Ξ_ℓ of the intensity ℓ as characteristics of the reciprocal class associated with $\mathbf{P}_\ell$.

Theorem 3. *Let $\mathbf{P}_\ell$ and $\mathbf{P}_{\tilde\ell}$ be two counting processes with intensities ℓ and $\tilde\ell$ as below.*

$$\mathfrak{R}_c(\mathbf{P}_\ell) = \mathfrak{R}_c(\mathbf{P}_{\tilde\ell}) \Leftrightarrow \Xi_\ell \equiv \Xi_{\tilde\ell}, \tag{4}$$

where $\Xi_\ell(t,x) := \partial_t \log \ell(t,x) + \big(\ell(t,x+1) - \ell(t,x)\big)$.

Proof. See [20] Theorem 6.58. □

Example 2. 1. The standard Poisson process $\mathbf{P} := \mathbf{P}_1$ has constant jump rate—or intensity—equal to 1 and initial deterministic condition equal to 0. Its reciprocal characteristics vanishes since $\ell \equiv 1 \Rightarrow \Xi_\ell \equiv 0$.
2. All Poisson processes are in the same reciprocal class since, for any constant jump rate $\lambda > 0$, $\ell \equiv \lambda \Rightarrow \Xi_\ell = \Xi_1 \equiv 0$.
3. For $x, y \in \mathbb{R}$ with $y - x \in \mathbb{N}$, the bridge $\mathbf{P}^{xy}$ of $\mathbf{P}$ is the Markov counting process starting at x with time–space-dependent intensity given by $\ell^{xy}(t,z) = \frac{\max(y-z,0)}{1-t}$, for any $t < 1$.
   One verifies, as in Example 1 (3), that $\Xi_{\ell^{xy}} = \Xi_1 = 0$.

3 Characterization Via Duality Formulae

Our aim is now to show that each reciprocal class coincides—in the frameworks we introduced below—with the set of random processes for which a perturbed duality relation holds between the stochastic integration and some derivative operator on the adequate path space.

3.1 Case of Brownian Diffusions

3.1.1 The Test Functions and the Operators

On Ω_c, we define a set of smooth cylindrical functionals by:

$$\mathcal{S} = \{\Phi : \Phi = \varphi(X_{t_1}, \ldots, X_{t_n}), \varphi \in \mathbf{C}_b^\infty(\mathbb{R}^n;\mathbb{R}), n \in \mathbb{N}^*, 0 \leq t_1 < \cdots < t_n \leq 1\}.$$

The *derivation operator* D_g in the direction $g \in L^2([0,1];\mathbb{R})$ is defined on $\mathcal{S}$ by

$$D_g\Phi(\omega) := \lim_{\varepsilon} \frac{1}{\varepsilon}\left(\Phi(\omega + \varepsilon \int_0^{\cdot} g(t)dt) - \Phi(\omega)\right) = \sum_{j=1}^{n} \int_0^{t_j} g(t)\, \frac{\partial \varphi}{\partial x_j}(\omega_{t_1}, \ldots, \omega_{t_n})\, dt.$$

$D_g\Phi$ is the Malliavin derivative of Φ in the direction $\int_0^{\cdot} g(t)dt$, element of the Cameron–Martin space. Furthermore,

$$D_g\Phi = \langle g, D_{\cdot}\Phi\rangle_{L^2([0,1];\mathbb{R})} \text{ where } D_t\Phi = \sum_{j=1}^{n} \frac{\partial \varphi}{\partial x_j}(X_{t_1}, \ldots, X_{t_n})\mathbf{1}_{[0,t_j]}(t).$$

The *integration operator* under the canonical process, denoted by δ_g, is defined as

$$\delta(g) := \int_0^1 g(t)\, dX_t.$$

It is always well defined if the test function g is simple, i.e. a linear combination of indicator functions of time intervals.

A *loop* on $[0,1]$ is a function g with vanishing integral: $\int_0^1 g(t)dt = 0$, that is, $g \in \{\mathbf{1}\}^{\perp}$ in $L^2([0,1];\mathbb{R})$.

3.1.2 Duality Formula Under the Wiener Measure and Its Reciprocal Class

We are now able to present the duality between the operators D and δ under all probability measures belonging to the reciprocal class of a Wiener measure. We denote by $\mathbb{P}$ the standard Wiener measure, which charges only paths with initial condition at 0.

Theorem 4. *Let Q be a probability measure on Ω_c such that $E_Q(|X_t|) < +\infty$ for all $t \in [0,1]$.*

$$Q \text{ is a Wiener measure} \Leftrightarrow \forall \Phi \in \mathcal{S},\, E_Q(D_g\Phi) = E_Q\big(\Phi\, \delta(g)\big),\, \forall g \text{ simple.} \quad (5)$$

$$Q \in \mathfrak{R}_c(\mathbb{P}) \Leftrightarrow \forall \Phi \in \mathcal{S},\, E_Q(D_g\Phi) = E_Q\big(\Phi\, \delta(g)\big),\, \forall g \text{ simple loop.} \quad (6)$$

Proof.

- Sketch of $\overset{(5)}{\Rightarrow}$: Using Girsanov formula,

$$E_{\mathbb{P}_0}(D_g\Phi) = E_{\mathbb{P}_0}\left(\lim_{\varepsilon\to 0} \frac{\Phi(\cdot + \varepsilon \int_0^{\cdot} g(t)dt) - \Phi}{\varepsilon}\right) = E_{\mathbb{P}_0}\left(\Phi\, \partial_\varepsilon Z_\varepsilon|_{\varepsilon=0}\right)$$

with $Z_\varepsilon := \exp(\varepsilon \int_0^1 g(t)dX_t - \frac{\varepsilon^2}{2}\int_0^1 g(t)^2 dt\ )$.

- $\overset{(5)}{\Leftarrow}$: With adequate choice of Φ and g, one can prove that the canonical process $X_t - X_0$ is a Q-martingale, as well as $(X_t - X_0)^2 - t$. This enables to conclude that Q is any Wiener measure. For details, see [24].
- First note that $Q \in \mathfrak{R}_c(\mathbb{P}) \Leftrightarrow Q = \int \mathbb{P}^{xy}\, Q_{01}(dx, dy)$.
  $\overset{(6)}{\Rightarrow}$: Take $\Phi(\omega) = \phi_0(\omega(0))\phi_1(\omega(1))\tilde{\Phi}(\omega)$ in (5). Then

$$E_{\mathbb{P}}\big(\phi_0\,\phi_1\,\tilde{\Phi}\,\delta(g)\big) = E_{\mathbb{P}}(D_g(\phi_0\,\phi_1\,\tilde{\Phi}))$$

  which implies that, for all smooth ϕ_0, ϕ_1,

$$\begin{aligned}
&E_{\mathbb{P}}\Big(\phi_0(X_0)\phi_1(X_1)\mathbb{P}(\tilde{\Phi}\delta(g)|X_0, X_1)\Big)\\
&= E_{\mathbb{P}}\Big(\phi_0(X_0)\phi_1(X_1)\mathbb{P}(D_g\tilde{\Phi}|X_0, X_1)\Big) + E_{\mathbb{P}}\Big(\phi_0(X_0)\phi_1'(X_1)\tilde{\Phi}\Big)\int_0^1 g(t)dt\\
&\Rightarrow E_{\mathbb{P}^{X_0X_1}}\Big(\tilde{\Phi}\delta(g)\Big) = E_{\mathbb{P}^{X_0X_1}}\Big(D_g\tilde{\Phi}\Big) \text{ if } \int_0^1 g(t)dt = 0.
\end{aligned}$$

  This identity holds for any mixture of Brownian bridges too.
- $\overset{(6)}{\Leftarrow}$: Q^{xy} satisfies (6) too, which leads to identify it as the unique Gaussian process with mean $x + t(y - x)$ and covariance $s(1-t)$, that is, $\mathbb{P}^{xy}$. For details, see [22].

∎

Remark 2. 1. Equation (5) is an infinite-dimensional generalization of the one-dimensional integration by parts formula, also called Stein's formula, satisfied by the standard Gaussian law:

$$\int_{\mathbb{R}} \varphi'(x)\,\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx = \int_{\mathbb{R}} \varphi(x)\,x\,\frac{e^{-x^2/2}}{\sqrt{2\pi}}dx.$$

Take $g \equiv 1$ and $\Phi = \varphi(X_1)$ in (5).

2. Equation (5) remains true under the Wiener measure $\mathbb{P}$, for random processes $g \in L^2(\Omega_c \times [0,1]; \mathbb{R})$ Skorohod integrable and for any general $\Phi \in \mathbf{D}^{1,2}$, closure of $\mathcal{S}$ under the norm $\|\Phi\|_{1,2}^2 := \int(\Phi^2 + \int_0^1 |D_t\Phi|^2 dt)d\mathbb{P}$. In such a generality, (5) shows the well-known duality between the Malliavin derivative D and the Skorohod integral δ under $\mathbb{P}$; see, e.g., *[2]*.
3. Since, for computing D_g, paths are not perturbed at time 0, it is clear that (5) characterizes only the Brownian dynamics (Wiener measure), but not the initial law of X_0 under Q.
4. Since, for computing D_g for a loop g, paths are perturbed neither at time 0 nor at time 1, the identity (6) characterizes only the dynamics of the bridges $Q^{X_0X_1}$.

3.1.3 Duality Formula Under the Reciprocal Class of Brownian Diffusions

We now investigate how the duality formula (6) is perturbed when the underlying reference process admits a drift b (satisfying the same smoothness assumptions as in Sect. 2.2.1). The transformed duality equation (7) we present below contains an additional term of order 0 in Φ, in which appears the reciprocal invariant F_b associated with $\mathbb{P}_b$.

Theorem 5. *Let Q be a probability measure on Ω_c such that, for all $t \in [0,1]$, $E_Q\Big(|X_t|^2 + \int_0^1 F_b^2(t, X_t)dt\Big) < +\infty$. Then,*

$$Q \in \mathfrak{R}_c(\mathbb{P}_b) \Leftrightarrow \forall \Phi \in \mathcal{S},\ \forall g \text{ simple loop,}$$

$$E_Q(D_g \Phi) = E_Q\big(\Phi\ \delta(g)\big) + E_Q\Big(\Phi \int_0^1 g(r) \int_r^1 F_b(t, X_t)\, dt\, dr\Big). \quad (7)$$

□

Proof.
- Sketch of $\Rightarrow$: First, the bridges of Q coincide with those of $\mathbb{P}_b$. Since $\mathbb{P}_b^{xy}$ is absolutely continuous with respect to $\mathbb{P}_b$ on any time interval $[0, 1-\varepsilon]$, $\varepsilon > 0$, one can use the Girsanov density to prove that $\mathbb{P}_b^{xy}$ satisfies (7), and thus, by linearity, Q satisfies (7) too.
- $\Leftarrow$: First, Q^{xy} satisfies (7) for a.a. x, y. This allows to prove that the canonical process is a Q^{xy}-quasi-martingale. Therefore, by Rao's theorem (see [9]), it is a Q^{xy}-semi-martingale. Its characteristics can be computed: The quadratic variation is t and the bounded variation part is of the form $t \mapsto \int_0^t b^{x,y}(s, X_s)ds$. One computes that $F_{b^{x,y}} = F_b$. Thus $Q^{xy} = \mathbb{P}_b^{xy}$ and $Q \in \mathfrak{R}_c(\mathbb{P}_b)$. For more details, see [22], Theorem 4.3. □

3.1.4 Some Applications

We first illustrate the use of the identity (7) to identify a process as element of some precise reciprocal class. Consider, as Markov reference process, the Ornstein–Uhlenbeck process denoted by P_{OU}, introduced in Example 1 (2), whose associated reciprocal characteristics is $F_{\text{OU}}(x) = \lambda^2 x$. Consider now the periodic Ornstein–Uhlenbeck process denoted by $P_{\text{OU}}^{\text{per}}$ and solution of the following stochastic differential equation with periodic boundary conditions on the time interval [0,1]:

$$dX_t = dB_t - \lambda X_t\, dt\ , \quad X_0 = X_1. \quad (8)$$

This process is Gaussian as the following representation shows:

$$X_t = \int_0^t \frac{e^{-\lambda(t-s)}}{1-e^{-\lambda}} dB_s + \int_t^1 \frac{e^{-\lambda(1+t-s)}}{1-e^{-\lambda}} dB_s =: \Psi(B)_t. \quad (9)$$

But it is not Markov as the following representation shows:

$$X_t = X_0 + B_t - \int_0^t \left(\lambda X_s - \lambda \frac{X_0 - e^{-\lambda(1-s)} X_s}{\sinh(\lambda(1-s))} \right) ds, \quad X_0 \sim \mathcal{N}\left(0, \frac{\coth(\lambda/2)}{2\lambda}\right).$$

A natural question is then to investigate if it is reciprocal. In [3] the authors analysed the form of its covariance kernel to deduce the reciprocality of $P_{\mathrm{OU}}^{\mathrm{per}}$. We proposed in [22] an alternative proof based on (7), which allows to conclude directly that $P_{\mathrm{OU}}^{\mathrm{per}} \in \mathfrak{R}_c(P_{\mathrm{OU}})$: Thanks to the representation (9), one notes that the shifted process $X + \varepsilon \int_0^{\cdot} g(t)dt$ can also be represented as the transform by Ψ of a shifted Brownian motion, if g is a loop. It remains to use Girsanov theorem by computing

$$E_{P_{\mathrm{OU}}^{\mathrm{per}}}(D_g \Phi) = E_{P_{\mathrm{OU}}^{\mathrm{per}}} \left(\lim_{\varepsilon \to 0} \frac{\Phi(\cdot + \varepsilon \int_0^{\cdot} g(t)dt) - \Phi}{\varepsilon} \right)$$

to obtain that $P_{\mathrm{OU}}^{\mathrm{per}}$ satisfies, for all $\Phi \in \mathcal{S}$ and g simple loops,

$$E_{P_{\mathrm{OU}}^{\mathrm{per}}}(D_g \Phi) = E_{P_{\mathrm{OU}}^{\mathrm{per}}}\big(\Phi\ \delta(g)\big) + E_{P_{\mathrm{OU}}^{\mathrm{per}}}\Big(\Phi \int_0^1 g(r) \int_r^1 \lambda^2 X_t\, dt\, dr\Big).$$

Let us now present a generalization of the famous result stated by Kolmogorov in [16]: A Brownian diffusion with values in $\mathbb{R}^d$ and time-homogeneous drift b is reversible (i.e. there exists an initial distribution such that $\mathbb{P}_b = \mathbb{P}_b^*$) if and only if the function b is a gradient.

In the next Theorem, whose proof is detailed in [23] Theorem 5.4, we obtain the same result under much weaker assumptions: We only require that there exists one reversible law in $\mathfrak{R}_c(\mathbb{P}_b)$ and we do not suppose that the drift is time-homogeneous. Its proof is based on the d-dimensional duality formula characterizing the reciprocal class $\mathfrak{R}_c(\mathbb{P}_b)$.

Theorem 6. *Let b be a d-dimensional smooth drift such that for any $i, j \in \{1, \ldots, d\}$, the function $\big(\partial_j b^i - \partial_i b^j\big)(t, x)$ is time-independent. Furthermore suppose there exists $Q \in \mathfrak{R}_c(\mathbb{P}_b)$ with finite entropy which is time-reversible. Then the drift b is of gradient type, i.e.*

$$\exists \varphi : [0, 1] \times \mathbb{R}^d \mapsto \mathbb{R} \text{ such that, for all } t, \quad b(t, \cdot) = -\nabla\varphi(t, \cdot).$$

Moreover, if Q is itself a Brownian diffusion with drift b, then b is time-independent and

$$Q(\cdot) = \frac{1}{c} \int_{\mathbb{R}^d} \mathbb{P}_b(\cdot | X_0 = x)\, e^{-2\varphi(x)}\, dx,$$

for some positive constant c.

3.2 *Case of Counting Processes*

3.2.1 The Test Functions and the Operators

Any path ω in Ω_j is characterized by its initial value x, the number of its jumps till time 1, say n, and the times of its jumps, $t_1, \dots, t_n$. We then define the ith jump-time of a path by the functional:

$$T_i(\omega) = T_i\left(x\,\delta_0 + \sum_{j=1}^{n} \delta_{t_j}\right) := t_i \mathbf{1}_{i \le n} + \mathbf{1}_{i>n}.$$

We now define a set of smooth test functionals on Ω_j by:

$$\mathcal{S} = \{\Phi : \Phi = \varphi\big(X_0, T_1, \dots, T_n\big), \varphi \in \mathbf{C}_b^\infty(\mathbb{R}^{n+1}; \mathbb{R}), n \in \mathbb{N}^*\}.$$

The *derivation operator* $\mathcal{D}_g$ in the direction $g \in L^2([0,1]; \mathbb{R})$ is based on the perturbation of the jump-times and defined on $\mathcal{S}$ by:

$$\mathcal{D}_g\Phi := \lim_{\varepsilon} \frac{1}{\varepsilon}\left(\varphi\Big(X_0, T_1 + \varepsilon \int_0^{T_1} g(t)dt, \dots, T_n + \varepsilon \int_0^{T_n} g(t)dt\Big) - \Phi\right).$$

It was introduced by Elliott and Tsoi in [10].

3.2.2 Duality Formula Under the Poisson Process and Its Reciprocal Class

We are now able to present the duality between $\mathcal{D}$ and an integration operator under all probability measures in the reciprocal class of the standard Poisson process. Recall the notations introduced in Example 2: $\mathbf{P}$ denotes the standard Poisson process on [0,1] and $\mathbf{P}_\lambda$ denotes a Poisson process on [0,1] with intensity λ and any marginal law at time 0.

Theorem 7. *Let Q be a probability measure on Ω_j such that $E_Q(|X_1 - X_0|) < +\infty$.*

$$Q = \mathbf{P}_\lambda \Leftrightarrow \forall \Phi \in \mathcal{S}, E_Q(\mathcal{D}_g\Phi) = E_Q\left(\Phi \int_0^1 g(s)(dX_s - \lambda ds)\right), \forall g \text{ simple} \tag{10}$$

$$Q \in \Re_c(\mathbf{P}) \Leftrightarrow \forall \Phi \in \mathcal{S}, E_Q(\mathcal{D}_g\Phi) = E_Q\left(\Phi \int_0^1 g(s)(dX_s - ds)\right) \forall g \text{ simple loop}. \tag{11}$$

Proof. • Sketch of $\overset{(10)}{\Leftrightarrow}$. The main tool is Watanabe's characterization: Q is a Poisson process with intensity λ on Ω_j if and only if $(X_t - X_0 - \lambda t)_t$ is a Q-martingale.

- Sketch of $\overset{(11)}{\Leftrightarrow}$. One fixes an initial value x and tries to identify the compensator of Q^x. Using (11) one shows that its compensator is absolutely continuous with respect to Lebesgue measure, with Markov intensity of the form $\ell^x(t, X_{t}-)$, and that $\Xi_{\ell^x} \equiv 0$. Thanks Theorem 3 one can conclude.
  For details, see [20] Theorem 6.39. □

Remark 3. 1. Equation (10) is an infinite-dimensional generalization of the formula characterizing the Poisson distribution $\mathcal{P}_\alpha$ on $\mathbb{N}$, known as Chen's lemma, see [5]: Let Z a real-valued random variable.

$$Z \sim \mathcal{P}_\alpha \Leftrightarrow \forall \varphi \text{ smooth}, \quad E(\varphi(Z)Z) = \alpha\, E(\varphi(Z+1)).$$

2. For loops g, the right side of (11) indeed reduces to $E_Q\big(\Phi \int_0^1 g(s)dX_s\big)$. Therefore one immediately recovers that all Poisson processes with any intensity are in a unique reciprocal class, the reciprocal class of the standard Poisson process **P**. In particular, the law of bridges of Poisson processes depends uniquely on their boundary conditions but does not depend of their original intensities.

3.2.3 Duality Formula Under the Reciprocal Class of a Counting Process

We now investigate how the duality formula (11) is perturbed when the underlying reference process $\mathbf{P}_\ell$ admits a jump intensity ℓ which is no more constant, but smooth enough, as in Theorem 3. Similar to Sect. 3.1.3, the transformed duality equation (12) presented below contains an additional term of order 0 in Φ, in which appears the reciprocal invariant Ξ_ℓ associated with $\mathbf{P}_\ell$.

Theorem 8. *Let Q be a probability measure on Ω_j such that* $E_Q(|X_1 - X_0|) < +\infty$.

$$Q \in \mathfrak{R}_c(\mathbf{P}_\ell) \Leftrightarrow \forall \Phi \in \mathcal{S},\ \forall g \text{ simple loop},$$
$$E_Q(D_g\Phi) = E_Q\Big(\Phi \int_0^1 g(s)(dX_s - ds)\Big) + E_Q\Big(\Phi \int_0^1 g(s) \int_s^1 \Xi_l(r, X_{r}-)dX_r ds\Big). \tag{12}$$

Such a duality formula can be used to several aims. One application is, e.g., the investigation of the time reversal of reciprocal processes belonging to the class $\mathfrak{R}_c(\mathbf{P}_\ell)$; see [20] for details.

The extension of these results to pure jump processes with general jumps is in preparation.

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References

1. Bernstein, S.: Sur les liaisons entre les grandeurs aléatoires. Vehr. des intern. Mathematikerkongress Zürich **I**, 288–309 (1932)
2. Bismut, J.M.: Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions. Z. Wahrsch. Verw. Gebiete **56**, 469–505 (1981)
3. Carmichael, J.P., Masse J.C., Theodorescu R.: Processus gaussiens stationnaires reciproques sur un intervalle. C. R. Acad. Sci. Paris Sér. I Math. **295**, 291–293 (1982)
4. Chaumont, L., Uribe Bravo, G.: Markovian bridges: weak continuity and pathwise constructions. Ann. Probab. **39**, 609–647 (2011)
5. Chen, L.H.Y.: Poisson approximation for dependent trials. Ann. Probab. **3**, 534–545 (1975)
6. Chung, K.L., Zambrini, J.C.: Introduction to Random Time and Quantum Randomness. World Scientific Publishing, Singapore (2003)
7. Clark, J.M.C.: A local characterization of reciprocal diffusions. Appl. Stoch. Anal. **5**, 45–59 (1991)
8. Cruzeiro, A.B., Zambrini, J.C.: Malliavin calculus and Euclidean quantum mechanics. J. Funct. Anal. **96**, 62–95 (1991)
9. Dellacherie, C., Meyer, P.A.: Probabilités et Potentiel. Théorie des martingales. Hermann, Paris (1980)
10. Elliott, R.J., Tsoi, A.H.: Integration by parts for Poisson processes. J. Multivar. Anal. **44**, 179–190 (1993)
11. Fitzsimmons, P., Pitman, J., Yor, M.: Markovian bridges: construction, palm interpretation, and splicing. Progr. Probab. **33**, 101–134 (1992)
12. Jamison, B.: Reciprocal processes: the stationary Gaussian case. Ann. Math. Stat. **41**, 1624–1630 (1970)
13. Jamison, B.: Reciprocal processes. Z. Wahrsch. verw. Geb. **30**, 65–86 (1974)
14. Jamison, B.: The Markov processes of Schroedinger. Z. Wahrsch. verw. Geb. **32**, 323–331 (1975)
15. Kallenberg, O.: Splitting at backward times in regenerative sets. Ann. Probab. **9**, 781–799 (1981)
16. Kolmogorov, A.: Zur Umkehrbarkeit der statistischen Naturgesetze. Mathematische Annalen **113**, 766–772 (1937)
17. Krener, A.J.: Reciprocal diffusions and stochastic differential equations of second order. Stochastics **24**, 393–422 (1988)
18. Léonard, C., Roelly, S., Zambrini, J.C.: Temporal Symmetry of Some Classes of Stochastic Processes. Preprints des Instituts für Mathematik der Universität Potsdam 2, no. 7, http://opus.kobv.de/ubp/volltexte/2013/6459/pdf/pre_math7.pdf (2013)
19. Mecke, J.: Stationäre zufällige Masse auf lokalkompakten Abelschen Gruppen. Z. Wahrsch. verw. Geb. **9**, 36–58 (1967)
20. Murr, R.: Reciprocal classes of Markov processes. An approach with duality formulae. Ph.D Thesis, Universität Potsdam (2012). http://opus.kobv.de/ubp/volltexte/2012/6301/pdf/premath26.pdf
21. Privault, N., Zambrini, J.C.: Markovian bridges and reversible diffusion processes. Annales I H Poincaré **40**, 599–633 (2004)
22. Roelly, S., Thieullen, M.: A characterization of reciprocal processes via an integration by parts formula on the path space. Probab. Theory Relat. Fields **123**, 97–120 (2002)

23. Roelly, S., Thieullen, M.: Duality formula for the bridges of a Brownian diffusion: application to gradient drifts. Stoch. Process. Appl. **115**, 1677–1700 (2005)
24. Roelly S., Zessin, H.: Une caractérisation des diffusions par le calcul des variations stochastiques. C. R. Acad. Sci. Paris Sér. I Math. **313-5**, 309–312 (1991)
25. Schrödinger, E.: Über die Umkehrung der Naturgesetze. Sitzungsberichte Preuss. Akad. Wiss. Berlin. Phys. Math. **144**, 144–153 (1931)
26. Slivnjak, I.M.: Some properties of stationary streams of homogeneous random events. Teor. Verojatnost. i Primenen. **7**, 347–352 (1962)
27. Thieullen, M., Zambrini, J.C.: Symmetries in the stochastic calculus of variations. Probab. Theory Relat. Fields **107**, 401–427 (1997)

Part II
Stochastic Equations

Probabilistic Counterparts of Nonlinear Parabolic Partial Differential Equation Systems

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Abstract We extend the results of the FBSDE theory in order to construct a probabilistic representation of a viscosity solution to the Cauchy problem for a system of quasilinear parabolic equations. We derive a BSDE associated with a class of quasilinear parabolic system and prove the existence and uniqueness of its solution. To be able to deal with systems including nondiagonal first order terms along with the underlying diffusion process, we consider its multiplicative operator functional. We essentially exploit as well the fact that the system under consideration can be reduced to a scalar equation in an enlarged phase space. This allows to obtain some comparison theorems and to prove that a solution to FBSDE gives rise to a viscosity solution of the original Cauchy problem for a system of quasilinear parabolic equations.

1 Introduction

Quasilinear systems of parabolic equations arise as mathematical models which describe various chemical and biological phenomena. They arise as well in financial mathematics and in differential geometry when one considers nonlinear parabolic equations in sections of vector bundles.

Let d, d_1 be given integers and $a(x) \in R^d, A(x) \in R^{d\times d}, B(x) \in R^{d\times d_1\times d_1}, c(x) \in R^{d_1\times d_1}, x \in R^d$, and $g : R^d \times R^{d_1} \times R^{d\times d_1} \to R_1^d$ be given functions. Consider a class of quasilinear parabolic equations of the form

$$\frac{\partial u_l}{\partial s} + \frac{1}{2}TrA^*\nabla^2 u_l A + \langle a, \nabla u_l\rangle + B^i_{lm}\nabla_i u_m + c_{lm}u_m + g_l(s, x, u, \nabla u) = 0,$$
$$u_l(T, x) = u_{0l}(x), \quad l = 1, \dots, d_1 \tag{1}$$

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with respect to R^{d_1}-valued function $u(s,x)$ defined on $[0,T]\times R^d$. Here and below, we assume a convention of summing up over repeating indices if the contrary is not mentioned and denote by $\langle\cdot,\cdot\rangle$ an inner product in R^d regardless of d. One can suggest at least a couple of probabilistic counterparts of the Cauchy problem (1). To derive them let us assume first that there exists a classical solution $u(s,x)$ to this problem. In this case one can prove applying the standard technique of the stochastic differential equation theory and especially the Ito formula that the function $u(s,x)$ satisfying (1) admits at least two probabilistic representations.

The first one was suggested in papers by Dalecky and Belopolskaya [1–3] and was aimed to develop a probabilistic approach to prove the existence and uniqueness of a classical solution to (1) and to much more general systems of the form

$$\frac{\partial u_l}{\partial s}+F(x,u,\nabla u,\nabla^2 u_l)=0,\quad u_l(T,x)=u_{l0}(x).$$

The second one suggested in papers by Pardoux and Peng [4–6] leads to the powerful backward stochastic differential equations (BSDE) theory. This approach allows to construct a viscosity solution to a quasilinear scalar parabolic PDE or to a diagonal system of PDEs (see [6, 7]). In terms of (1) this means that one has to set $B\equiv 0$ and $c\equiv 0$ and $g_l(x,u,A^*\nabla u)\equiv g_l(x,u,A^*\nabla u_l)$.

To present these approaches we fix a probability space $(\Omega,\mathcal{F},P)$ and denote by $w(t)\in R^d$ the standard Wiener process. Let $\mathcal{F}_t$ be a flow of σ-subalgebras of $\mathcal{F}$ generated by $w(t)$ and $E_{s,x}[f(\xi(T))]=E[f(\xi(T)|\xi(s)=x]$ denote the conditional expectation.

Assume that g in (1) does not depend on ∇u and all coefficients a,A,B,C depend on s,x and u. Assume that $u(s,x)$ is a smooth function satisfying (1) with these parameters. Then it was stated in [1] that this function admits a representation of the form

$$\langle h,u(s,x)\rangle=E_{s,x}\left[\langle\eta(T),u_0(\xi(T))\rangle+\int_s^T\langle\eta(\theta),g(\theta,\xi(\theta),u(\theta,\xi(\theta)))d\theta\rangle\right],\tag{2}$$

where stochastic processes $\xi(t)$ and $\eta(t)$ satisfy the stochastic equations

$$d\xi(t)=a(\xi(t),u(t,\xi(t)))dt+A(\xi(t),u(t,\xi(t)))dw(t),\quad \xi(s)=x,\tag{3}$$

and

$$d\eta(t)=c(\xi(t),u(t,\xi(t)))\eta(t)dt+C(\xi(t),u(t,\xi(t)))(\eta(t),dw(t)),\quad \eta(s)=h.\tag{4}$$

Note that a,A,c in (3), (4) are the same as in (1), while it is assumed that C in (4) and B in (1) satisfy an equality $B_k^{lm}=C_i^{lm}A_{ik}$.

Remark 1. Notice that when A is a nondegenerated matrix one can define C by $C_i^{lm}=B_k^{lm}A_{ki}^{-1}$, while when A is a degenerated matrix we assume that B has the above form. It is important that although in a general case $A = 0$ yields $B = 0$ and we do not obtain a general nonlinear system of hyperbolic equations as a vanishing viscosity limit of (1). Nevertheless one can state some restrictions on B such that given $A_\epsilon = \epsilon A$ and $C_\epsilon = \epsilon^{-1}C$ one can apply (2) to investigate the vanishing viscosity limit of (1) with these coefficients (see AB).

An important observation is the fact that we can consider (2)–(4) as a closed system of equations and state conditions on its data to ensure the existence and uniqueness of a solution to this system. If in addition it will be revealed that the function $u(s, x)$ given by (2) is twice differentiable in the spatial variable x, then one can verify that $u(s, x)$ is a unique classical solution of (1) with correspondent parameters. It should be mentioned that this approach can be essentially generated to give a possibility to study systems of quasilinear and even fully nonlinear parabolic equations. In other words one can consider (1) with coefficients a, A, c, C, g depending on $(x, u, \nabla u)$ or even $(x, u, \nabla u, \nabla^2 u)$. Note that to deal with these more complicated cases within a framework of this approach we require more strong assumptions concerning regularity of coefficients of (3)–(4) and the Cauchy data u_0. As a result we can prove on this way the existence and uniqueness of a classical solution to (1), possibly on a small time interval.

To describe the second approach which allows to construct a different class of solutions to the Cauchy problem

$$\frac{\partial u_l}{\partial s}+\frac{1}{2}TrA^*(x)\nabla^2 u_l A(x)+\langle a(x), \nabla u_l\rangle+g_l(x, u, A^*\nabla u_l)=0, \quad u_l(T, x)=u_{0l}(x), \tag{5}$$

we assume once again that there exists a classical solution $u_l(s, x)$ of (5).

Consider a stochastic process $\xi(t)$ satisfying (3) with coefficients $a(s, x, u) \equiv a(s, x)$, $A(s, x, u) \equiv A(s, x)$. Keeping in mind that $u_l(s, x)$ is a classical solution of (5), by Ito's formula, we derive an expression for a stochastic differential of $y(t) = u(t, \xi(t))$ in the form

$$dy(t) = -g(t, \xi(t), y(t), z(t))dt - zdw(t), \quad y(T) = u_0(\xi(T)), \tag{6}$$

where $z(t) = A^*(\xi(t))\nabla u(t, \xi(t))$. The equation (6) is called a BSDE.

In general one can forget about the process $\xi(t)$ and consider an independent BSDE of the form

$$dy(t) = -f(t, y(t), z(t))dt - zdw(t), \quad y(T) = \zeta, \tag{7}$$

where $f(t, y, z)$ is an $\mathcal{F}_t$-adapted random process meeting some additional requirements and ζ is an $\mathcal{F}_T$-measurable random variable. A general theory of BSDEs was developed by a number of authors (see, e.g., [7] for references). In addition the system (4), (6) shows a way to construct the so-called viscosity solution to (5) (defined in [8]) setting $u(s, x) = y(s)$).

To generalize this approach and apply it to (1), we observe that this system has a crucial property which can be easily revealed if one analyzes the probabilistic representation (2) of a smooth solution to (1). Namely, the Cauchy problem (1) can be reduced to the Cauchy problem for a scalar equation:

$$\frac{\partial \Phi}{\partial s}+\frac{1}{2}TrQ^*(x,h)\nabla^2\Phi Q(x,h)+\langle q(x,h),\nabla\Phi\rangle+G(s,h,x,\Phi,Q^*\nabla\Phi)=0, \quad (8)$$

$$\Phi(T,x)=\Phi_0(x,h)=\langle h,u_0(x)\rangle.$$

with respect to a scalar function $\Phi(s,x,h)=\langle h,u(s,x)\rangle$.

Here

$$\begin{aligned} TrQ^*\nabla^2\Phi(s,x,h)Q &= A^*_{ki}\frac{\partial^2\Phi(s,x,h)}{\partial x_i\partial x_j}A_{jk}+2C_k^{lm}h_l\frac{\partial^2\Phi(s,x,h)}{\partial x_j\partial h_m}A_{jk} \\ &+C_k^{qm}h_m\frac{\partial^2\Phi(s,x,h)}{\partial h_q\partial h_p}C_k^{pn}h_n \\ &= A^*_{ki}\frac{\partial^2\Phi(s,x,h)}{\partial x_i\partial x_j}A_{jk}+2C_k^{lm}h_l\frac{\partial^2\Phi(s,x,h)}{\partial x_j\partial h_m}A_{jk}, \end{aligned}$$

since, due to linearity of $\Phi(s,x,h)$ in h, we have $\frac{\partial^2\Phi(s,x,h)}{\partial h_q\partial h_p}\equiv 0$. In addition

$$\langle q,\nabla\Phi(s,x,h)\rangle = a_j\frac{\partial\Phi(s,x,h)}{\partial x_j}+c_{lm}h_m\frac{\partial\Phi(s,x,h)}{\partial h_l},$$

$$G(s,x,h)=\langle h,g(s,x,u,A^*\nabla u)\rangle.$$

Coming back to (4), we notice that its solution (provided it exists) gives rise to a multiplicative operator functional $\Gamma(t,s,\xi(\cdot))\equiv\Gamma(t,s)$ of the process $\xi(t)$ satisfying (3), that is, $\eta(t)=\Gamma(t,s)h$ and $\Gamma(t,s)h=\Gamma(t,\theta)\Gamma(\theta,s)$ a.s. for $0\le s\le\theta\le t\le T$. Hence to derive a forward–backward stochastic equation (FBSDE) associated with (1), we can proceed as follows.

Assume that there exists a classical solution to the Cauchy problem (1) or what is equivalent suppose that there exists a classical solution to (8) and compute a stochastic differential of a stochastic process $Y(t)=\langle\eta(t),u(t,\xi(t))\rangle$,

$$dY(t)=\langle d\eta(t),u(t,\xi(t))\rangle+\langle\eta(t),du(t,\xi(t))\rangle+\langle d\eta(t),du(t,\xi(t))\rangle.$$

Taking into account (3) and (4) by Ito's formula, we derive the relation

$$dY(t)=-F(t,Y(t),Z(t))dt+\langle Z(t),dW(t)\rangle,\quad Y(T)=\zeta=\langle h,u_0(\xi(T))\rangle, \quad (9)$$

where $W(t) = (w(t), w(t))^*$,

$$\langle Z(t), dW(t)\rangle = \langle C(\Gamma(t)h, dw(t)), u(t, \xi(t))\rangle + \langle \Gamma(t)h, \nabla u(t, \xi(t))Adw\rangle$$
$$= \langle h, \Gamma^*(t)[C^*u(t, \xi(t)) + A^*\nabla u(t, \xi(t))]dw(t)\rangle$$

and $\Gamma(t)h \equiv \Gamma(t,s)h = \eta_{s,h}(t)$. As a result we can rewrite (9) in the form

$$dy(t) = -f(t, y(t), z(t))dt + z(t)dw(t), \quad y(T) = u_0(\xi(T)), \tag{10}$$

where

$$\begin{aligned} f(t, y(t), z(t)) &= \Gamma^*(t)g\left(\xi(t), u(t, \xi(t)), C^*(t, \xi(t))u(t, \xi(t))\right. \\ &\quad \left. + A^*(t, \xi(t))\nabla u(t, \xi(t))\right) \\ &= \Gamma^*(t)g\Big(\xi(t), [\Gamma^*]^{-1}(t)y(t), C^*(\xi(t))[\Gamma^*]^{-1}(t)y(t) \\ &\quad + A^*(\xi(t))[\Gamma^*]^{-1}(t)z(t)\Big), \\ Z(t) &= ([\Gamma^*]^{-1}(t)C^*(t, \xi(t))u(t, \xi(t)), [\Gamma^*]^{-1}(t)A^*(\xi(t))\nabla u(t, \xi(t)))^*, \\ z(t)dw(t) &= [\Gamma^*]^{-1}(t)[C^*udw(t) + A^*\nabla udw(t)] \in R^{d_1} \end{aligned} \tag{11}$$

and $\langle h, z(t)dw(t)\rangle = \langle Z(t), dW(t)\rangle$.

When the solution $y(t)$ is a scalar process and a comparison theorem holds, one can prove that the function $u(s, x)$ defined by $y(s) = u(s, x)$ is a viscosity solution of the Cauchy problem for a corresponding quasilinear parabolic equation. In a multidimensional case it was shown in [9] that given a solution of the BSDE

$$dy_l(t) = -g_l(t, \xi(t), y(t), z_l(t))dt + \langle z_l(t), dw(t)\rangle, \quad y_l(T) = u_{0l}(\xi(T)), \tag{12}$$

where $\xi(t)$ satisfies (9) under some condition one can prove that the function $u(s, x) = y(s)$ is a viscosity solution to the Cauchy problem

$$\frac{\partial u_l}{\partial s} + \frac{1}{2}TrA^*\nabla^2 u_l A + \langle a, \nabla u_l\rangle + g_l(s, x, u, A^*\nabla u_l) = 0, \quad u_l(T, x) = u_{0l}(x). \tag{13}$$

In this paper we show that a certain combination of two approaches allows to extend the results of FBSDEs theory to construct a viscosity solution to the system of the form (1). In particular we define the very notion of a viscosity solution for (1) and prove a comparison theorem for solutions of multidimensional BSDEs which is a crucial point in construction of the viscosity solution via a solution to a BSDE.

In the next section we give a construction of an FBSDE required to construct a viscosity solution for (1), assuming that coefficients a, σ, C, c do not depend on u. We state here conditions on the BSDE parameters that ensure the existence and uniqueness of its solution. In Sect. 3 we prove a comparison theorem, and in Sect. 4 we state the notion of a viscosity solution of the Cauchy problem for (1) and prove that FBSDE solution gives rise to a viscosity solution for (1).

2 Forward–Backward Stochastic Differential Equations

In this section we introduce notations and present in a suitable form necessary results from FBSDE theory adapted to the case under consideration.

Given integers d, d_1 consider Euclidian spaces R^d, R^{d_1} and let $\|\cdot\|$ denote a norm in R^d and $\langle \cdot, \cdot \rangle$ denote an inner product regardless of d.

Given a Euclidian space X, let:

- $L_t^p(X)$ be a set of $\mathcal{F}_t$ -measurable X-valued random variables, $E\|\xi\|^p < \infty$.
- $\mathcal{H}_c^2(X)$ be a set of $\mathcal{F}_t$ -measurable X-valued semimartingales such that $E\left[\sup_{0\le t\le T}\|y(t)\|^2\right] < \infty$.
- $\mathcal{H}_t^2(X)$ be a set of $\mathcal{F}_{s,t}$ -measurable X-valued semimartingales such that $E\left[\sup_{0\le \theta\le t}\|y(\theta)\|^2\right] < \infty$.
- $\mathcal{H}^2(X)$ be a set of square integrable progressively measurable processes $z(t) \in X$ such that $E\left[\int_0^T \|z(\tau)\|^2 d\tau\right] < \infty$.
- $\mathcal{S}^2 = \mathcal{H}_c^2(R^{d_1}) \cup \mathcal{H}^2(R^{d\times d_1})$.
- $\mathcal{S}^3 = \mathcal{H}_c^2(R^d) \cup \mathcal{H}_c^2(R^{d_1}) \cup \mathcal{H}_T^2(R^{d\times d_1})$.
- $\mathcal{B}^2 = \mathcal{H}^2(R^{d_1}) \cup \mathcal{H}_T^2(R^{d\times d_1})$.
- $\mathcal{B}^3 = \mathcal{H}^2(R^d) \cup \mathcal{H}^2(R^{d_1}) \cup \mathcal{H}_T^2(R^{d\times d_1})$;

  $L(R^d)$ be the space of bounded linear maps acting in R^d;

  $L(R^d; R^{d_1}) \equiv R^{d\times d_1}$ be the space of bounded linear maps acting from R^d to R^{d_1}.
- Given $\beta > 0$ and $\phi \in \mathcal{H}_T^2(R^d)$, let $\|\phi\|_\beta^2 = E\left[\int_0^T e^{\beta t}\|\phi(t)\|^2 dt\right]$ and $\mathcal{H}_{T,\beta}^2(R^d)$ be the space $\mathcal{H}_T^{2,d}$ equipped with the norm $\|\cdot\|_\beta$.

Let $W(t) = (w(t), w(t)) \in R^d \times R^d$ and $\kappa(t) = (\xi(t), \eta(t)) \in R^d \times R^{d_1}$ be a solution of a system of SDEs

$$d\xi(t) = a(t, \xi(t))dt + A(t, \xi(t))dw(t), \quad \xi(s) = x \in R^d, \tag{14}$$

$$d\eta(t) = c(t, \xi(t))\eta(t)dt + C(t, \xi(t))(\eta(t), dw(t)), \quad \eta(s) = h \in R^{d_1}. \tag{15}$$

We say that condition **C 2.1** holds if coefficients $a : [0,\infty) \times R^d \to R^d$, $A : [0,\infty) \times R^d \to L(R^d)$, $c : [0,\infty) \times R^d \to L(R^{d_1})$, $C : [0,\infty) \times R^d \to$

$L(R^d; L(R^{d_1}))$ are continuous in $t \in [0, T]$ and there exist constants K_1, K_2, L_1, L_2 such that

$$\|a(t,x)\|^2 + \|A(t,x)\|^2 \le K_1[1 + \|x\|^2];$$

$$\|a(t,x_1) - a(t,x_2\|^2 + \|A(t,x_1) - A(t,x_2)\|^2 \le L_1\|x_1 - x_2\|^2;$$

$$\|c(t,x)h\|^2 + \|C(t,x)h\|^2 \le K_2\|h\|^2;$$

$$\|c(t,x_1) - c(t,x_2)h\|^2 + \|[C(t,x_1) - C(t,x_2)]h\|^2 \le L_2\|x_1 - x_2\|^2\|h\|^2.$$

Recall that we use notation $\|A\| = [\sum_{j,k=1}^{d} A_{kj}A_{jk}]^{\frac{1}{2}}$ for $A \in L(R^d)$.

Lemma 1. *Let condition* **C 2.1** *hold. Then there exists a unique solution $\kappa(t) = (\xi(t), \eta(t)) \in R^d \times R^{d_1}$ to (14), (15) such that $\xi(t) \in R^d$ is a Markov process with $E\|\xi(t)\|^2 < \infty$ and $\eta(t) \in R^{d_1}$ with $E\|\eta(t)\|^2 < \infty$ for any $t \in [0, T]$.*

It follows from **C 2.1** that coefficients of (14) and (15) satisfy classical conditions of the existence and uniqueness theorem for solutions of SDEs and hence the lemma statement results from this theorem.

Lemma 2. *Let condition* **C 2.1** *hold. Then the stochastic process $\eta(t)$ satisfying (15) gives rise to a multiplicative operator functional $\Gamma(t) \equiv \Gamma(t,s) : \mathcal{H}_s^2(R^{d_1}) \to \mathcal{H}_t^2(R^{d_1})$ satisfying the SDE*

$$d\Gamma(t) = c(t, \xi(t))\Gamma(t)dt + C(t, \xi(t))(\Gamma(t), dw(t)), \quad \Gamma(s,s) = I, \tag{16}$$

where I is the identity operator in R^{d_1}. Moreover there exists an inverse map $\Gamma^{-1}(s,t) : \mathcal{H}_t^2(R^{d_1}) \to \mathcal{H}_s^2(R^{d_1})$ satisfying

$$\begin{aligned} \Gamma^{-1}(s,t) = I - \int_s^t \Gamma^{-1}(\theta,t)[c(\theta,\xi(\theta)) - C^2(\theta,\xi(\theta))]d\theta \\ - \int_s^t \Gamma^{-1}(t)C(\theta,\xi(\theta))dw(\theta) \end{aligned} \tag{17}$$

with probability 1.

Proof. Under the condition **C 2.1**, we can state the existence and uniqueness of a solution to (17) and the corresponding properties of the map $\Gamma^{-1}(s,t)$. In particular we deduce from uniqueness of solutions to (15) and (17) that the map $\Gamma(t,s)$ defined by $\eta(t) = \Gamma(t,s)h$ is an evolution family, that is, $\Gamma(t,\theta)\Gamma(\theta,s) = \Gamma(t,s)$ with probability 1 and the map $\Gamma^{-1}(t,s)$ has the same property. Besides, by Ito's formula we can check that $\Gamma(t,s)\Gamma^{-1}(s,t) = I$ a.s. Let $\Gamma^*(s,t)$ be defined by $\langle\Gamma(t,s)h, u\rangle = \langle h, \Gamma^*(s,t)u\rangle$. We can verify that $\Gamma^*(s,t)$ is an invertible evolution map acting from $\mathcal{H}_t^2(R^{d_1})$ to $\mathcal{H}_s^2(R^{d_1})$. Here and below we identify the space R^d with its dual space $(R^d)^*$. □

Consider a BSDE of the form

$$dy(t) = -\Gamma^*(s,t)g([\Gamma^*]^{-1}(s,t)y(t), [\Gamma^*]^{-1}(s,t)z(t))dt + z(t)dw(t), \quad (18)$$

and state conditions on its parameters g and ζ to ensure that there exists a unique solution $(y(t) \in R_1^d, z(t) \in R^{d\times d_1})$ to (18) under an assumption that $y(T) = u_0(\xi(T))$.

We say that condition **C 2.2** holds when $g : [s,T] \times R^d \times R^{d_1} \times R^{d\times d_1} \to R^{d_1}$, $\zeta \in R^{d_1}$ be an $\mathcal{F}_T$-measurable square integrable random variable and there exist constants L, L_3, such that

$$\|g(t,x^1,y,z) - g(t,x^2,y,z)\| \le L_3\|x^1 - x^2\|,$$

$$\|g(t,x,y^1,z^1) - g(t,x,y^2,z^2)\| \le L[\,\|y^1 - y^2\| + \|z^1 - z^2\|\,],$$

$$\langle y - y_1, g(t,x,y^1,z) - g(t,x,y^2,z)\rangle \le \mu\|y - y^1\|^2.$$

There exists a constant $C_0 > 0$ such that for all $x, x' \in R^d$

$$\|u_0(x) - u_0(x')\| \le C_0\|x - x'\|.$$

Denote by $f(t,y,z) = \Gamma^*(t)g(\xi(t), [\Gamma^*]^{-1}(t)y, [\Gamma^*]^{-1}(t)z)$ and let $\zeta = u_0(\xi(T))$, where $\xi(t), t \in [s,T]$ is a solution to (14). Consider a BSDE

$$dy(t) = -f(t,\xi(t),y(t),z(t))dt + z(t)dw(t), \quad y(T) = \zeta \in R^{d_1}. \quad (19)$$

A couple of progressively measurable random processes $(y(t), z(t)) \in \mathcal{B}^2$ is called a solution of (19) if with probability 1

$$y(t) = \zeta + \int_t^T f(\theta,\xi(\theta),y(\theta),z(\theta))ds - \int_t^T z(\theta)dw(\theta), \quad 0 \le t \le T. \quad (20)$$

Lemma 3. *Let conditions* **C 2.1**, **C 2.2** *hold. Then*

$$\|f(t,x,y_1,z_1) - f(t,x,y_2,z_2)\| \le L[\,\|y_1 - y_2\| + \|z - z_1\|\,].$$

Proof. By Lipschitz continuity of g and the properties of $\Gamma(t)$, we have a.s.

$$\begin{aligned}
&\|f(t,y_1,z_1) - f(t,y_2,z_2)\| \\
&\quad = \|g(t,\xi(t),[\Gamma^*]^{-1}y_1,[\Gamma^*]^{-1}z_1) - g(t,\xi(t),[\Gamma^*]^{-1}y_2,[\Gamma^*]^{-1}z_2)\| \\
&\quad \le \|\Gamma^*\|L[\,\|[\Gamma^*]^{-1}y_1 - [\Gamma^*]^{-1}y_2\|\,] + [\,\|[\Gamma^*]^{-1}z_1 - [\Gamma^*]^{-1}z_2\|\,] \\
&\quad \le L[\,\|y - y_1\| + \|z_1 - z_2\|\,].
\end{aligned}$$

□

Given $(u, v) \in \mathcal{B}^2$, we define a map M by $(y, z) = M(u, v)$ as follows. Let ζ be R^{d_1}-valued $\mathcal{F}_T$-measurable random variable and given $f : [s, T] \times R^d \times R^{d_1} \times R^{d \times d_1} \to R^{d_1}$ set

$$y(t) = E\left[\zeta + \int_t^T f(\theta, \xi(\theta), u(\theta), v(\theta))d\theta|\mathcal{F}_t\right], \quad 0 \le t \le T. \tag{21}$$

We apply the Ito theorem about martingale representation of a square integrable random variable

$$\chi = \zeta + \int_0^T f(\theta, u(\theta), v(\theta))d\theta$$

to define the process $z(t)$ by the equality

$$\chi = E[\chi] + \int_0^T z(\theta)dw(\theta).$$

It is easy to check that the couple (y, z) defined in this way satisfies

$$y(t) = \zeta + \int_t^T f(\theta, \xi(\theta), u(\theta), v(\theta))d\theta - \int_t^T z(\theta)dw(\theta).$$

In a standard way we show that M acts in $\mathcal{B}^2$ and possesses a contraction property. To this end we denote by $\bar{f} = f_1 - f_2$ for $f = y, z, u, v$. By Ito's formula, we obtain

$$e^{\beta t} E\|\bar{y}(t)\|^2 + E\left[\int_t^T e^{\beta s}[\beta\|\bar{y}(s)\|^2 + \|\bar{z}(s)\|^2]ds\right]$$
$$= 2E\left[\int_t^T e^{\beta s}\langle\bar{y}(s), f(s, u_1(s), v_1(s)) - f(s, u_2(s), v_2(s))\rangle ds\right].$$

Taking into account Lipschitz continuity of f, we obtain

$$E[e^{\beta t}\|\bar{y}(t)\|^2] + E\left[\int_t^T e^{\beta s}[\beta\|\bar{y}(s)\|^2 + \|\bar{z}(s)\|^2]ds\right]$$

$$\le 2LE\left[\int_t^T e^{\beta s}\|\bar{y}(s)\|[\|\bar{u}(s)\| + \|\bar{v}(s)\|]ds\right]$$

and by the elementary inequality $2ab \le a^2\alpha^2 + \frac{b^2}{\alpha^2}$,

$$E[e^{\beta t}\|\bar{y}(t)\|^2] + E\left[\int_t^T e^{\beta s}\|\bar{z}(s)\|^2]ds\right]$$

$$\leq [2L^2\alpha^2 - \beta]E\left[\int_t^T e^{\beta s}\|\bar{y}(s)\|^2 ds\right]$$
$$+\frac{1}{\alpha^2}E\left[\int_t^T e^{\beta s}(\|\bar{u}(s)\|^2 + \|\bar{v}(s)\|^2)ds\right].$$

Choosing $\frac{1}{\alpha^2} = \frac{1}{2}$ and $\beta - 4L^2 = 1$, we obtain

$$e^{\beta t}E\|\bar{y}(t)\|^2 + E\left[\int_t^T e^{\beta s}\|\bar{z}(s)\|^2 ds\right] \leq \frac{1}{2}E\left[\int_t^T e^{\beta s}[\|\bar{u}(s)\|^2 + \|\bar{v}(s)\|^2]ds\right].$$

In a similar way we can check that $(y, z) = M(u, v) \in \mathcal{B}^2$. As a result we deduce that M is a contraction in $\mathcal{B}^2$ and the following statement holds.

Theorem 1. *Let condition* **C 2.2** *hold. Then there exists a unique solution* $(y, z) \in \mathcal{B}^2$ *of BSDE (19) and successive approximations* (y^n, z^n) *of the form*

$$y^{n+1}(t) = \zeta + \int_t^T f(\theta, \xi(\theta), y^n(\theta), z^n(\theta))d\theta - \int_t^T z^{n+1}(\theta)dw(\theta)$$

converges to the solution of (19) with probability 1.

Proof. The existence and uniqueness of a solution (y, z) to (19) follows from the fixed point theorem for the contraction $M : \mathcal{B}^2 \to \mathcal{B}^2$. By applying the above estimates to the successive approximations (y^n, z^n), we can verify that when $m, n \to \infty$,

$$E\left[\int_t^T e^{\beta\theta}\|y^n(\theta) - y^m(\theta)\|^2 ds|\mathcal{F}_t\right] + E\left[\int_t^T e^{\beta\theta}\|z^n(\theta) - z^m(\theta)\|^2 ds|\mathcal{F}_t\right] \to 0,$$

with probability 1. Hence, (y^n, z^n) is a Cauchy sequence in $\mathcal{B}^2$, and the limit $P - \lim_{n\to\infty}(y^n, z^n) = (y, z)$ exists and satisfies (17). □

Below along with a weakly coupled multidimensional FBSDE of the form

$$dy(t) = -f(t, \xi(t), y(t), z(t))dt + z(t)dw(t), \quad y(T) = u_0(\xi(T)), \tag{22}$$

where $\xi(t)$ is a solution of (17), we consider a weakly coupled scalar FBSDE which can be described as follows. Let

$$q(\kappa) = \begin{pmatrix} a(x) \\ c(x)h \end{pmatrix}, Q(\kappa) = \begin{pmatrix} A(x) & 0 \\ 0 & C(x)h \end{pmatrix}, \quad \tilde{G}(\kappa, y, z) = \langle h, f(x, y, z)\rangle. \tag{23}$$

Obviously, we can rewrite the system (14), (15) in the form

$$d\kappa(t) = q(t, \kappa(t))dt + Q(t, \kappa(t))dW(t), \quad \kappa(s) = \kappa = (x, h). \tag{24}$$

The required FBSDE can be presented in the form

$$dY(t) = -\tilde{G}(t, \kappa(t), Y(t), Z(t))dt + \langle Z(t), dW(t)\rangle, \quad Y(T) = \langle h, u_0(\xi(T)\rangle, \tag{25}$$

where $\kappa(t) = (\xi(t), \eta(t))$ solves (24), $W(t) = (w(t), w(t))^*$, and $\langle Z(t), dW(t)\rangle = \langle h, z(t)dw(t)\rangle$.

A triple of progressively measurable random processes $(\kappa(t), y(t), z(t)) \in \mathcal{B}^3$ is called a solution of (24), (25) if with probability 1 for all $0 \le s \le t \le T$

$$\kappa(t) = \kappa + \int_s^t q(\theta, \kappa(\theta))d\theta + \int_s^t Q(\theta, \kappa(\theta))dW(\theta), \tag{26}$$

$$Y(t) = \langle h, u_0(\xi(T))\rangle + \int_t^T \tilde{G}(\theta, \kappa(\theta), Y(\theta), Z(\theta))d\theta - \int_t^T \langle Z(\theta), dW(\theta)\rangle. \tag{27}$$

The FBSDEs (14), (15), (19), and (24), (25) are equivalent.

3 Comparison Theorem for Multidimensional BSDE

Comparison theorems present an important tool in the BSDE and FBSDE theory and in particular in the context of the connections between FBSDE theory and viscosity solutions of corresponding parabolic equations and systems. In this paper to prove a comparison theorem for a multidimensional BSDE, we use the special features of the BSDE under consideration.

Consider a couple of d_1-dimensional BSDEs

$$y^i(t) = \zeta^i + \int_t^T f^i(\theta, y^i(\theta), z^i(\theta))d\theta - \int_t^T z^i(\theta)dw(\theta), \quad i = 1, 2 \tag{28}$$

for $0 \le t \le T$ and use the specific features of these BSDEs investigated in the previous sections. Here $\zeta^i, f^i(\theta, y, z) \in R^{d_1}$ for $\theta \in [0, T]$, $y \in R^{d_1}, z \in R^{d\times d_1}$.

For any fixed nonzero vector $h \in R^{d_1}$ and $y^1, y^2 \in R^{d_1}$, we say that $y^1 \le_h y^2$ under h if $\langle h, y^1\rangle \le \langle h, y^2\rangle$. Without loss of generality we choose h to have $\|h\| = 1$.

Given two vectors $y^1, y^2 \in R^{d_1}$, we say $y^1 \le y^2$ if $y^1_m \le y^2_m$, $m = 1, \ldots, d_1$, where $y_m = \langle y, e_m\rangle$ and $(e_m)_{m=1}^{d_1}$ is a fixed orthonormal basis in R^{d_1}.

Given $f \in R^{d_1}$, we denote by $f_m^+ = \max[f_m, 0]$, $m = 1, \ldots, d_1$.

Consider a couple of BSDEs with parameters $\zeta^i, f^i, i = 1, 2$.

We say that condition **C 3.1** holds if

(i) $\zeta^1 \le \zeta^2$, $P-$a.s..

(ii) For each $m = 1, \dots, d_1$ inequality, $f_m^1(t, y^1, z^1) \le f_m^2(t, y^2, z^2)$ holds true when $y_l^1 \le y_l^2$ for all $l = 1, \dots, d_1$ except $l = m$ while $y_m^1 = y_m^2$, and $z_{mk}^1 = z_{mk}^2$ for each $k = 1, \dots, d$.

(iii) For all $y^1, y^2 \in R^{d_1}, z^1, z^2 \in R^{d\times d_1}$ and for each $m = 1, \dots, d_1$

$$|f_m^i(t, y^1, z^1) - f_m^i(t, y^2, z^2)| \le L[\|y^1 - y^2\| + \|z^1 - z^2\|], i = 1, 2.$$

Set $\bar{\alpha} = \alpha^1 - \alpha^2$ for $\alpha = y, \zeta, f$, and z as well.

Let us mention that within this section we do not assume summing up with respect to repeating indices.

Theorem 2. *Let (ζ^i, f^i), $i = 1, 2$ be parameters of BSDEs (28) satisfying conditions* **C 2.1** *and* **C 3.1**. *Assume that $(y^i(t), z^i(t))$, $i = 1, 2, \quad t \in [s, T]$ solve (28) with this parameters. Then $y^1(t) \le y^2(t)$ a.s. Moreover the comparison is strict, that is, if in addition $y^2(s) = y^1(s)$, then $\zeta^1 = \zeta^2$, $f^2(t, y^2(t), z^2(t)) = f^1(t, y^2(t), z^2(t))$ and $y^2(t) = y^1(t)$, $\forall t \in [s, T] P$-a.s. In particular whenever either $P(\zeta^1 < \zeta^2) > 0$ or $f^1(t, y^2(t), z^2(t)) < f^2(t, y^2(t), z^2(t))$ on a set of positive $dt \times dP$ measure, then $y^1(s) < y^2(s)$ a.s.*

Proof. Applying Ito's formula to $|\bar{y}_j(t)^+|^2$ where $j = 1, \dots, d_1$ and evaluating mean value, we get

$$\begin{aligned} E|\bar{y}_j(t)^+|^2 &= E|\bar{\zeta}_j^+|^2 \\ &\quad - E\left[\int_t^T 2I_{y_j^1(s) > y_j^2(s)} \bar{y}_j(s)[f_j(s, y^1(s), z^1(s)) - f_j(s, y^2(s), z^2(s))]ds\right] \\ &\quad - E\left[\int_t^T I_{\{y_j(s) > y_j^2(s)\}} \|\bar{z}_j(s)\|^2 ds\right] - E\left[\int_t^T \bar{y}_j^+ dL_j(s)\right], \end{aligned} \tag{29}$$

where $L_j(t)$ is the local time of $\bar{y}_j(s)$ at 0. Note that the last summand is equal to 0 and since $\zeta^1 \le \zeta^2$ a.s. we have $E[\|[\zeta^1 - \zeta^2]^+\|^2] = 0$, Obviously, $E\left[\int_t^T I_{y_j^1(s) > y_j^2(s)} \bar{y}_j(s) \bar{z}_j(s) dw(s)\right] = 0$. Hence,

$$\begin{aligned} E[\bar{y}_j(t)^+] &= E\left[\int_t^T I_{y_j^1(s) > y_j^2(s)} 2\bar{y}_j^+(s)[f_j^1(s, y^1(s), z^1(s)) - f_j^2(s, y^2(s), z^2(s))]ds\right] \\ &\quad - E\left[\int_t^T I_{\{y_j^1(s) > y_j^2(s)\}} \|\bar{z}_j(s)\|^2 ds\right]. \end{aligned}$$

Set

$$\begin{aligned} \bar{f}_j(s) &= f_j^1(s, y^1, z^1) - f_j^2(s, y^2, z^2) \\ &= f_j^1(s, y_1^1, \dots, y_j^1, \dots, y_{d_1}^1, z_1^1, \dots, z_j^1, \dots, z_{d_1}^1) \end{aligned}$$

$$
\begin{aligned}
&-f_j^2(s, y_1^2, \ldots, y_j^2, \ldots, z_1^2, \ldots, z_j^2, \ldots, z_{d_1}^2)\\
&= [f_j^1(s, y_1^1, \ldots, y_j^1, \ldots, y_{d_1}^1, z_1^1, \ldots, z_j^1, \ldots, z_{d_1}^1)\\
&\quad -f_j^2(s, y_1^1 + \bar{y}_1^+, \ldots, y_j^1, \ldots, y_{d_1}^1 + \bar{y}_{d_1}^+, z_1^2, \ldots, z_j^1, \ldots, z_{d_1}^2)]\\
&\quad +[f_j^2(s, y_1^2 + \bar{y}_1^+, \ldots, y_j^1, \ldots, y_{d_1}^2 + \bar{y}_{d_1}^+, z_1^2, \ldots, z_j^1, \ldots, z_{d_1}^2)\\
&\quad -f_j^2(s, y_1^2, \ldots, y_j^2, \ldots, z_1^2, \ldots, z_j^2, \ldots, z_{d_1}^2)]\\
&= \Pi_1 + \Pi_2.
\end{aligned}
$$

Since for any $m = 1, \ldots, d_1$ we have $y_m^1 \le y_m^2 + \bar{y}_m^+$ for $m \ne j$, taking into account (ii) in **C 3.1**, we get $\Pi_1 \le 0$.

Next, due to Lipschitz continuity of f^2, we have

$$
\Pi_2 \le L[|\bar{y}_1^+| + \cdots + |\bar{y}_{j-1}^+| + |\bar{y}_j| + \cdots + |\bar{y}_{d_1}^+| + \|\bar{z}_j\|].
$$

By applying Ito's formula due to generator properties, we deduce that

$$
\begin{aligned}
E|\bar{y}_j^+(t)|^2 &\le 2E\left[\int_t^T I_{y_j^1(s)>y_j^2(s)} \bar{y}_j^+(s)\bar{f}_j(s)ds\right]\\
&\quad -E\left[\int_t^T I_{y_j^1(s)>y_j^2(s)} \sum_{k=1}^{d} |\bar{z}_{jk}(s)|^2 ds\right]\\
&\le E\left[2\int_t^T I_{y_j^1(s)>y_j^2(s)}\, L\bar{y}_j^+(s)[|\bar{y}_1(s)| + \cdots + |\bar{y}_{j-1}^+|\right.\\
&\quad +|\bar{y}_j(s)| + \cdots + |\bar{y}_{d_1}^+| + \|\bar{z}_j(s)\|]\, ds]\\
&\quad -E\left[\int_t^T I_{\{y_j^1(s)>y_j^2(s)\}}\|\bar{z}_j(s)\|^2 ds\right]\\
&\le E\left[\int_t^T I_{\{y_j^1(s)>y_j^2(s)\}} L^2(d_1+1)|\bar{y}_j(s)|^2 ds\right]\\
&\quad +E\left[\int_t^T I_{\{y_j^1(s)>y_j^2(s)\}}[\sum_{k=1}^{d_1} |\bar{y}_k(s)|^2 + \|\bar{z}_j(s)\|^2] ds\right]\\
&\quad -E\left[\int_t^T I_{\{y_j^1(s)>y_j^2(s)\}}\|\bar{z}_j(s)\|^2 ds\right]\\
&= L^2(d_1+1)\int_t^T E[I_{\{y_j^1(s)>y_j^2(s)\}}|\bar{y}_j(s)|^2] ds\\
&\quad + \int_t^T E[I_{\{y_j^1(s)>y_j^2(s)\}} \sum_{k=1}^{d_1} |\bar{y}_k(s)|^2] ds. \qquad (30)
\end{aligned}
$$

Note that above we have used an elementary inequality of the form

$$2L\bar{y}_j^+(s)|\bar{y}_k(s)| \le L^2|\bar{y}_j^+(s)|^2 + |\bar{y}_k(s)|^2.$$

Summing up the left- and right-hand side in (30), we get that the function $m(t) = \sum_{j=1}^{d_1} E|\bar{y}_j^+(t)|^2$ satisfies inequality

$$m(t) \le (L^2(d_1+1)+d_1)\int_t^T m(s)ds.$$

Finally, due to results of the previous section, we know that for $t \in [0,T]$ the inequality $E|\bar{y}_j(t)^+|^2 < \infty$ holds for each $j = 1,\dots,m$; then by the Gronwall lemma, we know that $m(t) = 0$, and since m is a sum of positive summands, each summand should be equal to zero. Hence, $|\bar{y}_j^+(t)| = 0$ and thus $y_j^1(t) \le y_j^2(t)$ a.s. for all $j = 1,\dots,d_1$. □

At the end of this section, we come back to the one-dimensional BSDE (27) and derive the corresponding comparison theorem. Note that this theorem motivates our choice of comparison for vector functions in the case under consideration.

Consider the SDE of the form

$$\kappa(t) = \kappa + \int_s^t q(\kappa(\theta))d\theta + \int_s^t Q(\kappa(\theta))dW(\theta), \quad s \le t \le T, \tag{31}$$

introduced in the previous section and note that one can consider instead of the BSDE

$$y(t) = u_0(\xi(T)) + \int_t^T f(\theta,\xi(\theta),y(\theta),z(\theta))d\theta - \int_t^T z(\theta)dw(\theta), \quad s \le t \le T, \tag{32}$$

with respect to the process $y(t) \in R^{d_1}$ a new BSDE

$$dY(t) = -\tilde{G}(t,\kappa(t),Y(t),Z(t))dt + \langle Z(t), dW(t)(t)\rangle, \quad Y(T) = \zeta = \langle h, u_0(\xi(T))\rangle, \tag{33}$$

where $Y(t) = \langle \eta(t), u(t,\xi(t))\rangle$ is a scalar process. We denote $|Y| = \sup_{\|h\|=1}|\langle h,u\rangle| = \|u\|$.

Theorem 3. *Let $(Y^i, Z^i), i = 1,2$ be solutions of one-dimensional BSDEs*

$$dY^i(t) = -\tilde{G}^i(t,\kappa(t),Y^i(t),Z^i(t))dt + \langle Z^i(t), dW(t)\rangle, \quad Y^i(T) = \Upsilon^i = \langle h, u_0^i(\xi(T))\rangle. \tag{34}$$

Suppose that $\Upsilon^1 \le \Upsilon^2$ and $\tilde{G}^1(t,\kappa,Y^2,Z^2) \le \tilde{G}^2(t,\kappa,Y^2,Z^2)$ $dt \times dP$ - a.e. Then $Y^1(t) \le Y^2(t)$ a.s. for all $s \le t \le T$.

Proof. Define a scalar process

$$\mu(t) = \begin{cases} \frac{\tilde{G}^1(t,\kappa(t),Y^2(t),Z^1(t))-\tilde{G}^1(t,\kappa(t),Y^1(t),Z^1(t))}{Y^2(t)-Y^1(t)} & \text{if } Y^1(t) \neq Y^2(t), \\ 0 & \text{if } Y^1(t) = Y^2(t), \end{cases}$$

and a vector process $\nu(t) \in R^d$ such that

$$\nu_k(t) = \begin{cases} \frac{\tilde{G}^1(t,\kappa(t),Y^1(t),Z^{(k)}(t))-\tilde{G}^1(t,\kappa(t),Y^1(t),Z^{(k-1)}(t))}{Z_k^2(t)-Z_k^1(t)} & \text{if } Z_k^1(t) \neq Z_k^2(t) \\ 0 & \text{if } Z_k^1(t) = Z_k^2(t) \end{cases},$$

where $Z^{(k)}(t)$ denotes the d-dimensional vector such that its first k components are equal to corresponding components of Z^2 and the remaining $d-k$ components are equal to those of Z^1. Due to Lipschitz continuity of g, the processes $\mu(t)$ and $\nu(t)$ are bounded, and in addition, they are progressively measurable.

As above we use notation $\bar{f} = f^1 - f^2$ for $f = Y, Z, \Upsilon$ and observe that $(\bar{Y}(t), \bar{Z}(t))$ satisfies the BSDE

$$\bar{Y}(t) = \bar{\Upsilon} + \int_t^T [\mu(\theta)\bar{Y}(\theta) + \langle \nu(\theta), \bar{Z}(\theta)\rangle]d\theta + \int_t^T N(\theta)d\theta - \int_t^T \langle \bar{Z}(\theta), dW(\theta)\rangle,$$

where $N(t) = \tilde{G}^1(t,\kappa(t),Y^2(t),Z^2(t)) - \tilde{G}^2(t,\kappa(t),Y^2(t),Z^2(t))$. For $s \le t \le T$ we define

$$\rho_{s,t} = \exp\left[\int_s^t (\mu(\theta) - \frac{1}{2}\|\nu(\theta)\|^2)d\theta + \int_s^t \langle \nu(\theta), dW(\theta)\rangle\right].$$

By Ito's formula, we can verify that $(\bar{Y}(\theta), \bar{Z}(\theta))$ satisfy the BSDE

$$d[\rho_{s,\theta}\bar{Y}(\theta)] = \rho_{s,\theta}[\bar{Y}(\theta) + N(\theta)]d\theta + \rho_{s,\theta}\langle \bar{Z}(\theta) + \bar{Y}(\theta)\nu(\theta), dW(\theta)\rangle$$

for $\theta \in [s, T]$ and

$$\bar{Y}(\theta) = E\left[\rho_{s,T}\bar{\Upsilon} + \int_\theta^T \rho_{s,\vartheta}N(\vartheta)d\vartheta | \mathcal{F}_\theta\right].$$

The required assertion immediately follows from negativity of $\bar{\Upsilon}$ and $N(t)$. □

Let us mention a useful remark. Let Y^1, Z^1 be a solution of BSDE

$$Y^1(t) = \Upsilon^1 + \int_t^T \tilde{G}^1(\theta, Y^1(\theta), Z^1(\theta))d\theta - \int_t^T \langle Z^1(\theta), dW(\theta)\rangle$$

and (Y^2, Z^2) satisfy

$$Y^2(t) = \Upsilon^2 + \int_t^T M(\theta)d\theta - \int_t^T \langle Z^2(\theta), dW(\theta)\rangle,$$

where $M(\theta)$ is a scalar progressively $\mathcal{F}_\theta$-measurable process. Suppose that $\Upsilon^1 \leq \Upsilon^2$ and $\tilde{G}^1(t, Y^2(t), Z^2(t)) \leq M(t)$. Then we can choose

$$\tilde{G}^2(t,\kappa(t), Y^2, Z^2) = \tilde{G}^1(t,\kappa(t), Y^2, Z^2) + [M(t) - G^1(t,\kappa(t), Y^2(t), Z^2(t))]$$

and apply the result of Theorem 3 to deduce that $Y^1(t) \leq Y^2(t)$. If in addition $\tilde{G}^1(t,\kappa(t), Y^2, Z^2) < M(t)$ on a set of positive measure $dt \times dP$, then $Y^1(s) < Y^2(s)$.

4 Viscosity Solution to Nonlinear Parabolic System

In this section we show that a solution of a FBSDE generates a viscosity solution of the Cauchy problem for a system of quasilinear parabolic equations.

Let $(\xi(t) \in R^d, y(t) \in R^{d_1}, z(t) \in R^{d\times d_1})$ be a solution of the FBSDE

$$d\xi(t) = a(\xi(t))dt + A(\xi(t))dw(t), \quad \xi(s) = x, \tag{35}$$

$$dy(t) = -\Gamma^*(t)g([\Gamma^*]^{-1}(t)y(t), [\Gamma^*]^{-1}(t)z(t))dt + z(t)dw(t), \quad y(T) = u_0(\xi(T)), \tag{36}$$

where $\Gamma(t)$ is a multiplicative operator functional of the process $\xi(t)$ generated by the solution $\eta(t) \in R^{d_1}$ of the linear SDE

$$d\eta(t) = c(\xi(t))\eta(t)dt + C(\xi(t))(\eta(t), dw(t)), \quad \eta(s) = h, \tag{37}$$

and $u_0 : R^d \to R^{d_1}$ be a continuous bounded function.

Denote by $S_+^{d_1} = \{h \in R^{d_1} : h_m \geq 0, m = 1,\ldots,d_1 \text{ and } \|h\| = 1\}$, and let $e_1,\ldots,e_{d_1}$ be a fixed orthonormal basis in R^{d_1}.

In Sect. 2 we have shown that one can write (36) in the form

$$dy(t) = -f(t,\xi(t), y(t), z(t))dt + z(t)dw(t), \quad y(T) = u_0(\xi(T)), \tag{38}$$

and proved that given a solution $\xi(t)$ of (35), there exists a unique solution $(y(t), z(t))$ of this BSDE.

Assume that there exists a solution $(\xi_{s,x}(t), y^{s,x}(t), z^{s,x}(t))$ to (35), (36) and the comparison Theorem 2 is valid. The aim of this section is to prove that the function $u(s,x) = y^{s,x}(s)$ is a viscosity solution of the Cauchy problem

$$\frac{\partial u_l}{\partial s} + \frac{1}{2} TrA^*(x)\nabla^2 u_l A(x) + \langle a(x), \nabla u_l\rangle + B^i_{lm}(x)\nabla_i u_m + c_{lm}(x)u_m$$
$$+g_l(x, u, K(u, \nabla u)) = 0, \quad l = 1, \dots, d_1, \quad (39)$$
$$u(T, x) = u_0(x),$$

where $B^i_{lm} = \sum_{q=1}^d C^q_{lm} A^{qi}, \quad K(u, \nabla u) = C^* u + A^* \nabla u.$

As it was mentioned in Sect. 2, the system (39) can be easily reduced to a scalar parabolic equation

$$\tfrac{\partial V}{\partial s} + \tfrac{1}{2} TrQ^*(x,h)\nabla^2 V Q(x,h) + \langle q(x,h), \nabla V\rangle + G(h, x, V, Q^*\nabla V) = 0,$$
$$V(T, x) = V_0(x, h) = \langle h, u_0(x)\rangle \quad (40)$$

with respect to a scalar function V defined on $[0, T] \times R^d \times S^{d_1}_+$ [see (14)].

Hence we recall first the definition of a viscosity solution of the Cauchy problem for a general scalar nonlinear parabolic equation:

$$\frac{\partial V}{\partial s} + \Psi(s, z, V, \nabla V, \nabla^2 V) = 0. \quad V(T, z) = V_0(z), \quad (41)$$

where $z = (x, h)$.

A function $\Psi : [0, T] \times (R^d \times S^{d_1}_+) \times R \times (R^d \times R^{d_1}) \times R^d \otimes R^d \to R$ satisfying estimates

$$\Psi(s, z, V, p, q) \le \Psi(s, z, U, p, q) \quad \text{if} \quad V \le U,$$

and

$$\Psi(s, z, V, p, q) \le \Psi(s, z, V, p, q_1) \quad \text{if} \quad q_1 \le q$$

is called a proper function.

Given a proper function Ψ to define a viscosity solution of (41), one has to introduce notions of a sub- and a supersolution of this Cauchy problem.

Denote by $C^{1,2}_{d,d_1} \equiv C^{1,2}([0, T] \times R^d; R^{d_1})$ a set of functions $\psi : [0, T] \times R^d; R^{d_1}$ differentiable in $s \in [0, T]$ and twice differentiable in $x \in R^d$.

A continuous real-valued function $V(s, z)$ is called a subsolution of (41) if $V(T, z) \le V_0(z)$, $z \in R^{d_2}$, $d_2 = d + d_1$, and for any $\Phi \in C^{1,2}_{d_2,1}$ and a point $(s, z) \in [0, T] \times R^{d_2}$ which is a local maximum of $V(t, \tilde{z}) - \Phi(t, \tilde{z})$ (that may be assumed to be equal to zero), the inequality

$$\frac{\partial \Phi}{\partial s} + \Psi(s, z, V, \nabla\Phi, \nabla^2\Phi) \ge 0$$

holds.

A continuous function $V(s, z)$ is called a supersolution of (41) if $V(T, z) \ge V_0(z)$, $z \in R^{d_2}$, and for any $\phi \in C^{1,2}_{d_2,1}$ and $(s, x) \in [0, T] \times R^d$ which is a local minimum of $V(t, \tilde{z}) - \Phi(t, \tilde{z})$ (that may be assumed to be equal to zero), the inequality

$$\frac{\partial \Phi}{\partial s} + \Psi(s, z, V, \nabla\Phi, \nabla^2\Phi) \le 0$$

holds. A continuous function $V(s, z)$ is called a viscosity solution of (41) if it is both sub- and supersolution of this Cauchy problem. Hence, to prove that the function $V(s, z)$ is a viscosity solution to (41), one has to prove that V is both sub- and supersolution of (41).

To give a definition of a viscosity solution of the Cauchy problem to the system (39), we use a definition of a viscosity solution of the scalar Cauchy problem (40) and then rewrite the definition in terms of the solution to (39).

Given functions $\phi_m \in C^{1,2}_{d,d_1}$, $m = 1, \dots, d_1$, denote by

$$[\mathcal{A}\phi]_m(x) = \frac{1}{2} TrA^*(x)\nabla^2\phi_m A(x) + \langle a(x), \nabla\phi_m\rangle + B^i_{ml}(x)\nabla_i\phi_l + c_{ml}(x)\phi_l,$$

where $i = 1, \dots d$, $m, l = 1, \dots, d_1$.

Let $(s, x, \phi, p, q) \in [0, T] \times R^d \times R^{d_1} \times R^{d\times d_1} \times R^{d^2\times d_1}$ and

$$\mathcal{M}_m(s, x, \phi, p, q_m) = \frac{1}{2} TrA^*(x)q_m A(x) + \langle a(x), p_m\rangle$$
$$+ B^i_{ml}(x)\nabla_i p_l + c_{ml}(x)\phi_l + g_l(s, x, u, p). \tag{42}$$

Given $\mathcal{M}_m, m = 1, \dots, d_1$, of the form (42), the system

$$\frac{\partial u_m}{\partial s} + \mathcal{M}_m(s, x, u, \nabla u, \nabla^2 u_m) = 0 \tag{43}$$

coincides with (39).

A continuous function $u : [0, T] \times R^d \to R^{d_1}$ is called a subsolution of (43) if for each $m = 1, \dots, d_1$ an inequality $u_m(T, x) \le u_{0m}(x)$ holds and for any $\varphi_m \in C^{1,2}_{d,1}$ and a point $(s, x) \in [0, T] \times R^d$ which is a local maximum of $u_m(\tilde{s}, \tilde{x}) - \varphi_m(\tilde{s}, \tilde{x})$ an inequality

$$\frac{\partial \varphi_m}{\partial s} + \mathcal{M}_m(s, x, u, \nabla\varphi, \nabla^2\varphi_m) \ge 0 \tag{44}$$

holds.

A continuous function $u(s, x)$ is called a supersolution of (43) if for each $m = 1, \dots, d_1$ an inequality $u_m(T, \tilde{x})\rangle \ge u_{0m}(\tilde{x})$, $x \in R^d$ holds and for any $\varphi_m \in C^{1,2}_{d,1}$ and a point $(s, x) \in [0, T] \times R^d$ which is a local minimum of $u_m(\tilde{s}, \tilde{x}) - \varphi_m(\tilde{s}, \tilde{x})$ an inequality

$$\frac{\partial \varphi_m}{\partial \tilde{s}} + \mathcal{M}_m(s, x, u, \nabla\varphi, \nabla^2\varphi_m) \le 0, \tag{45}$$

holds.

A continuous function $u(s,x)$ is called a viscosity solution of (43) if it is both sub- and supersolution of this Cauchy problem. Hence, to prove that the function $u(s,x)$ is a viscosity solution to (43), one has to prove that u is both sub- and supersolution of (43).

Theorem 4. *Assume that conditions of Theorem 2 hold and* $(\xi_{s,x}(t), y^{s,x}(t), z^{s,x}(t), \eta^{s,x}(t))$ *is a solution to (35)–(37). Then* $u(s,x) = y^{s,x}(s)$ *is a continuous in* (s,x) *viscosity solution of (39).*

Proof. Under assumptions of Sect. 2, continuity of $u(s,x) = y^{s,x}(s)$ in spatial variable x and time variable s is granted by the BSDE theory results [5] which state that under **C 2.1** and **C 2.2** the solution of BSDE (38) is continuous with respect to parameters (s,x). To verify that $u(s,x)$ is a viscosity solution of (39), we have to prove that u is both a subsolution and a supersolution of (39). First, we check that u is a subsolution. To this end, for each $m = 1,\dots,d_1$, we can choose a function $\phi_m \in C^{1,2}_{d,1}$ and a point $(s,x) \in [0,T] \times R^d$ such that at the point (s,x) a function $u_m(s,x) - \phi_m(s,x)$ has a local maximum. Without loss of generality, we assume that $u_m(s,x) = \phi_m(s,x)$.

We have to prove that (44) holds. Denote by

$$K(u,\nabla\phi)(t,\xi_{s,x}(t)) = \begin{pmatrix} \Gamma^*(t)A^*(\xi(t))\nabla\phi(t,\xi_{s,x}(t)) \\ \Gamma^*(t)C^*(\xi_{s,x}(t))u(t,\xi_{s,x}(t)) \end{pmatrix}, \quad s \le t \le s+\alpha,$$

and assume on the contrary that there exists $m \in \{1,\dots,d_1\}$ such that

$$\mathcal{K}^{u,\phi}_m(s,x) = \frac{\partial\phi_m}{\partial s} + [\mathcal{A}\phi]_m(s,x) + g_m(s,x,u(s,x),K(u,\nabla\phi)(s,x)) < 0. \quad (46)$$

By continuity, there exists $0 < \alpha \le T - s$ such that for all $\theta \in [s, s+\alpha]$, $x_1 \in R^d$, $h_1 \in R^{d_1}$, $\|x - x_1\| \le \alpha$, $\|e_m - h_1\| \le \alpha$ the inequalities

$$\Phi^u(\theta,x_1,h_1) - \Phi^\phi(\theta,x_1,h_1) \le 0 \quad (47)$$

and

$$\left\langle h_1, \left(\frac{\partial\phi}{\partial\theta} + \mathcal{A}\phi\right)(\theta,x_1) + g(\theta,x_1,u(\theta,x_1),K(u,\nabla\phi)(\theta,x_1))\right\rangle < 0 \quad (48)$$

hold.

Given $(\xi_{s,x}(t), \eta_{s,h}(t))$ satisfying (35) and (37), define τ by

$$\tau = \inf\{t \ge s : \|\xi_{s,x}(t) - x\| \ge \alpha\} \wedge \inf\{t \ge s : \|\eta_{s,h}(t) - h\| \ge \alpha\} \wedge (s+\alpha). \quad (49)$$

It follows from results in [10, 11] that the pair

$$(\hat{y}(t), \hat{z}(t)) = (y^{s,x}(t\wedge\tau), I_{[s,\tau]}(t)z^{s,x}(t\wedge\tau)), \quad s \le t \le s+\alpha$$

satisfies BSDE

$$\hat{y}(t) = \Gamma^*(t,\tau)u([s+\alpha]\wedge\tau, \xi([s+\alpha]\wedge\tau)) + \int_t^{s+\alpha} I_{[s,\tau]}(\theta) f(\theta,\xi(\theta),\hat{y}(\theta),\hat{z}(\theta))d\theta - \int_t^{s+\alpha} \hat{z}(\theta)dw(\theta)\rangle, \quad s\le t\le s+\alpha. \tag{50}$$

On the other hand, by applying Ito's formula, we obtain that the couple

$$(\tilde{y}(t),\tilde{z}(t)) = (\Gamma^*(t,t\wedge\tau)\phi(t\wedge\tau,\xi_{s,x}(t\wedge\tau)), I_{[s,\tau]}(t)K(u,\nabla\phi)(t,\xi_{s,x}(t))$$

for $t\in[s,s+\alpha]$ satisfies a backward stochastic equation

$$\tilde{y}(t) = (\Gamma^*(\tau)\phi(\tau,\xi_{s,x}(\tau)) + \int_t^{s+\alpha} I_{[s,\tau]}(\theta)\left(\frac{\partial\phi}{\partial\theta} + [\mathcal{A}\phi]\right)(\theta,\xi_{s,x}(\theta))d\theta + \int_t^{s+\alpha}\tilde{z}(\theta)dw(\theta).$$

Notice that $\hat{y}_m(s) = \tilde{y}_m(s) = u_m(s,x)$.

Then for a stopping time $\tau\in[s,s+\alpha]$ given by (49) due to (47) and (48), we derive

$$0 \ge [\Phi^u(\tau,\kappa(\tau)) - \Phi^\phi(\tau,\kappa(\tau))] = \langle e_m, u(s,x)-\phi(s,x)\rangle - \int_s^\tau \langle e_m, [\frac{\partial\phi}{\partial\theta} + \mathcal{A}\phi](\theta,\xi_{s,x}(\theta))\rangle d\theta - \int_s^\tau \langle e_m, f(\theta,\xi_{s,x}(\theta),\hat{y}(\theta),\hat{z}(\theta))\rangle d\theta + + \langle e_m, \int_s^\tau [\hat{z}(\theta) - K(u,\nabla\phi)(\theta,\xi_{s,x}(\theta))]dw(\theta)\,\rangle.$$

Keeping in mind that by assumption for each $m=1,\dots,d_1$ at the point (s,x), we have $u_m(s,x)-\phi_m(s,x)=0$, and computing the expectation of both parts of the last inequality, we deduce

$$E\left[\int_s^\tau \left[\frac{\partial\phi}{\partial\theta} + \mathcal{A}\phi\right]_m (\theta,\xi(\theta))d\theta + \int_s^\tau f_m(\theta,\xi(\theta),\hat{y}(\theta),\hat{z}(\theta))d\theta\right] \ge 0. \tag{51}$$

Denote by $\mathcal{K}^{u,\phi}(t,x) = K(u,\nabla\phi)(t,x)$ and set

$$\gamma_1(s,\tau) = \int_s^\tau \mathcal{K}_m^{u,\phi}(\theta,\xi(\theta))d\theta,$$

$$\gamma_2(s,\tau) = \int_s^\tau [f_m(\theta,\xi(\theta),\tilde{y}(\theta),\tilde{z}(\theta)) \\ -g_m(\theta,\xi(\theta),u(\theta,\xi(\theta)),K(u,\nabla\phi)(\theta,\xi(\theta)))]d\theta,$$

$$\gamma_3(s,\tau) = \int_s^\tau \{f_m(\theta,\xi(\theta),\hat{y}(\theta),\hat{z}(\theta)) - f_m(\theta,\xi(\theta),\tilde{y}(\theta),\tilde{z}(\theta))\}d\theta$$

and rewrite (51) in the form

$$E[\gamma_1(s,\tau)+\gamma_2(s,\tau)+\gamma_3(s,\tau)] \geq 0.$$

Due to (46), we can assume that there exists a number $\delta_0 < 0$ such that $\mathcal{K}^{u,\phi}(s,x) < \delta_0$ and

$$\tau_1 = \inf\{\theta \in [s,s+\alpha] : \mathcal{K}^{y(\theta),z(\theta)}(\theta,\xi(\theta)) \leq \delta_0\} \wedge \tau.$$

By assumption, the inequality (51) holds for τ and hence for τ_1. But this leads to a contradiction since

$$0 > \delta_0 E(\tau_1 - s) \geq E\left[\int_s^{\tau_1} [\frac{\partial\phi}{\partial\theta} + \mathcal{A}\phi]_m(\theta,\xi(\theta))d\theta \right. \\ \left. + \int_s^{\tau_1} g_m(\theta,\xi(\theta),u(\theta,\xi(\theta)),K(u,\nabla\phi)(\theta,\xi(\theta)))d\theta\right] \geq 0.$$

It remains to check that $\gamma_2(s,s+\Delta s) \to 0$ and $\gamma_3(s,s+\Delta s) \to 0$ as $\Delta s \to 0$ a.s.

Note that $\gamma_2(s,s+\Delta s) \to 0$ a.s. by definition of f, properties of $\Gamma(s,t)$, and uniqueness of a BSDE solution.

Finally we check that $\gamma_3(s,s+\Delta s) \to 0$ as $\Delta s \to 0$ a.s. Note that the couple $(\tilde{y}(t),\tilde{z}(t)), s \leq t \leq s+\Delta s$ satisfies

$$\tilde{y}(t) = \Gamma^*(s+\Delta s)\phi(s+\Delta s,\xi_{s,x}(s+\Delta s)) \\ + \int_t^{s+\Delta s} f(\theta,\xi_{s,x}(\theta),\tilde{y}(\theta),\tilde{z}(\theta))d\theta - \int_t^{s+\Delta s}\tilde{z}(\theta)dw(\theta). \quad (52)$$

Given $\theta \in [s,s+\Delta s]$, set

$$\upsilon(\theta) = \tilde{y}(s+\Delta s) - \Gamma^*(s,\theta)\phi(\theta,\xi_{s,x}(\theta)) - \int_\theta^{s+\Delta s} \mathcal{K}^{u,\phi}(\vartheta,\xi_{s,x}(\vartheta))d\vartheta$$

and

$$\varpi(\theta) = \tilde{z}(\theta) - K(u,\nabla\phi)(\theta,\xi_{s,x}(\theta))).$$

Applying Ito's formula, we derive BSDE to govern the couple $(\upsilon(\theta), \varpi(\theta))$

$$\begin{aligned}\upsilon(\theta) &= \Gamma^*(s, s+\Delta s)\phi(s+\Delta s, \xi_{s,x}(s+\Delta s)) - \Gamma^*(s,\theta)\phi(\theta, \xi_{s,x}(\theta))\\ &\quad + \int_\theta^{s+\Delta s} f(\vartheta, \xi_{s,x}(\vartheta), \tilde{y}(\vartheta), \tilde{z}(\vartheta))d\vartheta - \int_\theta^{s+\Delta s} \mathcal{K}^{u,\phi}(\vartheta, \xi_{s,x}(\vartheta))d\vartheta\\ &\quad - \int_\theta^{s+\Delta s} \tilde{z}(\vartheta)dw(\vartheta) + \int_\theta^{s+\Delta s} K(u, \nabla\phi)(\vartheta, \xi_{s,x}(\vartheta)))dw(\vartheta)\\ &= \int_\theta^{s+\Delta s} f(\vartheta, \xi_{s,x}(\vartheta), \upsilon(\vartheta) + \Gamma^*(s,\vartheta)\phi(\vartheta, \xi_{s,x}(\vartheta))\\ &\quad + \int_\vartheta^{s+\Delta s} \mathcal{K}^{u,\phi}(r, \xi_{s,x}(r))dr,\ \varpi(\vartheta) + K(u, \nabla\phi)(\vartheta, \xi_{s,x}(\vartheta)))d\vartheta\\ &\quad + \int_\theta^{s+\Delta s} \left[\left(\frac{\partial\phi}{\partial\vartheta} + \mathcal{A}\phi\right)(\vartheta, \xi_{s,x}(\vartheta)) - \mathcal{K}^{u,\phi}(\vartheta, \xi_{s,x}(\vartheta))\right]d\vartheta\\ &\quad - \int_\theta^{s+\Delta s} \varpi(\vartheta)dw(\vartheta). \qquad (53)\end{aligned}$$

We verify that (υ, ϖ) converges to $(0,0)$ as $\Delta s \to 0$. Keeping in mind the estimates for the generator g by standard reasoning based on the Ito formula and the Burkholder inequality, we can prove that

$$E\left[\sup_{t\in[s,s+\Delta s]} |\upsilon(t)|^2\right] + E\left[\int_s^{s+\Delta s} \|\varpi(\theta)\|^2 d\theta\right] \le LE\left[\int_s^{s+\Delta s} \|m(\theta, \Delta s)\|^2 d\theta\right],$$

where

$$\begin{aligned}m(\theta, \Delta s) &= -\mathcal{K}^{u,\phi}(\theta, \xi_{s,x}(\theta)) + \left(\frac{\partial\phi}{\partial\theta} + \mathcal{A}\phi\right)(\theta, \xi_{s,x}(\theta))\\ &\quad + f(\theta, \xi_{s,x}(\theta), \upsilon(\theta) + \Gamma^*(s,\theta)\phi(\theta, \xi_{s,x}(\theta))\\ &\quad + \int_\theta^{s+\Delta s} \mathcal{K}^{u,\phi}(r, \xi_{s,x}(r))dr,\ \varpi(\theta) + K(u, \nabla\phi)(\theta, \xi_{s,x}(\vartheta)).\end{aligned}$$

Furthermore, since $\sup_{\theta\in[s,s+\Delta s]} E[\|\xi_{s,x}(\theta) - x\|^2] \to 0$ as $\Delta s \to 0$ and parameters of stochastic equations as well as the function ϕ and its derivatives are uniformly continuous in x, we obtain

$$\lim_{\Delta s\to 0} \sup_{s\le\theta\le s+\Delta s} E[\|m(\theta, \Delta s)\|^2] = 0.$$

Hence,

$$E\left[\sup_{s\le\theta\le s+\Delta s}|\upsilon(\theta)|^2\right]+E\left[\int_s^{s+\Delta s}\|\varpi(\theta)\|^2d\theta\right]$$
$$\le LE\left[\int_s^{s+\Delta s}\|m(\theta,\Delta t)\|^2d\theta\right]\le\varepsilon(\Delta s)\Delta s, \tag{54}$$

where $\varepsilon(\Delta s)\to 0$ as $\Delta s\to 0$. As a result we get that $\tilde{y}(\theta)$ converges to $\phi(s,x)$ and $\tilde{z}(\theta)$ converges to $[Cu](s,x)+[\nabla\phi A](s,x)$ a.s. as $\Delta s\to 0$.

This estimate does not satisfy yet our purposes. To get a more satisfactory estimate, we evaluate the conditional expectation of both sides of (53) that leads to $\upsilon(\theta)=E\left[\int_\theta^{s+\Delta s}n(\vartheta,\Delta s)d\vartheta|\mathcal{F}_\theta\right]$, where

$$\begin{aligned}n(\theta,\Delta s)&=-\mathcal{K}^{u,\phi}(\theta,\xi_{s,x}(\theta))+\left[\frac{\partial\phi}{\partial\theta}+\mathcal{A}\phi\right](\theta,\xi_{s,x}(\theta))\\&\quad+f(\theta,\xi_{s,x}(\theta),\tilde{y}(\theta),\tilde{z}(\theta))\\&=f(\theta,\xi_{s,x}(\theta),\tilde{y}(\theta),\tilde{z}(\theta))-f\left(\theta,\xi_{s,x}(\theta),\Gamma^*(\theta)\phi(\theta,\xi_{s,x}(\theta))\right.\\&\quad\left.+\int_\theta^{s+\Delta s}\mathcal{K}^{u,\phi}(\theta,\xi_{s,x}(\theta))d\theta,\;K(u,\nabla\phi)(\theta,\xi_{s,x}(\theta))\right).\end{aligned}$$

By Lipschitz continuity of f, we have for $s\le\theta\le s+\Delta s$, $\|n(\theta,\Delta s)\|\le L[\|\upsilon(\theta)\|+\|\varpi(\theta)\|]$, that is $\|n(\theta,\Delta s)\|\to 0$ a.s. as $\Delta s\to 0$.

Hence we have proved that $u(s,x)$ is a viscosity subsolution of the Cauchy problem (39). In a similar way we prove that $u(s,x)$ is a supersolution of (39) and hence a viscosity solution of this problem. □

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References

1. Belopolskaya, Ya., Dalecky, Yu.: Investigation of the Cauchy problem for systems of quasilinear equations via Markov processes. Izv. VUZ Mat. **12**, 6–17 (1978)
2. Belopolskaya, Ya., Dalecky, Yu.L.: Stochastic Equations and Differential Geometry. Kluwer, Boston (1990)
3. Belopolskaya, Ya., Dalecky, Yu.: Markov processes associated with nonlinear parabolic systems. DAN SSSR **250**(3), 268–271 (1980)
4. Pardoux, E., Peng. S.: Adapted solutions of backward stochastic equations. Syst. Control Lett. **14**, 55–61 (1990)

5. Pardoux, E., Peng, S.: Backward Stochastic Differential Equations and Quasilinear Parabolic Partial Differential Equations. Lecture Notes in CIS, vol. 176, pp. 200–217. Springer, New York (1992)
6. Peng, S.: Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. Stoch. Stoch. Rep. **37**, 61–74 (1991)
7. Ma, J., Yong, J.: Forward–Backward Stochastic Differential Equations and Their Applications. Lecture Notes in Mathematics, vol. 1702. Springer, New York (1999)
8. Albeverio, S., Belopolskaya, Ya.: Probabilistic approach to systems of nonlinear pdes and vanishing viscosity method. Markov Process. Relat. Filelds **12**(1), 59–94 (2006)
9. Crandall, M., Ishii, H., Lions, P.: User's guide to viscosity solutions of second order partial differential equations. Bull. Am. Math. Soc. **27**(1), 1–67 (1992)
10. Pardoux, E.: Backward stochastic differential equations and viscosity solutions of systems of semilinear parabolic and elliptic PDEs of second order. Stochastic Analysis and Relates Topics: The Geilo Workshop, pp. 79–127. Birkhäuser, Boston (1996)
11. Pardoux, E., Tang, S.: Forward–backward stochastic differential equations and quasilinear parabolic PDEs. Probab. Theory Relat. Fields **114**(2), 123–150 (1999)

Finite-Time Blowup and Existence of Global Positive Solutions of a Semi-linear Stochastic Partial Differential Equation with Fractional Noise

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Abstract We consider stochastic equations of the prototype

$$\mathrm{d}u(t,x) = \left(\Delta u(t,x) + \gamma u(t,x) + u(t,x)^{1+\beta}\right)\mathrm{d}t + \kappa u(t,x)\,\mathrm{d}B_t^H$$

on a smooth domain $D \subset \mathbb{R}^d$, with Dirichlet boundary condition, where $\beta > 0$, γ and κ are constants and $\{B_t^H, t \geq 0\}$ is a real-valued fractional Brownian motion with Hurst index $H > 1/2$. By means of the associated random partial differential equation, obtained by the transformation $v(t,x) = u(t,x)\exp\{\kappa B_t^H\}$, lower and upper bounds for the blowup time of u are given. Sufficient conditions for blowup in finite time and for the existence of a global solution are deduced in terms of the parameters of the equation. For the case $H = 1/2$ (i.e. for Brownian motion), estimates for the probability of blowup in finite time are given in terms of the laws of exponential functionals of Brownian motion.

1 Introduction and Background

In a classical paper [7], Fujita proved that for a bounded smooth domain $D \subset \mathbb{R}^d$, the equation

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + u^{1+\beta}(t,x), \quad x \in D,$$

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with Dirichlet boundary condition, where $\beta > 0$ is a constant, explodes in finite time for all nonnegative initial values $u(0,x) \in L^2(D)$ satisfying

$$\int_D u(0,x)\psi(x)\,\mathrm{d}x > \lambda_1^{1/\beta}. \tag{1}$$

Here $\lambda_1 > 0$ is the first eigenvalue of the Laplacian on D and ψ the corresponding eigenfunction normalized so that $\|\psi\|_{L^1} = 1$.

In this paper we consider a stochastic analog of the above equation, namely, we investigate the semi-linear SPDE

$$\begin{aligned} \mathrm{d}u(t,x) &= (\Delta u(t,x) + \gamma u(t,x) + G(u(t,x)))\,\mathrm{d}t + \kappa u(t,x)\,\mathrm{d}B_t^H, \quad t > 0, \\ u(0,x) &= f(x) \geq 0, \quad x \in D, \\ u(t,x) &= 0, \quad t \geq 0, \quad x \in \partial D, \end{aligned} \tag{2}$$

where $G : \mathbb{R} \to \mathbb{R}_+$ is locally Lipschitz and satisfies

$$G(z) \geq C z^{1+\beta} \quad \text{for all } z > 0, \tag{3}$$

$C > 0, \gamma$, $\beta > 0$ and κ are given numbers, $\{B_t^H,\ t \geq 0\}$ is a one-dimensional fractional Brownian motion with Hurst index $H > 1/2$ on a stochastic basis $(\Omega, \mathcal{F}, P)$, and $f : D \to \mathbb{R}_+$ is of class C^2 and not identically zero. We assume (3) in Sects. 1–3 only; it is replaced by (16) in Sects. 4 and 5.

The results on global solutions of parabolic equations perturbed by an additive or multiplicative time or space–time fractional noise established up to now are sufficient to state the existence and uniqueness of the variational (weak) and of the mild solution of (2) and the equivalence of both; see Maslowski and Nualart [8], Nualart and Vuillermot [11], and Sanz and Vuillermot [17], where the integral with respect to B^H is understood in the sense of fractional calculus (see, e.g., Zähle [19, 20]). Let us recall the notions of variational and mild solutions we are going to use here; see [11, 17]. Let $\alpha \in (1-H, \frac{1}{2})$, $t > 0$, and let $\mathcal{B}^{\alpha,2}([0,t], L^2(D))$ be the Banach space of all measurable mappings $u : [0,t] \to L^2(D)$ endowed with the norm $\|\cdot\|_{\alpha,2}$, defined by

$$\|u\|_{\alpha,2}^2 = \left(\operatorname{ess\,sup}_{s\in[0,t]} \|u(s,\cdot)\|_2\right)^2 + \int_0^t \mathrm{d}s \left(\int_0^s \mathrm{d}r \frac{\|u(s,\cdot)-u(r,\cdot)\|_2}{(s-r)^{\alpha+1}}\right)^2 < \infty,$$

where $\|\cdot\|_2$ is the usual norm in $L^2(D)$. An $L^2(D)$–valued random field $u = \{u(t,\cdot),\ t \geq 0\}$ is a *variational solution* of (2) on the interval $]0,\varrho[$ if, a.s.,

$$u \in L^2([0,t], H^1(D)) \cap \mathcal{B}^{\alpha,2}([0,t], L^2(D)) \tag{4}$$

for all $t < \varrho$ and if, for every $\varphi \in H^1(D)$ vanishing on ∂D,

$$\int_D u(t,x)\varphi(x)\,\mathrm{d}x = \int_D f(x)\varphi(x)\,\mathrm{d}x + \int_0^t \int_D \big[\langle \nabla u(s,x), \nabla\varphi(x)\rangle_{\mathbb{R}^d} + \gamma u(t,x)\varphi(x) + G(u(s,x))\varphi(x)\big]\,\mathrm{d}x\,\mathrm{d}s + \kappa\int_0^t\int_D u(s,x)\varphi(x)\,\mathrm{d}x\,\mathrm{d}B_s^H \quad P-\text{a.s.}$$

for all $t \in [0,\varrho[$. The requirement for u to belong to the $\mathcal{B}^{\alpha,2}$ spaces implies that the integral with respect to B^H exists as a generalized Stieltjes integral in the sense of [20], see Proposition 1 in [11]. Let $\{S_t,\, t \geq 0\}$ be the semigroup of d-dimensional Brownian motion with variance parameter 2, killed at the boundary of D. An $L^2(D)$-valued random field $u = \{u(t,\cdot),\, t \geq 0\}$ is a *mild solution* of (2) on the interval $]0,\tau[$ if (4) holds a.s. for all $t < \tau$, and if

$$u(t,x) = S_t f(x) + \int_0^t [\gamma S_{t-r}(u(r,\cdot))(x) + S_{t-r}(G(u(r,\cdot)))(x)]\,\mathrm{d}r + \kappa\int_0^t S_{t-r}(u(r,\cdot))(x)\,\mathrm{d}B_r^H \quad P-\text{a.s. and } x-\text{a.e. in } D$$

for all $t \in]0,\tau[$ (see, e.g., [14, Chap. IV]). Let us remark that the proof of the uniqueness of the mild solution and the equivalence of the variational and the mild solutions are carried out in [17] under the conditions $H \in (\frac{4d+1}{4d+2}, 1)$ and $\alpha \in (1-H, \frac{1}{4d+2})$, and for the more general case where B^H is a space-dependent fractional Brownian motion. For an approach based on stochastic integrals in the Wick sense, we refer to [12]. The positivity of the solution of (2) will be addressed in the next section.

Our aim in this communication is to study the blowup behaviour of u by means of the random partial differential equation of Sect. 2 [see (6) below]. The case of $H = 1/2$, in which $\{B_t^H\}$ is a standard one-dimensional Brownian motion, was investigated in [4]. There we obtained estimates of the probability of blowup and conditions for the existence of a global solutions of (2) with $H = 1/2$ and $\gamma = 0$. Following closely the approach in [4], here we are going to derive the same kind of bounds for the positive solutions of (2), in the case $H > 1/2$ and with a constant drift in the non-random linear part. Moreover, we obtain useful lower and upper bounds τ_*, τ^* for the explosion time ϱ of (2). We remark that both, the estimates we obtain and the distributions of the random times τ_*, τ^*, are given in terms of exponential functionals of B^H of the form

$$\int_0^t e^{(-\lambda_1+\gamma)\beta s+\kappa\beta B_s^H}\,\mathrm{d}s \quad\text{and}\quad \int_0^\infty e^{(-\lambda_1+\gamma)\beta s+\kappa\beta B_s^H}\,\mathrm{d}s. \tag{5}$$

When $H = 1/2$ the distribution of the integrals above can be obtained using Dufresne's and Yor's formulae [5, 18] or the method of Pintoux and Privault [15]. However, to our knowledge such precise results are not presently available for $H \neq 1/2$. It remains a challenge to obtain more accurate information on the explosion times of (2).

We describe in Sects. 3 and 4 the blowup behaviour of the solution v of the random partial differential equation (6) in terms of the first eigenvalue and the first eigenfunction of the Laplace operator on D. This is done by solving explicitly a stochastic equation in the time variable which is obtained from the weak form of (6). The solution of this differential equation can be written in terms of integrals of the exponential of fractional Brownian motion with drift. Near the end of the paper, sufficient conditions for v to be a global solution are given in terms of the semigroup of the Laplace operator using recent sharp results on its transition density. These conditions show in particular that the initial condition f has to be small enough in order to avoid for a given G the blowup of v, as well as a sufficiently small $|\gamma|$ and a sufficiently big β. The results presented here can be used to investigate the blowup behaviour of u for non-linearities satisfying (3) or (16).

2 Weak Solutions of a Random PDE

In this section we investigate the random partial differential equation

$$\begin{aligned} \frac{\partial v}{\partial t}(t,x) &= \Delta v(t,x) + \gamma v(t,x) + e^{-\kappa B_t^H} G(e^{\kappa B_t^H} v(t,x)), \quad t>0, \quad x \in D, \\ v(0,x) &= f(x), \quad x \in D, \\ v(t,x) &= 0, \quad x \in \partial D. \end{aligned} \tag{6}$$

This equation is understood trajectorywise and classical results for partial differential equations of parabolic type apply to show existence and uniqueness of a solution $v(t,x)$ up to eventual blowup (see, e.g., Friedman [6, Chap. 7, Theorem 9]). Moreover,

$$v(t,x) = e^{\gamma t} S_t f(x) + \int_0^t e^{\gamma(t-s)} S_{t-s} \left(e^{-\kappa B_s^H} G(e^{\kappa B_s^H} v(s,x)) \right) \mathrm{d}s, \tag{7}$$

and therefore $v(t,x) \geq e^{\gamma t} S_t f(x) \geq 0$.

Proposition 1. *Let u be a weak solution of (2). Then the function v defined by*

$$v(t,x) = e^{-\kappa B_t^H} u(t,x), \quad t \geq 0, \quad x \in D,$$

solves (6).

Remark 1. Proposition 1 implies in particular that (2) possesses a strong local solution $u(t,x)$. Moreover, $u(t,x) \geq 0$ due to (7).

Proof. By Itô's formula for B^H (see, e.g., [10, Lemma 2.7.1])

$$e^{-\kappa B_t^H} = 1 - \kappa \int_0^t e^{-\kappa B_s^H} \, \mathrm{d}B_s^H.$$

We notice that the last integral can be defined as a Riemann–Stieltjes integral. Let us write $u(t,\varphi) \equiv \int_D u(t,x)\varphi(x)\,\mathrm{d}x$. Then the weak solution of (2) can be written as

$$u(t,\varphi)=u(0,\varphi)+\int_0^t u(s,\Delta\varphi)\,\mathrm{d}s+\int_0^t [\gamma u(s,\varphi)+G(u)(s,\varphi)]\,\mathrm{d}s+\kappa\int_0^t u(s,\varphi)\,\mathrm{d}B_s^H.$$

By applying the integration by parts formula, which is a special case of the two-dimensional Itô's formula (see [10], p. 184), we get

$$\begin{aligned} v(t,\varphi) &:= \int_D v(t,x)\varphi(x)\,\mathrm{d}x \\ &= v(0,\varphi)+\int_0^t e^{-\kappa B_s^H}\,\mathrm{d}u(s,\varphi)+\int_0^t u(s,\varphi)\left(-\kappa e^{-\kappa B_s^H}\,\mathrm{d}B_s^H\right). \end{aligned}$$

Therefore,

$$\begin{aligned} v(t,\varphi) &= v(0,\varphi)+\int_0^t e^{-\kappa B_s^H}\,[u(s,\Delta\varphi)+\gamma u(s,\varphi)+G(u)(s,\varphi)]\,\mathrm{d}s \\ &\quad +\kappa\int_0^t e^{-\kappa B_s^H} u(s,\varphi)\,\mathrm{d}B_s^H-\kappa\int_0^t e^{-\kappa B_s^H} u(s,\varphi)\,\mathrm{d}B_s^H \\ &= v(0,\varphi)+\int_0^t \left[v(s,\Delta\varphi)+\gamma v(s,\varphi)+e^{-\kappa B_s^H} G(e^{\kappa B_\cdot^H} v)(s,\varphi)\right]\mathrm{d}s. \end{aligned}$$

Moreover, by self-adjointness of the Laplacian and the fact that $\varphi(x) = 0$ for $x\in\partial D$,

$$v(s,\Delta\varphi)=\int_D v(s,x)\Delta\varphi(x)\,\mathrm{d}x=\int_D \Delta v(s,x)\varphi(x)\,\mathrm{d}x=\Delta v(s,\varphi). \qquad \square$$

In what follows ϱ denotes the blowup time of (6). Due to Proposition 1 and to the a.s. continuity of $B_\cdot^H$, ϱ is also the explosion time of (2).

3 An Upper Bound for ϱ

Without loss of generality, let us assume that $C = 1$ in (3). Let ψ be the eigenfunction corresponding to the first eigenvalue λ_1 of the Laplacian on D, normalized by $\int_D \psi(x)\,\mathrm{d}x = 1$. It is well known that ψ is strictly positive on D. Due to Proposition 1, we have that

$$v(t,\psi)=v(0,\psi)+\int_0^t [v(s,\Delta\psi)+\gamma v(s,\psi)]\,\mathrm{d}s+\int_0^t e^{-\kappa B_s^H} G(e^{\kappa B_\cdot^H} v)(s,\psi)\,\mathrm{d}s.$$

Moreover,

$$v(s,\Delta\psi) = -\lambda_1 v(s,\psi),$$

and, due to (3),

$$\int_D e^{-\kappa B_s^H} G(e^{\kappa B_s^H} v(s,x))\psi(x)\,\mathrm{d}x \;\geq\; e^{\kappa\beta B_s^H}\int_D v(s,x)^{1+\beta}\psi(x)\,\mathrm{d}x.$$

By Jensen's inequality,

$$\int_D v(s,x)^{1+\beta}\psi(x)\,\mathrm{d}x \;\geq\; \left[\int_D v(s,x)\psi(x)\,\mathrm{d}x\right]^{1+\beta} \;=\; v(s,\psi)^{1+\beta},$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}v(t,\psi) \geq (-\lambda_1+\gamma)v(t,\psi) + e^{\kappa\beta B_s^H} v(t,\psi)^{1+\beta}. \tag{8}$$

Hence $v(t,\psi) \geq I(t)$ for all $t \geq 0$, where $I(\cdot)$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t}I(t) = (-\lambda_1+\gamma)I(t) \,+ e^{\kappa\beta B_t^H} I(t)^{1+\beta}\,,\;\; I(0) = v(0,\psi),$$

and is given by

$$I(t) \;=\; e^{(-\lambda_1+\gamma)t}\left[v(0,\psi)^{-\beta} - \beta\int_0^t e^{(-\lambda_1+\gamma)\beta s+\kappa\beta B_s^H}\,\mathrm{d}s\right]^{-\frac{1}{\beta}}, \quad 0 \leq t < \tau^*,$$

with

$$\tau^* := \inf\left\{t \geq 0 \,\middle|\, \int_0^t e^{(-\lambda_1+\gamma)\beta s+\kappa\beta B_s^H}\,\mathrm{d}s \geq \frac{1}{\beta}v(0,\psi)^{-\beta}\right\}. \tag{9}$$

It follows that I exhibits finite-time blowup on the event $[\tau^* < \infty]$. Due to $I \leqq v(\cdot,\psi)$, τ^* is an upper bound for the blowup time of $v(\cdot,\psi)$. Since by assumption $\int_D \psi(x)\,\mathrm{d}x = 1$, $v(t,x)$ cannot be bounded on $[\tau^* < \infty]$. Hence τ^* is also an upper bound for the blowup times of v and u.

We subsume the above argumentation into the following corollary.

Corollary 1. *The function $v(t,\psi) = \int_D v(t,x)\psi(x)\,\mathrm{d}x$ explodes in finite time on the event $[\tau^* < \infty]$, hence $u(t,x) = e^{\kappa B_t^H} v(t,x)$ also explodes in finite time if $\tau^* < \infty$, and the blowup times of u and v are the same.*

Remark 2. Notice that, from (9),

$$P[\tau^* = +\infty] = P\left[\int_0^t e^{(-\lambda_1+\gamma)\beta s+\kappa\beta B_s^H}\,\mathrm{d}s < \frac{1}{\beta}v(0,\psi)^{-\beta} \text{ for all } t>0\right]$$
$$= P\left[\int_0^\infty e^{(-\lambda_1+\gamma)\beta s+\kappa\beta B_s^H}\,\mathrm{d}s \leqq \frac{1}{\beta}v(0,\psi)^{-\beta}\right]. \tag{10}$$

Assume now that $\gamma > \lambda_1$, and recall the law of the iterated logarithm for B^H [1]:

$$\liminf_{t\to+\infty}\frac{B_t^H}{t^H\sqrt{2\log\log t}} = -1,\ \ \limsup_{t\to+\infty}\frac{B_t^H}{t^H\sqrt{2\log\log t}} = +1 \quad P-\text{a.s.}$$

It follows that the integral in (10) diverges. Therefore $P[\tau^* = +\infty] = 0$ and any nontrivial positive solution of (2) explodes in finite time a.s. If $\gamma < \lambda_1$ this is not true anymore, and it would be interesting to estimate this probability. As mentioned in the introduction, the law of these integrals is known only in the case $H = \frac{1}{2}$, i.e. for Brownian motion. After the following remark, we consider this case in more detail.

Remark 3. By putting $\kappa = \gamma = 0$ we get $v = u$ and, moreover, in (10) we obtain that $P[\tau^* = +\infty] = 0$ or 1 according to $\int_D f(x)\psi(x)\,\mathrm{d}x > \lambda_1^{1/\beta}$ or $\int_D f(x)\psi(x)\,\mathrm{d}x \leq \lambda_1^{1/\beta}$, which is a probabilistic counterpart to condition (1).

The case of $H = \frac{1}{2}$ was investigated in [4], where we obtained a lower bound for the probability of finite-time explosion of the solution of (2). For the reader's convenience, here we explain the calculations for this special case.

When $H = \frac{1}{2}$, Itô's formula contains a second order term and the associated random PDE therefore reads (we write W instead of $B^{1/2}$)

$$\begin{aligned}\frac{\partial v}{\partial t}(t,x) &= \Delta v(t,x) + \left(\gamma - \frac{\kappa^2}{2}\right)v(t,x) + e^{-\kappa W_t}G(e^{\kappa W_t}v(t,x)), \quad t>0, \quad x\in D,\\ v(0,x) &= f(x), \quad x \in D,\\ v(t,x) &= 0, \quad x\in\partial D.\end{aligned} \tag{11}$$

We get again a differential inequality for $v(t,\psi)$, and the blowup time of the associated differential equation for I is

$$\tilde{\tau}^* = \inf\left\{t\geq 0 \ \int_0^t e^{-(\lambda_1+\kappa^2/2-\gamma)\beta s+\kappa\beta W_s}\,\mathrm{d}s \geq \frac{1}{\beta}v(0,\psi)^{-\beta}\right\}. \tag{12}$$

Now

$$P[\tilde{\tau}^* = +\infty] = P\left[\int_0^\infty e^{-(\lambda_1+\kappa^2/2-\gamma)\beta s+\kappa\beta W_s}\,\mathrm{d}s \leqq \frac{1}{\beta}v(0,\psi)^{-\beta}\right]$$
$$= P\left[\int_0^\infty e^{2\hat{\beta}W_s^{(\mu)}}\,\mathrm{d}s \leqq \frac{1}{\beta}v(0,\psi)^{-\beta}\right], \qquad (13)$$

where $W_s^{(\mu)} := \mu s + W_s$, $\mu := -(\lambda_1 - \gamma + \kappa^2/2)/\kappa$, and $\hat{\beta} := \kappa\beta/2$. Setting $\hat{\mu} = \mu/\hat{\beta}$ we get by performing the time change $s \mapsto s(\hat{\beta})^2$,

$$P[\tilde{\tau}^* = +\infty] = P\left[\frac{4}{\kappa^2\beta^2}\int_0^\infty \exp\{2W_s^{(\hat{\mu})}\}\,\mathrm{d}s \leqq \frac{1}{\beta}v(0,\psi)^{-\beta}\right]. \qquad (14)$$

If $\tilde{\mu} := -(\lambda_1 - \gamma + \kappa^2/2)/\kappa\beta > 0$, it follows again that $P[\tilde{\tau}^* = +\infty] = 0$ and any nontrivial positive solution of (2) with B^H replaced by W explodes in finite time a.s.*;* see also [9, Proposition 6.4]*,* or [16, Sect. 2]. If $\tilde{\mu} < 0$, it follows from [18] (Chap. 6, Corollary 1.2) that

$$\int_0^\infty \exp\{2W_s^{(\hat{\mu})}\}\,\mathrm{d}s \;=\; \frac{1}{2Z_{-\hat{\mu}}}$$

in distribution, where $Z_{-\hat{\mu}}$ is a random variable with law $\Gamma(-\hat{\mu})$, i.e. $P(Z_{-\hat{\mu}} \in \mathrm{d}y) = \frac{1}{\Gamma(-\hat{\mu})}e^{-y}y^{-\hat{\mu}-1}\,\mathrm{d}y$. We get therefore (see also Formula 1.10.4(1) in [3])

$$P[\tilde{\tau}^* = +\infty] = \int_0^{\frac{1}{\beta}v(0,\psi)^{-\beta}} h(y)\,\mathrm{d}y,$$

where

$$h(y) \;=\; \frac{(\kappa^2\beta^2 y/2)^{-(2(\lambda_1-\gamma)+\kappa^2)/\kappa^2\beta}}{y\Gamma((2(\lambda_1-\gamma)+\kappa^2)/(\kappa^2\beta))}\exp\left\{-\frac{2}{\kappa^2\beta^2 y}\right\}.$$

In this way we have proved the following:

Proposition 2. *The probability that the solution of (2) with B^H replaced by W blows up in finite time is bounded from below by $\int_{\frac{1}{\beta}v(0,\psi)^{-\beta}}^{+\infty} h(y)\,\mathrm{d}y$.*

We end this section by reviewing another method to find upper estimates of the blowup time of the solution of (2). In [2] it is shown that the formula (12) for $\tilde{\tau}^*$can also be found by replacing the random differential inequality (8) by a stochastic differential inequality, whose associated equality can be solved explicitly. A comparison theorem for stochastic differential inequalities is needed for this, and since no such theorem seems to be known at present for inequalities with fractional Brownian motion, we have to restrict ourselves to Brownian motion where these theorems are classical.

Proceeding with the variational solution in Sect. 1 in the same way as with the random PDE at the beginning of this section, we get the stochastic differential inequality

$$u(t,\psi) \;\geq\; u(0,\psi) + \int_0^t \left[(\gamma-\lambda_1)u(s,\psi) + u(s,\psi)^{1+\beta}\right] \mathrm{d}s + \kappa \int_0^t u(s,\psi)\,\mathrm{d}W_s.$$

The corresponding stochastic differential equation

$$X_t = u(0,\psi) + \int_0^t \left[(\gamma-\lambda_1)X_s + X_s^{1+\beta}\right] \mathrm{d}s + \kappa \int_0^t X_s\,\mathrm{d}W_s$$

can be solved explicitly. In fact, by the ansatz $Y_t = h(X_t)$ and by Itô's formula, we then get

$$Y_t = Y_0 + \int_0^t \left[h'(Y_s)\left((\gamma-\lambda_1)Y_s + Y_s^{1+\beta}\right) + \frac{\kappa^2}{2}h''(Y_s)Y_s^2\right] \mathrm{d}s + \kappa \int_0^t h'(Y_s)Y_s\,\mathrm{d}W_s.$$

The function h can now be chosen in such a way that Y satisfies the linear stochastic differential equation

$$Y_t = Y_0 + \int_0^t (a+bY_s)\;\mathrm{d}s + \int_0^t (c+dY_s)\;\mathrm{d}W_s.$$

for suitable constants $a,b,c,d \in \mathbb{R}$. In fact, a comparison of the martingale parts of both representations of Y gives a differential equation for h whose solution is given by $h(Y_t) = kY_t^{d/\kappa} - \frac{c}{d}$ for any constant $k \in \mathbb{R}$. By comparing the finite variation parts of the representations of Y, we get

$$\frac{kd}{\kappa}Y_t^{d/\kappa+\beta} + \frac{kd}{\kappa}(\gamma-\lambda_1)Y_t^{d/\kappa} + \frac{1}{2}kd(d-\kappa)Y_t^{d/\kappa} = a + bkY_t^{d/\kappa} - \frac{bc}{d}.$$

We choose $d=-\beta\kappa$, $c=0$, $b=\beta(\frac{(1+\beta)\kappa^2}{2}-\gamma+\lambda_1)$, $a=-k\beta$ and get $Y_t = X_t^{-\beta}$. From the explicit formula for the solution of the linear equation for Y, we get

$$X_t = Y_t^{-1/\beta} = \left[u(0,\psi)^{-\beta}Y_t^0 - \beta Y_t^1\right]^{-1/\beta}, \tag{15}$$

where

$$Y_t^1 = Y_t^0 \int_0^t \exp\left\{-(\lambda_1+\kappa^2/2-\gamma)\beta s + \kappa\beta W_s\right\}\,\mathrm{d}s$$

with $Y_t^0 = \exp\left\{(\lambda_1+\kappa^2/2-\gamma)\beta t - \kappa\beta W_t\right\}$. It can easily be seen that (15) yields the same formula for the blowup time of X as (12) for I.

4 A Lower Bound for ϱ

We consider again equation (6), but we assume that $\kappa \neq 0$ and that $G : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $G(0) = 0$, $G(z)/z$ is increasing and

$$G(z) \leq \Lambda z^{1+\beta} \quad \text{for all } \; z > 0, \tag{16}$$

where Λ and β are certain positive numbers. Let $\{S_t,\ t \geq 0\}$ again denote the semigroup of d-dimensional Brownian motion killed at the boundary of D. Recall the integral form (7) of (6). We define

$$F(t) = \left(1 - \Lambda\beta \int_0^t e^{\kappa\beta B_r^H} \|e^{\gamma r} S_r f\|_\infty^\beta \,\mathrm{d}r\right)^{-\frac{1}{\beta}}, \quad 0 \leq t < \tau_*, \tag{17}$$

where

$$\tau_* = \inf\left\{t > 0 : \int_0^t e^{\kappa\beta B_r^H} \|e^{\gamma r} S_r f\|_\infty^\beta \,\mathrm{d}r \geq (\Lambda\beta)^{-1}\right\}. \tag{18}$$

Hence $F(0) = 1$ and

$$\frac{\mathrm{d}F}{\mathrm{d}t}(t) = \Lambda e^{\kappa\beta B_t^H} \|e^{\gamma t} S_t f\|_\infty^\beta F^{1+\beta}(t),$$

which implies

$$F(t) = 1 + \Lambda \int_0^t e^{\kappa\beta B_r^H} \|e^{\gamma r} S_r f\|_\infty^\beta F^{1+\beta}(r)\,\mathrm{d}r.$$

Let

$$R(V)(t,x) := e^{\gamma t} S_t f(x) + \int_0^t e^{-\kappa B_r^H} e^{\gamma(t-r)} S_{t-r}\left(G(e^{\kappa B_r^H} V_r(\cdot))\right)(x)\,\mathrm{d}r, \quad x \in D,\ t \geq 0,$$

where $(t,x) \mapsto V_t(x)$ is any nonnegative continuous function such that $V_t(\cdot) \in C_0(D)$, $t \geq 0$, and

$$V_t(x) \leq e^{\gamma t} S_t f(x) F(t), \quad 0 \leq t < \tau_*, \quad x \in D. \tag{19}$$

Then $e^{\gamma t} S_t f(x) \leq R(V)(t,x)$ and

$$
\begin{aligned}
&R(V)(t,x)\\
&= e^{\gamma t} S_t f(x) + \int_0^t e^{-\kappa B_r^H} e^{\gamma(t-r)} S_{t-r}\left(\frac{G(e^{\kappa B_r^H} V_r(\cdot))}{V_r(\cdot)} V_r(\cdot)\right)(x)\,\mathrm{d}r\\
&\le e^{\gamma t} S_t f(x) + \int_0^t e^{-\kappa B_r^H} e^{\gamma(t-r)} S_{t-r}\left(\frac{G(e^{\kappa B_r^H} F(r)\|e^{\gamma r} S_r f\|_\infty)}{F(r)\|e^{\gamma r} S_r f\|_\infty} V_r(\cdot)\right)(x)\,\mathrm{d}r\\
&\le e^{\gamma t} S_t f(x) + \Lambda \int_0^t e^{\kappa\beta B_r^H} F^{1+\beta}(r)\|e^{\gamma r} S_r f\|_\infty^\beta e^{\gamma(t-r)} S_{t-r}(e^{\gamma r} S_r f)(x)\,\mathrm{d}r\\
&= e^{\gamma t} S_t f(x)\left[1 + \Lambda \int_0^t e^{\kappa\beta B_r^H} F^{1+\beta}(r)\|e^{\gamma r} S_r f\|_\infty^\beta\,\mathrm{d}r\right] = e^{\gamma t} S_t f(x) F(t), \quad (20)
\end{aligned}
$$

where to obtain the first inequality we used (19) and the fact that $G(z)/z$ is increasing and to obtain the second inequality we used (16). Consequently,

$$
e^{\gamma t} S_t f(x) \le R(V)(t,x) \le e^{\gamma t} S_t f(x) F(t), \quad 0 \le t < \tau_*, \quad x \in D.
$$

Let

$$
v_t^0(x) := e^{\gamma t} S_t f(x) \quad \text{and} \quad v_t^{n+1}(x) = R(v^n)(t,x), \quad n = 0,1,2,\ldots.
$$

Using induction, one can easily prove that the sequence $\{v^n\}$ is increasing, and therefore the limit

$$
v_t(x) = \lim_{n\to\infty} v_t^{(n)}(x)
$$

exists for all $x \in D$ and all $0 \le t < \tau_*$. The monotone convergence theorem implies

$$
v_t(x) = R v_t(x) \text{ for } x \in D \text{ and } 0 \le t < \tau_*,
$$

i.e. the function $v_t(x)$ solves (7) on $[0,\tau_*) \times D$. Moreover, because of (20) and (17),

$$
v_t(x) \le \frac{e^{\gamma t} S_t f(x)}{\left(1 - \Lambda\beta \int_0^t e^{\kappa\beta B_r^H} \|e^{\gamma r} S_r f\|_\infty^\beta\,\mathrm{d}r\right)^{1/\beta}} < \infty
$$

as long as

$$
\int_0^t e^{\kappa\beta B_r^H} \|e^{\gamma r} S_r f\|_\infty^\beta\,\mathrm{d}r < (\Lambda\beta)^{-1}.
$$

In this way we have proved the following proposition.

Proposition 3. *The blowup time of (7) is bounded from below by the random variable τ_* defined in (18).*

5 Non-Explosion of v

An immediate consequence of the discussion in the preceding section is the following result.

Theorem 1. *Assume that the function $f \geq 0$ is such that*

$$\Lambda\beta \int_0^\infty e^{\kappa\beta B_r^H} \|e^{\gamma r} S_r f\|_\infty^\beta \, \mathrm{d}r < 1. \tag{21}$$

Then (6) admits a global solution $v(t,x)$ that satisfies

$$0 \leq v(t,x) \leq \frac{e^{\gamma t} S_t f(x)}{\left(1 - \Lambda\beta \int_0^t e^{\kappa\beta B_r^H} \|e^{\gamma r} S_r f\|_\infty^\beta \, \mathrm{d}r\right)^{\frac{1}{\beta}}}, \quad t \geq 0. \tag{22}$$

When the boundary of D is sufficiently smooth, it is possible to derive a sufficient condition for (21) in terms of the transition kernels $\{p_t(x,y), t > 0\}$ of $\{S_t, t \geq 0\}$ and the first eigenvalue λ_1 and corresponding eigenfunction ψ. We recall the following sharp bounds for $\{p_t(x,y), t > 0\}$, which we borrowed from Ouhabaz and Wang [13].

Theorem 2. *Let $\psi > 0$ be the first Dirichlet eigenfunction on a connected, bounded $C^{1,\alpha}$- domain in $\mathbb{R}^d$, where $\alpha > 0$ and $d \geq 1$, and let $p_t(x,y)$ be the corresponding Dirichlet heat kernel. There exists a constant $c > 0$ such that, for any $t > 0$,*

$$\max\left\{1, \frac{1}{c} t^{-(d+2)/2}\right\} \leq e^{\lambda_1 t} \sup_{x,y} \frac{p_t(x,y)}{\psi(x)\psi(y)} \leq 1 + c(1 \wedge t)^{-(d+2)/2} e^{-(\lambda_2 - \lambda_1)t},$$

where $\lambda_2 > \lambda_1$ are the first two Dirichlet eigenvalues. This estimate is sharp for both short and long times.

The above theorem is useful in verifying condition (21). Let the domain D satisfy the assumptions in Theorem 2, and let the initial value $f \geq 0$ be chosen so that

$$f(y) \leq K S_\eta \psi(y), \quad y \in D, \tag{23}$$

where $\eta \geq 1$ is fixed and $K > 0$ is a sufficiently small constant to be specified later on. Arguing as in [4] we obtain that condition (21) is satisfied provided that

$$\Lambda\beta \left[K(1+c)e^{-\lambda_1 \eta} \left(\sup_{x \in D} \psi(x)\right)^2 \int_D \psi(y) \, \mathrm{d}y \right]^\beta \int_0^\infty e^{\kappa\beta B_r^H + (-\lambda_1 + \gamma)\beta r} \, \mathrm{d}r < 1,$$

or

$$\int_0^\infty e^{\kappa\beta B_r^H+(-\lambda_1+\gamma)\beta r}\,\mathrm{d}r < \frac{e^{\lambda_1\beta\eta}}{\Lambda\beta\left[K(1+c)\left(\sup_{x\in D}\psi(x)\right)^2\int_D\psi(y)\,\mathrm{d}y\right]^\beta}, \tag{24}$$

which holds if K in (23) is sufficiently small. In this way we get the following:

Theorem 3. *Let G satisfy (16), and let D be a connected, bounded $C^{1,\alpha}$-domain in $\mathbb{R}^d$, where $\alpha > 0$. If (23) and (24) hold for some $\eta > 0$ and $K > 0$, then the solution of (7) is global.*

Remark 4. The integral on the left side of (24) coincides with the corresponding integral in Sect. 3. If $G(z) = \Lambda z^{1+\beta}$, the results of this section can be applied also to the solution u of (2) because $v(t,x) = e^{-\kappa B_t^H}u(t,x)$, $t \geq 0$, $x \in D$.

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References

1. Arcones, M.A.: On the law of the iterated logarithm for Gaussian processes. J. Theor. Probab. **8**, 877–903 (1995)
2. Bandle, C., Dozzi, M., Schott, R.: Blow up behaviour of a stochastic partial differential equation of reaction-diffusion type. In: Stochastic Analysis: Random Fields and Measure-Valued Processes. Israel Mathematical Conference Proceedings, vol. 10, pp. 27–33. Bar-Ilan University, Ramat Gan (1996)
3. Borodin, A.N., Salminen, P.: Handbook of Brownian Motion—Facts and Formulae, 2nd edn. Birkhäuser, Basel (2002)
4. Dozzi, M., López-Mimbela, J.A.: Finite-time blowup and existence of global positive solutions of a semi-linear SPDE. Stoch. Process. Appl. **120**(6), 767–776 (2010)
5. Dufresne, D.: The distribution of a perpetuity, with applications to risk theory and pension funding. Scand. Actuarial J. **9**, 39–79 (1990)
6. Friedman, A.: Partial Differential Equations of Parabolic Type. Prentice-Hall Inc., Englewood Cliffs (1964)
7. Fujita, H.: On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations. In: Nonlinear Functional Analysis. American Mathematical Society, Providence, RI (1970). Proceedings of Symposia in Pure Mathematics, vol. 18, Part 1, pp. 105–113, Chicago (1968)
8. Maslowski, B., Nualart, D.: Evolution equations driven by a fractional Brownian motion. J. Funct. Anal. **202**, 277–305 (2003)

9. Matsumoto, H., Yor, M.: Exponential functionals of Brownian motion. I. Probability laws at fixed time. Probab. Surv. **2**, 312–347 (2005)
10. Mishura, Y.: Stochastic calculus for fractional Brownian motion and related processes. Springer Lecture Notes in Mathematics, vol. 1929. Springer, Berlin (2008)
11. Nualart, D., Vuillermot, P.: Variational solutions for partial differential equations driven by a fractional noise. J. Funct. Anal. **232**, 390–454 (2006)
12. Ohashi, A.M.F.: Stochastic evolution equations driven by a fractional white noise. Stoch. Anal. Appl. **24**, 555–578 (2006)
13. Ouhabaz, E.M., Wang, F.-Y.: Sharp estimates for intrinsic ultracontractivity on $C^{1,\alpha}$-domains. Manuscripta Math. **122**(2), 229–244 (2007)
14. Pazy, A.: Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, vol. 44. Springer, New York (1983)
15. Pintoux, C., Privault, N.: A direct solution to the Fokker-Planck equation for exponential Brownian functionals. Anal. Appl. (Singapore) **8**, 287–304 (2010)
16. Salminen, P., Yor, M.: Properties of perpetual integral functionals of Brownian motion with drift. Ann. Inst. H. Poincaré Probab. Stat. **41**(3), 335–347 (2005)
17. Sanz-Solé, M., Vuillermot, P.: Mild solutions for a class of fractional SPDE's and their sample paths. J. Evol. Equat. **9**, 235–265 (2009)
18. Yor, M.: On some exponential functionals of Brownian motion. Adv. Appl. Probab. **24**, 509–531 (1992)
19. Zähle, M.: Integration with respect to fractal functions and stochastic calculus I. Probab. Theory Relat. Fields **111**, 333–374 (2001)
20. Zähle, M.: Integration with respect to fractal functions and stochastic calculus II. Math. Nachr. **225**, 145–183 (2001)

Hydrodynamics and Stochastic Differential Equation with Sobolev Coefficients

Shizan Fang

Abstract In this chapter, we will explain how the Brenier's relaxed variational principle for Euler equation makes involved the ordinary differential equations with Sobolev coefficients and how the investigation on stochastic differential equations (SDE) with Sobolev coefficients is useful to establish variational principles for Navier–Stokes equations. We will survey recent results on this topic.

1 Introduction

The Euler equation describes the velocity of incompressible nonviscous fluids on $\mathbf{R}^d$ or on a compact Riemannian manifold M:

$$\frac{d}{dt}u_t + (u_t \cdot \nabla)u_t = -\nabla p, \quad \mathrm{div}(u_t) = 0, \tag{1}$$

where $(u_t \cdot \nabla)u_t$ is understood as the covariant derivative $\nabla_{u_t} u_t$ in the case of manifolds.

If $x \to u_t(x)$ is smooth, there is a flow of diffeomorphisms g_t such that

$$\frac{d}{dt}g_t(x) = u_t(g_t(x)), \quad g_0(x) = x. \tag{2}$$

Fix a time $T > 0$, then $t \to g_t$ is a continuous curve on the group of diffeomorphisms $\mathrm{Diff}(M)$; in other words, $g_\cdot \in C([0,T];\mathrm{Diff}(M))$. A famous work by V. Arnold says that u is a solution to (1) if and only if g is a *geodesic* on $\mathrm{Diff}(M)$ equipped with L^2 metric. Precisely, $g_\cdot$ minimizes the functional $C([0,T];\mathrm{Diff}(M))$

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$$S[\varphi] = \frac{1}{2}\int_0^T \int_M \left|\frac{d}{dt}\varphi_t(x)\right|^2_{T_xM} dxdt. \tag{3}$$

To solve (1), it turns often to minimize (3) with constraints: $\varphi_0 = I, \varphi_T = h$; the pressure field p arises as a Lagrange multiplier from the incompressibility constraint. It is known that the solution u_t to (1) loses more and more regularity as t grows even the initial data u_0 is smooth (see, e.g., [9]). From a time t, u_t should be in some Sobolev space. In [8], Brenier has relaxed (3) by looking for probability measures η on the path space $C([0,T], M)$, which minimizes the functional

$$S[\eta] = \frac{1}{2}\int_{C([0,T],M)} \left[\int_0^T |\dot{\gamma}(t)|^2_{T_{\gamma(t)}M}\, dt\right] d\eta(\gamma), \tag{4}$$

with constraints: $(e_0, e_T)_*\eta = (I,h)_*(dx)$ and $(e_t)_*\eta = dx$, where $e_t : \gamma \to \gamma(t)$.

If $g.$ is a classical solution to (3) with $g_T = h$, then the probability measure η on the path space $C([0,T], M)$ defined by

$$\int_{C([0,T],M)} \psi(\gamma)\, d\eta(\gamma) = \int_M \psi(g.(x))\, dx$$

is a Brenier's solution. In this latter case, η can be expressed by

$$\eta = (\Phi_g)_*(dx),$$

with $\Phi_g : M \to C([0,T], M)$, $x \to g.(x)$. In other words, η is supported by the graph of $g \in C([0,T], \mathrm{Diff}(M))$ in above sense (see [2]). So it is interesting to know for which class of vector fields the above construction remains true.

In the viscous case, the velocity obeys the Navier–Stokes equation

$$\frac{d}{dt}u_t + (u_t\cdot\nabla)u_t - \nu\Delta u_t = -\nabla p,\ \mathrm{div}(u_t) = 0. \tag{5}$$

How do you establish the suitable variational principle for the Navier–Stokes equation? In parallel to the flow (2) associated with a velocity field in the Euler equation, a stochastic flow of diffeomorphisms having u, a solution to (5), as the drift should be considered. A straight generalization of (4) meets two difficulties: first, there is no canonical way to construct the Brownian motion on M, when the latter is not parallelizable, and second, solutions to SDE are not absolutely continuous with respect to the time t. Progresses have been done by A.B. Cruzeiro and others (see, e.g., [3] for a notion of generalized flow and [4] a survey for the related topics). See also related works by Le Jan and Raimond [17, 18]. In this survey based on our works [15, 16], we will do the exploration in an opposite direction; we want to solve strongly SDE with coefficients having Sobolev regularity.

In 1989, Di Perna and Lions [12] solved the ODE with coefficient in Sobolev space on $\mathbf{R}^d$:

$$\frac{dX_t(x)}{dt} = V_t(X_t(x)), \tag{6}$$

where $X_t : \mathbf{R}^d \to \mathbf{R}^d$ is a family of measurable maps, such that $(X_t)_*(dx)$ has a density, and V_t has to be supposed to have bounded divergence. A useful tool for solving (6) is the transport equation in distribution sense

$$\frac{du_t}{dt} + V_t \cdot \nabla u_t = 0. \tag{7}$$

In order to prove the well-posedness of (7), they introduced the notion of *renormalized solution*: if u is a solution to (7), then for any $\beta \in C_b^1(\mathbf{R})$, $\beta(u)$ still is a solution to (7), but with a different initial data.

Let's see this concept for SDE. Consider the Stratanovich SDE

$$dX_t = \sum_{i=1}^m A_i(X_t) \circ dw_t^i + A_0(X_t)\,dt,$$

where $t \to w_t = (w_t^1, \ldots, w_t^m)$ is the standard Brownian motion. When the coefficients are good enough, the above SDE defines a flow of homeomorphisms X_t. Let $\theta(t,x) = \theta_0(X_t^{-1}(x))$; then θ satisfies the stochastic transport equations

$$d_t\theta + \sum_{i=1}^m (A_i \cdot \nabla\theta) \circ dw_t^i + (A_0 \cdot \nabla\theta)\,dt = 0,$$

or in Itô form

$$d_t\theta = -\sum_{i=1}^m (A_i \cdot \nabla\theta) dw_t^i - (A_0 \cdot \nabla\theta)\,dt - \frac{1}{2}\sum_{i=1}^m \mathcal{L}_{A_i}^2 \theta\,dt, \tag{8}$$

where $\mathcal{L}_{A_i}$ denotes the derivative with respect to A_i. By Itô formula, it is easy to see that (8) is not stable under the left action by β; this means that $\beta(\theta)$ is no more a solution to (8).

2 Flat Case $\mathbf{R}^d$

To simplify things, we consider the standard Gaussian measure γ_d on $\mathbf{R}^d$ as the reference measure. We say that a measurable map $X\colon \Omega \times \mathbf{R}^d \to C([0,T], \mathbf{R}^d)$ is a solution to the Itô SDE

$$dX_t = \sum_{i=1}^m A_i(X_t)\,dw_t^i + A_0(X_t)\,dt, \quad X_0 = x, \tag{9}$$

if

1. For each $t \in [0, T]$ and almost all $x \in \mathbf{R}^d$, $w \to X_t(w, x)$ is measurable with respect to $\mathcal{F}_t$, i.e., the natural filtration generated by the Brownian motion $\{w_s : s \le t\}$.
2. For each $t \in [0, T]$, there exists $K_t \in L^1(\Omega \times \mathbf{R}^d)$ such that the push forward measure $(X_t(w, \cdot))_{\#}\gamma_d$ admits K_t as the density with respect to γ_d.
3. Almost surely

$$\sum_{i=1}^m \int_0^T |A_i(X_s(w,x))|^2\, ds + \int_0^T |A_0(X_s(w,x))|\, ds < +\infty.$$

4. For almost all $x \in \mathbf{R}^d$,

$$X_t(w,x) = x + \sum_{i=1}^m \int_0^t A_i(X_s(w,x))\, dw_s^i + \int_0^t A_0(X_s(w,x))\, ds.$$

5. The flow property holds

$$X_{t+s}(w,x) = X_t(\theta_s w, X_s(w,x)),$$

where θ_s denotes the shift operator: $(\theta_s w)(t) = w(t+s) - w(s)$.

For a vector field A on $\mathbf{R}^d$ and $p > 1$, we say that $A \in \mathbf{D}_1^p(\gamma_d)$ if $A \in L^p(\gamma_d)$ and there exists $\nabla A : \mathbf{R}^d \to \mathbf{R}^d \otimes \mathbf{R}^d$ in $L^p(\gamma_d)$ such that for any $v \in \mathbf{R}^d$

$$\nabla A(x)(v) = \lim_{\eta \to 0} \frac{A(x+\eta v) - A(x)}{\eta} \quad \text{holds in } L^{p-},$$

where L^{p-} means that the above convergence holds on all $L^q(\gamma_d)$ for $q < p$.

For such a vector field $A \in \mathbf{D}_1^p(\gamma_d)$, the divergence $\mathrm{div}_\gamma(A)$ with respect to the Gaussian measure γ_d exists in $L^p(\gamma_d)$ (see [14]) in the sense that

$$\int_{\mathbf{R}^d} \langle \nabla f, A\rangle\, d\gamma_d = -\int_{\mathbf{R}^d} f\, \mathrm{div}_\gamma(A)\, d\gamma_d, \quad \text{for any } f \in C_b^1(\mathbf{R}^d).$$

We have $\mathrm{div}_\gamma(A) = \sum_{i=1}^d (\partial A^i/\partial x_i - x_i A^i)$.

Theorem 1 (see [15]). *Assume that the diffusion coefficients $A_1, \ldots, A_m$ belong to the Sobolev space $\bigcap_{q>1} \mathbf{D}_1^q(\gamma_d)$ and the drift $A_0 \in \mathbf{D}_1^q(\gamma_d)$ for some $q > 1$. Assuming for a small λ_0 that*

$$\int_{\mathbf{R}^d} \exp\left[\lambda_0\Big(|\mathrm{div}_\gamma(A_0)| + \sum_{j=1}^m \big(|\mathrm{div}_\gamma(A_j)|^2 + |\nabla A_j|^2\big)\Big)\right] d\gamma_d < +\infty, \qquad (10)$$

and that the coefficients $A_0, A_1, \dots, A_m$ are of linear growth, then there is a unique stochastic flow of measurable maps $X\colon [0,T]\times\Omega\times\mathbf{R}^d \to \mathbf{R}^d$, which solves (9) for almost all initial $x\in\mathbf{R}^d$, and the push forward $(X_t(w,\cdot))_{\#}\gamma_d$ admits a density with respect to γ_d, which is in $L^1\log L^1$.

An immediate consequence of Theorem 1 is the following:

Theorem 2. *Let $A_0, A_1, \dots, A_m$ be globally Lipschitz continuous. Suppose that there exists a constant $C>0$ such that*

$$\sum_{j=1}^m \langle x, A_j(x)\rangle^2 \le C\,(1+|x|^2) \quad \textit{for all } x\in\mathbf{R}^d. \tag{11}$$

Then the stochastic flow of homeomorphisms X_t generated by SDE *(9) leaves the Lebesgue measure quasi-invariant.*

It is clear that the condition (11) implies (10). Note that A. Y. Pilipenko recently told me that the condition (11) could be removed using an early result by Bouleau and Hirsch [7], which says that $x\to X_t(x)$ is in $W_1^{p,loc}$, together with a generalized formula of change of variable. We refer to [5] for a new development.

For proving Theorem 1, we need two ingredients that we will explain in what follows.

In the case of ODE where V is smooth, the push forward measure $(X_t)_\sharp\gamma_d$ admits the density K_t with respect to γ_d and

$$K_t(x) = \exp\Big(\int_0^t -\mathrm{div}_\gamma(V)(X_{-s}(x))ds\Big),$$

and the Cruzeiro's estimate [11] in $L^p(\gamma_d)$ for $p>1$

$$||K_t||_{L^p}^p \le \int_{\mathbf{R}^d} \exp\Big(\frac{p^2 t}{p-1}|\mathrm{div}_\gamma(V)|\Big)\,d\gamma_d$$

holds. In the case of SDE (9), we obtained the following:

Theorem 3 ([15]). *Let $K_t(w,x)$ be the density of $(X_t)_\sharp\gamma_d$ with respect to γ_d. Then for $p>1$, we have*

$$\|K_t\|_{L^p(\mathbf{P}\times\gamma_d)} \le \Bigg[\int_{\mathbf{R}^d} \exp\Bigg(pt\Big[2|\mathrm{div}_\gamma(A_0)| + \sum_{j=1}^m \big(|A_j|^2+|\nabla A_j|^2+2(p-1)|\mathrm{div}_\gamma(A_j)|^2\big)\Big]\Bigg)d\gamma_d\Bigg]^{\frac{p-1}{p(2p-1)}}. \tag{12}$$

We see that the condition (10) in Theorem 1 comes from (12). The second ingredient we need is a method developed by Crippa–De Lellis [10] and Xicheng Zhang [19].

The absence of Lipschitz condition was filled by the following inequality: for $f \in W^{1,1}_{loc}(\mathbf{R}^d)$,

$$|f(x) - f(y)| \le C_d\, |x-y|\, \big(M_R|\nabla f|(x) + M_R|\nabla f|(y)\big)$$

holds for $x, y \in N^c$ and $|x-y| \le R$, where N is a negligible set of $\mathbf{R}^d$ and $M_R g$ is the local maximal function defined by

$$M_R g(x) = \sup_{0<r\le R} \frac{1}{\mathrm{Leb}_d(B(x,r))} \int_{B(x,r)} |g(y)|\, dy,$$

where $B(x,r)$ denotes the ball of center x and of radius r and Leb_d denotes the Lebesgue measure (see [15] for a complete proof); the classical moment estimate is replaced by estimating the quantity

$$\int_{B(0,r)} \log\left(\frac{|X_t(x) - \tilde{X}_t(x)|}{\sigma} + 1\right) dx,$$

where $\sigma > 0$ is a small parameter. Here is a basic estimate:

Lemma 1 ([15]). *Let $q > 1$. Suppose that $A_1, \ldots, A_m$ as well as $\hat{A}_1, \ldots, \hat{A}_m$ are in $\mathbf{D}_1^{2q}(\gamma_d)$ and $A_0, \hat{A}_0 \in \mathbf{D}_1^q(\gamma_d)$. Then, for any $T > 0$ and $R > 0$, there exist constants $C_{d,q,R} > 0$ and $C_T > 0$ such that for any $\sigma > 0$,*

$$\begin{aligned} &\mathbf{E}\Big[\int_{G_R} \log\Big(\frac{\sup_{0\le t\le T} |X_t - \hat{X}_t|^2}{\sigma^2} + 1\Big) d\gamma_d\Big] \\ &\le C_T \Lambda_{p,T} \Big\{ C_{d,q,R}\Big[\|\nabla A_0\|_{L^q} + \sum_{i=1}^m \|\nabla A_i\|^2_{L^{2q}}\Big] \\ &\quad + \frac{1}{\sigma^2}\sum_{i=1}^m \|A_i - \hat{A}_i\|^2_{L^{2q}} + \frac{1}{\sigma}\Big[\|A_0 - \hat{A}_0\|_{L^q}\Big]\Big\}, \end{aligned}$$

where p is the conjugate number of q: $1/p + 1/q = 1$, and

$$G_R(w) = \Big\{ x \in \mathbf{R}^d;\ \sup_{0\le t\le T} |X_t(w,x)| \vee |\hat{X}_t(w,x)| \le R \Big\}.$$

Proof of Theorem 1. Let A_j^n be a suitable regularization of A_j for $j = 0, 1, m$. Let X^n be the solution associated with A_j^n. Let

$$\sigma_{n,k} = ||A_0^n - A_0^k||_{L^q} + \left(\sum_{i=1}^m ||A_i^n - A_i^k||^2_{L^{2q}}\right)^{1/2},$$

and $G_{n,R} = \{x \in \mathbf{R}^d;\ \sup_{0\le t\le T} |X_t^n(w,x)| \le R\}$. By Lemma 1,

$$I_{n,k} := \mathbf{E}\Big[\int_{G_{n,R}\cap G_{k,R}} \log\Big(\frac{\|X^n - X^k\|^2_{\infty,T_0}}{\sigma^2_{n,k}} + 1\Big)\, d\gamma_d\Big]$$

is bounded with respect to n, k, where $||\cdot||_{\infty,T_0}$ denotes the uniform norm over $[0, T_0]$; from this fact, we deduce that $\{X^n; n \ge 1\}$ is a Cauchy sequence in L^0. Combined with boundedness in all L^p, we get the convergence of X^n to X in L^p.

□

Note that the above method does not work if $\mathbf{R}^d$ is replaced by a Riemannian manifold M.

3 Non-flat Case

Let M be a complete Riemannian manifold and Δ_M the Laplace operator on it. For a C^1 vector field Z on M, the diffusion process X_t on M of generator

$$\frac{1}{2}\Delta_M + Z$$

can be constructed intrinsically through the bundle of orthonormal frames $\pi : O(M) \to M$. More precisely, let $H_1, \ldots, H_d$ be canonical horizontal vector fields on M, that is, $\pi'(r)H_i(r) = r \cdot \varepsilon_i$, where $\{\varepsilon_1, \ldots, \varepsilon_d\}$ is the canonical basis of $\mathbf{R}^d$ and $r \in O(M)$ is an isometry $\mathbf{R}^d \to T_{\pi(r)}M$. We lift Z to a horizontal vector field $\tilde{Z}$ on $O(M)$:

$$\pi'(r)\tilde{Z}(r) = Z_{\pi(r)}.$$

Then we can consider the SDE on $O(M)$:

$$dr_t(w) = \sum_{i=1}^{d} H_i(r_t(w)) \circ dw_t^i + \tilde{Z}(r_t(w))dt, \quad r_0(w) = r_0 \tag{13}$$

which has a unique strong solution, up to the lifetime $\tau(w)$. Then the process

$$X_t(w) = \pi(r_t(w))$$

admits the generator $\frac{1}{2}\Delta_M + Z$.

This construction is well known, due to Eells-Elworthy, Malliavin in the year 1976. Now the question is how to construct such a process when Z in (13) is in a Sobolev space

$$Z \in \mathbf{D}_1^q?$$

where the Sobolev space is defined with respect to the Riemannian measure dx on M.

Note that the essential step to prove the well-posedness of the transport equation (7) on the Euclidean space $\mathbf{R}^d$

$$\frac{du_t}{dt} + V_t \cdot \nabla u_t = 0.$$

is to establish the following type of estimate:

$$||c_n(f, Z)||_{L^1} \leq C\, ||f||_{L^p}\, (||\nabla Z||_{L^q} + ||\mathrm{div}(Z)||_{L^q}), \tag{14}$$

where $c_n(f, Z) = (D_Z f) * \chi_n - D_Z(f * \chi_n)$, and χ_n is a sequence of smoothing functions.

For a function f on a Riemannian manifold M, it is natural to use the heat semigroup $T_t^M f$ to regularize it.

For the sake of simplicity, we will assume that M is compact. Consider the SDE on $O(M)$:

$$dr_t(w) = \sum_{i=1}^{d} H_i(r_t(w)) \circ dw_t^i.$$

Then the heat semigroup admits the representation:

$$T_t^M f(x) = \mathbf{E}(f(\pi(r_t))) = \mathbf{E}(f(x_t)), \quad \text{where } x_t = \pi(r_t).$$

Let V be a smooth vector field on M; for a function $f \in C^1(M)$, we set

$$c_t(f, V) = \mathcal{L}_V(T_t^M f) - T_t^M(\mathcal{L}_V f),$$

$\mathcal{L}_V$ denotes the Lie derivative along V.

In the sequel, the spaces L^p are defined with respect to the Riemannian measure and ∇V denotes the Riemannian covariant derivative of V and $\mathrm{div}(V)$ denotes the divergence of V with respect to the Riemannian metric.

In contrast to (14), we have the following result.

Theorem 4 ([16]). *We have*

$$||c_t(f, V)||_{L^1} \leq C\Big(||f||_{L^{2p}}||\nabla V||_{L^q} + \sqrt{t}||f||_{L^{2p}}||V||_{L^q} + ||f||_{L^p}||\mathrm{div}(V)||_{L^q}\Big). \tag{15}$$

To prove the result (15), we have used in [16] two representation formula: *Bismut formula* for backward derivative [6] and *Driver formula* for forward derivative [13] in stochastic differential geometry.

Now we will develop Di Perna–Lions method on manifolds: Let $V_t \in \mathbf{D}_1^q(M, TM)$ be a time-dependent vector field; consider the transport equation on M

$$\frac{du_t}{dt} + V_t \cdot \nabla u_t + \xi_t u_t = 0, \quad u|_{t=0} = u_0, \tag{16}$$

where $\xi : [0, T] \times M \to \mathbf{R}$ is a measurable function.

Theorem 5 ([16]). *Let* $V \in L^1([0, T], \mathbf{D}_1^q)$ *and assume* $\mathrm{div}(V), \xi \in L^1([0, T], L^\infty(M))$. *Then for any* $u_0 \in L^p \cap L^{2p}$, *(16) admits a unique solution* $u \in L^\infty([0, T], L^p \cap L^{2p})$.

Now we are going to solve the ODE

$$\frac{dX_t}{dt} = V_t(X_t).$$

We also have to regularize vector fields. To this end, we use de Rham–Hodge semigroup $T_\varepsilon^1 = e^{-\varepsilon\Delta^1}$ on differential forms, where $\Delta^1 = dd^* + d^*d$. Let ω be a differential form and V a vector field. We define the vector field $\omega^\sharp$ and the differential form V^* by

$$\langle \omega_x, V_x \rangle = \langle \omega^\sharp, V \rangle_{T_xM} = \langle \omega, V^* \rangle_{T_xM^*}.$$

We define

$$T_\varepsilon^1 V = (T_\varepsilon V^*)^\sharp.$$

In the case of compact manifold, the semigroup T_ε^1 has a good behavior. Let $-c$ be the lower bound of the Ricci tensor, that is,

$$\mathrm{Ric}_x \geq -c\,\mathrm{Id}.$$

Then it holds that

$$|T_\varepsilon^1 V| \leq e^{\varepsilon c} T_\varepsilon^M |V|, \ \mathrm{div}(T_\varepsilon^1 V) = T_\varepsilon^M(\mathrm{div}(V)). \tag{17}$$

Now consider

$$V_t^\varepsilon = T_\varepsilon^1 V_t.$$

Then for each $\varepsilon > 0$,

$$\int_0^T \|\mathrm{div}(V_t^\varepsilon)\|_\infty \, dt \leq \int_0^T \|\mathrm{div}(V_t)\|_\infty \, dt,$$

and for $0 \le \varepsilon \le 1$,

$$\int_0^T \int_M |V_t^\varepsilon(x)|^q \, dt dx \le e^{cq} \int_0^T \int_M |V_t(x)|^q \, dt dx.$$

Therefore, according to (17), the condition

$$\int_0^T \|\mathrm{div}(V_t)\|_\infty \, dt + \int_0^T \int_M |V_t(x)|^q \, dt dx < +\infty \tag{18}$$

implies the nonexplosion of following ODE associated with V_t^ε:

$$\frac{dX_t^\varepsilon}{dt} = V_t^\varepsilon(X_t^\varepsilon), \quad X_0^\varepsilon = x.$$

And the density k_t^ε of $(X_t^\varepsilon)_\#(dx)$ with respect to dx is bounded:

$$e^{-\int_0^t ||\mathrm{div}(V_s)||_\infty ds} \le k_t^\varepsilon \le e^{\int_0^t ||\mathrm{div}(V_s)||_\infty ds}. \tag{19}$$

Fix $T > 0$. Let η_ε be the push forward measure on the path space

$$W(M) = C([0, T], M)$$

of dx by the map $x \to X_\cdot^\varepsilon(x)$. Then the family of finite measures $\{\eta_\varepsilon;\ \varepsilon > 0\}$ on $W(M)$ is tight if $q > 2$. Let η be a limit point.

Let $e_t : W(M) \to M$ be the evaluation map : $e_t(\gamma) = \gamma(t)$ and $\nu_t = (e_t)_\#\eta$. Then

$$\int_M f \, d\nu_t = \lim_{n\to+\infty} \int_M f(X_t^{\varepsilon_n}(x)) \, dx.$$

So the density k_t of ν_t with respect to dx exists and by (19),

$$k_t \le e^{\int_0^T ||\mathrm{div}(V_s)||_\infty ds}.$$

Under the probability η on $W(M)$, we have

$$\frac{d\gamma(t)}{dt} = V_t(\gamma(t)), \quad \text{for a.e. } t \in [0, T]. \tag{20}$$

Now let η be any finite measure on $W(M)$ such that for η-a.s $\gamma \in W(M)$, (20) holds. If $k_t^\eta = d(e_t)_\#\eta)/dx$ exists, then k_t^η solves the transport equation

$$\frac{du_t}{dt} + V_t \cdot \nabla u_t + \mathrm{div}(V_t) u_t = 0. \tag{21}$$

Theorem 6. *Let V be a vector field in $L^1([0,T],\mathbf{D}_1^q)$ with $q>2$. Assume that $\int_0^T ||\mathrm{div}(V_t)||_\infty\, dt < +\infty$. Then there exists a unique flow of measurable maps $X_t : M \to M$ such that $\frac{dX_t}{dt} = V_t(X_t)$ and $(X_t)_\#(dx)$ admits a density.*

Proof. Let $d\eta = d\eta_x\, dx$. Then η_x is concentrated on integral curves γ such that $\gamma(0) = x$. The idea, due to Ambrosio [1], is to prove that η_x is supported by a graph $x \to X.(x)$, that is,

$$\eta_x = \delta_{X.(x)}.$$

Otherwise, we could construct two different solutions to the transport equation (21), with the same initial value, which is in contradiction with Theorem 5 . □

Now we will construct the diffusion process associated with $\frac{1}{2}\Delta_M + V$ with $V \in \mathbf{D}_1^q$ and with bounded divergence.

We first lift V to a horizontal vector field $\tilde{V}$ on $O(M)$ and consider the time-dependent random vector fields:

$$\tilde{V}_t(w,\cdot) = (U_t^{-1}(w,\cdot))_*\tilde{V},$$

where $U_t(w,r)$ is the stochastic flow of diffeomorphisms on $O(M)$ associated with

$$dr_t(w) = \sum_{i=1}^{d} H_i(r_t(w)) \circ dw_t^i.$$

Then we solve $\dfrac{dY_t}{dt} = \tilde{V}_t(w, Y_t)$ (see below). The projection on M of $Z_t = U_t(Y_t)$ gives the desired process.

In order to lift V to a horizontal vector field $\tilde{V}$ of Sobolev regularity, we have to choose a metric on $O(M)$. Let (θ,ω) be the 1- differential form taking values in $\mathbf{R}^d \oplus \mathrm{so}(d)$, called the parallelism.

Define, for a vector field A on $O(M)$,

$$|A(r)|^2_{T_r O(M)} = |\theta(A)|^2_{\mathbf{R}^d} + |\omega(A)|^2_{\mathrm{so}(d)}. \tag{22}$$

The divergence with respect to this metric (22) is identical to the one defined by

$$\int_{O(M)} \Phi\, \mathrm{div}(A)\, dr = -\int_{O(M)} \langle \nabla\Phi(r), A(r)\rangle_{T_r O(M)}\, dr, \tag{23}$$

where dr is the Liouville measure on $O(M)$. Then for a C^1 vector field V on M, we have

$$\mathrm{div}(\tilde{V}) = \mathrm{div}(V)\circ\pi \quad \text{and} \quad ||\tilde{V}||_{\mathbf{D}_1^q} \le C\, ||V||_{\mathbf{D}_1^q}.$$

Using (23) and the fact that U_t preserves the measure dr, we have

$$\operatorname{div}(\tilde{V}_t(w,\cdot)) = \operatorname{div}(\tilde{V})(U_t(w,\cdot)).$$

So that

$$\int_0^T ||\operatorname{div}(\tilde{V}_t)||_\infty dt \le T\,||\operatorname{div}(V)||_\infty.$$

Finally, we have

Theorem 7 ([16]). *Let M be a compact Riemannian manifold. Then for a vector field $V \in \mathbf{D}_1^q$ with $q > 2$ and $||\operatorname{div}(V)||_\infty < +\infty$, we have*

$$\int_{[0,T]\times O(M)} \Big[|\tilde{V}_t|^q + |\nabla^{O(M)}\tilde{V}_t|^q\Big]\,dt\,dr < +\infty, \tag{24}$$

and the SDE on $O(M)$

$$dZ_t = \sum_{i=1}^d H_i(Z_t)\circ dw^i(t) + \tilde{V}(Z_t)dt \tag{25}$$

has a strong solution, and $X_t = \pi(Z_t)$ has $\frac{1}{2}\Delta_M + V$ as generator.

Proof. Note that by (24), the condition (18) is satisfied for vector fields $\tilde{V}_t$ on $O(M)$. By Theorem 6, the differential equation $\dfrac{dY_t}{dt} = \tilde{V}_t(w, Y_t)$ admits a unique solution of measurable maps. So $Z_t = U_t(Y_t)$ solves (25). □

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References

1. Ambrosio, L.: Transport equation and Cauchy problem for BV vector fields. Invent. Math. **158**, 227–260 (2004)
2. Ambrosio, L., Figalli, A.: Geodesics in the space of measure-preserving maps and plans. Arch. Rat. Mech. Anal. **194**, 421–462 (2009)
3. Antoniouk, A., Arnaudon, M., Cruzeiro, A.B.: Generalized stochastic flows and applications to incompressible viscous fluids. Bulletin des Sciences Mathématiques, (2013 in Press)
4. Arnaudon, M., Cruzeiro, A.B.: Stochastic Lagrangian flows and Navier–Stokes equations (2012, preprint)
5. Aryasova, O.V., Pilipenko, A.Y.: On properties of a flow generated by a SDE with discontinuous drift. Electron. J. Probab. **17**(106), 1–20 (2012)

6. Bismut, J.M.: Large Deviation and Malliavin Calculus. Progress in Mathematics, vol. 45. Birkhäuser, Boston (1984)
7. Bouleau, N., Hirsch, F.: Dirichlet Forms and Analysis on Wiener Space. De Gruyter Studies in Mathematics, vol. 14. De Gruyter, Berlin (1991)
8. Brenier, Y.: The least action principle and the related concept of generalized flows for incompressible perfect fluids. J. Am. Math. Soc. **2**, 225–255 (1989)
9. Chemin, J.Y.: Fluides Parfaits Incompressibles. Astérisque, vol. 230. Société Mathématique de France, Paris (1995)
10. Crippa, G., De Lellis, C.: Estimates and regularity results for the Di Perna-Lions flows. J. Reine Angew. Math. **616**, 183–201 (2008)
11. Cruzeiro, A.B.: Equations différentielles sur l'espace de Wiener et formules de cameron-Martin non linéaires. J. Funct. Anal. **54**, 206–227 (1983)
12. Di Perna, R.J., Lions, P.L.: Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math. **98**, 511–547 (1989)
13. Driver, B.: Integration by parts for heat kernel measures revisited. J. Math. Pures Appl. **76**, 703–737 (1997)
14. Fang, S.: Introduction to Malliavin Calculus. Mathematical Series for Graduate Students, vol. 3. Tsinghua University Press, Springer, Beijing (2005)
15. Fang, S., Luo, D., Thalmarier, A.: Stochastic differential equations with coefficients in Sobolev spaces. J. Funct. Anal. **259**, 1129–1168 (2010)
16. Fang, S., Li, H., Luo, D.: Heat semi-group and generalized flows on complete Riemannian manifolds. Bull. Sci. Math. **135**, 565–600 (2011)
17. Le Jan, Y., Raimond, O.: Integration of Brownian vector fields. Ann. Probab. **30**, 826–873 (2002)
18. Le Jan, Y., Raimond, O.: Flows, coalescence and noise. Ann. Probab. **32**, 1247–1315 (2004)
19. Zhang, X.: Stochastic flows of SDEs with irregular coefficients and stochastic transport equations. Bull. Sci. Math. **134**, 340–378 (2010)

Elementary Pathwise Methods for Nonlinear Parabolic and Transport Type Stochastic Partial Differential Equations with Fractal Noise

Michael Hinz, Elena Issoglio, and Martina Zähle

Abstract We survey some of our recent results on existence, uniqueness, and regularity of function solutions to parabolic and transport type partial differential equations driven by non-differentiable noises. When applied pathwise to random situations, they provide corresponding statements for stochastic partial differential equations driven by fractional noises of sufficiently high regularity order. The approach is based on semigroup theory.

1 Introduction

In this survey we list several of our recent results on existence, uniqueness, and regularity of function solutions to linear and nonlinear parabolic stochastic partial differential equations such as abstract stochastic heat equations [16–18], stochastic transport-diffusion equations [23], and stochastic Burgers system [15]. Our approach combines semigroup theory [30, 39, 42] and fractional calculus [14, 31, 36]. This leads to an elementary and easily accessible formulation in the sense that more sophisticated techniques such as rough path theory [11, 25, 26]

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are avoided, and we obtain explicit formulas in terms of the semigroup. The basic idea of the studies surveyed here was to formulate a framework for stochastic partial differential equations using analogs of the pathwise techniques [24, 44–46] previously employed by the third named author to solve stochastic differential equations.

General information on stochastic partial differential equations can be found in [9, 19, 32, 41], results close to our discussion of parabolic equations are for instance [12, 20, 27, 38]. A transport type equation was investigated in [34], and some results on stochastic Burgers equation can be found in [3, 7, 8, 13]. Of course there are many more highly valuable references on these topics.

Classical stochastic calculus allows to integrate predictable processes against semimartingale integrators. In particular it can be used to study stochastic differential equations with respect to a given semimartingale process. From a practical viewpoint both semimartingale properties of the integrator and predictability of the integrand may be too restrictive in some situations. If the integrator is Gaussian, we may use methods from Malliavin calculus to define stochastic integrals for nonanticipating integrands [28]. Alternatively, if almost surely both the integrand and the integrator are of sufficiently high regularity (for instance in the sense of Hölder continuity, p-variation or fractional differentiability), then this regularity can be used to define stochastic integrals in a pathwise sense. In this case they are of Stieltjes type. By now the most popular approach to this idea probably is Young integration [43], which later inspired the development of rough path theory [25, 26]. Another way to go, although not entirely pathwise, is to use stochastic calculus via regularization [35]. Yet another technique was introduced in [44, 45] and is based on fractional calculus.

Let $I = (a, b)$ be a bounded interval and E be a Banach space. Given $\eta > 0$ and a function $\varphi \in L_1((a, b), E)$, consider the (forward and backward) *Riemann–Liouville fractional integrals of order* η by

$$\mathrm{I}^{\eta}_{a+}\varphi(t) := \frac{1}{\Gamma(\eta)} \int_a^t \frac{\varphi(\tau)}{(t-\tau)^{1-\eta}} \mathrm{d}\tau$$

and

$$\mathrm{I}^{\eta}_{b-}\varphi(t) := \frac{(-1)^{-\eta}}{\Gamma(\eta)} \int_t^b \frac{\varphi(\tau)}{(\tau-t)^{1-\eta}} \mathrm{d}\tau \ .$$

Here for $\eta > 0$ the powers are understood as usual in the sense of choosing the main branch of the analytic function ζ^{η}, $\zeta \in \mathbb{C}$, with the cut along the positive half axis, in particular, $(-1)^{\eta} = \mathrm{e}^{i\eta\pi}$. Here and in the following, the integrals are understood in the Bochner sense. Let $\mathrm{I}^{\eta}_{a+}(L_p((a, b), E))$ denote the space of functions $f = \mathrm{I}^{\eta}_{a+}\varphi$ with $\varphi \in L_p((a, b), E)$, similarly $\mathrm{I}^{\eta}_{b-}(L_p((a, b), E))$. For $0 < \eta < 1$ and functions $f \in \mathrm{I}^{\eta}_{a+}(L_p((a, b), E))$, respectively $f \in \mathrm{I}^{\eta}_{b-}(L_p((a, b), E))$, consider the *left-sided Weyl–Marchaud fractional derivatives of order* η,

$$\mathrm{D}_{a+}^{\eta} f(t) := \frac{\mathbf{1}_{(a,b)}(t)}{\Gamma(1-\eta)} \left(\frac{f(t)}{(t-a)^{\eta}} + \eta \int_a^t \frac{f(t)-f(\tau)}{(t-\tau)^{\eta+1}} \mathrm{d}\tau \right)$$

and the *right-sided Weyl–Marchaud fractional derivatives of order* η,

$$\mathrm{D}_{b-}^{\eta} f(t) := \frac{(-1)^{\eta}\mathbf{1}_{(a,b)}(t)}{\Gamma(1-\eta)} \left(\frac{f(t)}{(b-t)^{\eta}} + \eta \int_t^b \frac{f(t)-f(\tau)}{(\tau-t)^{\eta+1}} \mathrm{d}\tau \right),$$

the convergence of the principal values being pointwise almost everywhere if $p = 1$ and in $L_p((a,b),E)$ if $p > 1$. Under these assumptions $\mathrm{I}_{a+}^{\eta}\mathrm{D}_{a+}^{\eta} f = f$ in $L_p((a,b),E)$, while $\mathrm{D}_{a+}^{\eta}\mathrm{I}_{a+}^{\eta}\varphi = \varphi$ is true for any $\varphi \in L_1((a,b),E)$. In the case $\eta = 1$ set $\mathrm{D}_{a+}^{1} f = \mathrm{d}f/\mathrm{d}t$ and $\mathrm{D}_{b-}^{1} f = -\mathrm{d}f/\mathrm{d}t$ and in the case $\eta = 0$, define D_{a+}^{0} and D_{b-}^{0} to be the identity. See for instance [16, 36].

For a moment assume that $E = \mathbb{R}$ and consider real-valued functions f and g on (a,b) such that the limits $f(a+)$, $g(a+)$ and $g(b-)$ exist. Consider the regulated functions

$$f_{a+}(t) := \mathbf{1}_{(a,b)}(t)(f(t)-f(a+)) \quad \text{and} \quad g_{b-}(t) := \mathbf{1}_{(a,b)}(t)(g(t)-g(b-)) \; . \tag{1}$$

In [44] it had been shown that if $f_{a+} \in \mathrm{I}_{a+}^{\eta}(L_p(a,b))$ and $g_{b-} \in \mathrm{I}_{b-}^{1-\eta}(L_q(a,b))$ for some $1/p + 1/q \le 1$ and $0 \le \eta \le 1$, then the integral

$$\int_a^b f(s)\mathrm{d}g(s) := (-1)^{\eta} \int_a^b \mathrm{D}_{a+}^{\eta} f_{a+}(s)\mathrm{D}_{b-}^{1-\eta} g_{b-}(s)\mathrm{d}s + f(a+)(g(b-)-g(a+)) \tag{2}$$

is well defined, that is, the value of the right-hand side in (2) is a real number that is independent of the particular choice of η. Moreover, if f and g are sufficiently regular such that both (2) and the Lebesgue–Stieltjes integral $(LS)\int_a^b f\mathrm{d}g$ exist, then they agree. For instance, if f and g are Hölder continuous and the sum of their Hölder orders is greater than one, then (2) exists and equals the Riemann–Stieltjes integral $(RS)\int_a^b f\mathrm{d}g$. If f and g satisfy the above conditions with $0 \le \eta < 1/p$, then the correction terms in (2) may be dropped; more precisely, we have

$$\int_a^b f(s)\mathrm{d}g(s) = (-1)^{\eta} \int_a^b \mathrm{D}_{a+}^{\eta} f(s)\mathrm{D}_{b-}^{1-\eta} g_{b-}(s)\mathrm{d}s \; .$$

See [44, 45] for details. Integrals of type (2) may for instance be used to investigate differential equations of the form

$$\begin{cases} \mathrm{d}x(t) = a(x(t),t)\mathrm{d}z(t) + b(x(t),t)\mathrm{d}t \\ x(0) = x_0 \; , \end{cases} \tag{3}$$

where z is a non-differentiable function that is Hölder continuous of order greater than $1/2$ and a and b are coefficients that satisfy certain growth and smoothness assumptions. Equation (3) is made precise by saying that $x = (x(t))_{t\geq 0}$ solves (3) if

$$x(t) = x_0 + \int_0^t a(x(s), s)\mathrm{d}z(s) + \int_0^t b(x(s), s)\mathrm{d}s$$

for any $t > 0$, where the first integral is defined as in (2). As usual, the existence and uniqueness of solutions x to (3) is obtained by combining a priori estimates on the integral operator $x \mapsto \int_0^{\cdot} a(x(s), s)\mathrm{d}z(s)$ in suitable function spaces and fixed point arguments [24, 45]. If typical realizations of suitable random processes are used in place of z, such as for instance the paths of a fractional Brownian motion B^H with Hurst parameter $H > 1/2$, this yields a stochastic differential equation in the pathwise sense.

Equation (3) is an evolution problem subject to a perturbation z. Also parabolic partial differential equations of the form

$$\begin{cases} \frac{\partial u}{\partial t}(t) = -\mathrm{A}u(t) \\ u(0) = u_0 \end{cases}$$

are commonly viewed as evolution problems, now of course in abstract (Banach or Hilbert) spaces [30], and their behavior is completely governed by a related semigroup $(\mathrm{T}(t))_{t\geq 0}$ of evolution operators, that is, the solution u to the Cauchy problem will be of the form $u(t) = \mathrm{T}(t)u_0$, $t > 0$. We will use an analog of (2) to incorporate a noise signal z into the equation. A simple linear multiplicative perturbation would for instance lead to a Cauchy problem of the form

$$\begin{cases} \frac{\partial u}{\partial t}(t) = -\mathrm{A}u(t) + u(t) \cdot \dot{z}(t) \\ u(0) = u_0 \ . \end{cases}$$

If the noise z is random, this yields again a pathwise technique, now for stochastic partial differential equations. It allows to investigate problems perturbed by signals that lack semimartingale properties but have sufficiently high regularity in terms of Hölder and Sobolev norms.

To consider a version of (2) for vector-valued functions, let E and F be separable Banach spaces and let $L(E, F)$ denote the space of bounded linear operators from E into F. Given $0 \leq \eta \leq 1$, an E-valued function z on (a, b) and an $L(E, F)$-valued function U on (a, b) such that $\mathrm{D}_{b-}^{1-\eta} z_{b-} \in L_\infty((a, b), E)$ and $\mathrm{D}_{a+}^{\eta} U \in L_1((a, b), L(E, F))$, the integral

$$\int_a^b U(s)\mathrm{d}z(s) := (-1)^\eta \int_a^b \mathrm{D}_{a+}^{\eta} U(s) \mathrm{D}_{b-}^{1-\eta} z_{b-}(s)\mathrm{d}s \tag{4}$$

is well defined. More precisely, the right-hand side of (4) is an element of F and does not depend on the particular choice of η. The notation z_{b-} is to be understood as in (1).

2 Semigroups and Function Spaces

Let $(X, \mathcal{X}, \mu)$ be a σ-finite measure space and let $L_p(\mu)$, $1 < p < \infty$ and $L_\infty(\mu)$ denote the spaces of (equivalence classes of) p-integrable and essentially bounded functions on X, respectively.

We assume that $\mathrm{T} = (\mathrm{T}(t))_{t\geq 0}$ is a symmetric strongly continuous semigroup on $L_2(\mu)$, that is, $\mathrm{T}(t+s) = \mathrm{T}(t)\circ\mathrm{T}(s)$, $\mathrm{T}(0)u = u$, and $\langle \mathrm{T}(t)u, v\rangle_{L_2(\mu)} = \langle u, \mathrm{T}(t)v\rangle_{L_2(\mu)}$ for any $s,t \geq 0$ and any $u, v \in L_2(\mu)$ and $\lim_{t\to 0}\|\mathrm{T}(t)u - u\|_{L_2(\mu)} = 0$ for any $u \in L_2(\mu)$. We further assume that $(\mathrm{T}(t))_{t\geq 0}$ is Markovian, that is, for any $t \geq 0$ and any $u \in L_2(\mu)$ with $0 \leq u \leq 1$ μ-a.e., we have $0 \leq \mathrm{T}(t)u \leq 1$ μ-a.e. In this case the semigroup $(\mathrm{T}(t))_{t\geq 0}$ is automatically contractive, $\|\mathrm{T}(t)u\|_{L_2(\mu)} \leq \|u\|_{L_2(\mu)}$, $t \geq 0$, $u \in L_2(\mu)$.

Let $-\mathrm{A}$ denote the infinitesimal $L_2(\mu)$-generator of $(\mathrm{T}(t))_{t\geq 0}$,

$$-\mathrm{A}u = \lim_{t\to 0}\frac{1}{t}(\mathrm{T}(t)u - u) \text{ strongly in } L_2(\mu)$$

for members u of $\mathcal{D}(\mathrm{A})$, the dense subspace of $L_2(\mu)$ for whose members this limit exists. Both A and $\mathrm{T}(t)$ are nonnegative definite self-adjoint operators on $L_2(\mu)$. In particular, the fractional powers A^α, $\alpha \geq 0$, of A can be defined in the usual way using the spectral representation [39, 42].

For $0 < \alpha < 1$, we can characterize the domain $\mathcal{D}(\mathrm{A}^\alpha)$ of A^α in terms of the semigroup: $u \in L_2(\mu)$ is in $\mathcal{D}(\mathrm{A}^\alpha)$ if and only if

$$\mathrm{A}^\alpha u = \lim_{\varepsilon\to 0}\frac{1}{\Gamma(-\alpha)}\int_\varepsilon^\infty t^{-\alpha-1}(\mathrm{T}(t) - \mathrm{I})u\mathrm{d}t \tag{5}$$

converges in $L_2(\mu)$; see, e.g., [2]. This may be interpreted as a right-sided Weyl–Marchaud derivative D^α_- of $t \mapsto \mathrm{T}(t)u$ at $t = 0$, more precisely, $\mathrm{D}^\alpha_-(\mathrm{T}(\cdot)u))(t) = (-1)^\alpha \mathrm{A}^\alpha\mathrm{T}(t)u$. See [17] or [36]. Now let us temporarily assume that zero is not an eigenvalue of A. Then the negative fractional powers $\mathrm{A}^{-\alpha}$, $\alpha > 0$, can be expressed by

$$\mathrm{A}^{-\alpha}u = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1}\mathrm{T}(t)u\mathrm{d}t\ , \tag{6}$$

what may be read as a right-sided Riemann–Liouville integral I^α_- of order $\alpha > 0$ of the function $t \mapsto \mathrm{T}(t)u$, i.e., $\mathrm{I}^\alpha_-(T(\cdot)u)(t) = (-1)^{-\alpha}\mathrm{A}^{-\alpha}\mathrm{T}(t)u$. Thus, for semigroups the language of traditional fractional calculus just leads to special cases of the usual functional calculus (cf. [36, 37, 42]).

The contractivity implies that $(\mathrm{T}(t))_{t\geq 0}$ is analytic on $L_2(\mu)$ (cf. [10] or [37], Chap. III). It also defines analytic semigroups on the spaces $L_p(\mu)$, $1 \leq p < \infty$; see [10, Theorem 1.4.1] or [37, Chap. III]. We use the same notation $\mathrm{T} = (\mathrm{T}(t))_{t\geq 0}$ for these semigroups, but denote their $L_p(\mu)$-generators by $-\mathrm{A}_p$, such that $\mathrm{A}_2 = \mathrm{A}$. In these cases (5) and (6) may be used to define their fractional powers; see [42]. Analyticity implies further useful properties: For any $u \in L_p(\mu)$, any $\alpha \geq 0$, and any $t > 0$, we have

$$\mathrm{T}(t)u \in \mathcal{D}(\mathrm{A}_p^\alpha) \ . \tag{7}$$

The operators $\mathrm{T}(t)$ and A_p^α commute on $\mathcal{D}(\mathrm{A}_p^\alpha)$. Given $\omega > 0$, the bound

$$\left\| (\omega \mathrm{I} + \mathrm{A}_p)^\alpha \mathrm{T}(t) \right\| \leq c_\alpha \mathrm{e}^{\omega t} t^{-\alpha} \tag{8}$$

holds for $t > 0$ (in the operator norm on $L_p(\mu)$) and the continuity estimate

$$\|\mathrm{T}(t)u - u\|_{L_p(\mu)} \leq c_\alpha t^\alpha \left\| (\omega \mathrm{I} + \mathrm{A}_p)^\alpha u \right\|_{L_p(\mu)} + (1 - \mathrm{e}^{-\omega t}) \|u\|_{L_p(\mu)}$$

is valid for $0 \leq \alpha < 1$, $u \in \mathcal{D}(\mathrm{A}_p^\alpha)$ and $t > 0$. See [30].

Given $\alpha_1, \alpha_2 \geq 0$, we have $\mathrm{A}_p^{\alpha_1+\alpha_2} = \mathrm{A}_p^{\alpha_1}\mathrm{A}_p^{\alpha_2}$, $\mathrm{A}_p^{\alpha_1}\mathrm{A}_p^{-\alpha_1} = \mathrm{I}$ and $\mathrm{A}_p^{\alpha_1} : \mathcal{D}(\mathrm{A}_p^{\alpha_1+\alpha_2}) \to \mathcal{D}(\mathrm{A}_p^{\alpha_2})$ is an isomorphism between these domains endowed with the graph norm. For $\sigma \geq 0$ we may regard the negative power

$$\mathrm{J}_p^\sigma(\mu) := (\mathrm{A}_p + \mathrm{I})^{-\sigma/2} \ .$$

as a *generalized Bessel potential operator* on $L_p(\mu)$. Set

$$H_p^\sigma(\mu) := \mathrm{J}_p^\sigma(\mu)(L_p(\mu)) \ ,$$

$\sigma \geq 0$, equipped with the norms

$$\|u\|_{H_p^\sigma(\mu)} := \|u\|_{L_p(\mu)} + \left\| \mathrm{A}_p^{\sigma/2} u \right\|_{L_p(\mu)} \ .$$

Clearly $H_p^0(\mu) = L_p(\mu)$. If $p = 2$, we write $H^\sigma(\mu)$ for $H_2^\sigma(\mu)$. Note that $\mathcal{D}((\mathrm{I} + \mathrm{A}_p)^\alpha) = \mathcal{D}(\mathrm{A}_p^\alpha) = H_p^{2\alpha}(\mu)$ and that potential operators $\mathrm{J}_p^\sigma(\mu)$, $\sigma \geq 0$ define isomorphic mappings from $H_p^\alpha(\mu)$ onto $H_p^{\alpha+\sigma}(\mu)$, $\alpha \geq 0$. Subspaces of essentially bounded functions will be denoted by

$$H_\infty^\sigma(\mu) := H^\sigma(\mu) \cap L_\infty(\mu) \ ,$$

normed by $\|\cdot\|_{H_\infty^\sigma(\mu)} := \|\cdot\|_{H^\sigma(\mu)} + \|\cdot\|_{L_\infty(\mu)}$. We write

$$H_{p'}^{-\sigma}(\mu) := ((H_p^\sigma(\mu))^*$$

for the *duals of the spaces* $H_p^\sigma(\mu)$, $1 < p < \infty$, $\sigma \geq 0$, $1/p + 1/p' = 1$, equipped with the usual (operator) norm $\|\cdot\|_{H_{p'}^{-\sigma}(\mu)}$.

If $X = \mathbb{R}^n$ and μ is the n-dimensional Lebesgue measure, then the spaces $H_p^\sigma(\mu)$, $1 < p < \infty$, $\sigma \geq 0$, coincide with potential spaces defined in terms of Fourier analysis,

$$H_p^\sigma(\mathbb{R}^n) := \left\{\mathcal{S}'(\mathbb{R}^n) : (1 + |\xi|^2)^{\sigma/2}\hat{f})^\vee \in L_p(\mathbb{R}^n)\right\},$$

$1 < p < \infty$, $\sigma \in \mathbb{R}$. Here $f \mapsto \hat{f}$ and $f \mapsto \check{f}$ denote the Fourier transform and the inverse Fourier transform, and $\mathcal{S}'(\mathbb{R}^n)$ is the space of Schwartz distributions on $\mathbb{R}^n$. Given a smooth bounded domain $D \subset \mathbb{R}^n$, we also consider the spaces

$$\tilde{H}_p^\sigma(D) := \left\{f \in H_p^\sigma(\mathbb{R}^n) : \operatorname{supp} f \subset \overline{D}\right\},$$

which are defined as subspaces of $H_p^\sigma(\mathbb{R}^n)$ for any $\sigma > -1/p$. We write $\tilde{H}^\sigma(D)$ if $p = 2$. The spaces $\tilde{H}_p^\sigma(D)$ may be regarded as the potential spaces associated with the operator A_p given as the L_p-generator of the Dirichlet heat semigroup $(T^D(t))_{t\geq 0}$ for D. For $\alpha, \sigma \in \mathbb{R}$ with $-1/2 < \alpha$ and $\alpha - \sigma < 3/2$ the fractional power $\mathrm{A}^{\sigma/2}$ maps $\tilde{H}^\alpha(D)$ isomorphically onto $\tilde{H}^{\alpha-\sigma}$. If $0 \leq \alpha < \frac{3}{2}$ and $\alpha \neq \frac{1}{2}$ then $\mathcal{D}(\mathrm{A}^{\alpha/2}) = \tilde{H}^\alpha(D)$. See [39]. The analyticity of $(\mathrm{T}^D(t))_{t\geq 0}$ also implies that for $-1/2 < \alpha, \sigma, \sigma + \alpha < 3/2$ the semigroup operators $\mathrm{T}^D(t)$ map $\tilde{H}^\alpha(D)$ into $\tilde{H}^{\alpha+\sigma}(D)$. In particular, given $f \in \tilde{H}^\alpha(D)$ we will have $\operatorname{supp} \mathrm{T}^D(t)f \subset \overline{D}$.

In the following we will exclusively use subspaces consisting of real-valued functions and real-valued dual elements respectively distributions. For simplicity we will not emphasize this fact by introducing new notation and therefore ask the reader to keep it in mind.

As we are going to investigate semilinear and transport equations, we need some preliminaries on composition and multiplication. Let $0 \leq \sigma \leq 1$. If $F \in C(\mathbb{R})$ satisfies $F(0) = 0$ and is Lipschitz, then we have

$$\|F(u)\|_{H_\infty^\sigma(\mu)} \leq c\,\|u\|_{H_\infty^\sigma(\mu)}$$

for any $u \in H_\infty^\sigma(\mu)$. If $F \in C^1(\mathbb{R})$ is such that $F(0) = 0$ and its derivative F' is bounded and Lipschitz, then

$$\|F(u) - F(v)\|_{H_\infty^\sigma(\mu)} \leq c\,\|u - v\|_{H_\infty^\sigma(\mu)}\left(\|v\|_{H_\infty^\sigma(\mu)} + 1\right)$$

for any $u, v \in H_\infty^\sigma(\mu)$. Finally, if $F \in C^2(\mathbb{R})$ with $F(0) = 0$ and bounded and Lipschitz second derivative F'', then

$$\|F(u_1) - F(v_1) - F(u_2) + F(v_2)\|_{H_\infty^\sigma(\mu)} \leq c\left(\|u_1 - v_1 - u_2 + v_2\|_{H_\infty^\sigma(\mu)} + \|u_2 - v_2\|_{H_\infty^\sigma(\mu)}\right)$$

for all $u_1, v_1, u_2, v_2 \in H_\infty^\sigma(\mu)$ with $\|u_i\|_{H_\infty^\sigma(\mu)} \leq 1$ and $\|v_i\|_{H_\infty^\sigma(\mu)} \leq 1$ for $i = 1, 2$. These properties basically follow from the Markov property and the mean value theorem; see [18, Proposition 3.1]). If $u, v \in H_\infty^\sigma(\mu)$, $0 \leq \sigma \leq 1$, then again by the Markov property the pointwise product uv is again in $H_\infty^\sigma(\mu)$ and

$$\|uv\|_{H_\infty^\sigma(\mu)} \leq c\,\|u\|_{H_\infty^\sigma(\mu)}\,\|v\|_{H_\infty^\sigma(\mu)}\ .$$

Given $u \in H_\infty^\sigma(\mu)$ and $z \in (H_\infty^\sigma)^*$, $0 \leq \sigma \leq 1$, we can define the product $uz \in (H_\infty^\sigma(\mu))^*$ by

$$(uz)(v) := (z, uv)\,, \quad v \in H_\infty^\sigma(\mu)\ ,$$

where $(\cdot,\cdot)$ denotes the dual pairing. For $z \in H^{-\sigma}(\mu)$ we observe

$$\|uz\|_{(H_\infty^\sigma(\mu))^*} \leq \|z\|_{H^{-\sigma}(\mu)}\,\|u\|_{H_\infty^\sigma(\mu)}\,, \tag{9}$$

note that $H^{-\sigma}(\mu)$ is a subspace of $(H_\infty^\sigma(\mu))^*$.

The semigroup $(\mathrm{T}(t))_{t\geq 0}$ is called *(locally) ultracontractive with spectral dimension* $d_S > 0$ if there exist constants $c > 0$ and $0 < \omega \leq 1$ such that for any $t > 0$ we have

$$\|\mathrm{T}(t)\|_{L_2(\mu)\to L_\infty(\mu)} \leq c t^{-d_S/4} \mathrm{e}^{\omega t}. \tag{10}$$

The estimate (10) is equivalent to several functional inequalities of Nash and Sobolev type; see [5, 6, 10, 40]. If (10) holds, we can define $\mathrm{T}(t)z$ for $z \in (H_\infty^\sigma(\mu))^*$ by means of dual pairing,

$$(\mathrm{T}(t)z)(v) := (z, \mathrm{T}(t)v)\,, \quad v \in L_2(\mu)\ ,$$

where we have implicitly used (7). For $z \in H^{-\sigma}(\mu)$ we obtain

$$\|\mathrm{T}(t)z\|_{L_2(\mu)} \leq c\mathrm{e}^{\omega t}(t^{-\sigma/2} + t^{-d_S/4})\,\|z\|_{H^{-\sigma}(\mu)} \tag{11}$$

by (8) and (10).

3 Integral Operators

Using some of the facts from the preceding section allows to verify the existence of a version of (4) that is suitable to solve related parabolic problems [18].

Let $t > 0$ and assume that u is a function on $(0, t)$ taking values in $H_\infty^\delta(\mu)$ for some $0 < \delta < 1$. If moreover $w \in H^{-\beta}(\mu)$ with $0 < \beta \leq \delta$ and $G \in C(\mathbb{R})$ is Lipschitz with $G(0) = 0$, then $G(u(\cdot))w$ is a function on $(0, t)$ taking its values in $(H_\infty^\beta(\mu))^*$. By (11)

$$s \mapsto U(t; s)w := \mathrm{T}(t-s)G(u(s))w\ ,\ w \in H^{-\beta}(\mu),$$

is seen to define a function $s \mapsto U(t;s)$ that takes its values in $L(H^{-\beta}(\mu), H^{\delta}_{\infty}(\mu))$. If $s \mapsto u(s)$ is sufficiently regular,

$$\mathrm{D}^{\eta}_{0+}U(t;s) := \frac{\mathbf{1}_{(0,t)}(s)}{\Gamma(1-\eta)}\left(\frac{U(t;s)}{s^{\eta}} + \eta\int_0^s \frac{U(t;s)-U(t;\tau)}{(s-\tau)^{\eta+1}}\mathrm{d}\tau\right)$$

converges in an appropriate sense.

To make this more precise, we introduce some additional function spaces. Given a separable Banach space E with norm $\|\cdot\|_E$ and a number $0 < \eta < 1$ let $W^{\eta}([0,t_0],E)$ denote the space of E-valued functions v on $[0,t_0]$ such that

$$\|v\|_{W^{\eta}([0,t_0],E)} := \sup_{0\le t\le t_0}\left(\|v(t)\|_E + \int_0^t \frac{\|v(t)-v(\tau)\|_E}{(t-\tau)^{\eta+1}}\mathrm{d}\tau\right) < \infty\,.$$

Similarly, let $C^{\eta}([0,t_0],E)$, $0<\eta<1$, denote the space of η-Hölder continuous E-valued functions v on $[0,t_0]$ such that

$$\|v\|_{C^{\eta}([0,t_0],E)} := \sup_{0\le t\le t_0}\|v(t)\|_E + \sup_{0\le\tau<t\le t_0}\frac{\|v(t)-v(\tau)\|_E}{(t-\tau)^{\gamma}} < \infty\,.$$

Lemma 1. *Let $0<\eta<1$, $t\in(0,t_0)$ and let $G\in C^2(\mathbb{R})$ with $G(0)=0$ and bounded and Lipschitz second derivative G''. If $0<\beta\le\delta<1$, $u\in W^{\eta}([0,t],H^{\delta}_{\infty}(\mu))$ and*

$$\delta \vee \frac{d_S}{2} < 2-2\eta-\left(\beta\vee\frac{d_S}{2}\right)\,,$$

then $\mathrm{D}^{\eta}_{0+}U(t;\cdot)$ converges in $L_1([0,t],L(H^{-\beta}(\mu),H^{\delta}_{\infty}(\mu)))$ and admits the following representation in terms of the semigroup:

$$\begin{aligned}\mathrm{D}^{\eta}_{0+}U(t;s) =&\mathrm{D}^{\eta}_{0+}(\mathrm{T}(t-\cdot)G(u(\cdot))(s)\\ =&\mathbf{1}_{(0,t)}(s)\Big\{-\mathrm{A}^{\eta}\mathrm{T}(t-s)G(u(s))+c_{\eta}\mathrm{T}(t-s)\int_s^{\infty} r^{-\eta-1}\mathrm{T}(r)G(u(s))\mathrm{d}r\\ &+c_{\eta}\int_0^s r^{-\eta-1}\mathrm{T}(r+t-s)[G(u(s))-G(u(s-r))]\mathrm{d}r\Big\}\,.\end{aligned}$$

Here $c_{\eta} = \eta\Gamma(1-\eta)^{-1} = -\Gamma(-\eta)^{-1}$.

Given $z\in C^{1-\alpha}([0,t_0],H^{-\beta}(\mu))$ and η slightly bigger than α we may consider

$$\mathrm{D}^{1-\eta}_{t-}z_t(s) := \frac{(-1)^{1-\eta}\mathbf{1}_{(0,t)}(s)}{\Gamma(\eta)}\left(\frac{z(s)-z(t)}{(t-s)^{1-\eta}} + (1-\eta)\int_s^t \frac{z(s)-z(\tau)}{(\tau-s)^{(1-\eta)+1}}\mathrm{d}\tau\right),$$

where $z_t(s) := \mathbf{1}_{(0,t)}(s)(z(s) - z(t))$. Then

$$w(s) := \mathrm{D}_{t-}^{1-\eta} z_t(s)\ ,\ s \in [0,t] \tag{12}$$

defines a function in $L_\infty([0,t], H^{-\beta}(\mu))$.

The next definition introduces an integral operator that may be seen as a version of (4). Recall the notation $U(t;s) = \mathrm{T}(t-s)G(u(s))$.

Definition 1. Given $t \in [0, t_0]$, $0 < \eta < 1$ and sufficiently regular functions u and z on $[0,t]$, put

$$\int_0^t \mathrm{T}(t-s)G(u(s))\mathrm{d}z(s) := (-1)^\eta \int_0^t \mathrm{D}_{0+}^{\eta} U(t;s)\mathrm{D}_{t-}^{1-\eta} z_t(s)\mathrm{d}s\ . \tag{13}$$

This integral operator is well defined.

Lemma 2. *Let t and η be as in Definition 1. Assume u is such that $\mathrm{D}_{0+}^{\eta} U(t;\cdot) \in L_1([0,t], L(H^{-\beta}(\mu), H_\infty^{\delta}(\mu)))$ and z is such that $\mathrm{D}_{t-}^{1-\eta} z_t \in L_\infty([0,t], H^{-\beta}(\mu))$, where $0 < \beta \le \delta < 1$. Then the right-hand side of (13) exists as an element of $H_\infty^{\delta}(\mu)$ and is independent of the particular choice of η.*

The following contraction property can be used to prove the existence and uniqueness of function solutions to Cauchy problems related to perturbed parabolic equations. To establish it we use equivalent norms on the space $W^\eta([0,t_0], E)$, $0 < \eta < 1$, given by

$$\|v\|_{W^\eta([0,t_0],E)}^{(\varrho)} := \sup_{0 \le t \le t_0} \mathrm{e}^{-\varrho t} \left(\|v(t)\|_E + \int_0^t \frac{\|v(t) - v(\tau)\|_E}{(t-\tau)^{\eta+1}} \mathrm{d}\tau \right) < \infty\ ,$$

where $\varrho \ge 1$ is a parameter [17]. This standard technique had been used before in [27, 29].

Proposition 1. *Assume $0 < \alpha, \beta, \gamma, \delta < 1$, $\alpha < \gamma < 1 - \alpha$, $\delta \ge \beta$ and*

$$2\gamma + \left(\delta \vee \frac{d_S}{2} \right) < 2 - 2\alpha - \left(\beta \vee \frac{d_S}{2} \right)\ .$$

Let $z \in C^{1-\alpha}([0,t_0], H^{-\beta}(\mu))$ and let $G \in C^2(\mathbb{R})$ with $G(0) = 0$ and bounded and Lipschitz second derivative G''. Suppose that $R > 0$ is given. Then

$$\left\| \int_0^{\cdot} T(\cdot - s)G(u(s))\mathrm{d}z(s) \right\|_{W^\gamma([0,t_0], H_\infty^\delta(\mu))}^{(\varrho)} \le C(\varrho) \left(1 + \|u\|_{W^\gamma([0,t_0], H_\infty^\delta(\mu))}^{(\varrho)} \right)\ , \tag{14}$$

$u \in W^\gamma([0,t_0], H_\infty^\delta(\mu))$, where $C(\varrho) > 0$ tends to zero as ϱ goes to infinity. For sufficiently large $\varrho_0 \ge 1$, the closed ball

$$B^{(\varrho_0)}(0,R) = \left\{ v \in W^\gamma([0,t_0], H_\infty^\delta(\mu)) : \|v\|_{W^\eta([0,t_0], H_\infty^\sigma(\mu))}^{(\varrho_0)} \le R \right\}$$

is mapped into itself and for $\varrho \geq \varrho_0$ *large enough,*

$$\left\| \int_0^{\cdot} T(\cdot - s)G(u(s))\mathrm{d}z(s) - \int_0^{\cdot} T(\cdot - s)G(v(s))\mathrm{d}z(s) \right\|^{(\varrho)}_{W^{\gamma}([0,t_0],H^{\delta}_{\infty}(\mu))}$$

$$\leq C(\varrho)\, \|u - v\|^{(\varrho)}_{W^{\gamma}([0,t_0],H^{\delta}_{\infty}(\mu))}\ ,\quad u, v \in B^{(\varrho_0)}(0, R).$$

4 Parabolic Problems on Metric Measure Spaces

One of the classes of problems we are interested in are Cauchy problems associated with perturbed semilinear equations. Formulated in a general and abstract way, they read as

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = -\mathrm{A}u(t,x) + F(u)(t,x) + G(u)\cdot \dot{z}(t,x)\ ,\ t \in (0,t_0),\ \ x \in X \\ u(0,x) = u_0(x)\ , \end{cases} \tag{15}$$

where $-\mathrm{A}$ is the $L_2(\mu)$-generator of a strongly continuous symmetric Markovian semigroup $(\mathrm{T}(t))_{t\geq 0}$ on $L_2(\mu)$ as in Sect. 2, F and G are (generally nonlinear) functions on $\mathbb{R}$ and $\dot{z}$ denotes a space-time perturbation that may be seen as a formal space-time derivative of a non-differentiable deterministic function z on $(0, t_0] \times X$. It is possible to study (15) on general *σ-finite measure spaces* $(X, \mathcal{X}, \mu)$. We will focus on cases with $u(t, x)$ real valued. In a similar manner one can consider equations with $\mathbb{R}^k$-valued $u(t, x)$; see Example 1 (ii) and [17].

As mentioned before, we investigate the existence of function solutions to these equations. More precisely, we aim at results that confirm the existence and uniqueness of a vector valued function $t \mapsto u(t)$ that solves problem (15) in an evolution sense and takes its values in a space of (equivalence classes of) locally integrable functions on X. This is to be distinguished from distribution solutions which are also commonly used to study stochastic partial differential equations.

A function u on $(0, t_0] \times X$ is called a *mild solution* to (15) if seen as vector-valued function $u(t) := u(t, \cdot)$, it satisfies

$$u(t) = \mathrm{T}(t)u_0 + \int_0^t \mathrm{T}(t-s)F(u(s))\mathrm{d}s + \int_0^t \mathrm{T}(t-s)G(u(s))\mathrm{d}z(s),\ \ t \in (0, t_0)\ . \tag{16}$$

If for any fixed $t \in [0, t_0]$, $u(t)$ determines a locally integrable function on $(X, \mathcal{X}, \mu)$, we call u a *function solution*. The last term in (16) is the integral operator as defined in (13). It realizes a temporal differentiation of z by means of fractional calculus, a spatial differentiation is hidden in the fact that for fixed time s, $z(s)$ is an element of the dual of an appropriate potential space.

The following result is true without any further hypotheses. A proof is given in [18]; its main ingredient is Proposition 1, which allows to use a contraction principle.

Theorem 1. *Assume* $(X, \mathcal{X}, \mu)$ *is a* σ*-finite measure space and* $t_0 > 0$*. Let* $-\mathrm{A}$ *be the generator of a strongly continuous symmetric Markovian semigroup* $(\mathrm{T}(t))_{t\geq 0}$ *on* $L_2(\mu)$ *which is ultracontractive with spectral dimension* $d_S > 0$*.*

Suppose $0 < \alpha, \beta, \gamma, \delta, \varepsilon < 1$ *and* $z \in C^{1-\alpha}([0, t_0], H^{-\beta}(\mu))$*. Let* $F \in C^1(\mathbb{R})$*,* $F(0) = 0$*, have a bounded Lipschitz derivative* F' *and* $G \in C^2(\mathbb{R})$*,* $G(0) = 0$*, have a bounded Lipschitz second derivative* G''*. Assume* $f \in H^{2\gamma+\delta+\varepsilon}(\mu)$*. If* $\alpha < \gamma < 1 - \alpha$*,* $\delta \geq \beta$ *and*

$$2\gamma + \left(\delta \vee \frac{d_S}{2}\right) < 2 - 2\alpha - \left(\beta \vee \frac{d_S}{2}\right) . \tag{17}$$

Then problem (15) has a unique mild solution (16) in $W^\gamma([0, t_0], H^\delta_\infty(\mu))$*, which means in particular that the solution is a function.*

In many cases more structural knowledge about the space X and the semigroup $(\mathrm{T}(t))_{t\geq 0}$ is available. For instance, X may be a *metric measure space* and $(\mathrm{T}(t))_{t\geq 0}$ may possess transition densities that satisfy some typical estimates [21, 22]. Under the following assumptions we can improve our results.

Assumption 1. (X, d) *is a locally compact separable metric space,* $\mathcal{X} = \mathcal{B}(X)$ *the Borel-*σ*-field on* X *and* μ *a Radon measure on* (X, d)*.*

Assumption 2. *The semigroup* $(\mathrm{T}(t))_{t\geq 0}$ *admits transition densities* $p(t, x, y)$*, that is,*

$$\mathrm{T}(t)u(x) = \int_X p(t, x, y)u(y)\mu(\mathrm{d}y) ,$$

and the $p(t, x, y)$ *satisfy bounds of the form*

$$t^{-d_f/w}\Phi_1\left(t^{-1/w}d(x, y)\right) \leq p(t, x, y) \leq t^{-d_f/w}\Phi_2\left(t^{-1/w}d(x, y)\right)$$

for any $(x, y) \in X \times X$ *and* $t \in (0, R_0)$*, with bounded decreasing functions* Φ_i *on* $[0, \infty)$*. Here* $R_0 > 0$ *is a fixed number,* d_f *is the* Hausdorff dimension *of* (X, d) *and* $w \geq 2$ *satisfies* $d_S = 2d_f/w$*. For a given number* $\beta > 0$ *we further assume the validity of the integral condition*

$$\int_0^\infty s^{d_f+\beta/2-1}\Phi_2(s)\mathrm{d}s < \infty .$$

Under these circumstances we get the following improved result.

Theorem 2. *Let F and G be as in Theorem 1. Suppose $0 < \alpha, \beta, \gamma, \delta, \varepsilon < 1$ and Assumptions 1 and 2 are satisfied. Assume $\alpha < \gamma < 1 - \alpha$ and $0 < \beta < \delta < d_S/2$. If $f \in H^{2\gamma+\delta+\varepsilon}(\mu)$ and $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$ for $q = d_S/\delta$ and*

$$2\gamma + \frac{d_S}{2} < 2 - 2\alpha - \beta \,, \tag{18}$$

then problem (15) has a unique mild solution (16) in $W^{\gamma}([0, t_0], H_{\infty}^{\delta}(\mu))$, which means in particular that the solution is a function.

Theorem 2 is proved in a similar way as Theorem 1 by verifying the contractivity of the integral operator and applying a contraction argument. In particular, analogs of Lemmas 1 and 2 can be used. The only news is the following improved product estimate that replaces the former (9).

Proposition 2. *Let $0 < \beta < \delta < d_S/2 \wedge 1$ and $p = d_S/(d_S - \delta)$. Let the semigroup be ultracontractive with spectral dimension $d_S > 0$ and let Assumptions 1 and 2 be satisfied. Then we have*

$$\|uv\|_{H_p^{\beta}(\mu)} \leq c \; \|u\|_{\delta} \, \|v\|_{\delta}$$

for any $u, v \in H^{\delta}(\mu)$.

Theorem 2 requires $d_S < 4$. For symmetric diffusion semigroups on $\mathbb{R}^n$, we have $d_S = n$, hence need $n \leq 3$. This is typical, because to deal with the nonlinear transformations F and G we need the solution to be $L_{\infty}(\mu)$-bounded, but only in low dimensions the singularity of the semigroup at zero is small enough to provide $L_{\infty}(\mu)$-bounds. The special case of linear F and G allows to remove this restrictive condition.

Theorem 3. *Let F and G be linear. Suppose $0 < \alpha, \beta, \gamma, \delta, \varepsilon < 1$ and Assumptions 1 and 2 are satisfied. Assume $\alpha < \gamma < 1 - \alpha$ and $0 < \beta < \delta < d_S/2$. If $f \in H^{2\gamma+\delta+\varepsilon}(\mu)$ and $z \in C^{1-\alpha}([0, t_0], H_q^{-\beta}(\mu))$ for $q = d_S/\delta$ and*

$$2\gamma + \delta < 2 - 2\alpha - \beta \,,$$

then problem (15) has a unique mild solution (16) in $W^{\gamma}([0, t_0], H^{\delta}(\mu))$, which means in particular that the solution is a function.

Examples 1. To consider some examples of stochastic partial differential equations based on (15), let $0 < H, K < 1$ and consider the spatially isotropic fractional Brownian sheet $B^{H,K}$ on $[0, t_0]\times\mathbb{R}^n$ with Hurst indices H and K (see [1]), that is, the centered real-valued Gaussian random field $B^{H,K}$ on $[0, t_0] \times \mathbb{R}^n$ over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that for any $0 \leq s < t \leq t_0$ and $x, y \in \mathbb{R}^n$,

$$\mathbb{E}\left[B^{H,K}(t, y) - B^{H,K}(t, x) - B^{H,K}(s, y) + B^{H,K}(s, x)\right]^2 = c_{H,K}(t-s)^{2H}|x-y|^{2K},$$

where $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^n$. It is not difficult to see that for $\mathbb{P}$-a.e. $\omega \in \Omega$ and any $0 < \gamma < H$, $0 < \sigma < K$ and $1 < q < \infty$, the realization $B^{H,K}(\omega)$ is an element of $C^\gamma([0, t_0], H_q^\sigma(\mathbb{R}^n))$. The components of its distributional (spatial) gradient $\nabla B^{H,K}(\omega)$ are elements of $C^\gamma([0, t_0], H_q^{\sigma-1}(\mathbb{R}^n))$.

(i) Let $n = 1$, $X = (0, 1)$, let $(\mathrm{T}(t))_{t \geq 0}$ be the Dirichlet heat semigroup on $(0, 1)$ and Δ the Dirichlet Laplacian. Consider the one-dimensional semilinear heat equation on $(0, t_0) \times (0, 1)$ driven by a fractional Brownian sheet $B^{H,K}$,

$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + F(u(t, x)) + G(u(t, x)) \cdot \frac{\partial^2 B^{H,K}}{\partial t \partial x} .$$

It has a unique function solution if $1/2 < H < 1$ and $2H + K > 2$.

(ii) In [17] we have considered boundary initial value problems on smooth bounded domains $D \subset \mathbb{R}^n$ associated with parabolic equations of type

$$\frac{\partial u}{\partial t}(t, x) = -\mathrm{A}u(t, x) + F(u(t, x)) + \left\langle G(u), \frac{\partial}{\partial t} \nabla V \right\rangle_{\mathbb{R}^n} (t, x) .$$

Here V is a real-valued noise potential, G is an $\mathbb{R}^n$-valued nonlinearity on $\mathbb{R}$, and $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ denotes the scalar product in $\mathbb{R}^n$.

5 Transport Equations on Domains

In this section we consider transport–diffusion equations of the form

$$\begin{cases} \frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + \langle \nabla u, \nabla z \rangle_{\mathbb{R}^n} (t, x), & t \in (0, t_0],\ x \in D \\ u(t, x) = 0, & t \in (0, t_0],\ x \in \partial D \\ u(0, x) = u_0(x), & x \in D\ , \end{cases} \tag{19}$$

where $D \subset \mathbb{R}^n$ is a smooth bounded domain and z is a non-differentiable function on $\mathbb{R}^n$. Here Δ denotes the Dirichlet Laplacian for D and the gradient ∇ is interpreted in distributional sense. As before, $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$ denotes the scalar product in $\mathbb{R}^n$. In this model z is viewed as a temporally constant perturbation.

Problems of type (19) have been considered by the second named author in [23]. Again we are interested in the existence, uniqueness, and regularity of function solutions. Now $(\mathrm{T}(t))_{t \geq 0}$ will denote the Dirichlet heat semigroup $(\mathrm{T}^D(t))_{t \geq 0}$ for the domain D.

A function u on $(0, t_0] \times D$ is called a *mild solution* to (19) if seen as a vector-valued function $u(t) := u(t, \cdot)$ it satisfies

$$u(t) = \mathrm{T}(t)u_0 + \int_0^t \mathrm{T}(t-r) \langle \nabla u(r), \nabla z \rangle_{\mathbb{R}^n} \, \mathrm{d}r, \quad t \in (0, t_0] \, ,$$

and a *function solution* if in addition $u(t)$ is a locally integrable function on D for any t. By the following multiplication property (see [33]), the right-hand side in this definition admits a reasonable interpretation.

Lemma 3. *Let $w \in \tilde{H}_p^{1+\delta}(D)$, $z \in H_q^{1-\beta}(\mathbb{R}^n)$ with $1 < p, q < \infty$, $q > p \vee n/\delta$, $0 < \beta < 1/2$ and $\beta < \delta$. Then $\langle \nabla w, \nabla z \rangle_{\mathbb{R}^n}$ is a member of $\tilde{H}_p^{-\beta}(D)$ and*

$$\| \langle \nabla w, \nabla z \rangle_{\mathbb{R}^n} \|_{H_p^{-\beta}(\mathbb{R}^n)} \leq c \, \|w\|_{H_p^{1+\delta}(\mathbb{R}^n)} \, \|z\|_{H_q^{1-\beta}(\mathbb{R}^n)} \, .$$

The main result of [23] reads as follows.

Theorem 4. *Let $t_0 > 0$ and $0 < \beta < \delta < 1/2$ and $0 < 2\gamma < 1 - \beta - \delta$. Let $z \in H_q^{1-\beta}$ for some $q > 2 \vee d/\delta$. Then for any $u_0 \in \tilde{H}^{1+\delta+2\gamma}(D)$ there exists a unique mild solution $u \in C^\gamma([0, t_0], \tilde{H}^{1+\delta}(D))$ to (19), which means in particular that the solution is a function.*

The theorem follows by fixed point arguments and the following contractivity result. Similarly as before it is formulated in terms of equivalent norms. For $\varrho \geq 1$ we equip the space $C^\eta([0, t_0], E)$ of η-Hölder continuous E-valued functions v on $[0, t_0]$ with the equivalent norm

$$\|v\|_{C^\eta([0,t_0],E)}^{(\varrho)} := \sup_{0 \leq t \leq t_0} \mathrm{e}^{-\varrho t} \left(\|v(t)\|_E + \sup_{0 \leq \tau < t} \frac{\|v(t) - v(\tau)\|_E}{(t-\tau)^\gamma} \right) .$$

Proposition 3. *$0 < \beta < \delta < 1/2$ and $z \in H_q^{1-\beta}(\mathbb{R}^d)$ for some $q > 2 \vee d/\delta$. Then for any γ with $0 < 2\gamma < 1 - \beta - \delta$ we have*

$$\left\| \int_0^\cdot \mathrm{T}(\cdot - s) \langle \nabla u(s), \nabla z \rangle_{\mathbb{R}^n} \, ds \right\|_{C^\gamma([0,t_0],\tilde{H}^{1+\delta}(D))}^{(\varrho)} \leq C(\varrho) \, \|u\|_{C^\gamma([0,t_0],\tilde{H}^{1+\delta}(D))}^{(\varrho)}$$

for any $u \in C^\gamma([0, t_0], \tilde{H}^{1+\delta}(D))$ where $C(\varrho)$ tends to zero as ϱ goes to infinity.

Examples 2.

(i) If for instance B^H is a fractional Brownian field on $\mathbb{R}^n$ with Hurst parameter $1/2 < H < 1$, that is, a real-valued centered Gaussian random field on $\mathbb{R}^n$ with

$$\mathbb{E} \left[B^H(x) - B^H(y) \right]^2 = c_H |x - y|^{2H} \, ,$$

then we may consider a typical realization $B^H(\omega)$ in place of z to obtain results for stochastic transport equations with fractal noise:

$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x) + \left\langle \nabla u, \nabla B^H \right\rangle_{\mathbb{R}^n} (t, x) \, .$$

(ii) Combining the above with the results of the preceding section we can investigate a more general form of transport–diffusion equation:

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + \langle \nabla u, \nabla z\rangle_{\mathbb{R}^n}(t,x) + \left\langle F, \frac{\partial}{\partial t}\nabla V\right\rangle_{\mathbb{R}^n}, & t \in (0,t_0],\ x \in D \\ u(t,x) = 0, & t \in (0,t_0],\ x \in \partial D \\ u(0,x) = u_0(x), & x \in D\ , \end{cases}$$

where z is as above, F is a vector in $\mathbb{R}^n$, and $V = V(t,x)$ is a non-differentiable noise that may vary in space and time.

6 Some Remarks on Burgers System

We finish our survey with some brief look at a *Burgers-type equation* [4]. On $(0,t_0)\times\mathbb{R}^n$, $n \geq 1$, consider the equation

$$\frac{\partial u}{\partial t} = \Delta u - \langle u, \nabla\rangle_{\mathbb{R}^n} u + \frac{\partial}{\partial t}\nabla B \tag{20}$$

with some deterministic initial condition $u(0) = u_0$. Here $\langle\cdot,\cdot\rangle_{\mathbb{R}^n}$ denotes the Euclidean scalar product in $\mathbb{R}^n$ and $B = B(t,x)$ is a *fractional Brownian sheet* on $[0,t_0]\times\mathbb{R}^n$ over some probability space $(\Omega,\mathcal{F},\mathbb{P})$. Equation (20) is already a stochastic differential equation and the solution method considered by the first named author in [15] is not pathwise. However, it is made up from techniques very similar to those used in the preceding sections. Note that a solution u will be vector valued, i.e., $u(t,x) \in \mathbb{R}^n$ for fixed t and x.

We make (20) rigorous by defining weak and mild solutions. A process $u = u(t)$ is said to be a *distributional solution* to (20) on $(0,t_0)$ with initial condition u_0 if for any test function $\varphi \in \mathcal{D}(\mathbb{R}^n,\mathbb{R}^n)$ and any $t \in (0,t_0)$, we $\mathbb{P}$-a.s. have

$$(u(t),\varphi) = (u_0,\varphi) + \int_0^t (\Delta u(s),\varphi)\mathrm{d}s - \int_0^t (\langle u(s),\nabla\rangle_{\mathbb{R}^n} u(s),\varphi)\mathrm{d}s + (\nabla B(t),\varphi)\ . \tag{21}$$

Here Δ denotes the matrix Laplacian, that is, the $n\times n$ diagonal matrix with the usual Laplacian on the diagonal. $B(t) = B(t,\cdot)$ can be seen as a process taking values in a Sobolev space and the components $\partial B(t)/\partial x_i$ of $\nabla B(t) = (\partial B(t)/\partial x_1,\ldots,\partial B(t)/\partial x_n)$ are defined in the sense of distributions. As the name indicates, u is considered as a distribution-valued function. However, it turns out that it also provides a function solution. A process $u = u(t)$ is called a *mild solution* to (20) on $(0,t_0)$ with initial condition u_0 if for any $t \in (0,t_0)$, $\mathbb{P}$-a.s.

$$u(t) = \mathrm{T}(t)u_0 - \int_0^t \mathrm{T}(t-s)\langle u(s),\nabla\rangle_{\mathbb{R}^n} u(s)\mathrm{d}s + \int_0^t \mathrm{T}(t-s)\mathrm{d}(\nabla B)(s)\ . \tag{22}$$

Here $(\mathrm{T}(t))_{\geq 0}$ denotes the Brownian semigroup on $\mathbb{R}^n$, respectively its matrix version. The stochastic integral in (22) is of Wiener type, defined as the $L_p(\Omega)$-limit

$$\int_0^t \mathrm{T}(t-s)\mathrm{d}(\nabla B)(s) := \lim_{\varepsilon\to 0} \varepsilon \int_0^1 r^{\varepsilon-1} \int_0^t \mathrm{T}(t-s)\frac{\nabla B(s+r) - \nabla B(s)}{r}\mathrm{d}s\mathrm{d}r \;, \tag{23}$$

$p > 2$, of random variables taking values in a certain weighted Sobolev space. It is closely related to the pathwise integrals (4) and (13). We call u as in (22) a *function solution* to (20) if any $u(t)$, $t \in (0, t_0]$, is a locally integrable function on $\mathbb{R}^n$.

We follow the standard approach to Burgers equation and employ a stochastic variant of the *Cole–Hopf transformation*. First we consider a related *stochastic heat equation*,

$$\frac{\partial w}{\partial t}(t,x) = \Delta w(t,x) + w(t,x)\cdot \frac{\partial}{\partial t}B(t,x),\; t \in (0,T),\; x \in \mathbb{R}^n \;, \tag{24}$$

with some initial condition w_0. The term $\partial B/\partial t$ is a *half-noise*, similar as in [3]: For fixed $x \in \mathbb{R}^n$ and up to a constant, $t \mapsto B(t,x)$ behaves like a one-dimensional fractional Brownian motion with Hurst parameter H and in (24) we consider its formal time derivative $t \mapsto \partial B(t,x)/\partial t$. We say the random process $w : (0,t_0) \times \mathbb{R}^n \times \Omega \to \mathbb{R}$ is a *pointwise mild solution* to (24) if for fixed $t \in (0,t_0)$ and $x \in \mathbb{R}^n$, we have

$$w(t,x) = \mathrm{T}(t)w_0(x) + \lim_{\varepsilon\to 0} \varepsilon \int_0^1 r^{\varepsilon-1} \int_0^t \mathrm{T}(t-s)w(s,x)\frac{B(s+r,x) - B(s,x)}{r}\mathrm{d}s\mathrm{d}r \;. \tag{25}$$

The limit and the equality (25) are considered in $L_p(\Omega)$, $p > 1$. Our results are as follows:

Theorem 5. *Let $t_0 > 0$, $0 < K \leq 1/2$ and $2 < 2H + K$. Suppose that u_0 is of form $u_0(x) = -\nabla U_0(x)$, where U_0 is a real-valued function on $\mathbb{R}^n$ such that*

$$|U_0(x)| \leq b(1+|x|^\gamma) \;, \quad \left|\frac{\partial U_0}{\partial x_i}(x)\right| \leq \exp(b(1+|x|^\gamma)) \;, \quad i = 1,\ldots,n$$

for some $b > 0$, $2K \leq \gamma \leq 1$, and any $x \in \mathbb{R}^n$. Then there is a $\mathbb{P}$-a.s. strictly positive pointwise mild solution w to (24) with $w_0 = \exp(U_0/2)$ and $u := \nabla \log w$ is a distributional solution (21)–(20). The process u is also a function solution to (24).

Note that our hypothesis implies $H > 3/4$. For further details, we refer to [15].

References

1. Ayache, A., Leger, S., Pontier, M.: Drap brownien fractionnaire. Potential Anal. **17**, 31–43 (2002)
2. Berens, H., Butzer, P.L., Westphal, U.: Representation of fractional powers of infinitesimal generators of semigroups. Bull. Am. Math. Soc. **74**, 191–196 (1968)
3. Bertini, L., Cancrini, N., Jona-Lasinio, G.: The stochastic Burgers equation. Comm. Math. Phys. **165**, 211–232 (1994)
4. Burgers, J.M.: The Nonlinear Diffusion Equation. Reidel, Dordrecht (1974)
5. Carlen, E.A., Kusuoka, S., Stroock, D.W.: Upper bounds for symmetric Markov transition functions. Ann. Inst. H. Poincaré **23**, 245–287 (1987)
6. Coulhon, T.: Ultracontractivity and Nash type inequalitites. J. Funct. Anal. **141**, 510–539 (1996)
7. Da Prato, G., Debussche, A., Temam, R.: Stochastic Burgers equation. Nonlinear Diff. Equat. Appl. **1 (4)**, 389–402 (1994)
8. Da Prato, G., Gatarek, D.: Stochastic Burgers equation with correlated noise. Stoch. Stoch. Rep. **52**, 29–41 (1995)
9. Da Prato, G., Zabzcyk, J.: Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge (1992)
10. Davies, E.B.: Heat Kernels and Spectral Theory. Cambridge University Press, Cambridge (1989)
11. Friz, P.K., Victoir, N.: Multidimesnional Stochastic Processes as Rough Paths: Theory and Applications. Cambridge University Press, Cambridge (2010)
12. Gubinelli, M., Lejay, A., Tindel, S.: Young integrals and SPDE's. Potential Anal. **25**, 307–326 (2006)
13. Gyöngy, I., Nualart, D.: On the stochastic Burgers equation in the real line. Ann. Probab. **27**(2), 782–802 (1999)
14. Herrmann, R.: Fractional Calculus: An Introduction for Physicists. World Scientific, Singapore (2011)
15. Hinz, M.: Burgers system with a fractional Brownian random force. Stochastics **83**(1), 67–106 (2011)
16. Hinz, M., Zähle, M.: Gradient type noises I - partial and hybrid integrals. Complex Var. Elliptic Equat. **54**, 561–583 (2009)
17. Hinz, M., Zähle, M.: Gradient type noises II - systems of partial differential equations. J. Funct. Anal. **256**, 3192–3235 (2009)
18. Hinz, M., Zähle, M.: Semigroups, potential spaces and applications to (S)PDE. Potential Anal. **36**, 483–515 (2012)
19. Holden, H., Øksendal, B., Ubøe, J., Zhang, T.: Stochastic Partial Differential Equations: A Modelling, White Noise Functional Approach. Birkhäuser, Boston (1996)
20. Hu, Y., Nualart, D.: Stochastic heat equation driven by fractional noise and local time. Probab. Theory Relat. Fields. **143**, 285–328 (2009)
21. Hu, J., Zähle, M.: Potential spaces on fractals. Studia Math. **170**, 259–281 (2005)
22. Hu, J., Zähle, M.: Generalized Bessel and Riesz potential spaces on metric measure spaces. Potential Anal. **30**, 315–340 (2009)
23. Issoglio, E.: Transport equations with fractal noise: existence uniqueness and regularity of the solution. J. Anal. Appl. **32**(1), 37–53 (2013)
24. Klingenhöfer, F., Zähle, M.: Ordinary differential equations with fractal noise. Proc. Am. Math. Soc. **127**, 1021–1028 (1999)
25. Lyons, T.J.: Differential equations driven by rough signals I: an extension of an inequality by L.C. Young. Math. Res. Lett. **1**, 451–464 (1994)
26. Lyons, T.J.: Differential equations driven by rough signals II. Rev. Iberoam. **14** (2), 215–310 (1998)

27. Maslowski, B., Nualart, D.: Evolution equations driven by fractional Brownian motion. J. Funct. Anal. **202**, 277–305 (2003)
28. Nualart, D.: The Malliavin Calculus and Related Topics. Springer, New York (1995)
29. Nualart, D., Rascanu, A.: Differential equations driven by fractional Brownian motion. Collect. Math. **53**(1), 55–81 (2002)
30. Pazy, A.: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983)
31. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1998)
32. Prévôt, C., Röckner, M.: A Concise Course on Stochastic Partial Differential Equations. Lecture Notes in Mathematics, vol. 1905. Springer, New York (2007)
33. Runst, T., Sickel, W.: Sobolev Spaces of Fractional Order, Nemytskij Operators, and Non-linear Partial Differential Equations. De Gruyter, Berlin (1996)
34. Russo, F., Trutnau, G.: Some parabolic PDE whose drift is an irregular random noise in space. Ann. Probab. **35**(6), 2213–2262 (2007)
35. Russo, F., Vallois, P.: Forward, backward and symmetric stochastic integration. Probab. Theory Relat. Fields **97**, 403–421 (1993)
36. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach, Yverdon (1993)
37. Stein, E.M.: Topics in Harmonic Analysis Related to the Littlewood-Paley Theory. Annals of Mathematics Studies. Princeton University Press, Princeton (1970)
38. Tindel, S., Tudor, C.A., Viens, F.: Stochastic evolution equations with fractional Brownian motion. Probab. Theory Relat. Fields **127**, 186–204 (2003)
39. Triebel, H.: Interpolation Theory, Function Spaces, Differential Operators, 2nd edn. J.A. Barth, Heidelberg (1995)
40. Varopoulos, N.: Hardy-Littlewood theory for semigroups. J. Funct. Anal. **63**, 240–260 (1985)
41. Walsh, J.B.: An introduction to stochastic partial differential equations. École d'été de Probabilités de Saint-Flour, XIV-1984. Lecture Notes in Mathematics, vol. 1180. Springer, New York (1986)
42. Yosida, K.: Functional Analysis. Springer, New York (1980)
43. Young, L.C.: An inequality of Hölder type, connected with Stieltjes integration. Acta Math. **67**, 251–282 (1936)
44. Zähle, M.: Integration with respect to fractal functions and stochastic calculus I. Probab. Theory Relat. Fields **111**, (1998)
45. Zähle, M.: Integration with respect to fractal functions and stochastic calculus II. Math. Nachr. **225**, 145–183 (2001)
46. Zähle, M.: Forward integrals and stochastic differential equations. In: Dalang, R.C., Dozzi, M., Russo, F. (eds.) Seminar on Stochastic Analysis, Random Fields and Applications III. Progress in Probability, pp. 293–302. Birkhäuser, Basel (2002)

Stochastic Partial Differential Equations Driven by General Stochastic Measures

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Abstract Stochastic integrals of real-valued functions with respect to general stochastic measures are considered in the chapter. For the integrator we assume the σ-additivity in probability only. The chapter contains a review of recent results concerning Besov regularity of stochastic measures, continuity of paths of stochastic integrals, and solutions of stochastic partial differential equations (SPDEs) driven by stochastic measure. Some important properties of stochastic integrals are proved. The Riemann-type integral of random function with respect to the Jordan content is introduced. For the heat equation in $\mathbb{R}$, we consider the existence, uniqueness, and Hölder regularity of the mild solution. For a general parabolic SPDE in $\mathbb{R}^d$, we obtain the weak solution. Integrals of random functions with respect to deterministic measures in the equations are understood in Riemann sense.

1 Introduction

The aim of the chapter is to give a review of results on the regularity of general stochastic measures and on stochastic partial differential equations (SPDEs) driven by such measures. The stochastic term in these SPDEs is given by integral of real function with respect to (w.r.t.) a stochastic measure (SM) μ, and for μ we assume the σ-additivity in probability only (see Definition 1 below).

Integration of deterministic functions w.r.t. SMs was considered in [10, Chap. 7], [2, 13–15]. The results obtained in these publications and auxiliary Lemmas proved in Sect. 3 help to study SPDEs driven by SMs. Section 4 gives some regularity results for paths of SM.

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In Sect. 5 we will study the mild solution of a stochastic heat equation, which can formally be written as

$$du(x,t) = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} dt + f(x,t,u(x,t))dt + \sigma(x,t)d\mu(x),\ u(x,0) = u_0(x), \tag{1}$$

where $(x,t) \in \mathbb{R} \times [0,T]$, $a \in \mathbb{R}$, $a \neq 0$, and μ is an SM defined on Borel $\sigma-$algebra of $\mathbb{R}$.

In Sect. 7 we will consider the weak solution of stochastic parabolic equation

$$dX(x,t) = AX(x,t)\,dt + h(x,t)\,d\mu(t), \quad X(x,0) = \xi(x)\,, \tag{2}$$

where $(x,t) \in \mathbb{R}^d \times [0,T]$ and A is a second-order strongly elliptic differential operator.

Weak form of (2) includes an integral of random function w.r.t. a deterministic measure (the Jordan content). We interpret this integral as a limit in probability of Riemann integral sums (Sect. 6). This definition of the integral allows to interchange the order of integration w.r.t. deterministic and stochastic measures (Theorem 9), which is important for solving the equation.

Parabolic SPDEs driven by martingale measures had been introduced and discussed initially in [27]. This approach was developed in [1, 3]. In [5, 12] SPDEs were studied as stochastic equations in functional spaces. Hölder regularity of solutions of SPDEs of different types was considered in [4, 6], [27, Chap. 3]. In these and many other papers the stochastic noise satisfies some special conditions on distributions and moment existence or has martingale properties. In this chapter, we consider very general class of possible μ. On the other hand, the stochastic term in (1) and (2) is independent of u and X. A reason is that appropriate definition of integral of random function w.r.t. μ does not exist.

Hölder regularity of mild solution of the heat equation driven by SM on nested fractal was proved in [23]. Weak solutions of some SPDEs with SM were obtained in [18].

Some motivating examples for studying SPDEs may be found in [5, Introduction], [9, Sect. 13.2]. Equations (1) and (2) (for $A = \Delta$) describe the evolution in time of the density X of some quantity such as heat or chemical concentration in a system with random sources. In our model, the random influence can be rather general.

2 Preliminaries

Let $\mathsf{L}_0 = \mathsf{L}_0(\Omega, \mathcal{F}, \mathsf{P})$ be a set of (equivalence classes of) all real-valued random variables defined on a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Convergence in L_0 means the convergence in probability.

Let X be an arbitrary set and $\mathcal{B}$ be a σ-algebra of subsets of X.

Definition 1. Any σ-additive mapping $\mu : \mathcal{B} \to \mathsf{L}_0$ is called a *stochastic measure*.

In other words, μ is a vector measure with values in L_0. We do not assume positivity or integrability for an SM. In [10] such a μ is called a general stochastic measure. In the following, μ always denotes an SM.

Examples of SMs are the following. Let $\mathsf{X} = [0, T] \subset \mathbb{R}_+$, $\mathcal{B}$ be the σ-algebra of Borel subsets of $[0, T]$ and $Y(t)$ be a square integrable martingale. Then $\mu(\mathsf{A}) = \int_0^T \mathbf{1}_{\mathsf{A}}(t)\,\mathrm{d}Y(t)$ is an SM. If $W^H(t)$ is a fractional Brownian motion with Hurst index $H > 1/2$ and $f : [0, T] \to \mathbb{R}$ is a bounded measurable function, then $\mu(\mathsf{A}) = \int_0^T f(t)\mathbf{1}_{\mathsf{A}}(t)\,\mathrm{d}W^H(t)$ is also an SM, as follows from [11, Theorem 1.1]. Some other examples may be found in [10, Sect. 7.2]. Kwapień and Woycziński [10, Theorem 8.3.1] state the conditions under which the increments of a real-valued Lévy process generate an SM.

For deterministic measurable functions $g : \mathsf{X} \to \mathbb{R}$, an integral of the form $\int_{\mathsf{X}} g\,\mathrm{d}\mu$ is studied in [14] (see also [10, Chap. 7], [2, 13]). The construction of this integral is standard, uses an approximation by simple functions, and is based on results of [24–26]. In particular, every bounded measurable g is integrable w.r.t. any μ. An analogue of the Lebesgue dominated convergence theorem holds for this integral (see [10, Proposition 7.1.1] or [14, Corollary 1.2]).

3 Properties of the Integral w.r.t. Stochastic Measure

In this section we prove two technical lemmata for integrals w.r.t. SM. These statements and results of Kamont [8] will give us the connection of Besov spaces and sample paths of SMs.

Lemma 1 ([19, 20]). *Let $f_n : \mathsf{X} \to \mathbb{R}$, $n \geq 1$ be measurable functions such that $\bar{f}(x) = \sum_{n=1}^{\infty} |f_n(x)|$ is integrable w.r.t. μ. Then*

$$\sum_{n=1}^{\infty} \left(\int_{\mathsf{X}} f_n\,\mathrm{d}\mu \right)^2 < \infty \quad a.\ s.$$

Proof. Denote $\zeta_n(\omega) = \int_{\mathsf{X}} f_n\,\mathrm{d}\mu$. Suppose the statement were false. Then

$$\exists \varepsilon_0 > 0\ \forall M > 0\ \exists m \geq 1 :\ \mathsf{P}\left(\Omega_{m,M} = \left\{ \omega \in \Omega : \sum_{n=1}^{m} \zeta_n^2(\omega) \geq M \right\} \right) \geq \varepsilon_0.$$

Consider independent Bernoulli random variables ε_n defined on some other probability space $(\Omega', \mathcal{F}', \mathsf{P}')$, $\mathsf{P}'(\varepsilon_n = 1) = \mathsf{P}'(\varepsilon_n = -1) = 1/2$. The following is a consequence of the Paley–Zygmund inequality

$$\mathsf{P}'\left[\left(\sum_{n=1}^{m} \lambda_n \varepsilon_n \right)^2 \geq \frac{1}{4} \sum_{n=1}^{m} \lambda_n^2 \right] \geq \frac{1}{8}, \quad \lambda_n \in \mathbb{R}$$

(see, e.g., [10, Lemma 0.2.1] for $\lambda = 1/4$). Thus

$$\mathsf{P}'\left[\omega' : \left(\sum_{n=1}^{m} \varepsilon_n(\omega')\zeta_n(\omega)\right)^2 \geq \frac{M}{4}\right] \geq \frac{1}{8}$$

for all $\omega \in \Omega_{m,M}$. Integrating over set $\Omega_{m,M}$, we get

$$\mathsf{P} \times \mathsf{P}'\left[(\omega, \omega') : \left(\sum_{n=1}^{m} \varepsilon_n(\omega')\zeta_n(\omega)\right)^2 \geq \frac{M}{4}\right] \geq \frac{\varepsilon_0}{8} .$$

Hence there exists $\omega_0' \in \Omega'$ such that

$$\mathsf{P}\left[\omega : \left(\sum_{n=1}^{m} \varepsilon_n(\omega_0')\zeta_n(\omega)\right)^2 \geq \frac{M}{4}\right] \geq \frac{\varepsilon_0}{8} .$$

Since $\varepsilon_n(\omega_0') = \pm 1$, for the function $\bar{g}(x) = \sum_{n=1}^{m} \varepsilon_n(\omega_0') f_n(x)$, we have

$$|\bar{g}(x)| \leq \bar{f}(x), \quad \mathsf{P}\left[\left|\int_{\mathsf{X}} \bar{g}\, \mathrm{d}\mu\right| \geq \frac{\sqrt{M}}{2}\right] \geq \frac{\varepsilon_0}{8} .$$

Recall that $\varepsilon_0 > 0$ is fixed while M is arbitrary. By the dominated convergence theorem [10, Proposition 7.1.1], the set function $\int_{\mathsf{B}} \bar{f}\, \mathrm{d}\mu$, $\mathsf{B} \in \mathcal{B}$, is an SM. Applying [13, Lemma 1] (or [14, Theorem 1.2]), we observe a contradiction with the boundedness in probability of the set of values of the SM (see [25, Theorem A] or [10, Theorem B.2.1]). □

Further, we consider isotropic Besov spaces $B^{\alpha}_{pq}\left([0,1]^d\right)$, $1 \leq p, q < \infty, 0 < \alpha < 1$. For $f \in \mathsf{L}_p\left([0,1]^d\right)$, we put

$$\|f\|^{\alpha}_{p,q} = \|f\|_{\mathsf{L}_p([0,1]^d)} + \left(\int_0^1 \left(w_p(f,t)\right)^q t^{-\alpha q - 1}\, \mathrm{d}t\right)^{1/q},$$

where

$$w_p(f,t) = \sup_{|z| \leq t}\left(\int_{\mathsf{I}_z} |f(x+z) - f(x)|^p\, \mathrm{d}x\right)^{1/p},$$

$$\mathsf{I}_z = \left\{x \in [0,1]^d : x + z \in [0,1]^d\right\}.$$

We have

$$B^{\alpha}_{pq}\left([0,1]^d\right) = \left\{f \in \mathsf{L}_p\left([0,1]^d\right) : \|f\|^{\alpha}_{pq} < +\infty\right\}.$$

$\|\cdot\|^{\alpha}_{pq}$ is the norm in $B^{\alpha}_{pq}\left([0,1]^d\right)$.

A. Kamont obtained the following discrete characterization of Besov spaces. Let e_i be the ith unit vector in $\mathbb{R}^d$,

$$U(n,i) = \left\{ y = \left(\frac{k_1}{2^n}, \frac{k_2}{2^n}, \ldots, \frac{k_d}{2^n} \right) \middle| k_j = 0, 1, \ldots, 2^n, \right.$$
$$\left. 1 \le j \le d;\ y + 2^{-n} e_i \in [0,1]^d \right\}.$$

If $f \in \mathbb{C}\left([0,1]^d\right)$, then by [8, Corollary 3.3]

$$\|f\|_{pp}^{\alpha} \le C \left(f(0) + \sum_{i=1}^{d} \left(\sum_{n=1}^{\infty} 2^{n(\alpha p - d)} \sum_{y \in U(n,i)} |f(y + 2^{-n} e_i) - f(y)|^p \right)^{1/p} \right) \tag{3}$$

for some $C > 0$ independent of f. For $d = 1$ and $(1/p) < \alpha < 1$, the right-hand side of (3) is equivalent to the norm $\|f\|_{pp}^{\alpha}$ [8, Theorem 1.1]. In the sequel, C will denote a positive constant that is not important for our estimates.

For all $n \ge 0$, put

$$d_{kn} = k2^{-n}, \quad 0 \le k \le 2^n, \quad \Delta_{kn} = \left(d_{(k-1)n}, d_{kn}\right], \quad 1 \le k \le 2^n.$$

The following statement helps to prove the regularity of sample paths of integrals w.r.t. SM.

Lemma 2 ([19]). *Let μ be defined on Borel $\sigma-$algebra of $\mathbb{R}$, Z be an arbitrary set, and $q(z,x) : Z \times [0,1] \to \mathbb{R}$ be a function such that for some $1/2 < \alpha < 1$ and for each $z \in Z$ we have $q(z,\cdot) \in B_{22}^{\alpha}([0,1])$. Then the random function*

$$\eta(z) = \int_{[0,1]} q(z,x)\,\mathrm{d}\mu(x), \quad z \in Z,$$

has a version $\tilde{\eta}(z)$ such that for some constant C (independent of z, ω) and each $\omega \in \Omega$,

$$|\tilde{\eta}(z)| \le |q(z,0)\mu([0,1])| + C\,\|q(z,\cdot)\|_{B_{22}^{\alpha}([0,1])} \left\{ \sum_{n \ge 1} 2^{n(1-2\alpha)} \sum_{1 \le k \le 2^n} |\mu(\Delta_{kn})|^2 \right\}^{1/2}.$$

Proof. Consider the functions

$$q_n(z,x) = q(z,0)\,\mathbf{1}_{\{0\}}(x) + \sum_{1 \le k \le 2^n} q\left(z, d_{(k-1)n}\right) \mathbf{1}_{\Delta_{kn}}(x), \quad n \ge 0. \tag{4}$$

From the properties of Besov spaces, it follows that for $\alpha > 1/2$ we have $B_{22}^{\alpha}([0,1]) \subset \mathbb{C}([0,1])$. The dominated convergence theorem implies that

$$\int_{[0,1]} q_n(z,x)\,\mathrm{d}\mu(x) \xrightarrow{\mathsf{P}} \int_{[0,1]} q(z,x)\,\mathrm{d}\mu(x), \quad n\to\infty,$$

for each z. Therefore

$$\tilde{\eta}(z) = \int_{[0,1]} q_0(z,x)\,\mathrm{d}\mu(x) + \sum_{n\ge 1}\left(\int_{[0,1]} q_n(z,x)\,\mathrm{d}\mu(x) - \int_{[0,1]} q_{n-1}(z,x)\,\mathrm{d}\mu(x)\right)$$

is a version of $\eta(z)$. Using (4) and the Cauchy–Schwartz inequality, we obtain for any $\beta > 0$

$$\begin{aligned}
&\sum_{n\ge 1}\left|\int_{[0,1]} q_n(z,x)\,\mathrm{d}\mu(x) - \int_{[0,1]} q_{n-1}(z,x)\,\mathrm{d}\mu(x)\right| \\
&\quad\le \sum_{n\ge 1}\sum_{1\le k\le 2^n}\left|q\left(z, d_{(k-1)n}\right) - q\left(z, d_{(k'-1)(n-1)}\right)\right| \left|\mu\left(\Delta_{kn}\right)\right| \\
&\quad\le \left\{\sum_{n\ge 1}\sum_{1\le k\le 2^n} 2^{2n\beta}\left|q\left(z, d_{(k-1)n}\right) - q\left(z, d_{(k'-1)(n-1)}\right)\right|^2\right\}^{1/2} \\
&\qquad\times \left\{\sum_{n\ge 1}\sum_{1\le k\le 2^n} 2^{-2n\beta}\left|\mu\left(\Delta_{kn}\right)\right|^2\right\}^{1/2}.
\end{aligned}$$

(The number k' is chosen such that $\Delta_{kn}\subset\Delta_{k'(n-1)}$.) Applying [8, Theorem 1.1] for $\alpha = \beta + 1/2$, we get the statement of the lemma. □

Note that by Lemma 1 for $\beta > 0$,

$$\sum_{n\ge 1}\sum_{1\le k\le 2^n} 2^{-2n\beta}\left|\mu\left(\Delta_{kn}\right)\right|^2 < +\infty \text{ a. s.} \tag{5}$$

Further, we formulate two analogues of the Fubini theorem for the product of SM and real-valued measure.

Let $(\mathsf{X}, \mathcal{B}_{\mathsf{X}})$ and $(\mathsf{Y}, \mathcal{B}_{\mathsf{Y}})$ be measurable spaces, μ be an SM on $\mathcal{B}_{\mathsf{X}}$, and m be a finite real nonnegative measure on $\mathcal{B}_{\mathsf{Y}}$. Set $\mathsf{Z} = \mathsf{X}\times\mathsf{Y}$, and let $\mathcal{B}_{\mathsf{Z}} = \mathcal{B}_{\mathsf{X}}\otimes\mathcal{B}_{\mathsf{Y}}$ be the product σ-algebra.

Theorem 1 ([15]). *There exists a unique SM ν on $\mathcal{B}_{\mathsf{Z}}$ such that for all $\mathsf{A}_1\in\mathcal{B}_{\mathsf{X}}$, $\mathsf{A}_2\in\mathcal{B}_{\mathsf{Y}}$ holds $\nu(\mathsf{A}_1\times\mathsf{A}_2) = \mu(\mathsf{A}_1)\mathsf{m}(\mathsf{A}_2)$. If $f:\mathsf{Z}\to\mathbb{R}$ is integrable w.r.t. ν on Z, then the function $f(x,\cdot):\mathsf{Y}\to\mathbb{R}$ is integrable w.r.t. m on Y for each $x\in\mathsf{X}$ excluding μ-negligible set; the function $\int_{\mathsf{Y}} f(x,y)\,\mathsf{dm}(y)$ is integrable w.r.t. μ on X, and*

$$\int_{\mathsf{Z}} f(x, y)\,\mathrm{d}\nu = \int_{\mathsf{X}} \mathrm{d}\mu(x) \int_{\mathsf{Y}} f(x, y)\,\mathsf{dm}(y)\ .$$

Theorem 2 ([15]). *Let $\mathsf{X} = [a, b] \subset \mathbb{R}$ and $\mathcal{B}_{\mathsf{X}}$ be the Borel σ-algebra. Assume that $f : \mathsf{Z} \to \mathbb{R}$ be a bounded measurable function such that for some $\gamma > 1/2$ and $L > 0$ holds*

$$|f(x_1, y) - f(x_2, y)| \le L|x_1 - x_2|^{\gamma}\ . \tag{6}$$

Then

$$\int_{\mathsf{Z}} f(x, y)\,\mathrm{d}\nu = \int_{\mathsf{Y}} \mathsf{dm}(y) \int_{\mathsf{X}} f(x, y)\,\mathrm{d}\mu(x)\ . \tag{7}$$

Integral w.r.t. dm in (7) is found for each fixed $\omega \in \Omega$. Note that we need condition (6) in Theorem 2 because it helps to obtain the regularity of integral w.r.t. $\mathrm{d}\mu$ in (7).

4 Paths of Stochastic Measures and Stochastic Integrals

4.1 Besov Regularity of Paths of Stochastic Measures

From (3) and (5) we easily get the following result for an SM on subsets of $\mathbb{R}$.

Theorem 3 ([16, 20]). *Let $\mathsf{X} = [0, 1]$, $\mathcal{B}$ be the Borel σ-algebra, and the process*

$$\tilde{\mu}(t) = \mu([0, t])\ , \quad 0 \le t \le 1\ ,$$

has continuous paths. Then for any

$$1 \le p < +\infty\ , \quad 0 < \alpha < \min\{1/p, 1/2\}\ ,$$

the path of $\tilde{\mu}(t)$, $0 \le t \le 1$ with probability 1 belongs to the Besov space $B^{\alpha}_{pp}([0, 1])$.

For an SM on subsets of $\mathbb{R}^d$, $d > 1$, an analogous result was proved under additional assumption on integrability w.r.t. μ.

Theorem 4 ([20]). *Let $\mathsf{X} = [0, 1]^d$, $\mathcal{B}$ be the Borel σ-algebra and*

$$\tilde{\mu}(x) = \mu\left(\prod_{i=1}^{d} [0, x_i]\right), \quad x = (x_1, \ldots, x_d) \in [0, 1]^d,$$

be a pathwise continuous process. Suppose there exists a finite real nonnegative measure m *on* $\big([0,1]^d,\ \mathcal{B}\big)$ *such that each*

$$h : [0,1]^d \to \mathbb{R}, \quad \int_{[0,1]^d} h^2 \, \mathrm{d}\mathsf{m} < +\infty \,,$$

is integrable w.r.t μ *on* $[0,1]^d$.

Than for any

$$1 \le p < +\infty, \quad 0 < \alpha < \min\{1/p, 1/2\}$$

the path of $\tilde{\mu}(x), x \in [0,1]^d$, *with probability 1 belongs to the Besov space* $B^{\alpha}_{pp}\big([0,1]^d\big)$.

4.2 Continuity of Paths of Parameter Stochastic Integrals

Let $\mathsf{X} = [0,1]$, $\mathcal{B}$ be the Borel σ-algebra.

Theorem 5 ([17]). *Let* $\mathbb{T}$ *be a metric space, and function* $f : [0,1] \times \mathbb{T} \to \mathbb{R}$ *be such that for each* $x \in [0,1]$ $f(x,\cdot)$ *is continuous on* $\mathbb{T}$, *and for some* $\gamma > 1/2$ *and* $L > 0$ *holds*

$$|f(x,t) - f(y,t)| \le L|x-y|^{\gamma}.$$

Then the random function

$$\eta(t) = \int_{[0,1]} f(x,t) \, \mathrm{d}\mu(x) \,, \quad t \in \mathbb{T},$$

has a version with continuous on $\mathbb{T}$ *paths.*

The proof of Theorem 5 is based on Lemma 2. Similar considerations give the following result.

Theorem 6 ([17]). *Let the process* $\tilde{\mu}(x) = \mu([0,x]),\ 0 \le x \le 1$ *be continuous. Suppose* $h : [0,1] \to \mathbb{R}$ *is such that for some* $\gamma > 1/2$ *and* $L > 0$ *holds*

$$|h(x) - h(y)| \le L|x-y|^{\gamma}.$$

Then the random function

$$\tilde{\mu}_h(t) = \int_{[0,t]} h(x) \, \mathrm{d}\mu(x), \quad 0 \le t \le 1,$$

has a continuous version on $[0,1]$.

5 Mild Solution of the Heat Equation Driven by Stochastic Measure

Using the properties of integrals w.r.t. SM, we can study SPDEs with rather general integrator in stochastic term.

Let μ be an SM on Borel σ-algebra on $\mathbb{R}$. Consider the stochastic heat equation (1) in the following mild sense:

$$u(x,t) = \int_{\mathbb{R}} p(x-y,t)u_0(y)\,\mathrm{d}y + \int_0^t \mathrm{d}s \int_{\mathbb{R}} p(x-y,t-s) f(y,s,u(y,s))\,\mathrm{d}y$$
$$+ \int_{\mathbb{R}} \mathrm{d}\mu(y) \int_0^t p(x-y,t-s)\sigma(y,s)\,\mathrm{d}s, \quad x \in \mathbb{R},\ t \in (0,T], \tag{8}$$

where p is a Gaussian heat kernel,

$$p(x,t) = \frac{1}{2a\sqrt{\pi t}} \exp\left\{-\frac{|x|^2}{4a^2 t}\right\}.$$

Assumption 1. $u_0(y) = u_0(y,\omega) : \mathbb{R}\times\Omega \to \mathbb{R}$ is measurable and bounded, $|u_0(y,\omega)| \le C_{u_0}(\omega)$.

Assumption 2. $u_0(y)$ is Hölder continuous in $y \in \mathbb{R}$,

$$|u_0(y_1) - u_0(y_2)| \le L_{u_0}(\omega)\,|y_1 - y_2|^{\beta(u_0)}, \quad \beta(u_0) \ge 1/6.$$

Assumption 3. $f(y,s,z) : \mathbb{R}\times[0,T]\times\mathbb{R} \to \mathbb{R}$ is measurable and bounded.

Assumption 4. $f(y,s,z)$ is uniformly Lipschitz in $y,\ z \in \mathbb{R}$,

$$|f(y_1,s,z_1) - f(y_2,s,z_2)| \le L_f\,(|y_1-y_2| + |z_1-z_2|)\,.$$

Assumption 5. $\sigma(y,s) : \mathbb{R}\times[0,T] \to \mathbb{R}$ is measurable and bounded.

Assumption 6. $\sigma(y,s)$ is uniformly Hölder continuous in y,

$$|\sigma(y_1,s) - \sigma(y_2,s)| \le L_\sigma\,|y_1-y_2|^{\beta(\sigma)}, \quad \beta(\sigma) > 1/2.$$

Also denote

$$\tilde{\beta} = \min\{2\beta(\sigma), 3/2\}.$$

Conditions of the existence, uniqueness, and Hölder regularity of solution of (8) are given in the following theorem.

Theorem 7 ([19]). *Suppose Assumptions 1–6 hold.*

(1) Equation (8) has a solution $u(x,t)$. If $v(x,t)$ is another solution to (8), then for each t and x $u(x,t) = v(x,t)$ a.s.

(2) Let the function $|y|^\tau$ be integrable w.r.t. μ on $\mathbb{R}$ for some $\tau > 3/2$. Then, for any fixed

$$t \in [0,T], \quad K > 0, \quad \gamma_1 < \frac{\tilde{\beta}-1}{2\tilde{\beta}},$$

stochastic process $u(x,t)$, $x \in [-K,K]$, has a Hölder continuous version with exponent γ_1.

(3) Let the function $|y|^\tau$ be integrable w.r.t. μ on $\mathbb{R}$ for some $\tau > 5/2$. Then for any fixed

$$K > 0, \quad \delta > 0, \quad \gamma_1 < \frac{\tilde{\beta}-1}{2\tilde{\beta}}, \quad \gamma_2 < \frac{\tilde{\beta}-1}{6\tilde{\beta}},$$

the stochastic function $u(x,t)$ has a version $\tilde{u}(x,t)$ such that for some $L_{\tilde{u}}(\omega) > 0$

$$|\tilde{u}(x_1,t_1)-\tilde{u}(x_2,t_2)| \le L_{\tilde{u}}(\omega)\left(|x_1-x_2|^{\gamma_1} + |t_1-t_2|^{\gamma_2}\right), \quad x\in[-K,K], \quad t\in[\delta,T].$$

The stochastic term in (8) may be written in another form.

Theorem 8 ([21]). *Suppose Assumptions 5 and 6 hold, and function $|y|^\tau$ is integrable w.r.t. μ on $\mathbb{R}$ for some $\tau > 1/2$. Then*

$$\int_{\mathbb{R}} \mathrm{d}\mu(y) \int_0^t p(x-y,t-s)\sigma(y,s)\,\mathrm{d}s = \int_0^t \mathrm{d}s \int_{\mathbb{R}} p(x-y,t-s)\sigma(y,s)\,\mathrm{d}\mu(y)$$
$$= \int_{\mathbb{R}\times[0,t]} p(x-y,t-s)\sigma(y,s)\,\mathrm{d}\nu(y,s),$$

where ν is a product of μ and $\mathrm{d}s$ on $\mathbb{R}\times[0,T]$ (see Theorem 1).

6 Riemann Integral of a Random Function

If we consider the integral of a random function w.r.t. a real measure in pathwise sense, then the possibility to change the order of integration has been proved under some restrictive assumptions only (see Theorems 2 and 8). In order to avoid too restrictive conditions in Fubini-type theorems, we define the integral of random function w.r.t. real measure (Jordan content) in Riemann sense.

Let $\mathsf{B} \subset \mathbb{R}^d$ be a Jordan measurable set and $\xi : \mathsf{B} \to \mathsf{L}_0$ be a random function. We shall say that ξ has an integral on B if for any sequence of partitions

$$\mathsf{B} = \cup_{1\le k\le k_n} \mathsf{B}_{kn},\ n \ge 1,\ \max_k \operatorname{diam} \mathsf{B}_{kn} \to 0,\ n \to \infty,$$

and any $x_{kn} \in \mathsf{B}_{kn}$, the limit in probability

$$\mathrm{p}\lim_{n\to\infty} \sum_{1\le k\le k_n} \xi(x_{kn}) \mathsf{m}_d(\mathsf{B}_{kn}) = \int_{\mathsf{B}} \xi(x)\,\mathrm{d}x$$

exists. Here m_d denotes the Jordan content in $\mathbb{R}^d$; sets $\mathsf{B}_{kn},\ 1 \le k \le k_n$, are assumed to be Jordan measurable and have no common interior points.

Definition 2. Random function ξ is called *integrable on* B if ξ has an integral on B and the set of values $\{\xi(x),\ x \in \mathsf{B}\}$ is bounded in probability.

Let $\tilde{\mathsf{B}} \subset \mathbb{R}^d$ be an unbounded set for which there exists a sequence of Jordan measurable sets $\mathsf{B}^{(j)}$ such that

$$\mathsf{B}^{(j)} \uparrow \tilde{\mathsf{B}}, \quad \forall c > 0\ \exists j :\ \tilde{\mathsf{B}} \cap \{|x| \le c\} \subset \mathsf{B}^{(j)}. \tag{9}$$

We shall say that ξ is integrable (in improper sense) on $\tilde{\mathsf{B}}$, if ξ is integrable on each $\mathsf{B}^{(j)}$, and there exists the limit in probability

$$\mathrm{p}\lim_{j\to\infty} \int_{\mathsf{B}^{(j)}} \xi(x)\,\mathrm{d}x = \int_{\tilde{\mathsf{B}}} \xi(x)\,\mathrm{d}x,$$

that is independent of choice of $\mathsf{B}^{(j)}$.

Theorem 9 ([22]). *Let μ be an SM on $(\mathsf{X}, \mathcal{B})$ and $\tilde{\mathsf{B}} \subset \mathbb{R}^d$ be an unbounded set. Assume that $h(x,s) : \tilde{\mathsf{B}} \times \mathsf{X} \to \mathbb{R}$ is a measurable deterministic function which is Riemann integrable on $\tilde{\mathsf{B}}$ in improper sense for each fixed s, and $|h(x,s)| \le g(s)$, $\int_{\tilde{\mathsf{B}}} |h(x,s)|\,\mathrm{d}x = g_1(s)$, where $g,\ g_1 : \mathsf{X} \to \mathbb{R}$ are integrable on X w.r.t. μ. Then the random function $\xi(x) = \int_{\mathsf{X}} h(x,s)\,\mathrm{d}\mu(s)$ is integrable on $\tilde{\mathsf{B}}$ in improper sense, and*

$$\int_{\tilde{\mathsf{B}}} \mathrm{d}x \int_{\mathsf{X}} h(x,s)\,\mathrm{d}\mu(s) = \int_{\mathsf{X}} \mathrm{d}\mu(s) \int_{\tilde{\mathsf{B}}} h(x,s)\,\mathrm{d}x.$$

Some other properties of the Riemann integral of a random function are considered in [22].

7 Parabolic Equation with a General Stochastic Measure

Consider the differential operator

$$Ag(x) = \sum_{1\le i,j\le d} a_{ij}(x)\frac{\partial^2 g(x)}{\partial x_i \partial x_j} + \sum_{1\le i\le d} b_i(x)\frac{\partial g(x)}{\partial x_i} + c(x)g(x),$$

where $x \in \mathbb{R}^d$, $g : \mathbb{R}^d \to \mathbb{R}$, and $a_{ij} = a_{ji}$. Suppose that A is strongly elliptic (see [7, Equation (4.5)]).

Assumption 7. All functions $a_{ij},\ b_i,\ c,\ \frac{\partial a_{ij}}{\partial x_i},\ \frac{\partial^2 a_{ij}}{\partial x_i \partial x_j},\ \frac{\partial b_i}{\partial x_i}$ are bounded and Hölder continuous in $\mathbb{R}^d$.

Let μ be an SM on Borel subsets of $[0, T]$. We consider (2) in the weak sense, i.e.,

$$\begin{aligned}\int_{\mathbb{R}^d} X(x,t)\varphi(x)\,\mathrm{d}x = \int_{\mathbb{R}^d} \xi(x)\varphi(x)\,\mathrm{d}x \\ + \int_{\mathbb{R}^d} A^*\varphi(x)\,\mathrm{d}x \int_0^t X(x,s)\,\mathrm{d}s \\ + \int_{[0,t]} \mathrm{d}\mu(s) \int_{\mathbb{R}^d} h(x,s)\varphi(x)\,\mathrm{d}x \end{aligned} \tag{10}$$

for all test functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$ (rapidly decreasing Schwartz functions from $\mathbb{C}^\infty(\mathbb{R}^d)$). For each fixed $t \in [0, T]$, equality (10) holds a.s. Integrals of random functions w.r.t. real measures are considered in Riemann sense (see Sect. 6), and A^* denotes the adjoint operator of A.

Assumption 8. $\xi : \mathbb{R}^d \to \mathsf{L}_0$ such that $\xi(\cdot, \omega)$ is continuous and bounded in $\mathbb{R}^d$ for each fixed $\omega \in \Omega$.

Assumption 9. $h : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ is Borel measurable,

$$\sup_t |x|^{-k}|h(x,t)| \to 0,\ |x| \to \infty,\ \text{for some } k > 0,$$

$h(x, \cdot)$ is continuous and bounded in $\mathbb{R}^d$ for each fixed $t \in [0, T]$.

By [7, Theorem 1, Sect. 4], under Assumption 7, the equation $\partial g/\partial t = Ag$ has a fundamental solution $p_A(x, y, t - s)$ (recall that coefficients of A do not depend on t). Consider the semigroup

$$S(t)g(x) = \int_{\mathbb{R}^d} p_A(x,y,t)g(y)\,\mathrm{d}y, \quad t > 0, \quad S(0)g(x) = g(x).$$

Theorem 10 ([22]). *Suppose Assumptions 7–9 hold. Then the random function*

$$X(x,t) = S(t)\xi(x) + \int_{[0,t]} [S(t-s)h(x,s)]\,\mathrm{d}\mu(s) \tag{11}$$

is the solution to (10).

In addition, suppose the operator A is self-adjoint, $X(x,t)$ satisfies (10), is integrable on $\mathbb{R}^d \times [0,T]$ w.r.t. $\mathrm{d}x \times \mathrm{d}t$, is integrable on $\mathbb{R}^d$ w.r.t. $\mathrm{d}x$ for each fixed t, and is integrable on $[0,T]$ w.r.t. $\mathrm{d}t$ for each fixed x. Then $X(x,t)$ is given by (11).

Example. Let SM μ be generated by a continuous square integrable martingale Y, $\mu(\mathsf{A}) = \int_0^T \mathbf{1}_{\mathsf{A}}(t)\,\mathrm{d}Y(t)$, and λ be the Lebesgue measure in $\mathbb{R}^d$. Then $M_t(\mathsf{A}) = Y(t)\lambda(\mathsf{A})$, $0 \le t \le T$, $\mathsf{A} \subset \mathbb{R}^d$, is a worthy martingale measure with the dominating measure

$$K(\mathsf{A} \times \mathsf{B} \times (0,t]) = \left|\langle Y\rangle_t\right| \lambda(\mathsf{A})\,\lambda(\mathsf{B})$$

(we use the terminology of [27]). In this case, (11) leads to

$$\begin{aligned} X(x,t) &= \int_{\mathbb{R}^d} p_A(x,y,t)\xi(y)\mathrm{d}y + \int_{[0,t]} \mathrm{d}\mu(s) \int_{\mathbb{R}^d} p_A(x,y,t-s)h(y,s)\mathrm{d}y \\ &= \int_{\mathbb{R}^d} p_A(x,y,t)\xi(y)\mathrm{d}y + \int_{[0,t]\times\mathbb{R}^d} p_A(x,y,t-s)h(y,s)\,M(\mathrm{d}y\mathrm{d}s). \end{aligned} \tag{12}$$

The results of [27, Chap. 2] imply that the integral w.r.t. $M(\mathrm{d}y\,\mathrm{d}s)$ is well defined and is the limit of integrals of simple functions. For simple function, equality of two stochastic integrals in (12) is obvious. Further, we can use the dominated convergence theorem for integral with respect to $\mathrm{d}\mu(s)$. Thus, for self-adjoint A and $\xi = 0$, (12) coincides with the solution given in [27, Theorem 5.1].

Similar solution of parabolic SPDE w.r.t. a general martingale measure we have in Example 9 and Remark 20 [3].

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References

1. Albeverio, S., Wua, J.-L., Zhang, T.-S.: Parabolic SPDEs driven by Poisson white noise. Stoch. Process. Appl. **74**, 21–36 (1998)
2. Curbera, G.P., Delgado, O.: Optimal domains for L_0-valued operators via stochastic measures. Positivity **11**, 399–416 (2007)
3. Dalang, R.C.: Extending martingale measure stochastic integral with applications to spatially homogeneous s.p.d.e's, Electron. J. Probab. **4**, 1–29 (1999)
4. Dalang, R.C., Sanz-Solé, M.: Regularity of the sample paths of a class of second-order SPDE's. J. Funct. Anal. **227**, 304–337 (2005)

5. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions. Cambridge University Press, Cambridge (1992)
6. Dettweiler, J., Weis, L., van Neerven, J.: Space-time regularity of solutions of the parabolic stochastic Cauchy problem. Stoch. Anal. Appl. **24**, 843–869 (2006)
7. Ilyin, A.M., Kalashnikov, A.S., Oleynik, O.A.: Linear second-order partial differential equations of the parabolic type. J. Math. Sci. (N.Y.) **108**, 435–542 (2002)
8. Kamont, A.:A discrete characterization of Besov spaces. Approx. Theory Appl. **13**, 63–77 (1997)
9. Kotelenez, P.: Stochastic Ordinary and Stochastic Partial Differential Equations: Transition from Microscopic to Macroscopic Equations. Springer, Berlin (2007)
10. Kwapień, S., Woycziński, W.A.: Random Series and Stochastic Integrals: Single and Multiple. Birkhäuser, Boston (1992)
11. Memin, J., Mishura, Yu., Valkeila, E.: Inequalities for the moments of Wiener integrals with respect to a fractional Brownian motion. Stat. Probab. Lett. **27**, 197–206 (2001)
12. Peszat, S., Zabczyk, J.: Stochastic partial differential equations with Lévy noise: an evolution equation approach. Cambridge University Press, Cambridge (2007)
13. Radchenko, V.: Convergence of integrals of unbounded real functions with respect to random measures. Theory Probab. Appl. **42**, 310–314 (1998)
14. Radchenko, V.: Integrals with respect to general stochastic measures. Institute of Mathematics, Kyiv (1999) (in Russian)
15. Radchenko, V.: On the product of a random and of real measure. Theory Probab. Math. Stat. **70**, 161–166 (2005)
16. Radchenko, V.: Besov regularity of stochastic measures. Stat. Probab. Lett. **77**, 822–825 (2007)
17. Radchenko, V.: Parameter-dependent integrals with general random measures. Theory Probab. Math. Stat. **75**, 161–165 (2007)
18. Radchenko, V.: Heat equation and wave equation with general stochastic measures. Ukr. Math. J. **60**, 1968–1981 (2008)
19. Radchenko, V.: Mild solution of the heat equation with a general stochastic measure. Stud. Math. **194**, 231–251 (2009)
20. Radchenko, V.: Sample functions of stochastic measures and Besov spaces. Theory Probab. Appl. **54**, 160–168 (2010)
21. Radchenko, V.: Properties of integrals with respect to a general stochastic measure in a stochastic heat equation. Theory Probab. Math. Stat. **82**, 103–114 (2011)
22. Radchenko, V.: Riemann integral of a random function and the parabolic equation with a general stochastic measure. Teor. Imovir. Mat. Stat. **87**, 163–175 (2012)
23. Radchenko, V., Zähle, M.: Heat equation with a general stochastic measure on nested fractals. Stat. Prob. Lett. **82**, 699–704 (2012)
24. Ryll-Nardzewski, C., Woyczyński, W.A.: Bounded multiplier convergence in measure of random vector series. Proc. Am. Math. Soc. **53**, 96–98 (1975)
25. Talagrand, M.: Les mesures vectorielles à valeurs dans L_0 sont bornées. Ann. Sci. École Norm. Sup. (4) **14**, 445–452 (1981)
26. Turpin, Ph.: Convexités dans les espaces vectoriels topologiques généraux. Dissertationes Math. **131**, pp. 220 (1976)
27. Walsh, J.B.: An Introduction to Stochastic Partial Differential Equations. Lecture Notes in Matematics, vol. 1180, pp. 236–434. Springer, Berlin (1984)

Part III
Limit Theorems

Exponential Convergence of Degenerate Hybrid Stochastic Systems with Full Dependence

Svetlana V. Anulova and Alexander Yu. Veretennikov

Abstract This research stems from a control problem for a suspension device. For a general class of switching stochastic mechanical systems (including closed-loop control ones), we establish the following: (1) existence and uniqueness of a weak solution and its strong Markov property, (2) mixing property in the form of the local Markov–Dobrushin condition, and (3) exponentially fast convergence to the unique stationary distribution. These results are proved for discontinuous coefficients under nondegenerate disturbances in the force field; for (3) a stability condition is additionally imposed. Linear growth of coefficients is allowed.

1 Introduction

Convergence of marginal distributions of (Markov) stochastic systems to a stationary one has been thoroughly studied, and there are classic schemes for proving this property. At the level of ideas, if two facts are established—a version of Doeblin's condition and recurrence—then this provides convergence. As a version of the former we use the *local Markov–Dobrushin condition*. Quite often it is provided by the nondegeneracy of the Wiener disturbance. However, in this paper we deal with mechanical systems presented by highly degenerate stochastic differential equations of the Langevin–Smoluchowski type driven by a Wiener noise of "smaller

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dimension". The Wiener term presents a random force in the velocity component equation, while the equation for the state component has, naturally, no Wiener term. Thus, the system is in essence degenerate, and even standard existence and uniqueness results for it require revision.

The recurrence property of a stochastic system may often be reduced to the stability of the corresponding deterministic system (with the removed disturbance term). We establish it in terms of quadratic Lyapunov functions for deterministic systems with switching. The investigation of existence of such functions for general systems is still in progress nowadays; see [7]. Eventually, exponential convergence of marginal distributions in the total variation norm $\|\cdot\|_{TV}$ will be established.

This work, in fact, stems from the investigation of Campillo and Pardoux into the issue of a vehicle suspension device; see [8, 9].

Stochastic ergodic control—in particular, with expected average in time with infinite horizon as cost functional—proved to be a useful tool for constructing a closed-loop control of a vehicle suspension device; see [8, 12] and references therein. In [3] we have generalized the model of the suspension device to a multi-regime one. That is, we admitted several types of the road surface and assumed that the type of the road surface determines a gear box regime and hence also a working regime of the suspension device. This object may be described by a hybrid system (see [6]) with dynamics of a switching diffusion: position of the device X, its velocity Y and the type of the road surface V (the discrete component). Switchings constitute a Markov chain (see [15, 23, 24]). The novelty in comparison to the earlier works is degenerate diffusion and discontinuous coefficients; the former is due to the nature of the device, while the latter is caused by the control framework—optimal control is never smooth. Similar equations without switching have been studied in [1].

The crucial point in applying the technique of ergodic control is establishing the ergodicity property of the controlled process. Our result in [3] is ergodicity in the sense of Markov processes, the state space type (see [18], [17, Ch.6.3]). Moreover, we have shown that under every (homogeneous) admissible control policy, the distribution of the controlled process converges in time to its limit at an exponential rate. The rate of convergence is uniform over all admissible control policies and locally uniform with respect to initial conditions. We emphasize that *control problems* themselves are *not* addressed in the present paper.

We sketch briefly the contents of [3]—and simultaneously some results from [1] as a partial case—in the next section in order to make intelligible the motivation and reasoning of the present work and the investigation in progress.

The paper consists of Introduction (Sect. 1), Reminder about an earlier background model (Sect. 2), Main Results (Sect. 3) and Proofs of Theorems 1 and 2; Sect. 4 contains the proof of Theorem 3 on just three lines and the proof of Theorem 4 given as a sketch with references.

2 Case of Independent Markov Switchings: Reminder

Consider a two-dimensional stochastic differential equation,

$$\begin{aligned} dX_t &= Y_t\, dt, \qquad X_0 = x, \\ dY_t &= b(X_t, Y_t, V_t)\, dt + \sigma(V_t) dW_t, \qquad Y_0 = y. \end{aligned} \tag{1}$$

Here W and V are independent driving processes: the standard Wiener process and a Markov chain, taking values in a finite set $\mathcal{S} = \{1, 2, \dots, N\}$, $t \geq 0$. The generator of V is a matrix $Q = (q_{ij})_{N\times N}$, which determines transition probabilities over a small period of time $\Delta \downarrow 0$:

$$\mathbf{P}(V_{t+\Delta} = j | V_t = i) = \begin{cases} q_{ij}\Delta + o(\Delta) \text{ if } i \neq j, \\ 1 + q_{ii}\Delta + o(\Delta) \text{ if } i = j. \end{cases}$$

All intensities are positive, $q_{ij} > 0$ for $i \neq j$; for $j = i$ the value q_{ii} is defined as $q_{ii} = -\sum_{j:\, j\neq i} q_{ij}$. All trajectories of V are right-continuous step functions without accumulations of jumps (recall that $\mathcal{S}$ is finite and consequently $\max\{q_{ij}, i, j \in \mathcal{S}, i \neq j\} < \infty$).

Further,

$$b(x, y, v) = -u(x, y, v)y - \beta x - \gamma(v)\ \text{sign}(y). \tag{2}$$

Here a function u (the control policy) is Borel measurable and satisfies $u \in [u_1, u_2]$ with two constants $u_1 \leq u_2$. It is assumed that

$$u_1 > 0, \quad \beta > 0, \quad \min_v \gamma(v) > 0, \quad \min_v \sigma(v) > 0. \tag{3}$$

System (1) describes a mechanical "semi-active" suspension device in a vehicle under external stochastic perturbation forces treated as a white noise. The original model without switching V was suggested in [8]. In [3] it was extended to various road types by introducing switching.

In [3] the behavior of the stochastic system (1) under a fixed control policy u was studied, namely, how fast does the system approach its stationary regime. This may be measured by the distance in total variation. Important preliminary results about existence and uniqueness of solutions have been established. We have shown that under our assumptions the stationary regime exists and is unique. It is the discontinuity of u and the degeneracy of the equation that hinders the derivation of our results directly from the general theory of stochastic differential equations.

In the following theorems (quoted from [3]) we fix the values x, y, v— initial conditions for the system (1) and for the driving Markov chain. Existence and uniqueness are understood in a weak sense; see [14, Chap. IV, Definitions 1.2 and 1.4].

Proposition 1 ([3]). *Under the assumptions (3), the system (1) has a weak solution on* $[0,\infty)$ *unique in distribution. The joint process* (X, Y, V) *is also unique in distribution, and these distributions form a strong Markov process.*

Denote the marginal distribution of the triple (X_t, Y_t, V_t) with initial data x, y, v by $\mu_t^{x,y,v}$, $t \geq 0$.

Proposition 2 ([3]). *Under the assumptions (3), there exists a stationary probability distribution* μ_∞ *on* $\mathbb{R}^2 \times \mathcal{S}$ *and positive constants* $\bar{C}, \bar{c}$ *depending on* $\min\{\beta(v), v \in \mathcal{S}\}$, $\min\{\gamma(v), v \in \mathcal{S}\}$, $\min\{\sigma(v), v \in \mathcal{S}\}$, $\max\{\sigma(v), v \in \mathcal{S}\}$, $\min\{q_{ij}; i, j \in \mathcal{S}, i \neq j\}$, $\max\{q_{ij}; i, j \in \mathcal{S}, i \neq j\}$, u_1, u_2, N, *such that*

$$\|\mu_t^{x,y,v} - \mu_\infty\|_{TV} \leq \bar{C} \exp(-\bar{c}t)(1 + x^2 + y^2), \quad t \geq 0. \tag{4}$$

The specification of $\bar{C}, \bar{c}$ assures the rate of convergence to be uniform over all admissible control policies and locally uniform with respect to initial conditions, as stated in the Introduction. Note that although the fixed parameters x, y and v (initial values) are not shown in the left-hand side of (4), the measure μ_t does depend on them.

3 Main Results

3.1 The Model

We want to extend the results of [3] in two directions: (1) to consider general multidimensional mechanical systems and (2) to allow state- dependent switching.

From the theoretical mechanics point of view, we extend the model (1) from the case of one point mass to an ensemble of d point masses being under the influence of a combined force—the resultant of a force field, friction and interaction.

Let $d \geq 1$ and consider a system of stochastic differential equations in $\mathbb{R}^{2d}$: for given $x^1, x^2 \in \mathbb{R}^d$, and $t \geq 0$,

$$\begin{array}{ll} dX_t^1 = X_t^2 \, dt, & X_s^1 = x^1 \in \mathbb{R}^d, \\ dX_t^2 = b(X_t^1, X_t^2) \, dt + dW_t, & X_s^2 = x^2 \in \mathbb{R}^d. \end{array} \tag{5}$$

Here W is a d-dimensional Wiener process and the drift term b is a d-dimensional function. The value $d > 1$ corresponds to the multi-particle case.

Denote $X = (X^1, X^2) \in \mathbb{R}^{2d}$.

Let us now explain what is state-dependent switching. Consider a process X_t, $t \geq 0$, which is a solution of a stochastic differential equation with coefficients additionally depending on a process $(V_t, t \geq 0)$ taking values in a finite set $S = \{1, 2, \ldots, N\}$. The process V is, informally speaking, a conditional Markov chain: given a "frozen" value of $X_t = x$, its generator equals $Q(x) = (q_{ij}(x))_{N \times N}, x \in$

$\mathbb{R}^{2d}$. Informally, this matrix determines transition probabilities over a small period of time given $X_t = x$,

$$\mathbf{P}(V_{t+\Delta} = j | V_t = i, X_t = x) = \begin{cases} q_{ij}(x)\Delta + o(\Delta),\ i \neq j, \\ 1 + q_{ii}(x)\Delta + o(\Delta),\ i = j, \end{cases} \tag{6}$$

where $\Delta \downarrow 0$. For $j = i$ the value $q_{ii}(x)$ is defined as $q_{ii}(x) := -\sum_{j:\, j\neq i} q_{ij}(x)$.

Finally, consider a hybrid SDE system (X^1, X^2, V) in $\mathbb{R}^d \times \mathbb{R}^d \times \mathcal{S} = \mathbb{R}^{2d} \times \mathcal{S}$, $d \geq 1$:

for given $x^1, x^2 \in \mathbb{R}^d,\ v \in \mathcal{S}$

$$\begin{aligned} dX_t^1 &= X_t^2\, dt, \\ dX_t^2 &= b(X_t^1, X_t^2, V_t)\, dt + \sigma(V_t) dW_t, \\ &\quad t \geq 0, X_0^1 = x^1 \in \mathbb{R}^d, X_0^2 = x^2 \in \mathbb{R}^d,\ V_0 = v \in \mathcal{S}. \end{aligned} \tag{7}$$

Here W is a d-dimensional Wiener process, the drift term b is a d-dimensional function, and $\sigma(v)$ is a nondegenerate $d \times d-$matrix. In order to define this object rigorously, we should describe it through its two-component generator $L\phi(x, v) =$

$$\begin{aligned} \left(\frac{\partial \phi}{\partial x^1}(x^1, x^2, v), x^2\right) &+ \left(\frac{\partial \phi}{\partial x^2}(x^1, x^2, v), b(x^1, x^2, v)\right) \\ &+ \sum_{i,j=1}^{d} \sigma_{ij}(v) \frac{\partial^2 \phi}{\partial x_i^2\, \partial x_j^2}(x^1, x^2, v) \\ &+ \sum_{j \in S\setminus v} (\phi(x, j) - \phi(x, v)) q_{vj}(x). \end{aligned} \tag{8}$$

Here generator may be understood in the sense of the martingale problem (see Sect. 5.1 of [4], or [10]); in some papers it is called *extended* generator. This description also makes sense for discontinuous intensities q_{ij}.

Recall that due to the control origin of the model, no regularity may be assumed about the drift term b: it is just Borel measurable and of a no more than linear growth.

3.2 Standing Assumptions

The following assumptions are standing for the system (7).

The values d, N are natural numbers; the points of the euclidean space $\mathbb{R}^{2d}$ are denoted $x = (x^1, x^2)$ (the first and the last d coordinates); $\mathcal{S}$ is the set $\{1, \ldots, N\}$.

(SA 1) Dimension and measurability: $b(x,v) : \mathbb{R}^{2d} \times \mathcal{S} \to \mathbb{R}^d, \sigma(v) : \mathcal{S} \to \mathbb{R}^d \times \mathbb{R}^d$ and $Q : \mathbb{R}^{2d} \to \mathbb{R}^N \times \mathbb{R}^N$ are Borel measurable functions.
(SA 2) Nondegenerate diffusion: the matrix $\sigma(i)\sigma(i)^*$ is nondegenerate for any $i \in \mathcal{S}$.
(SA 3) Uniform linear growth : there exists a constant $C > 0$ such that for all $x \in \mathbb{R}^{2d}$ and $v \in \mathcal{S}$,

$$|b(x,v)| + \|\sigma(v)\| \le C(1+|x|). \tag{9}$$

(SA 4) Intensity bounds: there exist constants $0 < c_l \le c_u < \infty$ such that $c_l \le q_{ij}(x) \le c_u$ for all $x \in \mathbb{R}^{2d}$ and $i,j \in \mathcal{S}, i \ne j$; also, q_{ii} is defined as $q_{ii} := -\sum_{j:\,j\ne i} q_{ij}, i \in \mathcal{S}$.

3.3 Recurrence Assumption

This assumption about a Lyapunov function will be used only in the Sects. 3.6 and 3.7.

(RA 1) There exist a positive definite quadratic function $\phi : \mathbb{R}^{2d} \to [0,\infty)$ and positive constants c_1, c_2 such that

$$\left(\frac{\partial\phi}{\partial x^1}(x), x^2\right) + \left(\frac{\partial\phi}{\partial x^2}(x), b(x,v)\right) \le -c_1\phi(x) + c_2 \text{ for all } (x,v) \in \mathbb{R}^{2d} \times \mathcal{S}. \tag{10}$$

The class of systems satisfying (10) is non-empty: indeed, it includes the system (1)–(2) under the assumption (3) and other likewise models. Proposition 2 itself, actually, prompts why we wish to restrict Lyapunov functions to quadratic ones here. Another argument will be given after the Theorem 3.

3.4 Weak Existence and Uniqueness

Existence and uniqueness are understood in a weak sense; see [14, Chap. IV, Definitions 1.2 and 1.4].

Theorem 1. *Under the assumptions (SA 1)–(SA 4), the system (7) has a weak solution on* $[0,\infty)$ *unique in distribution. This solution forms a strong Markov process.*

Existence and uniqueness for the solution of the considered system may be explained as follows. Take a process with no switching (constructed in [19]) and attach to it a random moment, which is a minimum of all stopping times defined by the switching intensities of transitions to all other discrete states. That is, conditioned on the trajectory of the process, all distributions of these stopping times are "exponential" with corresponding (variable) intensities and independent of each other. Thus, the switched process is constructed up to the first switching. It is clear that its distribution up to the switching moment coincides with that of any solution of the system (7). This construction may be continued further inductively, from one switching moment to the next, and the scheme can be implemented in terms of stochastic differential equations with "rare" jumps—analogues of switchings. Such jumps can be generated with minimal restrictions on jump coefficients—only measurability is required; see [2, 5] and [14, Chap. IV, Sect. 9]. Strong Markov property in [19] was deduced from the Krylov selection method [16], more precisely, due to weak uniqueness. In the present paper, the same idea is helpful and the pasting construction used for establishing existence does preserve the strong Markov property. This procedure will be sketched in the proof of Lemma 2 in the Sect. 4.2.

3.5 Local Markov–Dobrushin Condition

This condition describes the following property of the process satisfying (7). For any two initial states at time zero, let us consider two corresponding processes; then, fix some moment of time and compare *marginal* distributions of the processes at this moment; then, they are *non-singular* and in a certain sense even uniformly in initial states.

This fact is non-trivial, but a simple "philosophical" background for such non-singularity is Girsanov's formula. However, the stochastic integral under the exponent with a non-bounded drift along with degeneracy makes the implementation of this idea technically involved. Namely, to make sure that expressions like $\exp(\int_0^T |b|^2(X_r, V_r))$ are bounded, we will need to consider restricted measures $\mu^{R'}$ with $R' < \infty$ instead of simple μ; see the next paragraph.

Let us define the following objects: $B_R = \{x \in \mathbb{R}^{2d} : |x| < R\}$; $\mu_{s,s+T}(x, v; dydu)$ denotes the transition measure from (s, x, v) to $(s + T; dydu)$; $\mu^{R'}_{s,s+T}(x, v; dydu)$ is the restriction of the transition measure $\mu_{s,s+T}(x, v; dydu)$ to trajectories, whose continuous component does not go beyond the boundary of $B_{R'}$ on $[s, s + T]$; by definition, $B_{+\infty} = \mathbb{R}^{2d}$.

The local Markov–Dobrushin condition, which we need, is formulated for a fixed triple $T > 0, R > 0, R' \in [R, +\infty]$:

$$\inf_{s\in[0,\infty)} \inf_{\substack{x,x'\in B_R,\\ v,v'\in\mathcal{S}}} \mu^{R'}_{s,s+T}(x, v) \wedge \mu^{R'}_{s,s+T}(x', v')(B_{R'} \times \mathcal{S}) > 0. \qquad (11)$$

Here the minimum $\mu \wedge \nu$ of two measures μ and ν is understood in the following way:

$$\mu \wedge \nu(A) := \int_A \left(\frac{d\mu}{d(\mu+\nu)} \wedge \frac{d\mu}{d(\mu+\nu)} \right) (\omega)\, (\mu+\nu)(d\omega).$$

Remark 1. The local Markov–Dobrushin condition, formulated for non-random initial conditions, implies immediately the same statement for distributed initial conditions. Definition (11) suits a nonhomogeneous case; in the homogeneous situation it suffices to take $s = 0$ and drop $\inf_{s\in[0,\infty)}$.

Theorem 2. *Under the assumptions (SA 1)–(SA 4), for any $R > 0$ there exist $T > 0$ and $R' > R$ such that the local Markov–Dobrushin condition (11) holds.*

3.6 Recurrence

Recurrence of stochastic systems is closely related to stability of deterministic systems and often may be reduced to it, although, in some cases, random perturbations may unexpectedly have a positive effect on the recurrence of the system; see [11, 13, 15]. (It is not unexpectedly that the opposite cases also occur.)

We shall conclude the recurrence from the existence of a quadratic Lyapunov function for our system with the removed stochastic term. It is interesting that the problem of existence of quadratic Lyapunov functions is yet unsolved in full generality even for linear deterministic switching systems.

Theorem 3. *Suppose the assumption (RA 1) from the Sect. 3.3 with a function ϕ is fulfilled. Then, there exist positive constants c_1', c_2' such that for the generator L given by (8) the following inequality holds:*

$$L\,\phi(x,v) \le -c_1'\phi(x) + c_2', \qquad x \in \mathbb{R}^{2d},\ v \in \mathcal{S}. \tag{12}$$

The constants c_1', c_2' depend on a function ϕ, constants c_1, c_2, a growth constant C from the inequality (9) and on dimension d.

The *proof* follows straightforward, as the second-order term in $L\,\phi$ adds a constant to the first-order expression and since $\lim_{|x|\to\infty} \phi(x) = +\infty$. Note that it shows that $c_1' = c_1$. □

Remark 2. One more reason why Lyapunov functions here are restricted to quadratic ones is our concern not to overcomplicate the presentation. Indeed, in the quadratic class, the inequality (12) follows easily, while for a general function, we would need strange additional assumptions; yet, clearly, such a class is wider than only quadratic functions.

Applying Ito's or Dynkin's formula to $\phi(X_t, V_t)$—the latter being equivalent to the martingale property, at least, for the appropriately stopped process—it is possible to show the following result. Let $\tau_R := \inf(t \ge s:\ |X_t| \le R)$, $R > 0$.

Corollary 1. *Under the assumptions (SA 1)–(SA 4) and (RA 1), there exist* $\alpha > 0$*,* $R_0 > 0$ *and* $C > 0$ *such that for any* $R \geq R_0$ *(and with* $(X_0, V_0) = (x, v)$*),*

$$\mathbf{E}_{x,v} \exp(\alpha\tau_R) \leq C(1 + |x|^2), \tag{13}$$

and also

$$\mathbf{E}_{x,v}|X_t|^2 1(t < \tau_R) \leq C|x|^2. \tag{14}$$

This statement admits some modifications: as an example, "for any $\alpha > 0$ there exist $R_0, C > 0$ such that for any $R \geq R_0$ the inequality (13) holds". The inequality (14) may be also stated without the indicator in the left-hand side (and with a right-hand side as in (13)), but the proof of this version is less elementary and is not necessary for the proof of the Theorem 4 in the next section.

We provide a brief sketch of the proof of the Corollary 1 for the reader's convenience. Dynkin's formula or, equivalently, the integral form of Ito's formula applied to the process $\exp(\alpha t)\phi(X_t)$ by virtue of (12) implies that

$$\mathbf{E}_{x,v} \exp(\alpha(t \wedge \tau_R))\phi(X_{t\wedge\tau_R}) - \phi(x) + \mathbf{E}_{x,v} \int_0^{t\wedge\tau_R} \exp(\alpha s)(c_1'\phi - c_2' - \alpha)(X_s)\, ds \leq 0. \tag{15}$$

If necessary, this procedure may be accomplished by an appropriate localization. Now, let us choose R so that

$$\inf_{|x|\geq R} (c_1'\phi - c_2')(x) \geq 1.$$

Then (13) follows by Fatou's lemma as $t \to \infty$, at least, if $\alpha \leq 1$.

Further, let $\alpha = 0$. Then it follows from (15) along with $c_1'\phi - c_2' \geq 1 > 0$ that

$$\mathbf{E}_{x,v}\phi(X_t)1(t < \tau_R) \leq \mathbf{E}_{x,v}\phi(X_{t\wedge\tau_R}) \leq \phi(x), \quad |x| > R,$$

and

$$\mathbf{E}_{x,v}\phi(X_t)1(t < \tau_R) = 0, \quad |x| \leq R,$$

the latter because $\tau_R = 0$ for $|x| \leq R$. Since quadratic form ϕ is positive definite, this suffices for (14), as required. □

3.7 Exponential Convergence

In this section, for the process (X, V) satisfying the system (7) with initial values (x, v), its marginal distribution at time is denoted by $\mu_t^{x,v}$, $t \geq 0$.

There is a routine scheme to deduce exponential—and also many others—convergence in total variation from two facts: (1) "minorization" condition of local Markov–Dobrushin type, here provided by the Theorem 2, and (2) recurrence property, regular returns of the trajectory to a certain set satisfying the "minorization" condition, here provided by the Corollary 1. This scheme is expounded in [20, 21], with the local Markov–Dobrushin condition called differently. Note that the Theorem 3 may also be used directly, without the Corollary 1.

Theorem 4. *Suppose the assumption (SA 1)–(SA 4) and (RA 1) are fulfilled. Then there exists a stationary probability distribution μ_∞ on $\mathbb{R}^{2d} \times \mathcal{S}$ and positive constants $\bar{C}, \bar{c}$, depending on d, C, ϕ, c_1, c_2, $\min\{\sigma(v), v \in \mathcal{S}\}$, $\min\{q_{ij}; i, j \in \mathcal{S}, i \neq j\}$, $\max\{q_{ij}; i, j \in \mathcal{S}, i \neq j\}$, N, such that*

$$\|\mu_t^{x,v} - \mu_\infty\|_{TV} \leq \bar{C} \exp(-\bar{c}t)(1 + x^2), \ t \in [0, \infty)$$

(all parameters are described in the Sect. 3.2).

4 Proofs of Theorems 1 and 2

4.1 Proof of Theorem 1

Proof. We shall establish existence on the basis of the paper [5]; another methodological source is the paper [10]. The paper [5] uses the language of stochastic differential equations; thus, we describe our system in such terms. Consider the system (5) with $b(X^1, X^2) = b(X^1, X^2, v)$ on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$, where it has a solution. Let the basis be extended if necessary, and let us add to the system (7) the equation for the discrete component

$$dV_t = \int_{\mathbb{R}^1} K(X_t, V_{t-}, z) N(dt, dz). \tag{16}$$

Here N is an $(\mathcal{F}_t)$-adapted Poisson random measure with a mean (compensator) measure $ds \times \frac{dz}{z^2}$ independent of the Wiener process. The coefficient K must be constructed so that it substitutes the intensities Q : for each $i \in \mathcal{S}$, it takes values in $\{j - i, j \in \mathcal{S}\}$ and

$$\int_{\{z: K(x,v,z) = j-i\}} \frac{dz}{z^2} = q_{ij}, \, j \in \mathcal{S} \setminus i.$$

We give now a description (slightly non-rigorous, although, hopefully, comprehensible) how to construct K. For each $i \in \mathcal{S}$, $K(x, i)$ takes value $1 - i$ on $[z_1(x), \infty)$; value $2 - i$ on $[z_2(x), z_1(x))$; ...; value -1 on $[z_{i-1}(x), z_{i-2}(x))$; value $+1$ on $[z_{i+1}(x), z_{i-1}(x))$; ...; value $N - i$ on $[z_N(x), z_{N-1}(x))$; value 0 on the rest of $\mathbb{R}^1$. The points z_j, $j \in \mathcal{S} \setminus \{i\}$, are defined by the relations

$$\int_{z_1}^{\infty} \frac{dz}{z^2} = q_{i1}, \int_{z_2}^{z_1} \frac{dz}{z^2} = q_{i2}, \ldots, \int_{z_N}^{z_{N-1}} \frac{dz}{z^2} = q_{iN},$$

where the term with index ii is *excluded.* Proposition 1 of [5] provides existence of a weak solution (its condition 2b is not needed in our case because the jump intensities are bounded and at the moments of jumps, the component X does not increase). In fact, Proposition 1 of [5] is proved in style of martingale problems, with pasting solutions at the moments of jumps.

To prove uniqueness, we use Lemma 2 of [5]. It uses a solution $(\tilde{X}_t,\ t \geq 0)$ of the equation (5) *without switching*. Given this trajectory, the first switching moment τ of the solution (X, V) of the system (7)–(16) has the following distribution (note that on $[0, \tau)$ the trajectories of X and $\tilde{X}$ coincide by construction):

Lemma 1. *Under the assumptions (SA 1)–(SA 4), given the trajectory* $\tilde{X}_t, t \in [0, \infty)$, *of the solution of (5) with* $b(X^1, X^2) = b(X^1, X^2, V_0)$, *the conditional probability of the event* $\{\tau > r\}$ *equals*

$$\exp\left\{-\int_0^r \sum_{j \in \mathcal{S} \setminus v} q_{vj}(\tilde{X}_u) du\right\}.$$

Finally, we give a sketch of the proof of strong Markov property. The solution of system (5) does possess a strong Markov property; see [1] and [19]. This entails a strong Markov property of the switched process (7). To prove it, adopt the method of [2, Sect. 4], where nondegenerate diffusions are considered. Instead of making sequentially infinite number of switchings, let us limit ourselves to the first k switchings and make no further ones. The result is a distribution on the space of trajectories (both continuous and discrete). Let us take an arbitrary stopping time τ and calculate the conditional distribution of this distribution given $\mathcal{F}_\tau$, restricted to $t \in [\tau, \infty)$. It equals the distribution of the process with initial conditions (τ, X_τ, V_τ) switched finitely many times—so many times how many out of the first k switchings took place after the moment τ. With $k \to \infty$ the proof is completed—the limiting conditional distribution is again that of a switching process, and due to its uniqueness, this provides the strong Markov property. □

4.2 Proof of Theorem 2

Proof. Here we shall explain how to deduce Theorem 2 for a general set $\mathcal{S}$ from the statement of this Theorem 2 for $\mathcal{S} = \{1\}$. Recall that it suffices to take $s = 0$ in (11).

Let us fix initial state (x, v); denote by ${}^vX_u, u \in [0, \infty)$, the corresponding solution of the equation (5) (without switching) with $b(X^1, X^2) \equiv b(X^1, X^2, v)$. In some cases it will be convenient to use a more sophisticated notation $({}^vX_u^{0,x},\ u \geq 0)$ for the same process where x is the initial data at 0. Respectively, $({}^vX_u^{r,x'},\ u \geq r)$ signifies a solution of the equation (5) with $b(X^1, X^2) = b(X^1, X^2, v)$ on the interval $[r, \infty)$ with initial value x' at r. Let us inspect what occurs on time interval $[0, T]$. The discrete component V is a point process with compensator intensities lying between the given lower and upper bounds. This implies that the probability that V_T equals 1 is bounded away from zero uniformly in *all* x, v.

We are now going to give a rigorous explanation of this fact, although its implementation may look a bit more complicated than it actually is, due to the inevitably involved notations. To simplify the latter a little bit, denote ${}^1\tilde{X}_u^{(r)} := {}^1X_u^{r, {}^vX_r^{0,x}}$, $u \geq r$ (this will be used only in this subsection); recall that $({}^vX_u^{0,x},\ u \geq 0)$ is a process *without switching* and emphasize that likewise without switching is the process $({}^1\tilde{X}_u^{(r)},\ u \geq r)$. Then for $v \neq 1$ the nonconditional probability of the event $\{V_T \text{ equals } 1\}$ is greater than or equal to the expectation of

$$\int_0^T \left(\exp\{-\int_0^r \sum_{j \in \mathcal{S}\setminus v} q_{vj}({}^vX_u) du\} \right.$$
$$\left. \times \exp\{-\int_r^T \sum_{j \in \mathcal{S}\setminus 1} q_{1j}({}^1\tilde{X}_u^{(r)}) du\} \right) q_{v1}({}^vX_r)\, dr,$$

or (notations p and q are defined below), equivalently, of

$$\int_0^T p(r) q(r) q_{v1}({}^vX_r) dr. \tag{17}$$

Here the conditional probability that the discrete component remains at state v on the time interval $[0, r)$ given $({}^vX_u,\ 0 \leq u < r)$ reads

$$p(r) = \exp\{-\int_0^r \sum_{j \in \mathcal{S}\setminus v} q_{vj}({}^vX_u) du\};$$

the conditional probability that the discrete component jumps from state v to state 1 on the time interval $[r, r + dr)$ given $({}^vX_u,\ 0 \leq u < r)$ equals

$$q_{v1}({}^vX_r) dr;$$

the conditional probability that the discrete component remains at state 1 on the time interval $[r + dr, T]$ given $({}^vX_u,\ 0 \le u < r)$ may be presented as

$$q(r) = \exp\{-\int_r^T \sum_{j \in \mathcal{S}\setminus 1} q_{1j}({}^1\tilde{X}_u^{(r)})du\}.$$

Due to the assumptions on all intensities q_{ij}, we get a lower bound $(\exp(-cT))$, the proof of which is based on the Lemma 1. Integration with respect to r in (17) is a complete probability formula, a rigorous justification for which may be given, for example, as in [22].

For $v = 1$ it is even easier to obtain a desired lower bound by virtue of the same Lemma 1, as in this case the probability in question is greater than or equal to the expectation of

$$\exp\{-\int_0^T \sum_{j \in \mathcal{S}\setminus 1} q_{1j}({}^1X_u)du\}.$$

Further, fix $R > 0$ and assume that $x \in B_R$. Copying the reasoning of [19] and [1], we obtain that there exists $R' \in (R, +\infty)$ such that the continuous component of the trajectory does not go beyond the boundary of $B_{R'}$ on $[0, T]$ with probability almost 1, and this is uniformly in initial conditions, belonging to $B_R \times \mathcal{S}$. Combining these two facts, we conclude that the probability that both events take place is bounded away from zero uniformly in $(x, v) \in B_R \times \mathcal{S}$.

The conditional distribution of (X, V) admits on these events a useful estimate. Take $T' > 0$ and consider the distribution of (X, V) on $[T, T + T']$, conditioned on the past time $[0, T]$ history. This distribution is minorized by the distribution of the solution of system (5) with initial conditions T, X_T, with a positive constant multiplier, which follows from the calculus in [1]—more precisely, from the proofs of the proofs of Lemmas 3 and 4 from [1]—accomplished by the Lemma 2 and its Corollaries. This suffices for the local Markov–Dobrushin condition (11). Note that in [1, 19] the initial conditions are assumed non-random; however, the Remark 1 removes this restriction.

To realize this plan, for $x \in \mathbb{R}^{2d}$, $v \in \mathcal{S}$, let us define the following:

$\mu(x, v; dXdV)$—the distribution of the solution of the system (7) on $[0, \infty)$ with initial conditions x, v;

${}^1\mu(x; dX)$—the distribution of the solution on $[0, \infty)$ of the system (5) with $b(X^1, X^2) = b(X^1, X^2, 1)$ and initial data x.

Lemma 2. *Under the assumptions (SA 1)–(SA 4), for any* $T > 0$ *there exists*

$$c(c_u,\ T) = e^{-T(N-1)c_u}$$

such that for any $x \in \mathbb{R}^{2d}$ *and any event* A *defined through the trajectory of* X *on the time interval* $[0,\ T]$, *the following inequality holds:*

$$\mu(x, 1; A \times \{V\big|_{[0,T]} \equiv 1\}) \geq c(c_u, T) \times {}^1\mu(x; A).$$

Proof. Let us fix x and construct on a stochastic basis the following objects:

(1) A process ${}^1X = ({}^1X(t), \ t \in [0, \infty))$—a solution of the system (5) with $b(X^1, X^2) = b(X^1, X^2, 1)$ and initial data x.
(2) A switching process (X, V) satisfying the system (7) with initial conditions $(x, 1)$ in such a way that they coincide up to the first switching. For this purpose take a process ${}^1X$ satisfying (5) and a random variable τ such that for $t \geq 0$, the probability of the event $\{\tau \leq t\}$ given $({}^1X_u, 0 \leq u < t)$ according to the Lemma 1 equals

$$1 - \exp\{-\int_0^t \sum_{j \in \mathcal{S} \setminus 1} q_{1j}({}^1X_u)du\}$$

(this can be done on the product space $\times[0, \infty)$). The moment τ is the moment of the first switching, and the value of $V_\tau = j$ is chosen proportionally to $q_{1j}, j \in \mathcal{S} \setminus 1$. At the moment τ the switched process X satisfying (7) acquires the corresponding conditional probability $\mu_{\tau, X_\tau, V_\tau}$, while the process ${}^1X$ satisfying (5) develops further in its dynamics. It is easy to see that the probability of the event $\{\tau > T\}$ conditioned on a trajectory of ${}^1X$ is uniformly bounded away from zero on the space of all trajectories: it is greater than or equal to $c(c_u, T)$. For any event A on time interval $[0, T]$, the probability that the switched process X lies in A is greater than or equal to the probability of the event {the process ${}^1X$ lies in A and the switched process X coincides with ${}^1X$}, which, in turn, is greater than or equal to $c(c_u, T)\times$ {the probability that ${}^1X \in A$}. Thus, it is also greater than or equal to $c(c_u, T)\times$ {the probability that ${}^1X \in A$ }. □

Corollary 2. *Under the assumptions (SA 1)–(SA 4), for any* $T > 0$ *there exists a constant*

$$c(c_u, T) = e^{-T(N-1)c_u}$$

such that for any $s \in [0, \infty)$, $x \in \mathbb{R}^{2d}$ *and any event* A *defined through trajectories of* X, V *on the time interval* $[0, T]$, *the following inequality holds:*

$$\mu(x, 1; A) \geq c(c_u, T) \times {}^1\mu(x, A \cap \{V\big|_{[0,T]} \equiv 1\}).$$

Proof. Indeed, $\mu(x, 1; A) \geq \mu(x, 1; A \cap \{V\big|_{[0,T]} \equiv 1\})$. □

Corollary 3. *Under the assumptions (SA 1)–(SA 4), for any* $T > 0$ *there exists a constant*

$$c(c_u, T) = e^{-T(N-1)c_u},$$

such that for any initial condition x, *a measurable* $\Gamma \subseteq \mathbb{R}^{2d} \times \mathcal{S}$ *and* $R > 0$ *the following inequality holds:*

$$\mu(x, 1; (X, V)_T \in \Gamma,\ X_u \in B_R,\ u \in [0, T]) \\ \ge c(c_u, T)^1 \mu\,(X_T \in \Gamma \cap \{V = 1\},\ X_u \in B_R,\ u \in [0, T])\,.$$

This completes the proof of the Theorem 2. □

5 Conclusion

We have proved for highly degenerate stochastic mechanical hybrid systems under quite general conditions (discontinuity and linear growth of coefficients and the Wiener process perturbations) the following properties:

– Existence and uniqueness theorem and a strong Markov property for solutions of such systems
– A local mixing property in the Markov–Dobrushin form for these solutions
– Exponential stochastic stability in total variation metric for solutions of such systems under the additional assumption (10)

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References

1. Abourashchi, N., Veretennikov, A. Yu.: On stochastic averaging and mixing. Theory of Stoch. Processes. **16**(1), 111–130 (2010)
2. Anulova, S., Pragarauskas, H.: Weak Markov solutions of stochastic equations. Lith. Math. J. **17**(3), 141–155 (1978)
3. Anulova, S., Veretennikov, A.: On ergodic properties of degenerate hybrid stochastic control systems. In: 49th IEEE Conference on Decision and Control (CDC), 2010, Issue Date: 15–17 Dec. 2010, pp. 2292–2297. Location: Atlanta, GA (2010) doi:10.1109/CDC.2010.5717412
4. Arapostathis, A., Borkar, V.S., Ghosh, M.K.: Ergodic control of diffusion processes. Encyclopedia of Mathematics and its Applications 143. xvi, 323 p. Cambridge University Press, Cambridge. (2012)

5. Athreya, K., Kliemann, W., Koch, G.: On sequential construction of solutions of stochastic differential equations with jump terms. Syst. Control Lett. **10**, 141–146 (1988)
6. Bujorianu, M., Lygeros, J.: Toward a general theory of stochastic hybrid systems. In: Blom, H., Lygeros, J. (eds.) Stochastic Hybrid Systems, pp. 3–30. Springer, Berlin (2006) http://www.springerlink.com/index/n62r24355788p6h7.pdf
7. Buyukkoroglu, T., Esen, O., Dzhafarov, V.: Common lyapunov functions for some special classes of stable systems. IEEE Trans. Automat Contr. **56**, 1963–1967 (2011)
8. Campillo, F.: Optimal ergodic control for a class of nonlinear stochastic systems—Application to semi-active vehicle suspensions. In: Proceedings of the 28th IEEE Conference on Decision Control, pp. 1190–1195. Tampa (1989)
9. Campillo, F., Pardoux, E.: Numerical methods in ergodic optimal stochastic control and application. In: Karatzas, I., Ocone, D. (eds.) Applied Stochastic Analysis, pp. 59–73. Springer, Berlin (1992) http://www.springerlink.com/index/y13p561655383ht7.pdf
10. Davis, M.H.A.: Piecewise-Deterministic Markov Processes: A General class of non-diffusion stochastic models. J. Royal Stat. Soc. Ser. B (Methodological) **4**6(3), 353–388 (1984)
11. Deng, F., Luo, Q., Mao, X., Pang, S.: Noise suppresses or expresses exponential growth. Syst. Control Lett. **5**7(3), 262–270 (2008)
12. Guglielmino, E., Sireteanu, T., Stammers, C., Ghita, G., Giuclea, M.: Semi-active Suspension Control: Improved Vehicle Ride and Road Friendliness. Springer, Berlin (2008)
13. Hu, G., Liu, M., Mao, X., Song, M.: Noise expresses exponential growth under regime switching Systems. Syst. Control Lett. **5**8(9), 691–699 (2009)
14. Ikeda, N., Watanabe, S.: Stochastic Differential Equations and Diffusion Processes. 2nd edn. North-Holland Mathematical Library, 24. xvi, 555 p. Amsterdam etc.: North-Holland; Tokyo: Kodansha Ltd. (1989)
15. Khasminskii, R., Zhu, C., Yin, G.: Stability of regime-switching diffusions. Stoch. Processes Appl. **1**17(8), 1037–1051 (2007)
16. Krylov, N.V.: On the selection of a Markov process from a system of processes and the construction of quasi-diffusion processes. Math. USSR, Izv. **7**, 691–709 (1973)
17. Nummelin, E.: General irreducible Markov chains and non-negative operators. Cambridge Tracts in Mathematics, 83. XI, 156 p. Cambridge etc.: Cambridge University Press. (1984)
18. Shur, M.: Ergodic theorems for Markov processes. In: Prohorov, Ju.V.(ed.) Probability and Mathematical Statistics: Encyclopedia, pp. 825–827. Moscow, The Big Russian Encyclopedia, 911 p. (1999) ISBN 5-85270-265-X
19. Veretennikov, A.: On weak solution of an SDE. In: 10th Vilnius conference on Probability Theory and Mathematical Statistics, 28.06-02.07.2010, Vilnius, Lithuania, Abstracts of Communications, vol. 287, TEV Publishers, Vilnius (2010)
20. Veretennikov, A. Yu.: Bounds for the mixing rate in the theory of stochastic equations. Theory Probab. Appl. **32**, 273–281 (1987)
21. Veretennikov, A. Yu.: On Polynomial mixing and convergence rate for stochastic difference and differential equations. Theory Probab. Appl. **44**(2), 361–374 (2000)
22. Veretennikov, A. Yu., Zverkina, G.A.: Simple proof of Dynkin's formula for single–server systems and polynomial convergence rates. arXiv:1306.2359v1 [math.PR]
23. Wu, L.: Large and moderate deviations and exponential convergence for stochastic damping Hamiltonian systems. Stoch. Processes Appl. **91**(2), 205–238 (2001)
24. Yuan, C., Mao, X.: Asymptotic stability in distribution of stochastic differential equations with Markovian switching. Stoch. Processes Appl. **103**(2), 277–291 (2003)

Asymptotic Behaviour of the Distribution Density of the Fractional Lévy Motion

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Abstract We investigate the distribution properties of the fractional Lévy motion defined by the Mandelbrot-Van Ness representation:

$$Z_t^H := \int_{\mathbb{R}} f(t,s) dZ_s,$$

where Z_s, $s \in \mathbb{R}$, is a (two-sided) real-valued Lévy process, and

$$f(t,s) := \frac{1}{\Gamma(H+1/2)} \left[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2} \right], \quad t,s \in \mathbb{R}.$$

We consider separately the cases $0 < H < 1/2$ (*short memory*) and $1/2 < H < 1$ (*long memory*), where H is the Hurst parameter, and present the asymptotic behaviour of the distribution density of the process. Some examples are provided, in which it is shown that the behaviour of the density in the cases $0 < H < 1/2$ and $1/2 < H < 1$ is completely different.

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1 Introduction

In this chapter we consider the distribution properties of the *fractional Lévy motion* (FLM, in the sequel). Various versions of the FLM have been used in a number of recent publications in order to interpret some experimental data. Apart from the rigorous mathematical definition, some modifications of the FLM are derived from the physical point of view; see, for example, [7, 11, 15]. The FLM driven by an α-stable Lévy process is used as a model for describing sub-diffusive effects in physics and biology (see [6, 26]), signal and traffic modelling [10, 18, 19], finance [5] and geophysics [13, 21, 25]. We refer to [25] for the discussion in which type of problems the FLM gives an adequate description for the observed phenomena. In the papers quoted above it was shown that the respective phenomena existing in nature can be better described by models, containing the FLM rather than the fractional Brownian motion (FBM). Finally, we refer to [8] for simulations of the FLM, which can be convenient in practical problems.

Similarly to the FBM, the FLM can be defined in two different ways: via the Mandelbrot-Van Ness representation (see [2] and [20]) or via the Molchanov-Golosov representation (see [3]). We also refer to [23] for a bit different definition of the fractional stable motion. These two representations, being equivalent in the Gaussian setting, in the Lévy setting lead, in general, to different processes; see [24]. Note that, in contrast to the FBM, in some cases, the FLM can even be a semi-martingale ([1, 4]).

In this paper we focus on the FLM Z_t^H defined by the Mandelbrot-Van Ness representation, i.e.

$$Z_t^H := \int_{\mathbb{R}} f(t,s)dZ_s, \tag{1}$$

where Z_s, $s \in \mathbb{R}$, is a (two-sided) real-valued Lévy process, $H \in (0,1)$ is the *Hurst parameter* and

$$f(t,s) := \frac{1}{\Gamma(H+1/2)}\left[(t-s)_+^{H-1/2} - (-s)_+^{H-1/2}\right], \quad t,s \in \mathbb{R}, \tag{2}$$

where $x_+ := \max(x,0)$. This definition gives a particularly important representative of the class of the so-called moving-average fractional Lévy motions. Since the FLM, according to the survey above, is an adequate model to some phenomena in nature, it would be appropriate to investigate deeply its properties. In particular, knowledge of the distribution properties of the FLM would naturally make it possible to solve various problems related to statistical inference, simulation, etc.

In this paper we concentrate on the asymptotic behaviour of the distribution density of the FLM. In contrast to the FBM case, the study of the distribution density of the FLM is much more complicated. In the recent paper [17], we presented the investigation of the distribution density of a FLM in the following cases: (i) $H = 1/2$, which means merely $Z_t^H \equiv Z_t$, and (ii) $1/2 < H < 1$, which corresponds to the so-called long memory case; see Definition 1.1 in [20]. Both cases can be treated in a unified way using a general result about the asymptotic behaviour of distribution densities of Lévy-driven stochastic integrals with deterministic kernels; see Theorem 2.1 in [17]. Since this theorem requires the respective kernel to be bounded, and the kernel (2) is unbounded when $0 < H < 1/2$, the case (iii) $0 < H < 1/2$ cannot be treated in the same way as in [17] and thus requires a completely different approach, which we present below. Further, we show that there is a substantial difference in the behaviour of the density in the cases (i), (ii) on the one hand and the case (iii) on the other hand, namely, the distribution density in these situations exhibits absolutely different types of the asymptotic behaviour. We also emphasize that in contrast to the situation studied in [17], for the case $0 < H < 1/2$, we do not require the existence of exponential moments of the tails of the Lévy measure.

Let us outline the rest of the paper. In Sect. 2 we set the notation and formulate our main results, Theorems 1 and 2. In Sect. 3 we formulate the general result, Theorem 3, on the asymptotic behaviour of the distribution density of Lévy functionals, on which Theorems 1 and 2 are based on. In Sect. 4 we give two examples which illustrate the effects that may happen in the "extremely heavy-tailed" case, i.e. when condition (17) (see below) fails. In Appendix we give some supplementary statements: the necessary and sufficient condition for the integral (1) to be well defined and the condition for the respective distribution to possess a density.

2 Settings and the Main Result

Let Z_t, $t \in \mathbb{R}$, be a real-valued (two-sided) *Lévy process* with *the characteristic exponent* ψ, which means that Z has stationary independent increments, and the characteristic function of an increment is given by

$$Ee^{iz(Z_t - Z_s)} = e^{(t-s)\psi(z)}, \quad t > s. \tag{3}$$

The characteristic exponent ψ admits the Lévy-Khinchin representation:

$$\psi(z) = iaz - bz^2 + \int_{\mathbb{R}} \left(e^{iuz} - 1 - izu1_{\{|u|\le 1\}}\right) \mu(du), \tag{4}$$

where $a \in \mathbb{R}$, $b \ge 0$ and $\mu(\cdot)$ is *a Lévy measure*, i.e. $\int_{\mathbb{R}}(1 \wedge u^2)\mu(du) < \infty$. To exclude the trivial cases, we assume that $b = 0$ and $\mu(\mathbb{R}) > 0$; that is, Z does not contain a diffusion part and contains a non-trivial jump part. To simplify the notation we also assume without loss of generality that $Z_0 = 0$ and $a = 0$.

We define the integral (1) as a limit in probability of the respective integral sums; see [22, Sect. 2]. When $H \neq 1/2$, the necessary and sufficient condition for this integral to be well defined is

$$\int_{|u|\ge 1} |u|^{2/(3-2H)}\mu(du) < \infty, \tag{5}$$

see Proposition 2 in Appendix I. Furthermore, it will be shown in Proposition 3 (see Appendix II) that under the same conditions and our standing assumption

$$\mu(\mathbb{R}) > 0, \tag{6}$$

the integral (1) possesses for any $t \neq 0$ a distribution density, which we denote by $p_t(x)$, and, moreover, $p_t \in C_b^\infty(\mathbb{R})$. Note that in the Lévy case, i.e. for $H = 1/2$, available sufficient conditions for the existence of the density $p_t \in C_b^\infty(\mathbb{R})$ are much stronger; see, for example, [14] for the *Hartman-Wintner condition*.

An important feature of the process Z_t^H is that one can explicitly write its characteristic function $\phi(t,z) := Ee^{izZ_t^H}$ (cf. [22, Theorem 2.7]):

$$\phi(t,z) = e^{\Psi(t,-z)}, \tag{7}$$

where

$$\Psi(t,z) = \int_{\mathbb{R}}\int_{\mathbb{R}} \left(e^{-izf(t,s)u} - 1 + izf(t,s)u1_{|u|\le 1}\right) \mu(du)ds, \quad z \in \mathbb{R}, \quad t > 0. \tag{8}$$

Observe, that if the measure μ possesses exponential moments, the function $\Psi(t,z)$ can be extended with respect to z to the complex plane. Moreover, one can see (cf. Sect. 3.3) under the assumptions that $\mu(\mathbb{R}_+) > 0$, the function

$$H(x,z) := izx + \Psi(1,z)$$

has a unique critical point $i\xi(x)$ on the line $i\mathbb{R}$. Put

$$D(x) := H(x, i\xi(x)), \quad K(x) := \frac{\partial^2}{\partial\xi^2} H(x,i\xi)\Big|_{\xi=\xi(x)}, \tag{9}$$

and

$$M_k(\xi) := \int_{\mathbb{R}} u^k e^{\xi u}\mu(du), \quad k \ge 2, \quad \xi \in \mathbb{R}.$$

Fix $t_0 > 0$. In what follows, we write $f \ll g$, if $f/g \to 0$, and $f \sim g$, if $f/g \to 1$.

Theorem 1. *Let Z_t^H, $1/2 < H < 1$, $t \ge t_0$, be a FLM defined by (1), where Z_t is a Lévy process with the associate Lévy measure μ. Suppose that the conditions below hold true:*

(1) $\mu(\mathbb{R}^+) > 0$*;*
(2) for all $C \in \mathbb{R}$

$$\int_{|y|\geq 1} e^{Cy}\mu(dy) < \infty; \tag{10}$$

(3) $\exists \gamma \in (0,1)$ *such that* $M_4(\xi) \ll M_2^2(\gamma\xi)$ *as* $\xi \to \infty$*;*
(4) $\ln\left(\frac{M_4(\xi)}{M_2(\xi)} \vee 1\right) + \ln\ln M_2(\xi) \ll \xi$, $\xi \to +\infty$.
Then the distribution density $p_t(x)$ *of* Z_t^H *exists,* $p_t \in C_b^\infty$*, and satisfies the asymptotic relation*

$$p_t(x) \sim \frac{1}{\sqrt{2\pi t^{2H} K(xt^{-H-1/2})}} e^{tD(xt^{-H-1/2})}, \quad t+x \to \infty, \quad (t,x) \in [t_0,\infty)\times\mathbb{R}_+, \tag{11}$$

where D and K are defined in (9).

To formulate the result for $0 < H < 1/2$ we need a bit more notation. Let $f(s) := f(1,s)$; see (2) for the definition of $f(t,s)$. Observe that $f(s)$ is strictly decreasing on $(-\infty,0)$ and maps $(-\infty,0)$ to $(-\infty,0)$, strictly increasing on $(0,1]$ and maps $(0,1]$ to $\left[\frac{1}{\Gamma(H+1/2)}, +\infty\right)$. In addition, the derivative $f'(s)$ is well defined and is continuous on $(-\infty,0)$ and $(0,1)$. Hence, we can put

$$\ell(y) := \begin{cases} (f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}, & y \in (-\infty,0) \cup \left[\frac{1}{\Gamma(H+1/2)}, +\infty\right), \\ 0, & \text{otherwise}. \end{cases} \tag{12}$$

Note that $\ell(y)$ is non-negative if $y \geq 0$ and is negative otherwise. Define

$$\mathfrak{m}(r) := \int_{-\infty}^{\infty} \frac{1}{y}\ell\left(\frac{r}{y}\right)\mu(dy), \quad r > 0. \tag{13}$$

Recall that (see Definition 4 in [16]) a function $g : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{L}d$ of *sub-exponential densities*, if $g(x) > 0$ for large enough positive x, and

$$\lim_{x\to+\infty} \frac{(g * g)(x)}{g(x)} = 2, \quad \text{and} \quad \lim_{x\to+\infty} \frac{g(x-y)}{g(x)} = 1 \text{ for any } y \in \mathbb{R}, \tag{14}$$

where $*$ is a usual definition for the convolution. Fix $t > 0$.

Theorem 2. *Let* Z_t^H*,* $0 < H < 1/2$*, be a FLM defined by (1), where* Z_t *is a Lévy process with the associate Lévy measure* μ*. Suppose that (5) holds true and*

$$\mu(\mathbb{R}_-) > 0. \tag{15}$$

Then for every $t > 0$ the value of FLM Z_t^H defined by (1) possesses a probability density $p_t \in C_b^\infty(\mathbb{R})$, which satisfies the following:

(i) If

$$\int_{|u|\geq 1} |u|^{2/(1-2H)}\mu(du) = \infty$$

and $\mathfrak{m} \in \mathcal{L}d$, then for all $t > 0$,

$$p_t(x) \sim t^{3/2-H}\mathfrak{m}(t^{1/2-H}x), \quad x \to +\infty. \tag{16}$$

(ii) If

$$\int_{|u|\geq 1} |u|^{2/(1-2H)}\mu(du) < \infty, \tag{17}$$

then

$$p_t(x) \sim c_H \left(\int_{\mathbb{R}} |u|^{2/(1-2H)}\mu(du)\right) x^{-(3-2H)/(1-2H)}, \quad x \to +\infty, \tag{18}$$

where

$$c_H = \frac{2}{1-2H}\Big(\Gamma\big(H+1/2\big)\Big)^{-2/(1-2H)}. \tag{19}$$

Remark 1. (a) Apparently, relation (16) holds true when both x and t tend to $+\infty$ in such a way that $xt^{-H-1/2} \to +\infty$. To prove such an extension of Theorem 2, one should have an extension of [16, Theorem 3.2] which applies to *a family* of random sums with the variable distribution of the number of summands.

(b) Clearly, results similar to (16) and (18) can be formulated for $x \to -\infty$. In that case, one should assume $\mu(\mathbb{R}_+) > 0$ instead of $\mu(\mathbb{R}_-) > 0$.

To illustrate the crucial difference between the cases treated in Theorem 1 and Theorem 2, consider two particular examples from [17] which concern the case $H > 1/2$. First, let the Lévy measure μ of the Lévy noise Z_t in (1) be supported in a bounded set. Then (see in [17, Corollary 5.1 and Corollary 5.2]) there exists a constant $c_*(\mu)$, defined in terms of the Lévy measure μ only, such that for any constants $c_1 > c_*(\mu)$ and $c_2 < c_*(\mu)$, there exists $y(c_1, c_2)$ such that for $x > y(c_1, c_2)t^{H+1/2}$, we have

$$p_t(x) \quad \begin{cases} \geq \exp\left(-\dfrac{c_1 x}{\Gamma(H+1/2)t^{H-1/2}} \ln\left(\dfrac{x}{t^{H+1/2}}\right)\right), \\ \leq \exp\left(-\dfrac{c_2 x}{\Gamma(H+1/2)t^{H-1/2}} \ln\left(\dfrac{x}{t^{H+1/2}}\right)\right). \end{cases} \tag{20}$$

Similar statement is available as well when the tails of the Lévy measure admit the following super-exponential estimates: for u large enough

$$\frac{1}{Q(u)}e^{-bu^\beta} \le \mu([u,+\infty)) \le Q(u)e^{-bu^\beta}, \tag{21}$$

where $b > 0$ and $\beta > 1$ are some constants and Q is some polynomial. In this case, instead of (20), we have for any constants $c_1 > c_*(\mu)$ and $c_2 < c_*(\mu)$

$$p_t(x) \quad \begin{cases} \ge \exp\left(-\frac{c_1 x}{\Gamma(H+1/2)t^{H-1/2}} \ln^{\frac{\beta-1}{\beta}}\left(\frac{x}{t^{H+1/2}}\right)\right) \\ \le \exp\left(-\frac{c_2 x}{\Gamma(H+1/2)t^{H-1/2}} \ln^{\frac{\beta-1}{\beta}}\left(\frac{x}{t^{H+1/2}}\right)\right), \end{cases} \tag{22}$$

for $x > y(c_1, c_2)t^{H+1/2}$ (again, $c_*(\mu)$ is defined in terms of the Lévy measure μ only).

Comparing (20) and (22), we see that the asymptotic behaviour of the tails of the Lévy measure μ is substantially involved in the estimates for $p_t(x)$. The case $0 < H < 1/2$ is completely different. In particular, if (17) holds true, then $p_t(x)$ satisfies (18), where the right-hand side is even independent of t, which is an interesting and quite an unexpected fact. We also emphasize that under (17) the polynomial "shape" of the expression in the right-hand side of (18) does not depend on μ and the only impact of μ is represented by the multiplier $\int_{\mathbb{R}} |u|^{2/(1-2H)}\mu(du)$. This means that in the case $0 < H < 1/2$, the asymptotic behaviour of $p_t(x)$ "mostly" does not depend on μ. However, when μ is "extremely heavy-tailed", there still remains a possibility for the density $p_t(x)$ to be more sensitive with respect to both the Lévy measure μ and the time parameter t. The dichotomy between the "regular" case (when (17) holds) and the "extremely heavy- tailed" case (when (17) fails) is illustrated in Sect. 4 below. Such a dichotomy can be informally explained by the competition between the impacts of the kernel $f(t,s)$ on the one hand and of the measure μ on the other hand.

3 Proofs

3.1 General Theorem

Before we proceed to the proofs, we formulate a central analytical result on the behaviour of the inverse Fourier transform for a certain class of functions. This result plays the key role in the proofs of Theorems 1 and 2.

Let $I \subset \mathbb{R}$ be some interval and $\mathbb{T}$ be some set of parameters. Consider a function $f : \mathbb{T} \times I \to \mathbb{R}$; a family of subsets $\mathcal{C}(t,s) \subset \mathbb{R}$, $t \in \mathbb{T}$, $s \in I$ and a Lévy measure μ such that

$$\int_I f^2(t,s)\,ds < \infty, \quad t \in \mathbb{T}, \tag{23}$$

$$\int_I \int_{\mathcal{C}(t,s)} \left(|f(t,s)u|^2 \wedge 1\right) \mu(du)ds < \infty, \tag{24}$$

$$\int_I \left| \int_{\mathcal{C}(t,s)} \left(f(t,s)u1_{|f(t,s)u|\le 1} - f(t,s)u1_{|u|\le 1}\right) \mu(du)\right| ds < \infty. \tag{25}$$

Then the following function is well defined:

$$\Psi(t,z) = \int_I \int_{\mathcal{C}(t,s)} \left(e^{-izf(t,s)u} - 1 + izf(t,s)u1_{|u|\le 1}\right)\mu(du)ds, \quad z \in \mathbb{R}. \tag{26}$$

Our aim is to investigate the asymptotic behaviour of the function (provided it exists)

$$q_t(x) = (2\pi)^{-1}\int_{\mathbb{R}} e^{ixz}\phi(t,z)dz = (2\pi)^{-1}\int_{\mathbb{R}} e^{-ixz+\Psi(t,z)}dz, \quad t>0, \quad x \in \mathbb{R}, \tag{27}$$

as (t,x) tend to infinity in some appropriate regions. Clearly, when $\mathcal{C}(t,s) \equiv \mathbb{R}$, the function $q_t(x)$ is nothing else but the distribution density of the Lévy functional

$$Y_t = \int_I f(t,s)dZ_s, \tag{28}$$

where Z_s is the Lévy process associated with measure μ, without a drift and a Gaussian component.

Assume in addition that for some $\lambda > 0$, we have

$$f(t,s)u \le \lambda, \quad t \in \mathbb{T}, \quad s \in I, \quad u \in \mathcal{C}(t,s). \tag{29}$$

Then it can be shown that the function $\Psi(t,\cdot)$ defined in (26) can be extended to the half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z \ge 0\}$, and respective extension (we denote it by the same letter Ψ) is continuous on $\mathbb{C}_+$ and analytical in the inner part of this half-plane.

Consider the function

$$H(t,x,z) := izx + \Psi(t,z), \quad z \in \mathbb{C}_+, \tag{30}$$

and observe that

$$\frac{\partial}{\partial \xi} H(t,x,i\xi) = -x + \int_I \int_{\mathcal{C}(t,s)} uf(t,s)\left(e^{\xi f(t,s)u} - 1_{|u|\le 1}\right)\mu(du)ds \to \infty, \quad \xi \to +\infty,$$

provided that

$$(\mu \times \mathrm{Leb})\Big(\{(u,s) : f(t,s)u > 0\}\Big) > 0. \tag{31}$$

Furthermore, under the same condition, $\frac{\partial^2}{\partial \xi^2} H(t,x,i\xi) > 0$ for all $\xi \in \mathbb{R}$. Hence,

$$(0,\infty) \ni \xi \mapsto \frac{\partial}{\partial \xi} H(t,x,i\xi)$$

is a continuous strictly increasing function with the range $(x_t - x, \infty)$, where

$$x_t = \lim_{\xi \to 0+} \Psi(t,i\xi) = \int_I \int_{\mathcal{C}(t,s)} u f(t,s) 1_{|u|>1} \mu(du) ds. \tag{32}$$

Note that by the above conditions on f, μ and $\mathcal{C}(t,s)$, the value x_t may equal $-\infty$, but is less than $+\infty$. Then for any $x > x_t$ there exists unique solution $\xi(t,x)$ to the equation

$$\frac{\partial}{\partial \xi} H(t,x,i\xi) = 0. \tag{33}$$

To formulate the result we need some extra notation:

$$\mathcal{M}_k(t,\xi) := \frac{\partial^k}{\partial \xi^k} \Psi(t,i\xi), \quad k \geq 1, \tag{34}$$

$$\mathcal{D}(t,x) := H(t,x,i\xi(t,x)), \quad \mathcal{K}(t,x) := \mathcal{M}_2(t,\xi(t,x)), \tag{35}$$

and

$$\Theta(t,z,B) := \int_I \int_{\{u:\, f(t,s)u \in B,\, u \in \mathcal{C}(t,s)\}} \Big(1 - \cos(f(t,s)zu)\Big) \mu(du) ds.$$

Consider a set $\mathcal{A} \subset \{(t,x) : t \in \mathbb{T}, x > x_t\} \subset \mathbb{T} \times \mathbb{R}$ and define

$$\mathcal{T} :=\{t : \exists x \in (x_t,\infty), (t,x) \in \mathcal{A}\}, \quad \mathcal{B} :=\{(t,\xi) : \exists (t,x) \in \mathcal{A}, (t,\xi)=(t,\xi(t,x))\}.$$

Finally, suppose that the function $\theta : \mathcal{T} \to (0,+\infty)$ is bounded away from zero on $\mathbb{T}$ and the function $\chi : \mathcal{T} \to (0,+\infty)$ is bounded away from zero on every set $\{t : \theta(t) \leq c\}$, $c > 0$.

Theorem 3. *Assume the following:*

H1 *Conditions (23)–(25), (29) and (31) hold true.*
H2 $\mathcal{M}_4(t,\xi) \ll \mathcal{M}_2^2(t,\xi)$, $\theta(t) + \xi \to \infty$, $(t,\xi) \in \mathcal{B}$.

H3 *For* $\theta(t)+\xi\to\infty$, $(t,\xi)\in\mathcal{B}$.

$$\ln\left(\left(\chi^{-2}(t)\frac{\mathcal{M}_4(t,\xi)}{\mathcal{M}_2(t,\xi)}\right)\vee 1\right)+\ln\left(\left(\ln\left((1\vee\chi^{-1}(t))\mathcal{M}_2(t,\xi)\right)\right)\vee 1\right)$$
$$\ll \ln\theta(t)+\chi(t)\xi.$$

H4 *There exist* $R>0$ *and* $\delta>0$ *such that*

$$\Theta(t,z,\mathbb{R}_+)\ge(1+\delta)\ln(\chi(t)|z|),\quad t\in\mathcal{T},\quad |z|>R. \tag{36}$$

H5 *There exists* $r>0$ *such that for every* $\epsilon>0$,

$$\inf_{|z|>\epsilon}\Theta(t,z,[r\chi(t),+\infty))\ge c\theta(t)\Big((\epsilon\chi(t))^2\wedge 1\Big),\quad t\in\mathcal{T},\quad c>0.$$

Then the function $q_t(x)$ *given by (27) is well defined and satisfies*

$$q_t(x)\sim\frac{1}{\sqrt{2\pi\mathcal{K}(t,x)}}e^{\mathcal{D}(t,x)},\quad \theta(t)+x\to\infty,\quad (t,x)\in\mathcal{A}. \tag{37}$$

Up to some straightforward and purely technical modifications, the proof of Theorem 3 coincides with the proof of [17, Theorem 2.1] and therefore is omitted. Here we only remark that the proof is based on an appropriate modification of the *saddle point method*; see [9] for details.

3.2 Outline of the Proofs

One can prove Theorem 1 using a simplified version of Theorem 3 with $\mathcal{C}(t,s)\equiv\mathbb{R}$ and the scaling property of the function $f(t,s)$:

$$f(t,s)=|t|^{H-1/2}f\left(\frac{s}{t}\right), \tag{38}$$

where

$$f(s)=f(1,s)=\frac{1}{\Gamma(H+1/2)}\left[(1-s)_+^{H-1/2}-(-s)_+^{H-1/2}\right],\quad s\in\mathbb{R};$$

see [17] for details.

Let us turn now to the proof of Theorem 2. To make the proof of Theorem 2 more transparent, we first sketch its main idea. In particular, we show how Theorem 3 applies in the situation when the function $f(t,s)$ is unbounded.

According to [22, Theorem 2.7], the characteristic function $\phi(t,z)$ (cf. (7)) can be decomposed for any fixed $\lambda > 0$ as

$$\phi(t,z) = \phi_1(t,z)\phi_2(t,z) = e^{\psi_1(t,z)}e^{\psi_2(t,z)},$$

where

$$\psi_1(t,z) := \int_{-\infty}^{t}\int_{\{u:\, uf(s/t)\le\lambda\}} \left(e^{izf(t,s)u} - 1 - izf(t,s)u1_{|u|\le 1}\right)\,\mu(du)ds, \tag{39}$$

$$\psi_2(t,z) := \int_{-\infty}^{t}\int_{\{u:\, uf(s/t)>\lambda\}} \left(e^{izf(t,s)u} - 1\right)\,\mu(du)ds + iza(t), \tag{40}$$

$$a(t) := \int_{-\infty}^{t}\int_{\{u:\, uf(s/t)>\lambda\}} f(t,s)u1_{|u|\le 1}\mu(du)ds = t^{H+1/2}a(1). \tag{41}$$

In the last identity we have used the scaling property (38) of the kernel f.

The function $|\phi_1(t,z)|$ is integrable with respect to z for any $t > 0$; see Remark 4 in the Appendix II below. Then there exists the distribution density

$$\tilde{p}_t(x) := \frac{1}{2\pi}\int_{\mathbb{R}} e^{-izx}\phi_1(t,z)dz, \tag{42}$$

and thus the required density $p_t(x)$ can be written as the convolution

$$p_t(x) = (\tilde{p}_t * P_t)(x), \tag{43}$$

where $P_t(dy)$ is the probability measure corresponding to the characteristic function $\phi_2(t,z)$. Define the measure $M_t(dy)$ by the relation

$$\int_{\mathbb{R}} g(y)M_t(dy) = \int_{-\infty}^{t}\int_{\{u:\, uf(s/t)>\lambda\}} g(f(t,s)u)\,\mu(du)ds, \tag{44}$$

where g is an arbitrary measurable and bounded function. Then, up to the shift by $a(t)$, the measure $P_t(dy)$ is equal to the distribution of the compound Poisson random variable with the intensity of the Poisson part equal to $M_t(dy)$. In other words,

$$P_t(dy) = \delta_{a(t)}(dy) * \left(e^{-M_t(\mathbb{R})}\delta_0(dy) + e^{-M_t(\mathbb{R})}\sum_{k=1}^{\infty}\frac{1}{k!}M_t^{*k}(dy)\right), \tag{45}$$

where $M_t^{*k}(dy)$ is the k-fold convolution of $M_t(dy)$.

It follows from the scaling property (38) that $M_t(\mathbb{R}) = t\Lambda$ with

$$\Lambda = \int_{-\infty}^{1}\int_{\{u:\, uf(s)>\lambda\}} \mu(du). \tag{46}$$

Furthermore, we prove that the measure $M_t(dy)$ is absolutely continuous with respect to the Lebesgue measure with the density

$$m_t(x) = t^{3/2-H}\mathfrak{m}(t^{1/2-H}x)1_{\{x>\lambda t^{H-1/2}\}}, \tag{47}$$

where the function $\mathfrak{m}$ is defined by (13). Then (43) can be written in the form

$$p_t(x) = e^{-\Lambda t}\tilde{p}_t(x-a(t)) + \int_{\mathbb{R}} \rho_t(x-y)\tilde{p}_t(y-a(t))dy, \tag{48}$$

where

$$\rho_t(x) := e^{-t\Lambda}\sum_{k=1}^{\infty}\frac{m_t^{*k}(x)}{k!}. \tag{49}$$

Clearly, ρ_t is the density of a random sum with the distribution of one term represented by m_t. Suppose that the function $\mathfrak{m}$ is sub-exponential. Then it follows from [16, Theorem 3.2] that the density ρ_t is sub-exponential as well and

$$\rho_t(x) \sim \left(e^{-t\Lambda}\sum_{k=1}^{\infty}\frac{k}{k!}(t\Lambda)^{k-1}\right)m_t(x) = m_t(x), \quad x\to+\infty, \tag{50}$$

where we used that $\int_0^\infty m_t(x)\,dx = M_t(\mathbb{R}) = t\Lambda$.

To estimate $\tilde{p}_t(x)$ we apply Theorem 3. Namely, in Proposition 1 below, we show that for a given $\varepsilon > 0$, there exists $y(\varepsilon) > 0$ such that

$$\exp\left(-\frac{(1+\varepsilon)x}{\lambda t^{H-1/2}}\ln\frac{x}{t^{H+1/2}}\right) \le \tilde{p}_t(x) \le \exp\left(-\frac{(1-\varepsilon)x}{\lambda t^{H-1/2}}\ln\frac{x}{t^{H+1/2}}\right), \quad xt^{-H-1/2} \ge y(\varepsilon). \tag{51}$$

Since a sub-exponential function decays slower than any exponential function (cf. [16]), the term m_t dominates both $\tilde{p}_t(x)$ and the integral term in (48). In such a way, (48), (50) and (51) provide the required relation (16).

Let us summarize the idea explained above. The distribution of Z_t^H is decomposed in two parts. For one part, the distribution density is controlled by means of the respective version of the saddle point method, while for the other part, the distribution can be evaluated in the form of the series of convolution powers with the explicitly given law of the first summand. Similarly to Theorem 2.1 in [17], Theorem 3 provides a flexible version of the saddle point method, which is applicable to a wide variety of integrals of the form (27). Thus, one can expect that the approach presented above can be extended to other processes of the form (1) with unbounded kernels $f(t,s)$. To keep the exposition reasonably tight, in this paper, we do not investigate this possibility in the whole generality and restrict our considerations to the important particular case of the FLM with $0 < H < 1/2$.

3.3 Properties of $\tilde{p}_t(x)$

Proposition 1. *Under (15), for a given $\varepsilon > 0$, there exists $y(\varepsilon) > 0$ such that $\tilde{p}_t(x)$ satisfies (51).*

Remark 2. Note that in the above Proposition, we do not assume that $t > 0$ is fixed.

Proof. We use Theorem 3 with $\theta(t) = t$, $\chi(t) = t^{H-\frac{1}{2}}$, $\mathbb{T} = [t_0, \infty)$, $I = (-\infty, t]$, $\mathcal{C}(t,s) = \{u : f(t,s)u \le \lambda\}$, and $\mathcal{A} = \{(t,x) \subset [t_0,\infty) \times \mathbb{R}_+ : \; xt^{-H-1/2} \ge c\}$. Here t_0, c and λ are some positive constants. Then condition **H1** is satisfied: (29) holds true by the construction, (31) holds true thanks to (15), and (23)–(25) can be proved using the same estimates as in the proof of Proposition 2 in Appendix I (we omit the details).

Recall that f possesses the self-similarity property (38). Then

$$\mathcal{M}_k(t,\zeta) = \chi^k(t) t \mathcal{M}_k(\chi(t)\zeta), \tag{52}$$

where

$$\mathcal{M}_k(\zeta) := \frac{\partial^k}{\partial \zeta^k} H(1,x,i\zeta(1,x)) = \begin{cases} \int_{-\infty}^{\lambda} u(e^{u\zeta} - 1) N(du), & k = 1, \\ \int_{-\infty}^{\lambda} u^k e^{u\zeta} N(du), & k \ge 2. \end{cases}$$

Here $N(du) := \int_{-\infty}^{1} \tilde{\mu}_s(du)ds$, and $\tilde{\mu}_s(du)$ is the image measure of $\mu(du)$ under the mapping $u \mapsto f(s)u$. The choice of λ above can be made in such a way that every segment $(\lambda - \varepsilon, \lambda)$ has a positive measure N. Then it can be shown (e.g. [17, Example 3.1]) that for any $\varepsilon > 0$,

$$e^{\zeta(\lambda-\varepsilon)} \ll \mathcal{M}_k(\zeta), \quad \mathcal{M}_k(\zeta) - \lambda^k N(\{\lambda\}) e^{\zeta\lambda} \ll e^{\zeta\lambda}, \quad \zeta \to +\infty. \tag{53}$$

Moreover, applying the Laplace method, we get

$$\mathcal{M}_k(\zeta) \sim \lambda^k \mathcal{M}_0(\zeta), \quad \zeta \to \infty, \tag{54}$$

where

$$\mathcal{M}_0(\zeta) = \int_{-\infty}^{\lambda} \Big(e^{\zeta u} - 1 - \zeta u\Big) N(du).$$

Note that the solution $\xi(t,x)$ to (33) satisfies

$$\xi(t,x) = \chi^{-1}(t)\zeta(xt^{-H-1/2}), \quad \text{where} \quad \zeta(x) := \xi(1,x). \tag{55}$$

Since $\zeta(x)$ is the solution to $\mathcal{M}_1(\zeta(x)) = x$, we have $\zeta(x) \to \infty$ as $x \to \infty$. Then by (55) and the definition of $\mathcal{A}$, we have $\chi(t)\xi \to \infty$ as $t + \xi \to \infty$, $(t,\xi) \in \mathcal{B}$, implying

$$\mathcal{M}_0(\chi(t)\xi) > 0, \quad \text{as } t + \xi \to \infty, \, (t,\xi) \in \mathcal{B}.$$

Therefore, by (52) and (54) we have **H2**:

$$\frac{\mathcal{M}_4(t,\xi)}{\mathcal{M}_2^2(t,\xi)} \sim \frac{1}{t\mathcal{M}_0(\chi(t)\xi)} \ll 1, \quad t+\xi \to \infty, \quad (t,\xi) \in \mathcal{B}.$$

Analogously, we have

$$\chi^{-2}(t)\frac{\mathcal{M}_4(t,\xi)}{\mathcal{M}_2(t,\xi)} \sim \lambda^2,$$

$$\ln\Big(\Big(\ln\big((1\vee\chi^{-1}(t))\mathcal{M}_2(t,\xi)\big)\Big)\vee 1\Big) \le C(\ln\ln t + \ln(\chi(t)\xi)) \ll \ln t + \chi(t)\xi,$$

as $t+\xi \to \infty$, $(t,\xi) \in \mathcal{B}$, which provides **H3**.

To show **H4**, take $b < 0$ and $\varepsilon > 0$ such that $0 < \mu([\lambda/f(b), -\varepsilon]) < \infty$. Then by (74) (see Appendix II) we have for $|z| \ge R$ with some R large enough

$$\begin{aligned}\Theta(t,z,\mathbb{R}_+) &\ge t\int_{-\infty}^0 \int_{\{u:\,0<f(s)u\le\lambda\}} (1-\cos(\chi(t)zuf(s)))\mu(du)ds\\ &\ge t_0\int_{-\infty}^b \int_{\lambda/f(b)}^{-\varepsilon} (1-\cos(\chi(t)zuf(s)))\mu(du)ds\\ &\ge ct_0\mu([\lambda/f(b),-\varepsilon])\ln(\chi(t)\varepsilon|z|).\end{aligned}$$

Since we can choose in (74) $c > 0$ arbitrary large, condition **H4** holds true. Finally, estimate (72) (see Appendix II) provides **H5**: since $\mu(\mathbb{R}_-) > 0$, there exists $(a,b) \subset (-\infty, 0)$, $q > 0$, such that $0 < \mu([q\lambda/f(b), \lambda/f(b)]) < \infty$ (note that $f(b) < 0$), and

$$\begin{aligned}\inf_{|z|\ge c}\Theta(t,z,[\chi(t)q,\infty)) &\ge t\inf_{|z|\ge c}\int_a^b \int_{q\lambda/f(s)}^{\lambda/f(s)} (1-\cos(\chi(t)zuf(s)))\mu(du)ds\\ &\ge t\inf_{|z|\ge c}\int_{q\lambda/f(b)}^{\lambda/f(b)} ((z\chi(t)u)^2\wedge 1)\mu(du)\\ &\ge c_1t((\chi(t)c)^2\wedge 1).\end{aligned}$$

Thus, all conditions of Theorem 3 are satisfied, and therefore (37) holds true.

By (53),

$$\ln\mathcal{M}_1(\zeta) \sim \lambda\zeta, \quad \zeta \to +\infty.$$

Since $\zeta(x)$ (cf. (55)) is the solution to $\mathcal{M}_1(\zeta(x)) = x$, this means that

$$\zeta(x) \sim \frac{\ln x}{\lambda}, \quad x \to +\infty.$$

Denote $\mathcal{D}(x) := H(1, x, i\xi(1, x))$, then using (54) with $k = 1, 2$, we get from the previous relation that

$$\mathcal{D}(x) \sim -\frac{x \ln x}{\lambda}, \quad \mathcal{M}_2(\zeta(x)) \sim \lambda x, \quad x \to +\infty. \tag{56}$$

From (37) and (56), we deduce that for a given $\varepsilon > 0$, there exists $y(\varepsilon) > 0$ such that (51) holds true.

3.4 Properties of $M_t(dx)$ and the Completion of the Proof

Lemma 1. *For every $t > 0$, we have $M_t(dx) = m_t(x)dx$ with m_t defined by (47).*

Proof. Let g be an arbitrary bounded measurable function. Using the scaling property (38) of the kernel $f(t, s)$, we can rewrite (44) as

$$\begin{aligned}
\int_{-\infty}^{\infty} g(y) M_t(dy) &= \int_{-\infty}^{t} \int_{\{yf(s/t)>\lambda\}} g(t^{H-1/2} f(s/t) y) \mu(dy) ds \\
&= t \int_{-\infty}^{0} \int_{\{y<-\lambda/f(s)\}} g_t(f(s)y) \mu(dy) ds \\
&\quad + t \int_{0}^{1} \int_{\{y>\lambda/f(s)\}} g_t(f(s)y) \mu(dy) ds \\
&=: I_- + I_+.
\end{aligned}$$

Here $g_t(y) := g(t^{H-1/2} y)$, and recall that the function $f(s) = f(1, s)$ is monotone on $(-\infty, 0)$ and $(0, 1)$, in particular, f is positive on $(0, 1)$ and negative on $(-\infty, 0)$. Let us transform the integrals I_+ and I_- separately.

Recall that the range of the restriction of f to $(0, 1)$ equals $\left[\frac{1}{\Gamma(H+1/2)}, +\infty\right)$. Then, making the change of variables $\tau = 1/f(s)$, we get

$$\begin{aligned}
I_+ &= t \int_0^{\Gamma(H+1/2)} \left(\int_{\{y>\lambda\tau\}} g_t\left(\frac{y}{\tau}\right) \mu(dy) \right) \frac{d\tau}{\tau^2 f'(f^{-1}(\frac{1}{\tau}))} \\
&= t \int_0^{+\infty} \left(\int_{\{y>\lambda\tau\}} g_t\left(\frac{y}{\tau}\right) \mu(dy) \right) \ell\left(\frac{1}{\tau}\right) \frac{d\tau}{\tau^2}.
\end{aligned}$$

In the second identity we take into account that, according to (12), the function ℓ vanishes on $\left(0, \frac{1}{\Gamma(H+1/2)}\right)$. Then the further change of variables $r = y/\tau$ and the Fubini theorem give

$$I_+ = t\int_\lambda^\infty g_t(r)\left(\int_0^\infty \frac{1}{y}\ell\left(\frac{r}{y}\right)\mu(dy)\right)dr.$$

Performing similar calculations, we get

$$I_- = t\int_\lambda^\infty g_t(r)\left(\int_{-\infty}^0 \frac{1}{y}\ell\left(\frac{r}{y}\right)\mu(dy)\right)dr.$$

Adding the expressions for I_+ and I_-, we get

$$\begin{aligned}\int_{-\infty}^\infty g(y)M_t(dy) &= t\int_\lambda^\infty g(t^{H-1/2}r)\mathfrak{m}(r)dr\\ &= \int_{\lambda t^{H-1/2}} g(y)\Big[t^{3/2-H}\mathfrak{m}(t^{1/2-H}y)\Big]\,dy.\end{aligned}$$

Let us summarize: we have (48) and (51); in addition, if $\mathfrak{m}$ is sub-exponential, we have (50) by Theorem 3.2 in [16]. In such a way, we obtain the proof of part (i) of Theorem 2. The following lemma completes the proof of the statement (ii).

Lemma 2. *If μ satisfies (17), then*

$$\mathfrak{m}(r) \sim c_H\left(\int_{\mathbb{R}} |u|^{2/(1-2H)}\mu(du)\right)r^{-(3-2H)/(1-2H)}, \quad r\to\infty, \tag{57}$$

where the constant c_H is defined in (19). In particular, $\mathfrak{m}\in\mathcal{L}d$.

Proof. Write $\mathfrak{m} = \mathfrak{m}_- + \mathfrak{m}_+$, where

$$\mathfrak{m}_-(r) := \int_{-\infty}^0 \frac{1}{y}\ell\Big(\frac{r}{y}\Big)\mu(dy), \quad \mathfrak{m}_+(r) := \int_0^{+\infty} \frac{1}{y}\ell\Big(\frac{r}{y}\Big)\mu(dy). \tag{58}$$

On the positive half-axis, the function ℓ can be calculated explicitly:

$$\ell(y) = c_H y^{-(3-2H)/(1-2H)}1_{\{y\ge 1/\Gamma(H+1/2)\}}, \tag{59}$$

where c_H is given by (19). Then

$$\mathfrak{m}_+(r) = c_H r^{-(3-2H)/(1-2H)}\int_0^{r\Gamma(H+1/2)} y^{2/(1-2H)}\mu(dy)$$

$$\sim c_H \left(\int_0^{+\infty} y^{2/(1-2H)} \mu(dy) \right) r^{-(3-2H)/(1-2H)}, \quad r \to +\infty.$$

Note that $2/(1-2H) > 2$ and μ is a Lévy measure (i.e. $\int_{|y|\le 1} y^2 \mu(dy) < \infty$), which together with (17) implies that $\int_0^{+\infty} y^{2/(1-2H)} \mu(dy) < \infty$.

On the negative half-axis, one has

$$\ell(y) \sim \begin{cases} -c_H (-y)^{-(3-2H)/(1-2H)}, & y \to -\infty, \\ -\hat{c}_H (-y)^{-(5-2H)/(3-2H)}, & y \to 0-, \end{cases} \tag{60}$$

with $\hat{c}_H = \frac{2}{3-2H} \left(\frac{1-2H}{2\Gamma(H+1/2)} \right)^{2/(3-2H)}$. Take arbitrary $\varepsilon > 0$ and choose $a_\varepsilon, b_\varepsilon > 0$ such that

$$-\ell(y) \le (\hat{c}_H + \varepsilon)(-y)^{-(5-2H)/(3-2H)}, \quad (-y) \in (0, a_\varepsilon),$$

$$-\ell(y) \le (c_H + \varepsilon)(-y)^{-(3-2H)/(1-2H)}, \quad (-y) > b_\varepsilon.$$

Then

$$\begin{aligned} \mathfrak{m}_-(r) &= \left[\int_{-\infty}^{-r/a_\varepsilon} + \int_{-r/a_\varepsilon}^{-r/b_\varepsilon} + \int_{-r/b_\varepsilon}^{0} \right] \frac{1}{y} \ell\Big(\frac{r}{y}\Big) \mu(dy) \\ &\le (\hat{c}_H + \varepsilon) r^{-(5-2H)/(3-2H)} \int_{-\infty}^{-r/a_\varepsilon} (-y)^{2/(3-2H)} \mu(dy) \\ &\quad + \sup_{y \in [-b_\varepsilon, -a_\varepsilon]} \Big(-\ell(y) \Big) \int_{-r/a_\varepsilon}^{-r/b_\varepsilon} \left(-\frac{1}{y} \right) \mu(dy) \\ &\quad + (c_H + \varepsilon) r^{-(3-2H)/(1-2H)} \int_{-r/b_\varepsilon}^{0} (-y)^{2/(1-2H)} \mu(dy) \\ &= I_1(r) + I_2(r) + I_3(r). \end{aligned}$$

By condition (17), one has

$$\begin{aligned} \int_{-\infty}^{-r/a_\varepsilon} (-y)^{2/(3-2H)} \mu(dy) &= \int_{-\infty}^{-r/a_\varepsilon} (-y)^{2/(1-2H)} (-y)^{-4/((1-2H)(3-2H))} \mu(dy) \\ &\le \left(\frac{a_\epsilon}{r} \right)^{4/((1-2H)(3-2H))} \int_{-\infty}^{-r/a_\varepsilon} (-y)^{2/(1-2H)} \mu(dy) \\ &\le c_1 r^{-4/((1-2H)(3-2H))}, \end{aligned}$$

which implies

$$r^{(3-2H)/(1-2H)} I_1(r) \to 0, \quad r \to \infty.$$

Further, since by (17) we have

$$r^{2/(1-2H)} \mu((-\infty, -r]) \le \int_{-\infty}^{-r} (-y)^{2/(1-2H)} \mu(dy) \to 0, \quad r \to \infty,$$

then

$$r^{(3-2H)/(1-2H)} I_2(r) \le c_2 r^{2/(1-2H)} \mu\Big(\big(-\infty, -r/b_\varepsilon\big]\Big) \to 0, \quad r \to +\infty.$$

Thus,

$$\limsup_{r\to+\infty} r^{(3-2H)/(1-2H)} \mathfrak{m}_-(r) \le (c_H + \varepsilon) \int_{-\infty}^{0} (-y)^{2/(1-2H)} \mu(dy).$$

The same argument provides the desired lower bound for $\liminf_{r\to+\infty}$ with $c_H - \varepsilon$ instead of $c_H + \varepsilon$. Since ε is arbitrary, these two estimates lead to the relation

$$\mathfrak{m}_-(r) \sim c_H \left(\int_{-\infty}^{0} (-y)^{2/(1-2H)} \mu(dy) \right) r^{-(3-2H)/(1-2H)}, \quad r \to +\infty,$$

which completes the proof.

4 Two Examples: The "Extremely Heavy-Tailed" Case

In this section we give two examples which illustrate the behaviour of $p_t(x)$ when condition (17) fails. In this case we say that the measure μ is "extremely heavy-tailed".

Example 1. Denote by $\mu_-(x) = \mu((-\infty, -x])$, $\mu_+(x) = \mu([x, +\infty))$, $x > 0$, the "tails" of the Lévy measure μ, and assume that μ_- and μ_+ are regularly varying at $+\infty$, that is, there exist $\alpha_\pm \in \mathbb{R}$ and slowly varying functions $L_\pm$, such that

$$\mu_\pm(x) = x^{-\alpha_\pm} L_\pm(x);$$

see, for example, [12, Chap. VIII, Sect. 8]. We investigate the behaviour of the functions $\mathfrak{m}_-$ and $\mathfrak{m}_+$ introduced in the proof of Lemma 2.

We assume

$$\alpha_\pm \in \Big(\frac{2}{3-2H}, \frac{2}{1-2H} \Big). \tag{61}$$

Note that condition $\alpha_\pm \geq 2/(3-2H)$ is necessary for (5) to hold true, and if $\alpha_\pm > 2/(1-2H)$, then (17) holds true and the required behaviour of $\mathfrak{m}_-$ and $\mathfrak{m}_+$ is already described in Lemma 2. In order to simplify the exposition, we exclude from the consideration the critical values $\alpha_\pm = 2/(3-2H)$ and $\alpha_\pm = 2/(1-2H)$.

The asymptotic behaviour of $\mathfrak{m}_+$ can be obtained almost straightforwardly using the standard result on the behaviour of the integrals w.r.t. the measures with regularly varying tails; see [12, Chap. VIII, Sect. 9, Theorem 2]:

$$\mathfrak{m}_+(r) \sim \left(\frac{\mu_+(r)}{r}\right) \int_0^{+\infty} \Phi(z)\, \mathfrak{p}_+(z)\, dz, \quad r \to \infty, \quad \text{with} \quad \mathfrak{p}_+(z) = \alpha_+ z^{-\alpha_+ - 1}. \tag{62}$$

The investigation of the behaviour of $\mathfrak{m}_-$ is slightly more complicated. However, the argument here is quite standard, and therefore we just sketch it.

Write $\mathfrak{m}_-$ in the form

$$\mathfrak{m}_-(r) = r^{-1} \int_{-\infty}^{0} \Phi\left(\frac{y}{r}\right) \mu(dy), \quad \Phi(x) := \frac{1}{x} \ell\left(\frac{1}{x}\right).$$

It follows from (60) that there exists a constant C such that $\Phi(x) \leq C(-x)^{2/(1-2H)}$ for $(-x)$ small enough and $\Phi(x) \leq C(-x)^{2/(3-2H)}$ for $(-x)$ large enough. Then, by [12, Chap. VIII, Sect. 9, Theorem 2], (see also Problem 30 in Sect. 10 of the same Chapter), we have for A small enough and B large enough

$$\begin{aligned}\int_{-\infty}^{-Br} \Phi\left(\frac{y}{r}\right) \mu(dy) &\leq C r^{-2/(3-2H)} \int_{-\infty}^{-Br} (-y)^{2/(3-2H)} \mu(dy) \\ &\leq C^1_{H,\alpha_-} B^{2/(3-2H)} \mu_-(Br),\end{aligned}$$

$$\begin{aligned}\int_{-Ar}^{0} \Phi\left(\frac{y}{r}\right) \mu(dy) &\leq C r^{-2/(1-2H)} \int_{-Ar}^{0} (-y)^{2/(1-2H)} \mu(dy) \\ &\leq C^2_{H,\alpha_-} A^{2/(1-2H)} \mu_-(Ar),\end{aligned}$$

with some explicitly given constant $C^i_{H,\alpha_-} \in (0,\infty)$, $i = 1, 2$. We have

$$\limsup_{r\to+\infty} A^{2/(1-2H)} \frac{\mu_-(Ar)}{\mu_-(r)} = A^{2/(1-2H)-\alpha_-} \limsup_{r\to+\infty} \frac{L_-(Ar)}{L_-(r)} = A^{2/(1-2H)-\alpha_-}$$

and, similarly,

$$\limsup_{r\to+\infty} B^{2/(3-2H)} \frac{\mu_-(Br)}{\mu_-(r)} = B^{2/(3-2H)-\alpha_-}.$$

Then, by condition (61), for every $\varepsilon > 0$, one can choose A and B in such a way that

$$\limsup_{r\to+\infty} \frac{1}{\mu_-(r)} \left(\int_{-\infty}^{0} \Phi\left(\frac{y}{r}\right)\mu(dy) - \int_{-Br}^{-Ar} \Phi\left(\frac{y}{r}\right)\mu(dy) \right) \le \varepsilon. \qquad (63)$$

Further, the function Φ is continuous and positive on $[-B, -A]$. Therefore, there exists a piece-wise constant function Φ_ε such that

$$(1-\varepsilon)\Phi_\varepsilon \le \Phi \le (1+\varepsilon)\Phi_\varepsilon. \qquad (64)$$

Clearly, one has for every segment $(a, b] \subset \mathbb{R}_-$

$$\mu((ra, rb]) \sim \mu_-(r) \int_a^b \mathfrak{p}_-(z)\, dz, \quad r \to \infty, \quad \text{with} \quad \mathfrak{p}_-(z) = \alpha_-(-z)^{-\alpha_- - 1}.$$

Therefore,

$$\frac{1}{\mu_-(r)} \int_{-Br}^{-Ar} \Phi_\varepsilon\left(\frac{y}{r}\right)\mu(dy) \to \int_{-B}^{-A} \Phi_\varepsilon(z)\, \mathfrak{p}_-(z)\, dz, \quad r \to +\infty.$$

Combined with (63) and (64), this gives

$$\limsup_{r\to+\infty} \frac{1}{\mu_-(r)} \int_{-\infty}^{0} \Phi\left(\frac{y}{r}\right)\mu(dy) \le \varepsilon + \frac{1+\varepsilon}{1-\varepsilon} \int_{-\infty}^{0} \Phi(z)\, \mathfrak{p}_-(z)\, dz.$$

One can write in the same fashion the lower bound for $\liminf_{r\to+\infty}$ (we omit the calculation). Then, since $\varepsilon > 0$ is arbitrary, we finally arrive at

$$\mathfrak{m}_-(r) \sim \left(\frac{\mu_-(r)}{r}\right) \int_{-\infty}^{0} \Phi(z)\, \mathfrak{p}_-(z)\, dz, \quad r \to \infty.$$

This and (62) give that the function

$$\mathfrak{m}(r) \sim \frac{1}{r} \left(\mu_-(r) \int_{-\infty}^{0} \Phi(z)\, \mathfrak{p}_-(z)\, dz + \mu_+(r) \int_0^{+\infty} \Phi(z)\, \mathfrak{p}_+(z)\, dz \right), \quad r \to +\infty,$$

clearly belongs to the class $\mathcal{L}d$, and thus the statement (i) of Theorem 2 holds true.

Consider, for instance, the "α-stable-like" case

$$\mu_-(r) \sim C_- r^{-\alpha}, \quad \mu_+(r) \sim C_+ r^{-\alpha}, \quad \text{with} \quad \alpha \in \left(\frac{2}{3-2H}, \frac{2}{1-2H}\right).$$

Then (16) and the above calculations give

$$p_t(x) \sim t^{1-\alpha(1/2-H)} x^{-\alpha-1} \int_{\mathbb{R}} \Phi(z) \mu_{\alpha, C_-, C_+}(dz), \quad x \to +\infty \qquad (65)$$

with

$$\mu_{\alpha,C_-,C_+}(dz) = \alpha|z|^{-\alpha-1}\Big(C_-1_{\mathbb{R}_-}(z) + C_+1_{\mathbb{R}_+}(z)\Big)\,dz.$$

Note that the formal expression for μ_{α,C_-,C_+} coincides with that for the Lévy measure of an α-stable distribution, although for $\alpha \in (2, 2/(1-2H))$, an "α-stable distribution" itself does not exist.

In contrast to (18), formula (65) contains explicitly the time parameter t. In addition, the polynomial "shape" of the expression in the right-hand side of (65) depends on α, i.e. on the "shape" of the tails of the Lévy measure μ.

Example 2. When the Lévy measure μ is "extremely heavy-tailed" in the sense explained above, the function $\mathfrak{m}$ may fail to belong to the class $\mathcal{L}d$ at all. Consider the measure

$$\mu(dx) = \delta_{-1}(dx) + \sum_{k\geq 0} 2^{k-2k/(1-2H)}\delta_{2^k}(dy).$$

Then $\mu(\mathbb{R}_-) > 0$, (17) fails, whereas (5) is satisfied:

$$\int_{\mathbb{R}} |y|^{2/(3-2H)}\mu(dy) = 1 + \sum_{k\geq 0} 2^{2k/(3-2H)}2^{k-2k/(1-2H)} = 1 + \sum_{k\geq 0} 2^{-k\cdot\frac{1+4H(2-H)}{(3-2H)(1-2H)}} < \infty.$$

Using (59), we can write $\mathfrak{m}(r)$ explicitly:

$$\mathfrak{m}(r) = -\ell(-r) + c_H r^{-(3-2H)/(1-2H)} \sum_{k:\,2^k \leq r\Gamma(H+1/2)} 2^k, \quad r \geq 0,$$

where c_H is defined in (19). To shorten the notation, put $c := (\Gamma(H+1/2))^{-1}$. We have for $r_n := 2^n c$ and $r_n' := (2^n - 1)c$, respectively,

$$\mathfrak{m}(r_n) = \mathfrak{m}(2^n c) = -\ell(-2^n c) + c_H(2^n c)^{-(3-2H)/(1-2H)} \sum_{k=0}^{n} 2^k$$

and

$$\mathfrak{m}(r_n') = \mathfrak{m}((2^n-1)c) = -\ell(-(2^n-1)c) + c_H((2^n-1)c)^{-(3-2H)/(1-2H)} \sum_{k=0}^{n-1} 2^k.$$

Note that

$$\ell(-(2^n-1)c) \sim \ell(-2^n c) \sim -c_H(-2^n c)^{-(3-2H)/(1-2H)} \quad \text{and} \quad \sum_{k=0}^{n} 2^k \sim 2^{n+1}$$

as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \frac{\mathfrak{m}((2^n-1)c)}{\mathfrak{m}(2^n c)} = \frac{1}{2},$$

and $\mathfrak{m} \notin \mathcal{L}d$.

From these two examples one can see that in the "extremely heavy-tailed" case, the asymptotic behaviour of the distribution density of Z_t^H is more sensitive with respect to the behaviour of the "tails" of the Lévy measure μ than in the case where the integrability condition (17) holds true. If these tails are regularly varying, then (16) holds true with the right-hand side depending both on t and on the "shape" of the "tails" of μ. On the other hand, when the "tails" of μ are both "heavy" and "irregular", the function $\mathfrak{m}$ may fail to belong to the class $\mathcal{L}d$, which means that we cannot apply Theorem 2 at all.

Appendix I: Existence of Integral (1)

Proposition 2. *Let* $0 < H < 1, H \neq 1/2$. *Then the integral* (1) *is well defined for every* $t \in \mathbb{R}$ *if, and only if, the Lévy measure* μ *satisfies (5).*

Proof. We consider the case $0 < H < 1/2$; the calculations in the case $1/2 < H < 1$ are analogous. We check the necessary and sufficient condition for the existence of (1) formulated in [22, Theorem 2.7]. In our case these conditions can be rewritten as

$$\int_{-\infty}^{1} f^2(s)ds < \infty, \tag{66}$$

$$I_1 := \int_{-\infty}^{1} \int_{\mathbb{R}} \left(1 \wedge |f(s)x|^2\right) \mu(dx)ds < \infty, \tag{67}$$

and

$$I_2 := \int_{-\infty}^{1} \left| \int_{\mathbb{R}} (\tau(f(s)x) - f(s)\tau(x))\, \mu(dx)\right| ds < \infty, \tag{68}$$

where

$$\tau(x) = \begin{cases} x, & \text{if} \quad |x| \le 1, \\ \frac{x}{|x|}, & \text{if} \quad |x| > 1, \end{cases} \tag{69}$$

and $f(s) := f(1, s)$. Clearly, (66) is satisfied. We show that (a) (67) and (5) are equivalent, and (b) (68) follows from (5).

(a) Split

$$I_1 = I_{11} + I_{12} + I_{13} + I_{14}, \tag{70}$$

where

$$I_{11} := \int_{-\infty}^{-1} \int_{|uf(s)|\le 1} \ldots, \quad I_{12} := \int_{-1}^{1} \int_{|uf(s)|\le 1} \ldots,$$

$$I_{13} := \int_{-\infty}^{-1} \int_{|uf(s)|> 1} \ldots, \quad I_{14} := \int_{-1}^{1} \int_{|uf(s)|> 1} \ldots,$$

and estimate the integrals I_{1i}, $i = 1, .., 4$, separately.

Since $f(s) \sim -\frac{2H-1}{2\Gamma(H+1/2)}|s|^{H-3/2}$ as $s \to -\infty$, $f(s) \sim -\frac{1}{\Gamma(H+1/2)}|s|^{H-1/2}$, as $s \to 0-$, and $f(s) = \frac{1}{\Gamma(H+1/2)}(1-s)^{H-1/2}$ for $0 \le s < 1$, to check the finiteness of I_1, it is enough to substitute $f(s)$ in the regions $(-\infty, -1]$ and $(-1, 0)$ with, respectively, $-|s|^{H-3/2}$ and $-|s|^{H-1/2}$, and to check the finiteness of the integrals $\tilde{I}_{11} := \int_1^\infty \int_{|x|\le s^{3/2-H}} \ldots$, $\tilde{I}_{12} := \int_0^1 \int_{|x|\le s^{1/2-H}} \ldots$, $\tilde{I}_{13} := \int_1^\infty \int_{|x|> s^{3/2-H}} \ldots$, and $\tilde{I}_{14} := \int_0^1 \int_{|x|> s^{1/2-H}} \ldots$.

We get:

$$\begin{aligned}
\tilde{I}_{11} &= \int_1^\infty \frac{1}{s^{3-2H}} \int_{|x|\le s^{3/2-H}} |x|^2 \mu(dx)ds \\
&= \int_1^\infty \frac{1}{s^{3-2H}} \int_{|x|\le 1} |x|^2 \mu(dx)ds + \int_1^\infty \frac{1}{s^{3-2H}} \int_{1<|x|\le s^{3/2-H}} |x|^2 \mu(dx)ds \\
&= \frac{1}{2-2H}\left(\int_{|x|\le 1} |x|^2 \mu(dx) + \int_{|x|\ge 1} |x|^{2/(3-2H)} \mu(dx)\right);
\end{aligned}$$

$$\tilde{I}_{12} = \int_0^1 s^{2H-1} \int_{|x|\le s^{1/2-H}} |x|^2 \mu(dx)ds \le \frac{1}{2H} \int_{|x|\le 1} |x|^2 \mu(dx);$$

$$\tilde{I}_{13} = \int_1^\infty \int_{|x|\ge s^{3/2-H}} \mu(dx)ds = \int_{|x|\ge 1} (|x|^{2/(3-2H)} - 1)\mu(dx);$$

$$\tilde{I}_{14} = \int_0^1 \int_{|x|\ge s^{1/2-H}} \mu(dx)ds = \int_{|x|\le 1} |x|^{2/(1-2H)} \mu(dx) + \int_{|x|\ge 1} \mu(dx).$$

Therefore, $I < \infty$ if and only if (5) holds true.

(b) Split $I_2 := I_{21} + I_{22}$, where $I_{21} := \int_{-\infty}^{1}\int_{|uf(s)|\le 1}\dots$ and $I_{22} := \int_{-\infty}^{1}\int_{|uf(s)|\ge 1}\dots$. Observe that

$$\int_{|x|\le 1/|u|}(ux-u\tau(x))\mu(dx)=\int_{1\le|x|\le 1/|u|}(ux-\frac{ux}{|x|})\mu(dx)\le 2\int_{1\le|x|\le 1/|u|}|ux|\mu(dx).$$

Then

$$I_{21}\le 2\int_{-\infty}^{-1}\int_{1\le|x|\le 1/|f(s)|}|f(s)x|\mu(dx)ds. \tag{71}$$

To estimate the right-hand side of (71) it is enough to estimate

$$\tilde{I}_{21}:=\int_1^{\infty}\int_{1\le|x|\le s^{3/2-H}}\frac{|x|}{s^{3/2-H}}\mu(dx)ds=\frac{2}{1-2H}\int_{|x|\ge 1}|x|^{2/(3-2H)}\mu(dx).$$

Thus, (5) implies the finiteness of $\tilde{I}_{21}$, and, consequently, of (71).

To estimate I_{22} observe that

$$\begin{aligned}&\int_{|x|\ge 1/|u|}(\frac{xu}{|xu|}-u\tau(x))\mu(dx)\\&\qquad=\int_{|x|\ge\max(1/|u|,1)}(\frac{xu}{|xu|}-u\frac{x}{|x|})\mu(dx)+\int_{1/|u|\le|x|\le 1}(\frac{xu}{|xu|}-ux)\mu(dx).\end{aligned}$$

Then

$$\begin{aligned}I_{22}&\le 2\Bigg(\int_{-\infty}^{-1}\int_{|x|\ge 1/|f(s)|}\mu(dx)ds+\int_{-1}^{1}\int_{|x|\ge 1}|f(s)|\mu(dx)ds\\&\qquad+\int_{-1}^{1}\int_{1/|f(s)|\le|x|\le 1}|f(s)|\mu(dx)ds\Bigg)\\&\le C_1\Bigg(\int_1^{\infty}\int_{|x|\ge s^{3/2-H}}\mu(dx)ds+\int_{-1}^{1}|f(s)|ds\int_{|x|\ge 1}\mu(dx)\\&\qquad+\int_0^1\int_{s^{1/2-H}\le|x|\le 1}s^{H-1/2}\mu(dx)ds\Bigg)\\&\le C_2\Bigg(\int_{|x|\ge 1}|x|^{2/(3-2H)}\mu(dx)+\int_{|x|\ge 1}\mu(dx)+\int_{|x|\le 1}|x|^{(1+2H)/(1-2H)}\mu(dx)\Bigg),\end{aligned}$$

and the finiteness of the right-hand side is implied by (5).

Appendix II: Existence of the Distribution Density

Proposition 3. *Let* $0 < H < 1, H \neq 1/2$. *Then, under (5) and our standing assumption (6), the integral (1) possesses for any* $t \neq 0$ *the distribution density* $p_t \in C_b^\infty(\mathbb{R})$.

Remark 3. In the non-Markov case $H \neq 1/2$, the kernel $f(t,s)$ provides a strong "smoothifying" effect in the sense that the weakest possible non-degeneracy assumption (6) is already sufficient for the integral (1) to possess a smooth distribution density. We refer to [17], Sect. 3, for the detailed discussion of the various forms of the "smoothifying" effect for Lévy-driven stochastic integrals with deterministic kernels.

For the proof we use the following statement; see [17, Lemma 3.3].

Proposition 4. *(a) For a positive function* $h(s)$ *having a continuous non-zero derivative on some interval* $[a,b] \subset \mathbb{R}$, *one has*

$$\int_a^b \big(1-\cos(h(s)x)\big)ds \geq c(x^2 \wedge 1). \tag{72}$$

(b) For a positive convex on $(-\infty, b) \subset \mathbb{R}$ *function* $h(s)$, *satisfying*

$$\lim_{s\to-\infty} e^{-\gamma s}h(s) = +\infty \quad \textit{for all} \quad \gamma > 0, \tag{73}$$

one has

$$\int_{-\infty}^b (1-\cos(xh(s)))\,ds \geq c\ln|x| \tag{74}$$

for all $c > 0$ *and* $|x|$ *big enough.*

Proof (Proof of Proposition 3). Recall (cf. (7)) that the characteristic function of Z_t^H is of the form $\phi(t,z) = e^{\Psi(t,-z)}$. For a fixed t, the function $h(s) = -t^{H-1/2}f(s)$ satisfies (73) with $b = 0$. Since $\mu(\mathbb{R}) > 0$ (cf. (6)), there exists $q > 0$ such that

$$Q := \max\{\mu((-\infty,-q]), \mu([q,\infty)\} > 0.$$

Then using (74) for $|z|$ large enough, we get

$$-\mathrm{Re}\,\Psi(t,-z) \geq t\int_{-\infty}^0 \int_{|u|\geq q} (1-\cos(t^{H-1/2}f(s)uz))\mu(du)ds \geq tcQ\ln|qt^{H-1/2}z|.$$

Since $c > 0$ is arbitrary, the function $|z|^n|\phi(t,z)| = e^{\mathrm{Re}\Psi(t,-z)+n\ln|z|}$ is integrable in z for any $n \geq 1$. Therefore, the density p_t is well defined and belongs to C_b^∞ as the inverse Fourier transform of $\phi(t,z)$:

$$p_t(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-izx}\phi(t,z)dz. \tag{75}$$

Remark 4. Literally the same argument implies that if the truncation level $\lambda > 0$ is chosen small enough, then $|z|^n |\phi_1(t, z)| = e^{\mathrm{Re}\Psi_1(t,-z)+n \ln |z|}$ is integrable in z for any $a > 0$, $t > 0$; see (39). This implies the existence of $\tilde{p}_t(x)$; see (42).

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References

1. Basse, A., Pedersen, J.: Lévy driven moving averages and semimartingales. Stochastic Processes Appl. **119**, 2970–2991 (2009)
2. Benassi, A., Cohen, S., Istas, J.: On roughness indices for fractional fields. Bernoulli **10(2)**, 357–373 (2004)
3. Bender, C., Marquardt, T.: Integrating volatility clustering into exponential Lévy models. J. Appl. Probab. **46(3)**, 609–628 (2009)
4. Bender, C., Lindner, A., Schicks, M.: Finite variation of fractional Lévy processes. J. Theor. Probab. **25(2)**, 594–612 (2012)
5. Bender, C., Sottinen, T., Valkeila, E.: Pricing by hedging and no-arbitrage beyond semimartingales. Finance Stoch. **12(4)**, 441–468 (2008)
6. Burnecki, K., Weron. A.: Fractional Lévy stable motion can model subdiffusive dynamics. Phys. Rev. **E82**, 021130 (2010)
7. Calvo, I., Sánchez, R., Carreras, A.: Fractional Lévy motion through paths integrals. J. Phys. **A 42**, 055003 (2009)
8. Chechkin, A.V., Gonchar, V. Yu.: A model for persistent Lévy motion. Physica **A277**, 312–326 (2000)
9. Copson, E.T.: Asymptotic Expansions. Cambridge University Press, Cambridge (1965)
10. Dimakis, A. G., Maragos, P.: Phase-modulated resonances modeled as self-similar processes with application to turbulent sounds. IEEE Trans. Signal Process. **53(11)** (2005)
11. Eliazar, I.I., Shlesinger, M.F.: Langevin unification of fraction motions. J. Phys. **A45**, 162002 (2012)
12. Feller, W.: An Introduction to Probability Theory and Its Applications, Vol. II. 2nd edn. John Wiley & Sons, Inc., New York (1971)
13. Gagnon, J.-S., Lovejoy, S., Schertzer, D.: Multifractal earth topography. Nonlinear process. geophys. **13 (5)**, 541–570 (2006)
14. Hartman, P., Wintner, A.: On the infinitesimal generators of integral convolutions. Am. J. Math. **64**, 273–298 (1942)
15. Huillet, T.: Fractional Lévy motions and related processes. J. Phys. **A 32**, 7225 (1999)
16. Klüppelberg, C.: Subexponential distributions and characterizations of related classes. Probab. Th. Rel. Fields. **82**, 259–269 (1989)
17. Knopova, V., Kulik, A.: Exact asymptotic for distribution densities of Lévy functionals. Electronic J. Probab. **16**, 1394–1433 (2011)
18. Kogon, S.M. Manolakis, D.G.: Signal modeling with self-similar α-stable processes: the fractional levy stable motion model. IEEE Trans. Signal Process. **44(4)**, 1006–1010 (1996)
19. Laskin, N., Lambadaris, I., Harmantzis, F., Devetsikiotis, M.: Fractional Lévy motion and its application to traffic modeling. Computer Networks (Special Issue on Long-Range Dependent Traffic Engineering) **40(3)**, 363–375 (2002)
20. Marquardt, T.: Fractional Lévy processes with an application to long memory moving average processes. Bernoulli **12(6)**, 1099–1126 (2006)

21. Painter, S., Patterson. L.: Fractional Lévy motion as a model for spatial variability in sedimentary rock. Geophys. Res. Let. **21(25)**, 2857–2860 (1994)
22. Rajput, B.S., Rosinski, J.: Spectral representations of infinitely divisible processes. Probab. Th. Rel. F. **82**, 451–487 (1989)
23. Samorodnitsky, G., Taqqu, M.S.: Stable Non-Gaussian Random Processes. Chapman Hall, New York (1994)
24. Tikanmäki, H. Mishura, Yu.: Fractional Lévy processes as a result of compact interval integral transformation. Stoch. Anal. Appl. **29**, 1081–1101 (2011)
25. Watkins, N. W., Credgington, D., Hnat, B., Chapman, S.C., Freeman, M.P., Greenhough. J.: Towards synthesis of solar wind and geomagnetic scaling exponents: A fractional Lévy motion model. Space Sci. Rev. **121 (1–4)**, 271–284 (2005)
26. Weron, A., Burnecki, K., Mercik, Sz., Weron, K.: Complete description of all self-similar models driven by Lévy stable noise. Phys. Rev. **E71** (2005), 016113 (2005)

Large Deviations for Random Evolutions in the Scheme of Asymptotically Small Diffusion

Volodymyr S. Korolyuk and Igor V. Samoilenko

Abstract The theory of large deviations deals with the asymptotic estimations for probabilities of rare events. The method, used in majority of classical works, is based on the change of measure and application of the variational formula to the cumulant of the process under study. Here the large deviations problem is considered for random evolutions in the scheme of asymptotically small diffusion. The method of asymptotic analysis for the exponential generator of the Markov process is used. The limit exponential generators are calculated for random evolution with the ergodic Markov switching (Sect. 3) and with the split-and-double merging switching Markov process (Sect. 4). The method proposed here may have applications for the finite dimensional models arising in the theory of random evolutions in $\mathbf{R}^d$, queuing theory, etc.

1 Introduction

The theory of large deviations had arisen in the work of H. Cramér [2] and deals with the asymptotic estimations for probabilities of rare events. The main problem in the large deviations theory is the construction of the rate functional to estimate probabilities of rare events. The method, used in the majority of classical works, is based on the change of measure and application of variational formula to the cumulant of the process under study. Different aspects and applications of this problem were investigated by many mathematicians. We discuss the Markov processes with independent increments, so it is natural to refer a reader to the fundamental works [3, 16] and [7].

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Another approach arises in works [8] and [1] and is applied to the large deviations problem in [6]. It is based on the asymptotic analysis of the nonlinear Hamilton-Jacobi equation corresponding to the process under study. Then, the solution of the limit nonlinear Hamilton-Jacobi equation is given by the variational formula that defines the rate functional of the prelimit process. The main problem here is to prove the uniqueness of the solution of the limit nonlinear equation.

All the technical problems connected to the application of the last method to different classes of Markov processes are solved in the monograph [5]. The main idea of this monograph is the following.

Let $\eta(t), t \geq 0$ be a Markov process in Euclidean space $\mathbf{R}$, defined by its linear generator $\mathbf{L}$. The function $\varphi(u) \in \mathcal{B}_{\mathbf{R}}$. Unlike the classical martingale characterization of the Markov processes

$$\mu_t = \varphi(\eta(t)) - \varphi(\eta(0)) - \int_0^t \mathbf{L}\varphi(\eta(s))ds,$$

the large deviations theory is based on the exponential martingale characterization (see [5, Chap. 1]). Namely,

$$\tilde{\mu}_t = \exp\{\varphi(\eta(t)) - \varphi(\eta(0)) - \int_0^t \mathbf{H}\varphi(\eta(s))ds\}$$

is a martingale.

The exponential (nonlinear) operator $\mathbf{H}$ is connected with the linear generator $\mathbf{L}$ of the Markov process $\eta(t), t \geq 0$ in a following way:

$$\mathbf{H}\varphi(u) = e^{-\varphi(u)}\mathbf{L}e^{\varphi(u)}, \quad e^{\varphi(u)} \in \mathcal{D}(\mathbf{L}).$$

The large deviations problem may be formulated as a limit theorem in the scheme of series with a small series parameter $\varepsilon \to 0 (\varepsilon > 0)$. Namely (compare with [5, Chap. 1])

$$\mathbf{H}^\varepsilon\varphi^\varepsilon \to \mathbf{H}\varphi, \quad \varphi^\varepsilon \to \varphi, \quad \varepsilon \to 0.$$

Here by definition

$$\mathbf{H}^\varepsilon\varphi(u) := e^{-\varphi(u)/\varepsilon}\varepsilon\mathbf{L}^\varepsilon e^{\varphi(u)/\varepsilon}.$$

The generator $\mathbf{L}^\varepsilon, \varepsilon > 0$ defines Markov process $x^\varepsilon(t), t \geq 0, \varepsilon > 0$ in the scheme of series under some scaling transform.

Example 1. The asymptotically small diffusion process is given by $\sqrt{\varepsilon}\sigma w(t), t \geq 0$ with the standard Brownian motion process $w(t), t \geq 0$. The generator of such a process is the following:

$$\mathbf{L}^\varepsilon\varphi(u) = \varepsilon\frac{1}{2}B\varphi''(u), \quad B = \sigma^2, \quad \varphi''(u) := \partial^2\varphi(u)/\partial u^2.$$

The exponential generator of the asymptotically small diffusion process may be easily calculated:

$$\mathbf{H}^{\varepsilon}\varphi(u) = \frac{1}{2}B[\varphi'(u)]^2 + \varepsilon B\varphi''(u).$$

Hence, the limit exponential operator is represented as

$$\mathbf{H}\varphi(u) = \frac{1}{2}B[\varphi'(u)]^2. \tag{1}$$

Remark 1. The exponential operator (1) in Euclidean space $\mathbf{R}^d, d \geq 2$ is represented by the quadratic form

$$\mathbf{H}\varphi(u) = \frac{1}{2}\sum_{k,r=1}^{d} B_{kr}\varphi'_k(u)\varphi'_r(u), \quad \varphi'_k(u) := \partial\varphi(u)/\partial u_k,$$

$B = [B_{kr}; 1 \leq k, r \leq d]$ is the variance matrix of $w(t)$.

To simplify the notations, we present all the following results in $\mathbf{R}$.

The aim of our investigation is the asymptotic analysis of the large deviations problem for the random evolutions in the scheme of asymptotically small diffusion.

At the beginning (Sect. 2) the large deviations problem is considered for the processes with locally independent increments under the scaling proposed by A.A. Mogulskii [12]:

$$\eta^{\varepsilon}(t) = \varepsilon^2\eta(t/\varepsilon^3), \quad t \geq 0, \quad \varepsilon > 0. \tag{2}$$

The generator of the Markov process (2) is given by

$$\Gamma^{\varepsilon}\varphi(u) = \varepsilon^{-3}\int_{\mathbf{R}}[\varphi(u+\varepsilon^2 v) - \varphi(u)]\Gamma(u, dv), \quad u \in \mathbf{R}, \quad \varphi(u) \in \mathcal{B}_{\mathbf{R}}. \tag{3}$$

Usually we assume that the Lévy measure $\Gamma(u, dv)$ satisfies the condition

$$\int_{\mathbf{R}} e^{av}\Gamma(u, dv) < \infty, \quad a > 0, \quad u \in \mathbf{R}. \tag{4}$$

In the Sect. 3 the large deviations problem is considered for the random evolution process with Markov switching [9, Chap. 2]. The scheme of asymptotically small diffusion is considered under additional balance conditions (local and total).

The large deviations problem in the scheme of phase merging is investigated in Sect. 4.

2 Processes with Locally Independent Increments

In this section we consider the compound Poisson processes which is supposed to be defined by the generator (3) under the condition (4) for simplicity.

The balance condition (local) formulates as

$$\Lambda B: \qquad b(u) := \int_{\mathbf{R}} v \Gamma(u, dv) \equiv 0.$$

The main part of the asymptotic representation of the generator (3) on smooth enough test functions is

$$\Gamma^{\varepsilon}\varphi(u) = \varepsilon \frac{1}{2} B(u)\varphi''(u) + \varepsilon\delta^{\varepsilon}(u)\varphi(u),$$

where

$$B(u) = \int_{\mathbf{R}} v^2 \Gamma(u, dv)$$

and the negligible term converges uniformly by u on the functions $\varphi(u) \in C^3(\mathbf{R})$:

$$|\delta^{\varepsilon}(u)\varphi(u)| \to 0, \ \varepsilon \to 0. \tag{5}$$

The large deviations problem for the processes (2) may be solved using the limit approximation of the exponential generator [5, Part 1]:

$$\mathbf{H}^{\varepsilon}\varphi(u) = e^{-\varphi(u)/\varepsilon} \varepsilon \Gamma^{\varepsilon} e^{\varphi(u)/\varepsilon} = \varepsilon^{-2} \int_{\mathbf{R}} [e^{\Delta^{\varepsilon}\varphi} - 1]\Gamma(u, dv),$$

$$\Delta^{\varepsilon}\varphi := \varepsilon^{-1}[\varphi(u + \varepsilon^2 v) - \varphi(u)] = \varepsilon v \varphi'(u) + \varepsilon\delta^{\varepsilon}\varphi(u).$$

Hence, due to the ΛB condition,

$$\mathbf{H}^{\varepsilon}\varphi(u) = \varepsilon^{-2} \int_{\mathbf{R}} [\varepsilon v \varphi'(u) + \varepsilon^2 \frac{1}{2} v^2 [\varphi'(u)]^2]\Gamma(u, dv) + \delta^{\varepsilon}(u)\varphi(u) =$$

$$= \frac{1}{2} B(u)[\varphi'(u)]^2 + \delta^{\varepsilon}(u)\varphi(u)$$

with the negligible term (5).

Conclusion (comp. with [12]): The limit exponential operator for the processes with locally independent increments in the scheme of asymptotically small diffusion is given by

$$\mathbf{H}\varphi(u) = \frac{1}{2} B(u)[\varphi'(u)]^2. \tag{6}$$

3 Random Evolutions in the Scheme of Ergodic Phase Merging

In this section we investigate the random evolutions with locally independent increments and switching, so we should note that random evolutions with switching are also studied in Chap. 11 of [5] by the classical methods of averaging and homogenization. This approach involves perturbed PDEs operators and perturbed test functions and arises in the works [11, 13]. Recent monographs [14, 15] include large bibliography on this problem. Application of this method for the nonlinear case may also be found in the work [4]. This approach is important for the infinite dimensional state space models like interacting particles or stochastic PDEs. But in this case a lot of additional problems appear: correct description of the functional space for the solutions, the domain of the infinitesimal operators, etc.

We use the generators of Markov processes with a locally compact vector state space (see [9] for more details). This simplifies the analysis because the generators are defined for all bounded measurable functions. We lose generality, but can present obvious algorithms for verification of convergence conditions and calculation of the limit generators. This approach is important for finite dimensional models arising in the theory of random evolutions in $\mathbf{R}^d$, queuing theory, etc.

The Markov random evolution process in the scheme of series with a small series parameter $\varepsilon \to 0 (\varepsilon > 0)$ is considered as *the stochastic additive functional* [9, Sect. 3.4.2]:

$$\xi^\varepsilon(t) = \xi_0 + \int_0^t \eta^\varepsilon(ds; x(s/\varepsilon^2)), \quad t \geq 0 \tag{7}$$

in the case of local balance condition or

$$\xi^\varepsilon(t) = \xi_0 + \int_0^t \eta^\varepsilon(ds; x(s/\varepsilon^3)), \quad t \geq 0 \tag{8}$$

in the case of total balance condition.

The family of the processes with locally independent increments $\eta^\varepsilon(t; x), t \geq 0$, $x \in E$ is determined by the generators

$$\Gamma^\varepsilon(x)\varphi(u) = \varepsilon^{-3} \int_{\mathbf{R}} [\varphi(u + \varepsilon^2 v) - \varphi(u)]\Gamma(u, dv; x), \quad \varphi(u) \in \mathcal{B}_{\mathbf{R}}. \tag{9}$$

The switching Markov process $x(t), t \geq 0$ is given on the standard phase space $(E, \mathcal{E})$ by the generator

$$Q\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)], \quad \varphi(u) \in \mathcal{B}_{\mathbf{E}}. \tag{10}$$

The random evolution process is considered as the two-component Markov process $\xi^\varepsilon(t), x^\varepsilon(t) := x(t/\varepsilon^2), t \geq 0$, given by the generator [9, Sect. 5.3.2]

$$\mathbf{L}^\varepsilon_A \varphi(u, x) = [\varepsilon^{-2} Q + \Gamma^\varepsilon(x)]\varphi(u, x) \tag{11}$$

in the case of the local balance condition or as the two-component Markov process $\xi^\varepsilon(t), x^\varepsilon(t) := x(t/\varepsilon^3), t \geq 0$, given by the generator

$$\mathbf{L}_T^\varepsilon \varphi(u,x) = [\varepsilon^{-3} Q + \Gamma^\varepsilon(x)]\varphi(u,x) \tag{12}$$

in the case of the total balance condition.

The main assumption in the scheme of ergodic phase merging is *the uniform ergodicity* of the switching Markov process $x(t)$.

EA: There exists the stationary distribution $\pi(dx)$ on $(E, \mathcal{E})$ which defines the projector

$$\Pi\varphi(x) := \int_E \pi(dx)\varphi(x), \quad \varphi(x) \in \mathcal{B}_E$$

on the null-space of the generator Q:

$$\Pi Q = Q\Pi = 0.$$

The main assumption EA provides that the potential operator R_0 exists:

$$QR_0 = R_0 Q = \Pi - I.$$

So, the Poisson equation

$$Q\varphi(x) = \psi(x), \quad \Pi\psi(x) = 0$$

may be solved as follows:

$$\varphi(x) = R_0\psi(x), \quad \Pi\varphi(x) = 0.$$

The scheme of asymptotically small diffusion is considered under additional balance condition (local or total):

ΛB: $$b(u;x) := \int_{\mathbf{R}} v\Gamma(u,dv;x) \equiv 0.$$

TB: $$b(u) := \int_E \pi(dx) b(u;x) \equiv 0.$$

Lemma 1 ([10]). *The generator (11) of the random evolution (7) admits the following asymptotic representation:*

$$\mathbf{L}_\Lambda^\varepsilon \varphi(u,x) = [\varepsilon^{-2} Q + \varepsilon \mathbf{B}(x)]\varphi(u,x) + \delta^\varepsilon(u,x)\varphi(u),$$

$$\mathbf{B}(x)\varphi(u) = \frac{1}{2} B(u;x)\varphi''(u), \quad B(u;x) = \int_{\mathbf{R}} v^2 \Gamma(u,dv;x)$$

under the local balance condition ΛB.

The generator (12) of the random evolution (8) admits the following asymptotic representation:

$$\mathbf{L}_T^\varepsilon \varphi(u,x) = [\varepsilon^{-3}Q + \varepsilon^{-1}\Gamma(x) + \varepsilon\mathbf{B}(x)]\varphi(u,x) + \delta^\varepsilon(u,x)\varphi(u),$$

under the total balance condition TB and the negligible terms converge uniformly by u, x on the functions $\varphi(u) \in C^3(\mathbf{R})$:

$$|\delta^\varepsilon(u,x)\varphi(u)| \to 0.$$

The large deviations problem for the random evolutions in the scheme of ergodic phase merging is solved by the exponential generators described in the following theorem.

Theorem 1 ([10]). *The exponential generators of the large deviations for the random evolutions (7)–(12) are determined by the relations*

$$\mathbf{H}\varphi(u) = \frac{1}{2}B_*(u)[\varphi'(u)]^2. \tag{13}$$

The variation $B_*(u)$ is determined by

$$B_\Lambda(u) = \int_E \pi(dx)B(u;x), \quad B(u;x) = \int_{\mathbf{R}} v^2\Gamma(u,dv) \tag{14}$$

under the local balance condition ΛB, and by

$$B_T(u) = B_\Lambda(u) + B_0(u), \tag{15}$$

$$B_0(u) = \int_E \pi(dx)B_0(u;x), \quad B_0(u;x) = 2b(u;x)R_0 b(u;x),$$

under the total balance condition TB.

Remark 2. The exponential generators of the large deviations for the random evolutions in the scheme of asymptotically small diffusion are determined exactly as the exponential generator of the processes with independent increments (compare (2)–(4), (6) with (7), (8), (13)–(15)).

The proof of the Theorem 1 is based on the following lemma:

Lemma 2 ([10]). *The exponential generator on the perturbed test function admits the following asymptotic representations:*

(1) In the case of the local balance condition ΛB on the perturbed test function $\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon \ln[1 + \varepsilon\varphi_1(u,x)]$,

$$\mathbf{H}^\varepsilon \varphi^\varepsilon(u,x) = Q\varphi_1 + \tilde{\mathbf{B}}(x)\varphi(u) + \delta^\varepsilon(u,x)\varphi(u).$$

Here the operator

$$\tilde{\mathbf{B}}(x)\varphi(u) = \frac{1}{2}B(u;x)[\varphi'(u)]^2.$$

(2) In the case of the total balance condition TB on the perturbed test function $\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon \ln[1 + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x)]$*:*

$$\mathbf{H}^\varepsilon\varphi^\varepsilon(u,x) = \varepsilon^{-1}[Q\varphi_1 + \Gamma(x)\varphi(u)] + [Q\varphi_2 - \varphi_1 Q\varphi_1 + \tilde{\mathbf{B}}(x)\varphi(u)] + \delta^\varepsilon(u,x)\varphi(u).$$

In this case the operator

$$\Gamma(x)\varphi(u) := b(u;x)\varphi'(u).$$

The negligible terms converge uniformly by u, x *on the functions* $\varphi(u) \in C^3(\mathbf{R})$*:*

$$|\delta^\varepsilon(u,x)\varphi(u)| \to 0, \varepsilon \to 0.$$

4 Large Deviations in the Scheme of Split-and-Double Merging [9, Sect. 5.7.2]

4.1 Split-and-Double Merging Scheme

We introduce the switching Markov process $x^\varepsilon(t), t \geq 0$ on the standard phase (state) space $(E, \mathcal{E})$ in the series scheme with a small series parameter $\varepsilon \to 0, \varepsilon > 0$ on the split phase space

$$E = \bigcup_{k=1}^{N} E_k, \quad E_k \cap E_{k'} = \emptyset, \quad k \neq k'.$$

The Markov kernel is

$$Q^\varepsilon(x,B,t) = P^\varepsilon(x,B)[1 - e^{-q(x)t}], \quad x \in E, \quad B \in \mathcal{E}, \quad t \geq 0.$$

We also introduce the following assumptions:

ME1: The transition kernel of the embedded Markov chain $x_n^\varepsilon, n \geq 0$ has the following representation:

$$P^\varepsilon(x,B) = P(x,B) + \varepsilon P_1(x,B).$$

The stochastic kernel $P(x, B)$ is coordinated with the split phase space as follows:

$$P(x, E_k) = \mathbf{1}_k(x) := \begin{cases} 1, x \in E_k, \\ 0, x \notin E_k. \end{cases}$$

The stochastic kernel $P(x, B)$ determines the support Markov chain $x_n, n \geq 0$ on the separate classes $E_k, 1 \leq k \leq N$. Moreover, the perturbing signed kernel $P_1(x, B)$ satisfies the conservative condition

$$P_1(x, E) = 0,$$

which is a direct consequence of $P^\varepsilon(x, E) = P(x, E) = 1$.

ME2: The associated Markov process $x^0(t), t \geq 0$, given by the generator

$$Q\varphi(x) = q(x) \int_E P(x, dy)[\varphi(y) - \varphi(x)]$$

is uniformly ergodic in every class $E_k, 1 \leq k \leq N$, with the stationary distributions $\pi_k(dx), 1 \leq k \leq N$, satisfying the relations:

$$\pi_k(dx)q(x) = q_k \rho_k(dx), \quad q_k := \int_{E_k} \pi_k(dx)q(x).$$

ME3: The average exit probabilities

$$\hat{p}_k := \int_{E_k} \rho_k(dx) P_1(x, E \backslash E_k) > 0, \quad 1 \leq k \leq N$$

are positive and

$$0 < q(x) < +\infty.$$

The perturbing signed kernel $P_1(x, B)$ defines the transition probabilities between classes $E_k, 1 \leq k \leq N$. So, the relation $P^\varepsilon(x, B) = P(x, B) + \varepsilon P_1(x, B)$ means that the embedded Markov chain $x_n^\varepsilon, n \geq 0$ spends a long time in every class E_k and jumps from one class to another with the small probabilities $\varepsilon P_1(x, E \backslash E_k)$.

Under Assumptions **ME1–ME3** the following weak convergence holds [9, Chap. 5]:

$$v(x^\varepsilon(t)) \Rightarrow \hat{x}(t), \ \varepsilon \to 0, \quad v(x) = k \in \hat{E} = \{1, \ldots, N\}, \ x \in E_k.$$

The limit Markov process $\hat{x}(t), t \geq 0$ on the merged phase space $\hat{E} = \{1, \ldots, N\}$ is determined by the generating matrix

$$\hat{Q}_1 = (\hat{q}_{kr}, 1 \leq k, r \leq N),$$

where:

$$\hat{q}_{kr} = \hat{q}_k \hat{p}_{kr}, \quad k \neq r, \quad \hat{q}_k = \hat{p}_k q_k, \quad 1 \leq k \leq N.$$

$$\hat{p}_{kr} = p_{kr}/\hat{p}_k, \quad p_{kr} = \int_{E_k} \rho_k(dx) P_1(x, E_r), \quad 1 \leq k, \quad r \leq N, \quad k \neq r,$$

$$\hat{p}_k = -\int_{E_k} \rho_k(dx) P_1(x, E_k).$$

ME4: The merged Markov process $\hat{x}(t), t \geq 0$ is ergodic, with the stationary distribution $\hat{\pi} = (\pi_k, k \in \hat{E})$.

Thus, the operator Q^ε may be presented as

$$Q^\varepsilon = Q + \varepsilon Q_1, \quad Q_1(x) = q(x) \int_E P_1(x, dy)\varphi(y).$$

Let Π be the projector onto the null-space of the reducible-invertible operator Q acting as follows on the test functions φ:

$$\Pi\varphi(x) = \sum_{k=1}^{N} \hat{\varphi}_k \mathbf{1}_k(x), \quad \hat{\varphi}_k := \int_{E_k} \pi_k(dx)\varphi(x).$$

The contracted operator $\hat{Q}_1$ is defined by the relation

$$\hat{Q}_1 \Pi = \Pi Q_1 \Pi.$$

Let $\hat{\Pi}$ be the projector onto the null-space of the reducible-invertible contracted operator $\hat{Q}_1$:

$$\hat{\Pi}\hat{\varphi} := \sum_{k \in \hat{E}} \hat{\pi}_k \hat{\varphi}_k.$$

We define the potential matrix $\hat{R}_0 = [\hat{R}^0_{kl}; 1 \leq k, l \leq N]$ by the following relations:

$$\hat{Q}_1 \hat{R}_0 = \hat{R}_0 \hat{Q}_1 = \hat{\Pi} - I.$$

4.2 Large Deviations Under the Local Balance Condition ΛB

The random evolutions are studied under the condition

ΛB: $$b(u; x) := \int_{\mathbf{R}} v\Gamma(u, dv; x) \equiv 0$$

with the following scaling:

$$\xi^\varepsilon(t) = \varepsilon^2 \xi(t/\varepsilon^3), \quad x^\varepsilon_t := x^\varepsilon(t/\varepsilon^3). \tag{16}$$

The generator of the random evolution is given by

$$\mathbf{L}^{\varepsilon}_{\Lambda}\varphi(u,x) = [\varepsilon^{-3}Q + \varepsilon^{-2}Q_1 + \Gamma^{\varepsilon}(x)]\varphi(u,x), \tag{17}$$

$$\Gamma^{\varepsilon}(x)\varphi(u) = \varepsilon^{-3}\int_{\mathbf{R}}[\varphi(u+\varepsilon^2 v) - \varphi(u)]\Gamma(u,dv;x).$$

The generator (17) has the following asymptotic representation:

$$\mathbf{L}^{\varepsilon}_{\Lambda}\varphi(u,x) = [\varepsilon^{-3}Q + \varepsilon^{-2}Q_1 + \varepsilon\mathbf{B}(x)]\varphi(u,x) + \varepsilon\delta^{\varepsilon}(u,x)\varphi(u,x).$$

Here

$$\mathbf{B}(x)\varphi(u) = \frac{1}{2}B(u;x)\varphi''(u), \quad B(u;x) = \int_{\mathbf{R}} v^2\Gamma(u,dv;x).$$

Theorem 2. *The exponential generator of the large deviations for the random evolutions (16) under the conditions **ME1–ME4** and ΛB is determined by the relation*

$$\mathbf{H}\varphi(u) = \frac{1}{2}\hat{\hat{B}}(u)[\varphi'(u)]^2,$$

$$\hat{\hat{B}}(u) = \sum_{k=1}^{N}\hat{\pi}_k\int_{E_k}\pi_k(dx)B(u;x), \quad B(u;x) = \int_{\mathbf{R}} v^2\Gamma(u,dv;x).$$

The proof follows from Lemma 3.

Lemma 3. *The exponential generator on the perturbed test function*

$$\varphi^{\varepsilon}(u,x) = \varphi(u) + \varepsilon\ln[1 + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x)]$$

admits the following asymptotic representation:

$$\mathbf{H}^{\varepsilon}\varphi^{\varepsilon}(u,x) = \varepsilon^{-1}Q\varphi_1 + Q\varphi_2 + Q_1\varphi_1 - \varphi_1 Q\varphi_1 + \tilde{\mathbf{B}}(x)\varphi(u) + \delta^{\varepsilon}_H(u,x)\varphi(u),$$

and the negligible term converges uniformly by u, x on the functions $\varphi(u) \in C^3(\mathbf{R})$:

$$|\delta^{\varepsilon}_H(u,x)\varphi(u)| \to 0, \ \varepsilon \to 0.$$

Here the operator

$$\tilde{\mathbf{B}}(x)\varphi(u) = \frac{1}{2}B(u;x)[\varphi'(u)]^2. \tag{18}$$

Proof. The proof of the lemma is based on the asymptotic analysis of the items

$$H_Q^\varepsilon \varphi^\varepsilon(u,x) = e^{-\varphi(u)/\varepsilon}[1+\varepsilon\varphi_1+\varepsilon^2\varphi_2]^{-1}[\varepsilon^{-2}Q+\varepsilon^{-1}Q_1][1+\varepsilon\varphi_1+\varepsilon^2\varphi_2]e^{\varphi(u)/\varepsilon}$$
$$= e^{-\varphi(u)/\varepsilon}[1-\varepsilon\varphi_1][\varepsilon^{-2}Q+\varepsilon^{-1}Q_1][1+\varepsilon\varphi_1+\varepsilon^2\varphi_2]e^{\varphi(u)/\varepsilon}+\delta^\varepsilon(x)\varphi(u)$$
$$= \varepsilon^{-1}Q\varphi_1 + Q\varphi_2 + Q_1\varphi_1 - \varphi_1 Q\varphi_1 + \delta^\varepsilon(x)\varphi(u)$$

and

$$H_\Gamma^\varepsilon \varphi^\varepsilon(u,x) = e^{-\varphi(u)/\varepsilon}[1+\varepsilon\varphi_1+\varepsilon^2\varphi_2]^{-1}\varepsilon\Gamma^\varepsilon(x)[1+\varepsilon\varphi_1+\varepsilon^2\varphi_2]e^{\varphi(u)/\varepsilon}$$
$$= e^{-\varphi(u)/\varepsilon}[1-\varepsilon\varphi_1]\varepsilon\Gamma^\varepsilon(x)[1+\varepsilon\varphi_1+\varepsilon^2\varphi_2]e^{\varphi(u)/\varepsilon} + \delta^\varepsilon(u,x)\varphi(u)$$
$$= \varepsilon^{-2}\int_{\mathbf{R}}[e^{\Delta_v^\varepsilon\varphi(u)}-1]\Gamma(u,dv;x) + \delta^\varepsilon(u,x)\varphi(u).$$

Here

$$\Delta_v^\varepsilon\varphi(u) = \varepsilon^{-1}[\varphi(u+\varepsilon^2 v)-\varphi(u)] = \varepsilon v\varphi'(u) + \varepsilon^3\hat{\varphi}_v''(u),$$

and due to the ΛB condition, we obtain

$$\varepsilon^{-2}\int_{\mathbf{R}}[e^{\Delta_v^\varepsilon\varphi(u)}-1]\Gamma(u,dv;x) = \varepsilon^{-2}\int_{\mathbf{R}}\left[\varepsilon v\varphi'(u)+\frac{1}{2}(\varepsilon v)^2[\varphi'(u)]^2\right]\Gamma(u,dv;x)$$
$$+ \delta^\varepsilon(u,x)\varphi(u) = \tilde{\mathbf{B}}(x)\varphi(u) + \delta^\varepsilon(u,x)\varphi(u).$$

Thus,

$$\mathbf{H}_\Gamma^\varepsilon\varphi^\varepsilon(u,x) = \tilde{\mathbf{B}}(x)\varphi(u) + \delta^\varepsilon(u,x)\varphi(u)$$

with the main term (18). □

Proof of Theorem 2. To finish the proof of the theorem we should apply the solution of the singular perturbation problem for the equations:

$$Q\varphi_1(u,x) = 0$$

$$Q\varphi_2 + Q_1\varphi_1 + \tilde{\mathbf{B}}(x)\varphi(u) = \hat{\hat{B}}\varphi(u).$$

It follows from the first equation that $\varphi_1(u,x) = \varphi_1(u,\hat{x}) \in N_Q$; thus, from the solvability condition for the second equation, we obtain a new relation

$$\Pi Q_1\Pi\varphi_1 + \Pi\tilde{\mathbf{B}}(x)\Pi\varphi(u) = \hat{\hat{B}}\varphi(u),$$

or

$$\hat{Q}_1\hat{\varphi}_1 + \widehat{\tilde{\mathbf{B}}(x)}\hat{\varphi}(u) = \hat{\hat{B}}\varphi(u).$$

The solvability condition for the averaged equation gives finally

$$\hat{\Pi}\widehat{\tilde{\mathbf{B}}(x)}\hat{\Pi}\hat{\varphi}(u) = \hat{\hat{B}}\varphi(u).$$

Thus, the relation

$$\mathbf{H}^{\varepsilon}\varphi^{\varepsilon}(u,x) = \mathbf{H}\varphi(u) + \delta_H^{\varepsilon}(u,x)\varphi(u)$$

finishes the proof of the theorem. □

4.3 *Large Deviations Under the Total Balance Condition TB*

Under the total balance condition:

$$\text{TB:} \qquad \begin{array}{c} b(u;x) = \int_{\mathbf{R}} v\Gamma(u,dv;x) \not\equiv 0, \\ \sum_{k=1}^{N} \hat{\pi}_k \hat{b}_k(u) = 0, \quad \hat{b}_k(u) = \int_{E_k} \pi_k(dx) b(u;x), \quad 1 \le k \le N \end{array}$$

we use the following scaling for the random evolutions:

$$\xi^{\varepsilon}(t) = \varepsilon^2 \xi(t/\varepsilon^3), \quad x_t^{\varepsilon} := x^{\varepsilon}(t/\varepsilon^4). \tag{19}$$

The generator of the random evolution is given by

$$\mathbf{L}_T^{\varepsilon}\varphi(u,x) = [\varepsilon^{-4}Q + \varepsilon^{-3}Q_1 + \Gamma^{\varepsilon}(x)]\varphi(u,x), \tag{20}$$

where

$$\Gamma^{\varepsilon}(x)\varphi(u) = \varepsilon^{-3}\int_{\mathbf{R}} [\varphi(u+\varepsilon^2 v) - \varphi(u)]\Gamma(u,dv;x).$$

The generator (20) has the following asymptotic representation:

$$\mathbf{L}_T^{\varepsilon}\varphi(u,x) = [\varepsilon^{-4}Q + \varepsilon^{-3}Q_1 + \varepsilon^{-1}\Gamma(x) + \varepsilon\mathbf{B}(x)]\varphi(u,x) + \varepsilon\delta^{\varepsilon}(u,x)\varphi(u,x).$$

Here

$$\Gamma(x)\varphi(u) := b(u;x)\varphi'(u).$$

Theorem 3. *The exponential generator of the large deviations for the random evolutions defined by (19) under the conditions **ME1–ME4** and TB is determined by the relation*

$$\mathbf{H}\varphi(u) = \frac{1}{2}\hat{\hat{B}}_T(u)[\varphi'(u)]^2, \quad \hat{\hat{B}}_T(u) = \hat{\hat{B}}(u) + \hat{\hat{B}}_0(u).$$

Here

$$\hat{\hat{B}}(u) := \sum_{k=1}^{N} \hat{\pi}_k \int_{E_k} \pi_k(dx) B(u;x), \quad B(u;x) = \int_{\mathbf{R}} v^2 \Gamma(u, dv; x),$$

$$\hat{\hat{B}}_0(u) := \hat{\Pi} \hat{b}(u, \hat{x}) \hat{R}_0 \hat{b}(u, \hat{x}) \hat{\Pi} = \sum_{k,l=1}^{N} \hat{\pi}_k \hat{b}_k \hat{R}^0_{kl} \hat{b}_l.$$

Remark 3. The limit exponential generator consists of two parts: the first one is the averaged diffusion coefficient and the second one is defined by the merged first moments of jumps, averaged with the potential of the limit merged Markov switching process.

The proof is based on the following Lemma.

Lemma 4. *The exponential generator on the perturbed test function*

$$\varphi^\varepsilon(u,x) = \varphi(u) + \varepsilon \ln[1 + \varepsilon\varphi_1(u,x) + \varepsilon^2\varphi_2(u,x) + \varepsilon^3\varphi_3(u,x)]$$

admits the following asymptotic representation:

$$\begin{aligned} \mathbf{H}^\varepsilon \varphi^\varepsilon(u,x) &= \varepsilon^{-2} Q\varphi_1 + \varepsilon^{-1}[Q\varphi_2 + Q_1\varphi_1 - \varphi_1 Q\varphi_1 + \Gamma(x)\varphi(u)] \\ &\quad + [Q\varphi_3 + Q_1\varphi_2 - \varphi_1 Q\varphi_2 - \varphi_2 Q\varphi_1 - \varphi_1 Q_1\varphi_1 + \tilde{\mathbf{B}}(x)\varphi(u)] \\ &\quad + \delta^\varepsilon(u,x)\varphi(u), \end{aligned}$$

and the negligible term converges uniformly by u, x on the functions $\varphi(u) \in C^3(\mathbf{R})$:

$$|\delta^\varepsilon(u,x)\varphi(u)| \to 0, \ \varepsilon \to 0.$$

Here the operators

$$\Gamma(x)\varphi(u) := b(u;x)\varphi'(u), \quad \tilde{\mathbf{B}}(x)\varphi(u) := \frac{1}{2} B(u;x)[\varphi'(u)]^2. \tag{21}$$

Proof. The proof of lemma is based on the asymptotic analysis of the items

$$\begin{aligned} H^\varepsilon_Q \varphi^\varepsilon(u,x) &= e^{-\varphi(u)/\varepsilon}[1 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3]^{-1}[\varepsilon^{-3}Q + \varepsilon^{-2}Q_1] \\ &\quad \times [1 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3] e^{\varphi(u)/\varepsilon} \\ &= e^{-\varphi(u)/\varepsilon}[1 - \varepsilon\varphi_1 - \varepsilon^2\varphi_2 + \varepsilon^2\varphi_1^2 - \varepsilon^3\varphi_3] \\ &\quad \times [\varepsilon^{-3}Q + \varepsilon^{-2}Q_1][1 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \varepsilon^3\varphi_3] e^{\varphi(u)/\varepsilon} + \delta^\varepsilon(x)\varphi(u) \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon^{-2}Q\varphi_1 + \varepsilon^{-1}[Q\varphi_2 + Q_1\varphi_1 - \varphi_1 Q\varphi_1] \\
&\quad + [Q\varphi_3 + Q_1\varphi_2 - \varphi_1 Q\varphi_2 - -\varphi_2 Q\varphi_1 + \varphi_1^2 Q\varphi_1 - \varphi_1 Q_1\varphi_1] \\
&\quad + \delta^\varepsilon(x)\varphi(u)
\end{aligned}
$$

and

$$
\begin{aligned}
H_\Gamma^\varepsilon \varphi^\varepsilon(u,x) &= e^{-\varphi(u)/\varepsilon}[1+\varepsilon\varphi_1+\varepsilon^2\varphi_2+\varepsilon^3\varphi_3]^{-1}\varepsilon\Gamma^\varepsilon(x)[1+\varepsilon\varphi_1+\varepsilon^2\varphi_2 \\
&\quad + \varepsilon^3\varphi_3]e^{\varphi(u)/\varepsilon} \\
&= e^{-\varphi(u)/\varepsilon}[1-\varepsilon\varphi_1-\varepsilon^2\varphi_2+\varepsilon^2\varphi_1^2]\varepsilon\Gamma^\varepsilon(x)[1+\varepsilon\varphi_1+\varepsilon^2\varphi_2+\varepsilon^3\varphi_3]e^{\varphi(u)/\varepsilon} \\
&\quad + \delta^\varepsilon(u,x)\varphi(u) \\
&= \varepsilon^{-2}\int_{\mathbf{R}}[e^{\Delta_v^\varepsilon\varphi(u)}-1]\Gamma(u,dv;x) \\
&\quad + \varepsilon^2 e^{-\varphi(u)/\varepsilon}[\Gamma^\varepsilon(x)\varphi_1 e^{\varphi(u)/\varepsilon} - \varphi_1\Gamma^\varepsilon(x)e^{\varphi(u)/\varepsilon}] + \delta_\Gamma^\varepsilon(u,x)\varphi(u).
\end{aligned}
$$

Here

$$
\Delta_v^\varepsilon\varphi(u) = \varepsilon^{-1}[\varphi(u+\varepsilon^2 v)-\varphi(u)] = \varepsilon v\varphi'(u) + \varepsilon^3\hat{\varphi}_v''(u),
$$

and due to the TB condition, we obtain

$$
\begin{aligned}
\varepsilon^{-2}\int_{\mathbf{R}}[e^{\Delta_v^\varepsilon\varphi(u)}-1]\Gamma(u,dv;x) &= \varepsilon^{-2}\int_{\mathbf{R}}\left[\varepsilon v\varphi'(u)+\frac{1}{2}(\varepsilon v)^2[\varphi'(u)]^2\right]\Gamma(u,dv;x) \\
&\quad + \delta^\varepsilon(u,x)\varphi(u) \\
&= \varepsilon^{-1}\Gamma(x)\varphi(u) + \tilde{\mathbf{B}}(x)\varphi(u) + \delta^\varepsilon(u,x)\varphi(u).
\end{aligned}
$$

Each of the terms in the square brackets is not negligible, for instance,

$$
\begin{aligned}
\varepsilon^2\varphi_1 e^{-\varphi(u)/\varepsilon}\Gamma^\varepsilon(x)e^{\varphi(u)/\varepsilon} = \varepsilon\varphi_1 e^{-\varphi(u)/\varepsilon}\varepsilon\Gamma^\varepsilon(x)e^{\varphi(u)/\varepsilon} &= \varphi_1\Gamma(x)\varphi(u) \\
&\quad +\delta^\varepsilon(u,x)\varphi(u).
\end{aligned}
$$

But their difference is equal to 0 due to the relation

$$
\begin{aligned}
\Gamma^\varepsilon(x)e^{\varphi(u)/\varepsilon}\varphi_1 &= \varepsilon^{-3}\int_{\mathbf{R}}[e^{\varphi(u+\varepsilon^2 v)/\varepsilon}\varphi_1(u+\varepsilon v,x) - e^{\varphi(u)/\varepsilon}\varphi_1(u,x)]\Gamma(u,dv;x) \\
&= \varphi_1(u,x)\Gamma^\varepsilon(x)e^{\varphi(u)/\varepsilon} + o(\varepsilon^2).
\end{aligned}
$$

Thus,

$$
H_\Gamma^\varepsilon\varphi^\varepsilon(u,x) = \varepsilon^{-1}\Gamma(x)\varphi(u) + \tilde{\mathbf{B}}(x)\varphi(u) + \delta^\varepsilon(u,x)\varphi(u),
$$

with the main terms (21). □

Proof of Theorem 3. To finish the proof of the theorem, we should apply the solution of the singular perturbation problems for the equations

$$Q\varphi_1 = 0,$$

$$Q\varphi_2 + Q_1\varphi_1 + b(u;x)\varphi'(u) = 0,$$

$$Q\varphi_3 + Q_1\varphi_2 - \varphi_1 Q\varphi_2 - \varphi_1 Q_1\varphi_1 + \tilde{\mathbf{B}}(x)\varphi(u) = \hat{\hat{B}}\varphi(u).$$

It follows from the first equation that $\varphi_1(u,x) = \varphi_1(u,\hat{x}) \in N_Q$; thus, from the solvability condition for the second equation

$$\widehat{Q\varphi_2} + \hat{Q}_1\hat{\varphi}_1 + \hat{b}(u;\hat{x})\varphi'(u) = 0, \quad \widehat{Q\varphi_2} = 0, \tag{22}$$

we obtain a new relation:

$$\hat{Q}_1\hat{\varphi}_1 + \hat{b}(u,\hat{x})\varphi'(u) = 0, \quad \hat{\Pi}\hat{b}(u;\hat{x}) \equiv 0,$$

from which we have

$$\hat{\varphi}_1(u,\hat{x}) = \hat{R}_0\hat{b}(u;\hat{x})\varphi'(u), \quad \hat{Q}_1\hat{\varphi}_1 = -\hat{b}(u,\hat{x})\varphi'(u). \tag{23}$$

Then, the solvability condition for the equation

$$Q\varphi_3 + Q_1\varphi_2 - \varphi_1 Q\varphi_2 - \varphi_1 Q_1\varphi_1 + \tilde{\mathbf{B}}(x)\varphi(u) = \hat{\hat{B}}\varphi(u)$$

gives

$$\hat{Q}_1\hat{\varphi}_2 - \hat{\varphi}_1\widehat{Q\varphi_2} - \hat{\varphi}_1\hat{Q}_1\hat{\varphi}_1 + \hat{\tilde{\mathbf{B}}}(x)\hat{\varphi}(u) = \hat{\hat{B}}\varphi(u),$$

but from (22)

$$\widehat{Q\varphi_2} = -[\widehat{Q_1\varphi_1} + \hat{b}(u,\hat{x})\varphi'(u)] = 0,$$

and using the solution (23), we have

$$\hat{Q}_1\hat{\varphi}_2 + \hat{B}_T(x)\hat{\varphi}(u) = \hat{\hat{B}}\varphi(u).$$

Application of the solvability condition for this equation finishes the proof of the theorem. □

References

1. Cole, J.D.: On a quasi-linear parabolic equation occuring in aerodynamics. Quart. Appl. Math. **9**, 225–236 (1951)
2. Cramér, H.: Sur un nouveau théoreme-limite de la théorie des probabilités. Acta. Sci. et Ind. **736**, 5–23 (1938)
3. Donsker, M.D., Varadhan, S.R.S.: Asymptotic evaluation of certain markov process expectations for large time, I,II,III. Comm. Pure Appl. Math. **27**, 1–47, **28**, 279–301, **29**, 389–461 (1975, 1976)
4. Evans, L.C.: The perturbed test function method for viscosity solutions of nonlinear PDE. Proc. Roy. Soc. Edinburgh Sect. A. **111**(3–4), 359–375 (1989)
5. Feng, J., Kurtz, T.G.: Large Deviation for Stochastic Processes, vol. 131, AMS, RI (2006)
6. Fleming, W.H., Souganidis, P.E.: A PDE approach to some large deviations problems. Nonlinear Systems of Partial Differential Equations in Applied Mathematics, Part 1 (Santa Fe, N.M., 1984), vol. 23, pp. 441–447. Lectures in Applied Mathematics, American Mathematical Society, Providence, RI (1986)
7. Freidlin, M.J., Wentzel, A.D.: Random Perturbations of Dynamical Systems. Springer, New York (1998)
8. Hopf, E.: The partial differential equation $u_t + uu_x = \mu u_{xx}$. Comm. Pure Appl. Math. **3**, 201–230 (1950)
9. Koroliuk, V.S., Limnios, N.: Stochastic Systems in Merging Phase Space. WSP, Singapore (2005)
10. Koroliuk, V.S.: Random evolutions with locally independent increments on increasing time intervals. J. Math. Sci. Vol. **179**(2), 273–289 (2011)
11. Kurtz, T.G.: A limit theorem for perturbed operator semigroups with applications to random evolutions. J. Funct. Anal. **12**, 55–67 (1973)
12. Mogulskii, A.A.: Large deviation for processes with independent increments. Ann.Prob. **21**, 202–215 (1993)
13. Papanicolaou, G.C., Stroock, D., Varadhan, S.R.S.: Martingale approach to some limit theorems. Duke Univ. Math. Ser. **21**, 202–215 (1977)
14. Pavliotis, G.A., Stuart, A.M.: Multiscale Methods: Averaging and Homogenization. Springer Science+Business Media, New York, LLC (2008)
15. Skorokhod, A.V., Hoppensteadt F.C. and Salehi, H.: Random Perturbation Methods with Applications in Science and Engineering. Springer, New York (2002)
16. Wentzell, A.D.: Rough limit theorems on large deviations for Markov stochastic processes, I, II, III. Theory Probab. Appl. **21**, 227–242 (1976), **24**, 675–672 (1979), **27**, 215–234 (1982)

Limit Theorems for Excursion Sets of Stationary Random Fields

Evgeny Spodarev

Abstract We give an overview of the recent asymptotic results on the geometry of excursion sets of stationary random fields. Namely, we cover a number of limit theorems of central type for the volume of excursions of stationary (quasi-, positively or negatively) associated random fields with stochastically continuous realizations for a fixed excursion level. This class includes in particular Gaussian, Poisson shot noise, certain infinitely divisible, α-stable, and max-stable random fields satisfying some extra dependence conditions. Functional limit theorems (with the excursion level being an argument of the limiting Gaussian process) are reviewed as well. For stationary isotropic C^1-smooth Gaussian random fields similar results are available also for the surface area of the excursion set. Statistical tests of Gaussianity of a random field which are of importance to real data analysis as well as results for an increasing excursion level round up the paper.

1 Introduction

Geometric characteristics such as Minkowski functionals (or intrinsic volumes, curvature measures, etc.) of excursions of random fields are widely used for data analysis purposes in medicine (brain fMRI analysis; see, e.g., [5, 55, 60, 62]), physics and cosmology (microwave background radiation analysis; see, e.g., [41] and references therein), and materials science (quantification of porous media; see, e.g., [42, 61]), to name just a few. Minkowski functionals include the volume, the surface area, and the Euler–Poincaré characteristic (reflecting porosity) of a set with a sufficiently regular boundary.

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Among the possible abundance of random field models, Gaussian random fields are best studied due to their analytic tractability. A number of results starting with explicit calculation of the moments of Minkowski functionals are available for them since the mid-1970s. We briefly review these results in Sect. 4. However, our attention is focused on the asymptotic arguments for (mainly non-Gaussian) stationary random fields. There has been a recent breakthrough in this domain starting with the paper [15] where a central limit theorem (CLT) for the volume of excursions of a large class of quasi-associated random fields was proved. We also cover a number of hard-to-find results from recent preprints and PhD theses.

The paper is organized as follows: After introducing some basic facts on excursions and dependence structure of stationary random fields in Sect. 2, we briefly review the limit theorems for excursions of stationary Gaussian processes ($d = 1$) in the next section. However, our focus is on the recent results in the multidimensional case $d > 1$ which is considered in Sects. 5 and 6. Thus, Sect. 5 gives (uni- and multivariate as well as functional) central limit theorems for the volume of excursion sets of stationary (in general, non-Gaussian) random fields over fixed, variable, or increasing excursion levels. In Sect. 6, a similar scope of results is covered for the surface area of the boundary of excursion sets of stationary (but possibly anisotropic) Gaussian random fields in different functional spaces. The paper concludes with a number of open problems.

2 Preliminaries

Fix a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Let $X = \{X(t,\omega),\ t \in \mathbb{R}^d,\ \omega \in \Omega\}$ be a stationary (in the strict sense) real-valued measurable (in $(t,\omega) \in \mathbb{R}^d \times \Omega$) random field. Later on we suppress ω in the notation. For integrable X we assume X to be centered (i.e., $\mathsf{E}X(o) = 0$ where $o \in \mathbb{R}^d$ is the origin point). If the second moment of $X(o)$ exists, then we denote by $C(t) = \mathsf{E}\,(X(o)X(t))$, $t \in \mathbb{R}^d$ the covariance function of X.

Let $\|\cdot\|_2$ be the Euclidean norm in $\mathbb{R}^d$ and dist_2 the Euclidean distance: for two sets $A, B \subset \mathbb{R}^d$, we put $\mathsf{dist}_2(A,B) = \inf\{\|x-y\|_2 : x \in A, y \in B\}$. Denote by $\|\cdot\|_\infty$ the supremum norm in $\mathbb{R}^d$ and by dist_∞ the corresponding distance function.

Let $\xrightarrow{d}$ mean convergence in distribution. Denote by A^c the complement and by $\mathrm{int}(A)$ the interior of a set A in the corresponding ambient space which will be clear from the context. Let $\mathrm{card}(A)$ be the cardinality of a finite set A. Denote by $B_r(x)$ the closed Euclidean ball with center in $x \in \mathbb{R}^d$ and radius $r > 0$. Let $\mathcal{H}^k(\cdot)$ be the k-dimensional Hausdorff measure in $\mathbb{R}^d$, $0 \le k \le d$. In the sequel, we use the notation $\kappa_j = \mathcal{H}^j(B_1(o))$, $j = 0, \ldots, d$.

To state limit theorems, one has to specify the way of expansion of windows $W_n \subset T$, where the random field $X = \{X(t),\ t \in T\}$ is observed, to the whole index space $T = \mathbb{R}^d$ or $\mathbb{Z}^d$. A sequence of compact Borel sets $(W_n)_{n\in\mathbb{N}}$ is called a *Van Hove sequence* (*VH*) if $W_n \uparrow \mathbb{R}^d$ with

$$\lim_{n\to\infty} V_d(W_n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{V_d(\partial W_n \oplus B_r(o))}{V_d(W_n)} = 0, \quad r > 0.$$

A sequence of finite subsets $U_n \subset \mathbb{Z}^d$, $n \in \mathbb{N}$ is called *regular growing* if

$$\mathrm{card}(U_n) \to \infty \quad \text{and} \quad \mathrm{card}(\delta U_n)/\mathrm{card}(U_n) \to 0 \ \text{ as } \ n \to \infty$$

where $\delta U_n = \{j \in \mathbb{Z}^d \setminus U_n : \mathsf{dist}_\infty(j, U_n) = 1\}$ is the discrete boundary of U_n in $\mathbb{Z}^d$.

2.1 Excursion Sets and Their Intrinsic Volumes

The *excursion set* of X at level $u \in \mathbb{R}$ in the compact observation window $W \subset \mathbb{R}^d$ is given by $A_u(X, W) = \{t \in W : X(t) \geq u\}$. The *sojourn set* under the level u is $S_u(X, W) = \{t \in W : X(t) \leq u\}$, respectively.

Due to measurability of X, $A_u(X, W)$ and $S_u(X, W)$ are random Borel sets. If X is a.s. upper (lower) semicontinuous, then $A_u(X, W)$ ($S_u(X, W)$, respectively) is a random closed set (cf. [45, Sect. 5.2.1]).

A popular way to describe the geometry of excursion sets is via their *intrinsic volumes* V_j, $j = 0, \ldots, d$. They can be introduced for various families of sets such as convex and polyconvex sets [54, Chap. 4], sets of positive reach, and their finite unions [22], unions of basic complexes [4, Chap. 6]. One possibility to define $V_j(K)$, $j = 0, \ldots, d$ for a set K belonging to the corresponding family is given by the *Steiner formula* (see, e.g., [53, Sect. 13.3]) as the coefficients in the polynomial expansion of the volume of the tubular neighborhood $K_r = \{x \in \mathbb{R}^d : \mathsf{dist}_2(x, K) \leq r\}$ of K with respect to the radius $r > 0$ of this neighborhood:

$$\mathcal{H}^d(K_r) = \sum_{j=0}^{d} \kappa_{d-j} V_j(K) r^{d-j}$$

for admissible $r > 0$ (for convex K, these are all positive r). The geometric interpretation of intrinsic volumes $V_j(K)$, $j = 1, \ldots, d-2$ can be given in terms of integrals of elementary symmetric polynomials of principal curvatures for convex sets K with C^2-smooth boundary, cf. [53, Sects. 13.5–6]. Without going into details here, let us discuss the meaning of some of $V_j(A_u(X, W))$, $j = 0, \ldots, d$ in several dimensions.

For $d = 1$, $V_1(A_u(X, W))$ is the length of excursion intervals and $V_0(A_u(X, W))$ is the number of upcrossings of level u by the random process X within W.

For dimensions $d \geq 2$, $V_d(A_u(X, W))$ is always the volume (i.e., the Lebesgue measure) of $A_u(X, W)$ and $V_{d-1}(A_u(X, W))$ is half the surface area, i.e., $1/2 \cdot \mathcal{H}^{d-1}(\partial A_u(X, W))$. The *Euler characteristic* $V_0(A_u(X, W))$ is a topological measure of "porosity" of excursion set $A_u(X, W)$. For "basic" sets A (e.g., nonempty convex sets or sets of positive reach), we set $V_0(A) = 1$. Then V_0 is defined for unions of basic sets by additivity. One can show that for $d = 2$, it holds

$$V_0(A) = \text{card}\{\text{connected components of } A\} - \text{card}\{\text{holes of } A\}.$$

The existence of $V_j\ (A_u(X, W))$, $j = d, d-1$, is clear since $A_u(X, W)$ is a Borel set whose Lebesgue and Hausdorff measures are well defined. Intrinsic volumes V_j of lower orders $j = 0, \ldots, d-2$ are well defined, e.g., for excursion sets of sufficiently smooth (at least C^2) deterministic functions (cf. [4, Theorem 6.2.2]) and Gaussian random fields (cf. [4, Theorem 11.3.3]) satisfying some additional conditions.

2.2 Dependence Concepts for Random Fields

To prove limit theorems for a random field X, some conditions have to be imposed on the structure of the dependence of X. Mixing conditions that are usually required (cf., e.g., [13, 20]) are however rather difficult to check for a particular random field under consideration. For this practical reason, we follow the books [16], [58, Chap. 10] and introduce *association* as well as related dependence concepts.

A random field $X = \left\{X(t),\ t \in \mathbb{R}^d\right\}$ is called *associated (**A**)* if

$$\text{cov}\left(f\ (X_I), g\ (X_I)\right) \geq 0$$

for any finite subset $I \subset \mathbb{R}^d$, and for any bounded coordinatewise non-decreasing functions $f : \mathbb{R}^{\text{card}(I)} \to \mathbb{R}$, $g : \mathbb{R}^{\text{card}(I)} \to \mathbb{R}$ where $X_I = \{X(t),\ t \in I\}$.

A random field $X = \left\{X(t),\ t \in \mathbb{R}^d\right\}$ is called *positively (**PA**)* or *negatively (**NA**) associated* if

$$\text{cov}\left(f\ (X_I), g\ (X_J)\right) \geq 0 \quad (\leq 0,\ \text{resp.})$$

for all finite disjoint subsets $I, J \subset \mathbb{R}^d$, and for any bounded coordinatewise non-decreasing functions $f : \mathbb{R}^{\text{card}(I)} \to \mathbb{R}$, $g : \mathbb{R}^{\text{card}(J)} \to \mathbb{R}$. It is clear that if $X \in \mathbf{A}$, then $X \in \mathbf{PA}$.

Subclasses of **A** (**PA**, **NA**)-fields are certain infinitely divisible (e.g., max-stable and α-stable) random fields. In particular, a Gaussian random field with nonnegative covariance function is associated.

A random field $X = \left\{X(t), t \in \mathbb{R}^d\right\}$ with finite second moments is called *quasi-associated* (**QA**) if

$$|\text{cov}\left(f\ (X_I), g\ (X_J)\right)| \leq \sum_{i \in I} \sum_{j \in J} \text{Lip}_i\ (f)\ \text{Lip}_j\ (g)\ |\text{cov}\left(X\ (i), X\ (j)\right)|$$

for all finite disjoint subsets $I, J \subset \mathbb{R}^d$, and for any Lipschitz functions $f : \mathbb{R}^{\text{card}(I)} \to \mathbb{R}$, $g : \mathbb{R}^{\text{card}(J)} \to \mathbb{R}$ where $\mathsf{Lip}_i(f)$ is the Lipschitz constant of function f for coordinate i. It is known that if square-integrable $X \in \mathbf{A}(\mathbf{PA}, \mathbf{NA})$, then $X \in \mathbf{QA}$, cf. [16, Theorem 5.3].

A real-valued random field $X = \{X(t), t \in \mathbb{R}^d\}$ is called (BL, θ)-*dependent* if there exists a nonincreasing sequence $\theta = \{\theta_r\}_{r \in \mathbb{R}_0^+}$, $\theta_r \downarrow 0$ as $r \to \infty$ such that for any finite disjoint sets $I,\ J \subset \mathbb{R}^d$ with $\mathsf{dist}_\infty(I, J) = r \in \mathbb{R}_0^+$ and any bounded Lipschitz functions $f : \mathbb{R}^{\text{card}(I)} \to \mathbb{R}$, $g : \mathbb{R}^{\text{card}(J)} \to \mathbb{R}$, one has

$$|\mathsf{cov}(f(X_I), g(X_J))| \leq \sum_{i \in I} \sum_{j \in J} \mathsf{Lip}_i(f)\, \mathsf{Lip}_j(g)\, |\mathsf{cov}(X(i), X(j))|\, \theta_r.$$

It is often possible to choose θ as the *Cox–Grimmett coefficient*

$$\theta_r = \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus B_r^\infty(y)} |\mathsf{cov}(X(y), X(t))|\, dt$$

where $B_r^\infty(y) = \{x \in \mathbb{R}^d : \|x - y\|_\infty \leq r\}$. It can be easily seen that if $X \in \mathbf{QA}$ and its covariance function is absolutely integrable on $\mathbb{R}^d$, then X is (BL, θ) dependent.

3 Excursions of Stationary Gaussian Processes

Excursions of stochastic processes is a popular research topic in probability theory since many years; see, e.g., [10] and references in [27]. The vast literature on this subject for different classes of processes such as Lévy, diffusion, stable, and Gaussian ones can be hardly covered by one review. For this reason, we concentrate on the excursions of (mainly stationary) Gaussian processes here.

Let $X = \{X(t), t \geq 0\}$ be a centered real-valued Gaussian process. If X is a polynomial of degree n with iid $N(0, 1)$-distributed coefficients, then the mean number of real roots of the equation $X(t) = 0$ was first obtained by M. Kac [28]. It initiated a substantial amount of papers on the roots of random algebraic polynomials; see [12] for a review. For C^1-smooth stationary Gaussian processes X, expectation of the number of upcrossings of a level u by X in time interval $[0, 1]$ has been studied in [14, 50, 51], etc. Higher-order factorial moments are considered in [17]; see also references therein and [7, 8]. For reviews (also including results on non-Gaussian stationary processes), see [33, Sects. 7.2 and 7.3] and [6, Chap. 3]. In [1] and [2], the notion of the number of upcrossings of level u for random processes has been generalized to the Euler–Poincaré characteristic of excursion sets of random fields.

The first proof of a central limit theorem for the number of zeros of a stationary Gaussian process within an increasing time interval was given in [40]. Cuzick [18] refined the assumptions given in [40] and proved a central limit theorem for the number of zeros $N_X(T) = 2V_0(A_0(X; [0, T]))$ of a centered separable stationary Gaussian process $X = \{X(t), t \geq 0\}$ in the time interval $[0, T]$ as well as analogous results for integrals $\int_0^T g\big(X(t)\big)\,dt$ as $T \to \infty$. He used approximations by m-dependent random processes with spectral representation as a method borrowed from [40]. In more detail, let $C(t)$ be twice differentiable with $C(0) = 1$, $C''(0) = -\lambda_2$, and variogram γ of X' be given by $\gamma(h) = C''(h) - C''(0) = 1/2\mathsf{E}(X'(h) - X'(0))$, $h \geq 0$.

Theorem 1 ([18]). *If C, C'' are square integrable on $\mathbb{R}_+$, $\int_0^\varepsilon \gamma(t)/t\,dt < \infty$ for some $\varepsilon > 0$ and*

$$\mathsf{Var} N_X(T)/T \to \sigma_2 > 0 \;\; as \;\; T \to +\infty \tag{1}$$

then

$$T^{-1/2}\,(N_X(T) - \mathsf{E}\,N_X(T)) \xrightarrow{d} N(0, \sigma^2) \;\; as \;\; T \to +\infty$$

where

$$\sigma^2 = \pi^{-1}\left(\lambda_2^{1/2} + \int\limits_0^\infty \left(\frac{\mathsf{E}\;(|X'(0)X'(t)||X(0) = X(t) = 0)}{\sqrt{1 - C^2(t)}} - (\mathsf{E}\,|X'(0)|)^2\right) dt\right).$$

Condition (1) is difficult to check and is substituted in [18, Lemma 5] by a more tractable sufficient condition involving C and λ_2. Piterbarg [48] managed to prove the above theorem by substituting condition (1) with

$$\int\limits_0^\infty t\,\big(|C(t)| + |C'(t)| + |C''(t)|\big)\;dt < \infty.$$

He approximates the point process of upcrossings of X of level u by a strongly mixing point process.

Theorem 2 ([18]). *Let X be a stationary Gaussian process with covariance function C being integrable on $\mathbb{R}_+$. For any measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $\mathsf{E}\,g^2(X(0)) < \infty$ and $g(x) - g(0)$ is not odd, it holds*

$$T^{-1/2}\left(\int_0^T g\big(X(t)\big)\,dt - T\,\mathsf{E}\,g\big(X(0)\big)\right) \xrightarrow{d} N(0, \sigma^2) \;\; as \;\; T \to +\infty \tag{2}$$

where $\sigma^2 > 0$.

It is clear that the choice $g(x) = \mathbf{1}\{x \in \mathbb{R} : x \geq u\}$ for any $u \in \mathbb{R}$ leads to the central limit theorem for the length $V_1(A_u(X; [0, T]))$ of excursion intervals of X in $[0, T]$.

Elizarov [21] first proved a functional central limit theorem for the sojourn times of the stationary Gaussian process under the level u, in our terms, for $V_1(S_u(X;[0,T]))$ if excursion level u is allowed to vary within $\mathbb{R}$. Additionally, an analogous result for local times

$$\lim_{\varepsilon\to+0}\frac{1}{2\varepsilon}\left(V_1(S_{u+\varepsilon}(X;[0,T]))-V_1(S_{u-\varepsilon}(X;[0,T]))\right)$$

was given. Both results were proved in the functional space $C[0,1]$ after the substitution $u\mapsto f(x)$, $x\in[0,1]$ where $f\in C[0,1]$ is a monotonously increasing function with $f(0)=-\infty$, $f(1)=\infty$.

Belyaev and Nosko [9] proved limit theorems for $V_1(A_u(X;[0,T]))$, $T\to\infty$ as $u\to\infty$ for stationary ergodic processes X satisfying a number of additional (quite technical) assumptions. In particular, these assumptions are satisfied if X is an ergodic Gaussian stationary process with twice continuously differentiable covariance function such that

$$\left|C''(t)-C''(0)\right|\le a/|\log|t||^{1+\varepsilon},\quad t\downarrow 0$$

for some constants $a,\varepsilon>0$.

Slud [57] gave a multiple Wiener- Itô representation for the number of crossings of a C^1-function ψ by X. In [31], methods of [40] and [18] are generalized to the case of functionals of X, X', and X''. CLTs for the number of crossings of a smooth curve ψ by a Gaussian process X as well as for the number of specular points of X (if X is a Gaussian process in time and space) are given in [32]. For a review of results on moments and limit theorems for different characteristics of stationary Gaussian processes, see [30]. In [27], CLTs for the multivariate nonlinear weighted functionals (similar to those in (2)) of Gaussian stationary processes with multiple singularities in their spectra, having a covariance function belonging to a certain parametric family, are proved.

4 Moments of $V_j\ (A_u(X,W))$ for Gaussian Random Fields

We briefly review the state of the art for $\mathsf{E}\,V_j\ (A_u(X,W))$ of Gaussian random fields X. For recent extended surveys, see the books [4] and [6]. For stationary (isotropic) Gaussian fields X, stratified C^2-smooth compact manifolds $W\subset\mathbb{R}^d$, and any $u\in\mathbb{R}$, formulae for $\mathsf{E}\,V_j\ (A_u(X,W))$, $j=0,\ldots,d$ are given in [4, Theorems 13.2.1 and 13.4.1].

Apart from obtaining exact (or asymptotic as $u\to\infty$) formulae for $\mathsf{E}\,V_j\ (A_u(X,W))$, $j=0,\ldots,d$, the possibility of an estimate

$$\left|\mathsf{P}\left(\sup_{t\in W}X(t)>u\right)-\mathsf{E}\,V_0\ (A_u(X,W))\right|\le g(u)\qquad(3)$$

(the so-called *Euler–Poincaré heuristic*) with $g(u) = o(1)$ as $u \to \infty$ is of special interest. It has been proved in [4, Theorem 14.3.3] with $g(u) = c_0 \exp\{-u^2(1+\alpha)/2\}$ for some positive constants c_0 and α if X is a (non)stationary Gaussian random field with constant variance on a stratified manifold W as $u \to \infty$. Lower and upper bounds for the density of supremum of stationary Gaussian random fields X (which imply relation (3)) for any $u \in \mathbb{R}$ are given in [6, Theorem 8.4]. Similar bounds are proven in [6, Theorem 8.10] for nonstationary Gaussian random fields X with a unique point of maximum of variance in $\mathrm{int}(W)$ as $u \to \infty$.

In [59], asymptotic behavior of $\mathsf{E}\, V_j\left(A_u(X,[a,b]^d)\right)$, $j = 0, d-1, d$ of nonstationary sufficiently smooth Gaussian random fields is studied as the excursion level $u \to \infty$. The variance of these fields is assumed to attain a global maximum at a vertex of $[a,b]^d$. It is shown that the heuristic (3) still holds true.

An interesting rather general formula for the mean surface area of Gaussian excursions is proven in [24]. Let W be a compact subset of $\mathbb{R}^d$ with a nonempty interior and a finite Hausdorff measure of the boundary. Let $X = \{X(t),\ t \in W\}$ be a Gaussian random field with mean $\mu(t) = \mathsf{E} X(t)$ and variance $\sigma^2(t) = \mathsf{Var} X(t)$. For an arbitrary (but fixed) excursion level $u \in \mathbb{R}$, introduce the zero set $\nabla_X^{-1}(0)$ of the gradient of the normalized field $(X-u)/\sigma$ by $\nabla_X^{-1}(0) = \{t \in W : \nabla\left((X(t)-u)/\sigma(t)\right) = 0\}$.

Theorem 3 ([24]). *Assume that* $X \in C^1(W)$ *a.s.,* $\mathsf{E} V_{d-1}\left(\nabla_X^{-1}(0)\right) < \infty$ *and* $\sigma(t) > 0$ *for all* $t \in W$*. Then*

$$\mathsf{E} V_{d-1}\left(\partial A_u(X,W)\right) = \frac{1}{2\sqrt{2\pi}} \int_W \exp\left[-\frac{(\mu(t)-u)^2}{2\sigma^2(t)}\right] \mathsf{E} \left\| \nabla\left((X(t)-u)/\sigma(t)\right) \right\|_2 dt.$$

Asymptotic formulae for $\mathsf{E}\, V_j\left(A_u(X,W)\right)$, $j = 0, \ldots, d$ as $u \to \infty$ of three subclasses of stable random fields (subgaussian, harmonizable, concatenated-harmonizable ones) are given in [3].

5 Volume of Excursion Sets of Stationary Random Fields

The first limit theorems of central type for the volume of excursion sets (over a fixed level u) of stationary isotropic Gaussian random fields were proved in [26, Chap. 2]. There, the case of short- and long-range dependence (Theorem 2.2.4 and Example 2.2.1, Theorem 2.4.6) was considered. The CLT followed from a general Berry–Esséen-type bound for the distribution function of properly normed integral functionals

$$\int_{B_r(o)} G\big(X(t)\big)\, dt \tag{4}$$

as $r \to \infty$ where $G : \mathbb{R} \to \mathbb{R}$ is a function such that $\mathsf{E}\, G^2(X(o)) < \infty$ satisfying some additional assumptions, cf. also [36]. To get the volume $V_d\,(A_u\,(X, B_r(o)))$ out of (4), set $G(x) = \mathbf{1}(x \geq u)$. The isotropy of X was essential as one used expansions with respect to the basis of Chebyshev–Hermite polynomials in the proofs. The cases of

$$G(x) = \mathbf{1}(|x| \geq u),\ \max\{0, x\},\ |x|$$

as well as of G depending on a parameter and of weighted integrals in (4) are considered as well.

In a remark [26, p. 81], it was noticed that similar CLTs can be expected for non-Gaussian mixing random fields. The aim of this section is to review the recent advances in proving such CLTs for various classes of stationary random fields that include also the (not necessarily isotropic) Gaussian case.

For instance, random fields with singularities of their spectral densities are considered in [37]. In Sect. 3.2 of that book, noncentral limit theorems for the volume of excursions of stationary isotropic Gamma-correlated and χ^2-random fields over a radial surface (i.e., the level u is not constant anymore, but a function of $\|t\|_2$, where $t \in \mathbb{R}^d$ is the integration variable in (4)) are proved. (Non)central limit theorems for functionals (4) of stationary isotropic vector-valued Gaussian random fields are given in the recent preprint [34]. There, the case of long- and short-range dependence is considered as well as applications to F- and t-distributed random fields.

The asymptotic behavior of tail probabilities

$$\mathsf{P}\left(\int_W e^{X(t)}\, dt > x\right), \quad x \to \infty$$

for a homogeneous smooth Gaussian random field X on a compact $W \subset \mathbb{R}^d$ is considered in [38]; see [39] for further extensions.

5.1 Limit Theorems for a Fixed Excursion Level

The main result (which we call a *methatheorem*) can be formulated as follows:

Theorem 4 (Methatheorem). *Let X be a strictly stationary random field satisfying* some additional conditions *and $u \in \mathbb{R}$ fixed. Then, for any sequence of VH-growing sets $W_n \subset \mathbb{R}^d$, one has*

$$\frac{V_d\,(A_u\,(X, W_n)) - \mathsf{P}(X(o) \geq u) \cdot V_d\,(W_n)}{\sqrt{V_d\,(W_n)}} \xrightarrow{\mathsf{d}} \mathcal{N}\left(0, \sigma^2(u)\right) \tag{5}$$

as $n \to \infty$. Here

$$\sigma^2(u) = \int_{\mathbb{R}^d} \mathsf{cov}\left(\mathbf{1}\{X(o) \geq u\}, \mathbf{1}\{X(t) \geq u\}\right) dt. \tag{6}$$

Depending on the class of random fields, these additional conditions will vary. First we consider the family of square-integrable random fields.

5.1.1 Quasi-Associated Random Fields

Theorem 5 ([15]). *Let $X = \{X(t), t \in \mathbb{R}^d\} \in$ **QA** be a stationary square-integrable random field with a continuous covariance function C such that $|C(t)| = \mathcal{O}\left(\|t\|_2^{-\alpha}\right)$ for some $\alpha > 3d$ as $\|t\|_2 \to \infty$. Let $X(o)$ have a bounded density. Then $\sigma^2(u) \in (0, \infty)$ and Theorem 4 hold true.*

Let us give an idea of the proof. Introduce the random field $Z = \{Z(j),\ j \in \mathbb{Z}^d\}$ by

$$Z(j) = \int_{j+[0,1]^d} \mathbf{1}\big\{X(t) \geq u\big\}\, dt - \Psi(u), \quad j \in \mathbb{Z}^d. \tag{7}$$

Here $\Psi(u) = \mathsf{P}\big(X(o) > u\big)$ is the tail distribution function of $X(o)$. It is clear that the sum of $Z(j)$ over indices $j \in W_n \cap \mathbb{Z}^d$ approximates the numerator in (5). One has to show that Z can be approximated by a sequence of (BL, θ)-dependent stationary centered square-integrable random fields Z_γ, $\gamma \downarrow 0$, on $\mathbb{Z}^d$. The proof finishes by applying the following CLT to Z_γ for each $\gamma > 0$.

Theorem 6 ([16], Theorem 3.1.12). *Let $Z = \{Z(j), j \in \mathbb{Z}^d\}$ be a (BL, θ)-dependent strictly stationary centered square-integrable random field. Then, for any sequence of regularly growing sets $U_n \subset \mathbb{Z}^d$, one has*

$$S(U_n) / \sqrt{card(U_n)} \xrightarrow{\mathsf{d}} \mathcal{N}\left(0, \sigma^2\right)$$

as $n \to \infty$, with

$$\sigma^2 = \sum_{j \in \mathbb{Z}^d} \mathsf{cov}\left(Z(o), Z(j)\right).$$

We give two examples of random fields satisfying Theorem 5.

Example 1 ([15]). Let $X = \{X(t),\ t \in \mathbb{R}^d\}$ be a stationary *shot noise random field* given by $X(t) = \sum_{i \in \mathbb{N}} \xi_i \varphi(t - x_i)$ where $\Pi_\lambda = \{x_i\}$ is a homogeneous Poisson point process in $\mathbb{R}^d$ with intensity $\lambda \in (0, \infty)$ and $\{\xi_i\}$ is a family of i.i.d. nonnegative random variables with $\mathsf{E}\,\xi_i^2 < \infty$ and characteristic function φ_ξ. Assume that Π_λ and $\{\xi_i\}$ are independent. Moreover, let $\varphi : \mathbb{R}^d \to \mathbb{R}_+$ be a bounded and uniformly continuous Borel function with $\varphi(t) \leq g_0(\|t\|_2) = \mathcal{O}\left(\|t\|_2^{-\alpha}\right)$ as $\|t\|_2 \to \infty$ for a function $g_0 : \mathbb{R}_+ \to \mathbb{R}_+$, $\alpha > 3d$, and

$$\int_{\mathbb{R}^d} \left| \exp\left\{ \lambda \int_{\mathbb{R}^d} \left(\varphi_\xi(s\varphi(t)) - 1 \right) dt \right\} \right| ds < \infty.$$

Then Theorem 5 holds true.

Example 2 ([15]). Consider a stationary Gaussian random field $X = \{X(t), t \in \mathbb{R}^d\}$ with a continuous covariance function $C(\cdot)$ such that $|C(t)| = \mathcal{O}\left(\|t\|_2^{-\alpha}\right)$ for some $\alpha > d$ as $\|t\|_2 \to \infty$. Let $X(o) \sim \mathcal{N}\left(a, \tau^2\right)$. Then, Theorem 5 holds true with

$$\sigma^2(u) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \int_0^{\rho(t)} \frac{1}{\sqrt{1-s^2}} e^{-\frac{(u-a)^2}{\tau^2(1+s)}} ds\, dt,$$

where $\rho(t) = \mathsf{corr}(X(o), X(t))$. In particular, for $u = a$ one has

$$\sigma^2(a) = \frac{1}{2\pi} \int_{\mathbb{R}^d} \arcsin\left(\rho(t)\right) dt.$$

5.1.2 PA- or NA-Random Fields

What happens if the field X does not have the finite second moment? In this case, another set of conditions for our methatheorem to hold was proven in [29, Theorem 3.59].

Theorem 7. *Let $X = \{X(t),\ t \in \mathbb{R}^d\} \in$ **PA**(**NA**) be stochastically continuous satisfying the following properties:*

1. *The asymptotic variance $\sigma^2(u) \in (0, \infty)$ (cf. its definition in (6)),*
2. $\mathsf{P}(X(o) = u) = 0$ *for the chosen level $u \in \mathbb{R}$.*

Then Theorem 4 holds.

The idea of the proof is first to show that the random field $Z = \{Z(j),\ j \in \mathbb{Z}^d\}$ defined in (7) is **PA** (**NA**). Second, use [16, Theorem 1.5.17] to prove that Z is (BL, θ)–dependent. Then apply Theorem 6 to Z.

A number of important classes of random fields satisfy Theorem 7. For instance, stationary infinitely divisible random fields $X = \{X(t),\ t \in \mathbb{R}^d\}$ with spectral representation

$$X(t) = \int_E f_t(x)\, \Lambda(dx), \quad t \in \mathbb{R}^d,$$

where Λ is a centered independently scattered infinitely divisible random measure on space E and $f_t : E \to \mathbb{R}_+$ are Λ-integrable kernels, are associated, and hence **PA** by [16, Chap. 1, Theorem 3.27]. The finite susceptibility condition

$\sigma^2(u) \in (0,\infty)$ can be verified by [29, Lemma 3.71]. Further examples of random fields satisfying Theorem 7 are *stable* random fields which we consider in more detail following [29, Sect. 3.5.3].

Max-Stable Random Fields

Let $X = \left\{X(t),\ t \in \mathbb{R}^d\right\}$ be a stationary max-stable random field with spectral representation

$$X(t) = \max_{i\in\mathbb{N}} \xi_i f_t(y_i), \quad t \in \mathbb{R}^d,$$

where $f_t : E \to \mathbb{R}_+$ is a measurable function defined on the measurable space (E,μ) for all $t \in \mathbb{R}^d$ with

$$\int_E f_t(y)\,\mu(dy) = 1, \quad t \in \mathbb{R}^d,$$

and $\{(\xi_i, y_i)\}_{i\in\mathbb{N}}$ is the Poisson point process on $(0,\infty)\times E$ with intensity measure $\xi^{-2}d\xi \times \mu(dy)$. It is known that all max-stable distributions are associated and hence **PA** by [49, Proposition 5.5.29]. The field X is stochastically continuous if $\|f_s - f_t\|_{L^1} \to 0$ as $s \to t$ (cf. [23, Lemma 2]). Condition $\sigma^2(u) \in (0,\infty)$ is satisfied if

$$\int_{\mathbb{R}^d}\int_E \min\{f_0(y), f_t(y)\}\,\mu(dy)\,dt < \infty.$$

α-Stable Random Fields

Let $X = \left\{X(t),\ t \in \mathbb{R}^d\right\}$ be a stationary α-stable random field ($\alpha \in (0,2)$, for simplicity $\alpha \neq 1$) with spectral representation

$$X(t) = \int_E f_t(x)\,\Lambda(dx), \quad t \in \mathbb{R}^d,$$

where Λ is a centered independently scattered α-stable random measure on space E with control measure m and skewness intensity $\beta : E \to [-1,1]$, $f_t : E \to \mathbb{R}_+$ is a measurable function on (E,m) for all $t \in \mathbb{R}^d$. By [52, Proposition 3.5.1], X is stochastically continuous if $\int_E |f_s(x) - f_t(x)|^\alpha\, m(dx) \to 0$ as $s \to t$ for any $t \in \mathbb{R}^d$. Condition $\sigma^2(u) \in (0,\infty)$ is satisfied if

$$\int_{\mathbb{R}^d}\left(\int_E \min\{|f_0(x)|^\alpha, |f_t(x)|^\alpha\}\, m(dx)\right)^{1/(1+\alpha)} dt < \infty.$$

5.2 A Multivariate Central Limit Theorem

If a finite number of excursion levels $u_k \in \mathbb{R}$, $k = 1, \ldots, r$ is considered simultaneously, a multivariate analogue of Theorem 4 can be proven. Introduce the notation

$$S_{\mathbf{u}}(W_n) = \left(V_d\left(A_{u_1}(X, W_n)\right), \ldots, V_d\left(A_{u_r}(X, W_n)\right)\right)^\top, \quad \Psi(\mathbf{u}) = (\Psi(u_1), \ldots, \Psi(u_r))^\top.$$

Theorem 8 ([15,29]). *Let X be the above random field satisfying Theorem 4. Then, for any sequence of VH-growing sets $W_n \subset \mathbb{R}^d$, one has*

$$V_d\,(W_n)^{-1/2}\,(S_{\mathbf{u}}(W_n) - \Psi(\mathbf{u})\,V_d\,(W_n)) \xrightarrow{d} \mathcal{N}(0, \Sigma(\mathbf{u}))$$

as $n \to \infty$. Here, $\Sigma(\mathbf{u}) = (\sigma_{lm}(\mathbf{u}))_{l,m=1}^r$ with

$$\sigma_{lm}(\mathbf{u}) = \int_{\mathbb{R}^d} \operatorname{cov}\left(\mathbf{1}\{X\,(0) \geq u_l\}, \mathbf{1}\{X\,(t) \geq u_m\}\right)\, dt.$$

If X is Gaussian as in Example 2, we have

$$\sigma_{lm}(\mathbf{u}) = \frac{1}{2\pi}\int_{\mathbb{R}^d}\int_0^{\rho(t)} \frac{1}{\sqrt{1-s^2}} \exp\left\{-\frac{(u_l-a)^2 - 2r(u_l-a)(u_m-a) + (u_m-a)^2}{2\tau^2(1-s^2)}\right\}\, ds\, dt.$$

However, the explicit computation of the elements of matrix Σ for the majority of fields X (except for Gaussianity) seems to be a very complex task. In order to overcome this difficulty in statistical applications of the methatheorem to testing, the matrix Σ can be (weakly) consistently estimated from one observation of a stationary random field X; see [47], [58, Sect. 9.8.3] and references therein.

Statistical Version of the CLT and Tests

Let X be a random field satisfying Theorem 4, $u_k \in \mathbb{R}$, $k = 1, \ldots, r$, and $(W_n)_{n\in\mathbb{N}}$ be a sequence of VH-growing sets. Let $\hat{C}_n = (\hat{c}_{nlm})_{l,m=1}^r$ be a weakly consistent estimator for the nondegenerate asymptotic covariance matrix $\Sigma(\mathbf{u})$, i.e., for any $l, m = 1, \ldots, r$

$$\hat{c}_{nlm} \xrightarrow{P} \sigma_{lm}(\mathbf{u}) \text{ as } n \to \infty.$$

Then

$$\hat{C}_n^{-1/2} V_d\,(W_n)^{-1/2}\,(S_{\mathbf{u}}(W_n) - \Psi(\mathbf{u})\,V_d\,(W_n)) \xrightarrow{d} \mathcal{N}(0, I). \tag{8}$$

Based on the latter relation, an asymptotic test for the following hypotheses can be constructed:
H_0 : *X is a random field satisfying Theorem 4 with tail distribution function* $\Psi(\cdot)$ vs. H_1 : *negation of* H_0. As a test statistic, we use

$$T_n = V_d\,(W_n)^{-1}\,(S_{\mathbf{u}}(W_n) - \Psi(\mathbf{u})\,V_d\,(W_n))^\top\,\hat{C}_n^{-1}\,(S_{\mathbf{u}}(W_n) - \Psi(\mathbf{u})\,V_d\,(W_n))$$

which is asymptotically χ^2_r distributed by continuous mapping theorem and relation (8): $T_n \xrightarrow{\mathsf{d}} \chi^2_r$ as $n \to \infty$. Hence, reject the null hypothesis at a confidence level $1-\nu$ if $T_n > \chi^2_{r,1-\nu}$ where $\chi^2_{r,1-\nu}$ is the $(1-\nu)$-quantile of χ^2_r-law.

5.3 Functional Limit Theorems

A natural generalization of multivariate CLTs is a functional CLT where the excursion level $u \in \mathbb{R}$ is treated as a variable, which also appears as a ("time") index in the limiting Gaussian process. In order to state the main results, introduce the *Skorokhod space* $D(\mathbb{R})$ of càdlàg functions on $\mathbb{R}$ endowed with the usual Skorokhod topology, cf. [11, Sect. 12]. Denote by $\Rightarrow$ the weak convergence in $D(\mathbb{R})$.

Define the stochastic processes $Y_n = \{Y_n(u),\ u \in \mathbb{R}\}$ by

$$Y_n(u) = \frac{1}{n^{d/2}}\left(V_d\left(A_u(X,[0,n]^d)\right) - n^d\Psi(u)\right), \quad u \in \mathbb{R}. \tag{9}$$

Introduce the following condition:

($\star$) For any subset $T = \{t_1, \dots, t_k\} \subset \mathbb{R}^d$ and its partition $T = T_1 \cup T_2$, there exist some constants $c(T), \gamma > 0$ such that

$$\mathsf{cov}\left(\prod_{t_i \in T_1} \phi_{a,b}\left(X(t_i)\right), \prod_{t_j \in T_2} \phi_{a,b}\left(X(t_j)\right)\right) \le c(T)\,(1+\mathsf{dist}_\infty(T_1,T_2))^{-(3d+\gamma)},$$

where $\phi_{a,b}(x) = \mathbf{1}(a < x \le b) - \mathsf{P}(a < X(o) \le b)$ for any real numbers $a < b$.

The following functional CLT is proven in [43, Theorem 1 and Lemma 1].

Theorem 9. *Let $X = \{X(t),\ t \in \mathbb{R}^d\}$ be a real-valued stationary random field with a.s. continuous sample paths and a bounded density of the distribution of $X(o)$. Let condition ($\star$) and Theorem 4 be satisfied. Then $Y_n \Rightarrow Y$ as $n \to \infty$ where $Y = \{Y(u),\ u \in \mathbb{R}\}$ is a centered Gaussian stochastic process with covariance function*

$$C_Y(u,v) = \int_{\mathbb{R}^d} \mathsf{cov}\left(\mathbf{1}\{X(0) \geq u\}, \mathbf{1}\{X(t) \geq v\}\right) dt, \quad u, v \in \mathbb{R}.$$

In particular, condition ($\star$) is satisfied if $X \in \mathbf{A}$ is square integrable with covariance function C that admits a bound

$$|C(t)| \leq \zeta\,(1 + \|t\|_\infty)^{-\lambda}$$

for all $t \in \mathbb{R}^d$ and some $\zeta > 0$, $\lambda > 9d$. The proofs are quite technical involving a Móricz bound for the moment of a supremum of (absolute values of) partial sums of random fields on $\mathbb{Z}^d$, cf. [46, Theorem 2].

For max-stable random fields introduced in Sect. 5.1.2, condition ($\star$) is satisfied if for any $T = \{t_1, \ldots, t_k\} \subset \mathbb{R}^d$ and its partition $T = T_1 \cup T_2$, there exist some constants $c(T), \gamma > 0$ such that

$$\int_E \min\left\{\max_{t_i \in T_1} f_{t_i}(y), \max_{t_j \in T_2} f_{t_j}(y)\right\} \mu(dy) \leq c(T)\,(1 + \mathsf{dist}_\infty(T_1, T_2))^{-(3d+\gamma)}. \tag{10}$$

For α-stable moving averages, i.e., α-stable random fields from Sect. 5.1.2 with $f_t(\cdot) = f(t - \cdot)$ for any $t \in \mathbb{R}^d$, condition (10) should be replaced by

$$\left(\int_{\mathbb{R}^d} \min\left\{\max_{t_i \in T_1} f(t_i - y), \max_{t_j \in T_2} f(t_j - y)\right\}^\alpha m(dy)\right)^{1/(1+\alpha)}$$
$$\leq c(T)\,(1 + \mathsf{dist}_\infty(T_1, T_2))^{-(3d+\gamma)}.$$

These results are proven (under slightly more general assumptions) in [29, Sect. 3.5.5] together with analogous conditions for infinitely divisible random fields (that are too lengthy to give them in a review paper) as well as examples of random fields satisfying them.

Theorem 9 together with the continuous mapping theorem can be used to test hypotheses of Sect. 5.2 with test statistic

$$T_n = \frac{\sup_{u \in \mathbb{R}} Y_n(u)}{\sqrt{\mathsf{E}\, Y_n^2(0)}}$$

if a large deviation result for the limiting Gaussian process Y is available, cf. [43, Corollary 1].

5.4 Limit Theorem for an Increasing Excursion Level

If the level $u \to \infty$, one may also expect that a CLT for the volume of the corresponding excursion set holds, provided that a particular rate of convergence of r to infinity is chosen in accordance with the expansion rate of the observation window.

First, results of this type were proven in [26, Theorems 2.7.1, 2.7.2, 2.8.1] for stationary isotropic Gaussian random fields with short- or long-range dependence. A generalization to the case of stationary **PA**-random fields is given in a recent preprint [19]:

Theorem 10. *Let* $X = \{X(t),\ t \in \mathbb{R}^d\} \in \mathbf{PA}$ *be a stationary random field with a continuous covariance function* C *such that* $|C(t)| = \mathcal{O}\left(\|t\|_2^{-\alpha}\right)$ *for some* $\alpha > 3d$ *as* $\|t\|_2 \to \infty$. *Let* $X(o)$ *have a bounded density* $p_{X(o)}$. *Assume that the variance of* $V_d\left(A_{u_n}\left(X, [0,n]^d\right)\right)$ *being equal to*

$$\sigma_n^2 = \int\limits_{[0,n]^d} \int\limits_{[-x,n-x]^d} \mathrm{cov}\left(\mathbf{1}\{X(o) \geq u_n\}, \mathbf{1}\{X(t) \geq u_n\}\right)\, dt\, dx$$

satisfies

$$\sigma_n^2 \to \infty, \quad n \to \infty. \tag{11}$$

Introduce $\gamma(x) = \sup_{y \geq x} p_{X(o)}(y)$, $x \in \mathbb{R}$. *Choose a sequence of excursion levels* $u_n \to \infty$ *such that*

$$\frac{n^d \gamma^{2/3}(u_n)}{\sigma_n^{2(\alpha+3)/3}} \to 0, \quad n \to \infty. \tag{12}$$

Then it holds

$$\frac{V_d\left(A_{u_n}\left(X, [0,n]^d\right)\right) - n^d \mathsf{P}s(X(o) \geq u_n)}{\sigma_n} \xrightarrow{\mathrm{d}} \mathcal{N}(0,1) \tag{13}$$

as $n \to \infty$.

Conditions (11), (12) are checked in [19] explicitly for stationary (non-isotropic) Gaussian as well as shot noise random fields leading to quite tractable simple expressions. For instance, it suffices to choose $u_n = O(\sqrt{\log n})$, $n \to \infty$ in the Gaussian case.

Student and Fisher–Snedecor random fields are considered in the recent preprint [34, Sect. 7]. CLTs for spherical measures of excess

$$\int_{\partial B_r(o)} \mathbf{1}\{X(t) > u(r)\}\, \mathcal{H}^{d-1}(dt)$$

of a stationary Gaussian isotropic random field X over the moving level $u(r) \to \infty$, $r \to \infty$ are proved in [37, Sect. 3.3]. For yet another type of geometric measures of excess over a moving level, see [35].

6 Surface Area of Excursion Sets of Stationary Gaussian Random Fields

Limit theorems for $V_{d-1}(A_u(X, W_n))$ have been first proven for one fixed level u and a stationary isotropic Gaussian random field X in [31] in dimension $d = 2$. There, the expansion of $V_{d-1}(A_u(X, W_n))$ in Hermite polynomials is used. In higher dimensions, a multivariate analogue of this result can be proven along the same guidelines; see [56, Proof of Theorem 1] for a shorter proof. A CLT for the integral of a continuous function along a level curve $\partial A_u(X, W)$ for an a.s. C^1-smooth centered mixing stationary random field $X = \{X(t),\ t \in \mathbb{R}^2\}$ in a rectangle W is proved in [25].

6.1 Functional Limit Theorems

Let us focus on functional LTs for $V_{d-1}(\partial A_u(X, W_n))$ proven in [44] for the phase space $L^2(\mathbb{R}, \nu)$ (where ν is a standard Gaussian measure in $\mathbb{R}$) and in [56] for the phase space $C(\mathbb{R})$.

Let $X = \{X(t),\ t \in \mathbb{R}^d\}$, $d > 1$, be a centered stationary and isotropic Gaussian random field with a.s. C^1-smooth paths and covariance function $C \in C^2(\mathbb{R}^d)$ satisfying $C(o) = 1$ as well as

$$|C(t)| + \frac{1}{1 - C(t)} \sum_{i=1}^{d} \left| \frac{\partial C(t)}{\partial t_i} \right| + \sum_{i,j=1}^{d} \left| \frac{\partial^2 C(t)}{\partial t_i \partial t_j} \right| < g(t) \tag{14}$$

for large $\|t\|_2$ (where $t = (t_1, \dots, t_d)^\top$) and a bounded continuous function $g : \mathbb{R}^d \to \mathbb{R}_+$ such that $\lim_{\|t\|_2 \to \infty} g(t) = 0$ and

$$\int_{\mathbb{R}^d} \sqrt{g(t)}\, dt < \infty.$$

Denote by $\nabla X(t)$ the gradient of $X(t)$. Assume that the $(2d + 2)$-dimensional random vector $(X(o), X(t), \nabla X(o), \nabla X(t))^\top$ is nondegenerate for all $t \in \mathbb{R}^d \setminus \{o\}$. Let $\lambda^2 = -\partial^2 C(o)/\partial t_1^2$.

Introduce the sequence of random processes $\{Y_n\}$, $n \in \mathbb{N}$ by

$$Y_n(u) = \frac{2\lambda^{d/2-1}}{n^{d/2}} \left(V_{d-1}\left(\partial A_u(X, [0, n]^d)\right) - \mathsf{E}\, V_{d-1}\left(\partial A_u(X, [0, n]^d)\right) \right) \tag{15}$$

where $u \in \mathbb{R}$. They will be interpreted as random elements in $L^2(\mathbb{R}, \nu)$. Let $\rightharpoonup$ denote the weak convergence of random elements in $L^2(\mathbb{R}, \nu)$. Let

$$\kappa(t) = f\left(X(t)\right) \exp\{-X^2(t)/2\} \|\nabla X(t)\|_2, \quad t \in \mathbb{R}^d.$$

Theorem 11 ([44]). *Under the above assumptions on X and C, it holds $Y_n \rightharpoonup Y$ as $n \to \infty$ where Y is a centered Gaussian random element in $L^2(\mathbb{R},\nu)$ with covariance operator*

$$\mathsf{Var}\langle Y, f\rangle_{L^2(\mathbb{R},\nu)} = \frac{1}{2\pi}\int_{\mathbb{R}^d} \mathsf{cov}\,(\kappa(o),\kappa(t))\; dt, \quad f \in L^2(\mathbb{R},\nu).$$

For $d \geq 3$, processes Y_n have a continuous modification $\tilde{Y}_n$ if conditions on X starting from (14) are replaced by the following ones:

1. Covariance function C as well as all its first- and second- order derivatives belong to $L^1(\mathbb{R})$
2. There exist $\tau \in (0,1)$ and $\beta > 0$ such that for all $h \in [-\tau,\tau]$ and $e_h = (h,0,0\ldots,0)^\top \in \mathbb{R}^d$, the determinant of the covariance matrix of the vector

$$\left(X(o), X(e_h), \frac{\partial X(o)}{\partial t_1}, \frac{\partial X(e_h)}{\partial t_1}\right)^\top$$

is not less than $|h|^\beta$.

Let $\rightharpoondown$ denote the weak convergence of random elements in $C(\mathbb{R})$. Denote by $p_{X(t)}$ ($p_{X(o),X(t)}$) the density of $X(t)$ ($(X(o),X(t))^\top$), $t \in \mathbb{R}^d$, respectively. Set

$$H_t(u,v) = \mathsf{E}\left(\|\nabla X(o)\|_2 \|\nabla X(t)\|_2 \;|X(o) = u,\; X(t) = v\right), \quad u,v \in \mathbb{R}, \quad t \in \mathbb{R}^d.$$

In definition (15), assume $\lambda = 1$.

Theorem 12 ([56]). *Under the above assumptions on X and C, it holds $\tilde{Y}_n \rightharpoondown Y$ as $n \to \infty$ for $d \geq 3$ where Y is a centered Gaussian random process with covariance function*

$$\mathsf{cov}\,(Y(u),Y(v)) = \int_{\mathbb{R}^d} \Big(H_t(u,v)p_{X(o),X(t)}(u,v) - (\mathsf{E}\|\nabla X(o)\|_2)^2\, p_{X(o)}(u)p_{X(t)}(v)\Big)\; dt$$

for $u,v \in \mathbb{R}$.

The case $d = 2$ is still open.

7 Open Problems

It is a challenging problem to prove the whole spectrum of limit theorems for $V_j\,(A_u(X,W_n))$ of lower orders $j = 0,\ldots,d-2$ for isotropic C^2-smooth stationary Gaussian random fields. Functional limit theorems and the case of increasing level $u \to \infty$ are therein of special interest. Further perspective of research is the generalization of these (still hypothetic) results to non-Gaussian random fields.

Another open problem is to prove limit theorems for a large class of functionals of non-Gaussian stationary random fields that includes the volume of excursion sets. It is quite straightforward to do this for

$$\int_{W_n} g(X(t))\,dt$$

for a measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $\mathsf{E}\, g^2(X(o)) < \infty$. For more general classes of functionals of the field X and the observation window W_n, it is still *terra incognita*.

References

1. Adler, R.: On generalizing the notion of upcrossings to random fields. Adv. Appl. Probab. **8**(4), 789–805 (1976)
2. Adler, R.: The Geometry of Random Fields. J. Wiley & Sons, Chichester (1981)
3. Adler, R., Samorodnitsky, G., Taylor, J.: Excursion sets of three classes of stable random fields. Adv. Appl. Probab. **42**, 293–318 (2010)
4. Adler, R., Taylor, J.: Random Fields and Geometry. Springer, New York (2007)
5. Adler, R.J., Taylor, J.E., Worsley, K.J.: Applications of random fields and geometry: Foundations and case studies. Springer Series in Statistics Springer, New York (2009) (In preparation). Http://webee.technion.ac.il/people/adler/hrf.pdf
6. Azais, J.M., Wschebor, M.: Level Sets and Extrema of Random Processes and Fields. Wiley, New York (2009)
7. Belyaev, Y.K.: On the number of intersections of a level by a Gaussian stochastic process, I. Theor. Probab. Appl. **11**(1), 106–113 (1966)
8. Belyaev, Y.K.: On the number of intersections of a level by a Gaussian stochastic process, II. Theor. Probab. Appl. **12**(3), 392–404 (1967)
9. Belyaev, Y.K., Nosko, V.P.: Characteristics of excursions above a high level for a Gaussian process and its envelope. Theor. Probab. Appl. **14**(2), 296–309 (1969)
10. Berman, S.M.: Sojourns and extremes of stochastic processes. Wadsworth & Brooks, CA (1992)
11. Billingsley, P.: Convergence of probability measures, 2nd edn. Wiley Series in Probability and Statistics: Probability and Statistics. Wiley, New York (1999)
12. Bordenave, C., Chafaï, D.: Around the circular law. Probab. Surv. **9**, 1–89 (2012)
13. Bradley, R.C.: Introduction to Strong Mixing Conditions. Vol. 1,2,3. Kendrick Press, Heber City, UT (2007)
14. Bulinskaya, E.V.: On the mean number of crossings of a level by a stationary Gaussian process. Theor. Probab. Appl. **6**, 435–438 (1961)
15. Bulinski, A., Spodarev, E., Timmermann, F.: Central limit theorems for the excursion sets volumes of weakly dependent random fields. Bernoulli **18**, 100–118 (2012)
16. Bulinski, A.V., Shashkin, A.P.: Limit theorems for associated random fields and related systems. World Scientific, Singapore (2007)
17. Cramér, H., Leadbetter, M.: The moments of the number of crossings of a level by a stationary normal process. Ann. Math. Statist. **36**(6), 1656–1663 (1965)
18. Cuzick, J.: A central limit theorem for the number of zeros of a stationary Gaussian process. Ann. Probab. **4**(4), 547–556 (1976)
19. Demichev, V., Schmidt, J.: A central limit theorem for the volume of high excursions of stationary associated random fields. Preprint (2013). Ulm University

20. Doukhan, P.: Mixing: properties and examples. Lecture Notes in Statistics, vol. 85, Springer, Berlin (1994)
21. Elizarov, A.: Central limit theorem for the sojourn time and local time of a stationary process. Theory Probab. Appl. **33**(1), 161–164 (1988)
22. Federer, H.: Curvature measures. Trans. Amer. Math. Soc. **93**, 418–491 (1959)
23. de Haan, L.: A spectral representation for max–stable processes. Ann. Probab. **12**(4), 1194–1204 (1984)
24. Ibragimov, I., Zaporozhets, D.: On the area of a random surface. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **312**, 154–175 (2010) arXiv:1102.3509
25. Iribarren, I.: Asymptotic behaviour of the integral of a function on the level set of a mixing random field. Probab. Math. Stat. **10**(1), 45–56 (1989)
26. Ivanov, A.V., Leonenko, N.N.: Statistical Analysis of Random Fields. Kluwer, Dordrecht (1989)
27. Ivanov, A.V., Leonenko, N.N., Ruiz-Medina, M.D., Savich, I.N.: Limit theorems for weighted nonlinear transformations of Gaussian stationary processes with singular spectra. Ann. Probab. **41**(2) (2013)
28. Kac, M.: On the average number of real roots of a random algebraic equation. Bull. Amer. Math. Soc. **49** 314-320 (1943)
29. Karcher, W.: On infinitely divisible random fields with an application in insurance. Ph.D. thesis, Ulm University, Ulm (2012)
30. Kratz, M.: Level crossings and other level functionals of stationary Gaussian processes. Probab. Surv. **3**, 230–288 (2006)
31. Kratz, M., Léon, J.: Central limit theorems for level functionals of stationary Gaussian processes and fields. J. Theor. Probab. **14**, 639–672 (2001)
32. Kratz, M., Léon, J.: Level curves crossings and applications for Gaussian models. Extremes **13**(3), 315–351 (2010)
33. Leadbetter, M.R., Lindgren, G., Rootzen, H.: Extremes and related properties of random sequences and processes, 1st edn. Springer Series in Statistics. Springer, New York (1983)
34. Leonenko, N., Olenko, A.: Sojourn measures of Student and Fischer–Snedecor random fields. Bernoulli (2013) (in press)
35. Leonenko, N.N.: Limit distributions of characteristics of exceeding a level by a Gaussian field. Math. notes of the Acad. Sci. of the USSR **41**(4), 339–345 (1987)
36. Leonenko, N.N.: Sharpness of the normal approximation of functionals of strongly correlated Gaussian random fields. Math. notes Acad. Sci. USSR **43**(2), 161–171 (1988)
37. Leonenko, N.N.: Limit theorems for random fields with singular spectrum. Kluwer, Dordrecht (1999)
38. Liu, J.: Tail approximations of integrals of Gaussian random fields. Ann. Probab. **40**(3), 1069–1104 (2012)
39. Liu, J., Xu, G.: Some asymptotic results of Gaussian random fields with varying mean functions and the associated processes. Ann. Statist. **40**(1), 262–293 (2012)
40. Malevich, T.L.: Asymptotic normality of the number crossings of level zero by a Gaussian process. Theor. Probab. Appl. **14**, 287–295 (1969)
41. Marinucci, D., Peccati, G.: Random Fields on the Sphere. Representation, Limit Theorems and Cosmological Applications, Lecture Notes of the London Mathematical Society, vol. 389. Cambridge University Press, Cambridge (2011)
42. Mecke, K., Stoyan, D. (eds.): Morphology of condensed matter. Physics and Geometry of Spatially Complex Systems, Lecture Notes in Physics, vol. 600. Springer, Berlin (2002)
43. Meschenmoser, D., Shashkin, A.: Functional central limit theorem for the volume of excursion sets generated by associated random fields. Statist. Probab. Lett. **81**(6), 642–646 (2011)
44. Meschenmoser, D., Shashkin, A.: Functional central limit theorem for the measures of level surfaces of the Gaussian random field. Theor. Probab. Appl. **57**(1), 162–172 (2013)
45. Molchanov, I.: Theory of Random Sets. Springer, London (2005)
46. Móricz, F.: A general moment inequality for the maximum of the rectangular partial sums of multiple series. Acta Math. Hung. **41**, 337–346 (1983)

47. Pantle, U., Schmidt, V., Spodarev, E.: On the estimation of integrated covariance functions of stationary random fields. Scand. J. Statist. **37**, 47–66 (2010)
48. Piterbarg, V.I.: The central limit theorem for the number of level crossings of a stationary Gaussian process. Theor. Probab. Appl. **23**(1), 178–182 (1978)
49. Resnik, S.I.: Extreme values, regular variation and point processes. Springer, Berlin (2008)
50. Rice, S.: Mathematical analysis of random noise. Bell Syst. Tech. J. **23**, 282–332 (1944)
51. Rice, S.: Mathematical analysis of random noise. Bell Syst. Tech. J. **24**, 46–156 (1945)
52. Samorodnitsky, G., Taqqu, M.: Stable non-Gaussian random processes. Chapman & Hall/CRC, London (1994)
53. Santaló, L.: Integral Geometry and Geometric Probability. Addison-Welsey, Reading, Mass (1976)
54. Schneider, R.: Convex Bodies. The Brunn–Minkowski Theory. Cambridge University Press, Cambridge (1993)
55. Schwartzman, A., Dougherty, R.F., Taylor, J.E.: False discovery rate analysis of brain diffusion direction maps. Ann. Appl. Stat. **2**(1), 153–175 (2008)
56. Shashkin, A.: A functional central limit theorem for the level measure of a Gaussian random field. Statist. Probab. Lett. **83**(2), 637–643 (2013)
57. Slud, E.: MWI representation of the number of curve–crossings by a differentiable Gaussian process, with applications. Ann. Probab. **22**(3), 1355–1380 (1994)
58. Spodarev, E. (ed.): Stochastic Geometry, Spatial Statistics and Random Fields. Asymptotics Methods, Lecture Notes in Mathematics, vol. 2068. Springer, Berlin (2013)
59. Spodarev, E., Zaporozhets, D.: Asymptotics of the mean Minkowski functionals of Gaussian excursions (2012). Preprint
60. Taylor, J.E., Worsley, K.J.: Detecting sparse signal in random fields, with an application to brain mapping. J. Am. Stat. Assoc. **102**(479), 913–928 (2007)
61. Torquato, S.: Random Heterogeneous Materials: Microstructure and Macroscopic Properties. Springer, New York (2002)
62. Worsley, K.J., Taylor, J.: Detecting fMRI activation allowing for unknown latency of the hemodynamic response. Neuroimage **29**, 649–654 (2006)

Part IV
Finance and Risk

Ambit Processes, Their Volatility Determination and Their Applications

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Abstract In this chapter we try to review the research done so far about ambit processes and their applications. The notion of ambit process was introduced by Barndorff-Nielsen and Schmiegel in 2007. Since then, many papers have been written studying their properties and applying them to model different natural or economic phenomena. As it is shown in the paper, these processes share their mathematical structure with the solutions of random evolution equations allowing them great flexibility for modelling. The goal of this paper is fourfold: to show the main characteristics of these processes; how to determine their main structural component: their volatility; how they can be used for modelling different random phenomena like turbulence or financial prices; and last but not least the mathematics behind.

1 Introduction

The notion of ambit process was introduced by Barndorff-Nielsen and Schmiegel in 2007; see [12]. Since then, many papers have been written studying their properties and applying them to model in different natural or economic phenomena; see [5, 7, 8, 12, 23], among others. In the present paper we try to review all this work and to enlighten the notion of ambit process and its flexibility for modelling. Before giving the definition of ambit processes, let us justify the generality and, consequently, the flexibility of such processes. Here we follow [6].

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Let L be a partial differential operator, for instance, the wave operator in dimension one

$$Lf = \frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2},$$

then, it is well known that there is a function G in $(\mathbf{R}_+, \mathbf{R})$ such that the solution of the PDE

$$Lu = \varphi, u(0,x) = 0,$$

where φ is a *test* function, can be written as

$$u(t,x) = \int_{\mathbf{R}_+ \times \mathbf{R}} G(t-s, x-y)\varphi(s,y)\mathrm{d}s\mathrm{d}y.$$

Imagine now we have the SPDE

$$Lu = W, u(0,x) = 0 \tag{1}$$

where W is an L^2-noise in $\mathbf{R}_+ \times \mathbf{R}$ that is a map

$$\begin{aligned} \mathcal{B}(\mathbf{R}_+ \times \mathbf{R}) &\longrightarrow L^2(\Omega, \mathcal{F}, \mathbf{P}) \\ A &\longmapsto W(A), \end{aligned}$$

such that

1. $W(\emptyset) = 0$ a.s.
2. For all disjoint and bounded sets $A_1, A_{2,} \ldots$ in $\mathcal{B}(\mathbf{R}_+ \times \mathbf{R})$, $W(A_i)$ are independent and

$$W(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} W(A_i), a.s.$$

and where the convergence of the series is in $L^2(\mathbf{P})$. Then it is natural to consider that the solution of (1) is given by

$$u(t,x) = \int_{\mathbf{R}_+ \times \mathbf{R}} G(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y). \tag{2}$$

This kind of solution is named a *mild* solution. In general, if we have a random phenomenon with a certain dynamics, the tempo-spatial derivatives of the magnitude in a point will be connected with the *driving* noise at that point, and this will imply that the value of the magnitude is related with the value of the driving noise in other points of the space–time set, as it can be appreciated in (2). Then, when modelling random phenomena, we can opt for proposing a kind of global dependency directly instead of a point-wise dynamical dependency. This is the motivation for the following definition:

Definition 1. A tempo-spatial *ambit field* is defined as

$$Y(t,x) = \mu + \int_{A(t,x)} g_{(t,x)}(s,\xi)\sigma(s,\xi)W(\mathrm{d}s,\mathrm{d}\xi) + \int_{B(t,x)} q_{(t,x)}(s,\xi)a(s,\xi)\mathrm{d}s\mathrm{d}\xi, \quad t \geq 0, x \in \mathbf{R}^n$$

where $\mu \in \mathbf{R}$, $\xi \in \mathbf{R}^n$, W is a σ-finite, L^2-valued measure, $g_{(t,x)}(\cdot)$ and $q_{(t,x)}(\cdot)$ are deterministic kernels, $\sigma(\cdot,\cdot) \geq 0$, and $a(\cdot,\cdot)$ are predictable random fields and $A(t,x) \subseteq \mathbf{R}^{n+1}$ and $B(t,x) \subseteq \mathbf{R}^{n+1}$ are *ambit sets*. Then, $X_t := Y_t(x(t))$, for a curve $x(t)$, is called an *ambit process*.

In this definition the stochastic integral is assumed in the sense of Walsh; see, for instance, [39] and the more recent reference [30]. However a slight extension of this integral is considered here; in fact, in the integral, time coordinate moves in $\mathbf{R}$ more than in $\mathbf{R}_+$. This extension has been studied recently in [19]. Another extension, now for the case when ξ is infinite dimensional and $W_s(\mathrm{d}\xi) := W([0,s],\mathrm{d}\xi)$, $s \geq 0$ is a cylindrical Brownian motion, can be found in [17].

The paper is organized as follows. Section 2 contains some properties and particularities of the ambit processes. Section 3 is devoted to see the application of ambit processes to modelling in turbulence and to study their statistical properties in the context on infill asymptotics. Section 4 is devoted to study their applications in quantitative finance to modelling term structures and energy markets.

2 Ambit Processes

The general concept of ambit field consists of a stochastic field $(Y(t,x))$ in space–time, $t \in \mathbf{R}$, $x \in \mathbf{R}^n$, where the values of $Y(t,x)$ depend on *innovations* prior to or at time t and that happened in a certain subset of $\mathbf{R}^n$. In other words, $Y(t,x)$ depends on what happened in a time–space subset (the so-called ambit set), $A(t,x) = \{(s,y) \in \mathbf{R}^{n+1}, s \in \Upsilon \subseteq (-\infty,t], y \in \Lambda_s \subseteq \mathbf{R}^n\}$. Then, if we take a curve $x(t)$ in $\mathbf{R}^n$, we have an ambit *process* $Y_t := Y(t,x(t))$. Evidently we can substitute a more abstract space, like a Hilbert space, for $\mathbf{R}^n$ to get a more general object. Another natural extension is to assume that Y takes values in $\mathbf{R}^n$, or even a Banach space. In any case we need further mathematical structure if we want to say something concrete about Y. The structure considered is that given in the Definition 1,

$$Y(t,x) = \mu + \int_{A(t,x)} g_{(t,x)}(s,\xi)\sigma(s,\xi)W(\mathrm{d}s,\mathrm{d}\xi) + \int_{B(t,x)} q_{(t,x)}(s,\xi)a(s,\xi)\mathrm{d}s\mathrm{d}\xi, \quad t \geq 0, x \in \mathbf{R}^n,$$

where $\mu \in \mathbf{R}$, $\xi \in \mathbf{R}^n$, W is a σ-finite, L^2-noise, $g_{(t,x)}(\cdot)$ and $q_{(t,x)}(\cdot)$ are deterministic kernels, $\sigma(\cdot,\cdot) \geq 0$, and $a(\cdot,\cdot)$ are predictable random fields and $A(t,x) \subseteq \mathbf{R}^{n+1}$ and $B(t,x) \subseteq \mathbf{R}^{n+1}$ are *ambit sets*. Ambit sets can be seen as *areas of influence* or *causality* and this part of the structure could be seen as the only dynamic condition in these kind of processes or fields. The condition is that future cannot influence the past. Nevertheless the ambit fields used in practice are of the form

$$Y(t,x) = \mu + \int_{A(t,x)} g_x(t-s,\xi)\sigma(s,\xi)W(\mathrm{d}s,\mathrm{d}\xi) + \int_{B(t,x)} q_x(t-s,\xi)a(s,\xi)\mathrm{d}s\mathrm{d}\xi, \quad t \geq 0, x \in \mathbf{R}^n,$$

where $A(t,x) = A + (t,x)$, with A involving only negative time coordinates, in agreement with the *causality principle,* and analogously for $B(t,x)$. In such a situation this class of fields include the class of stationary fields in time and, by this reason, they are called *semistationary*. If W is a Lévy noise, the field (or process) is called Lévy semistationary field (or process) ($\mathcal{LSS}$) and for the particular case where W is a Gaussian noise is called Brownian semistationary ($\mathcal{BSS}$). It is also said that

$$X_t := \int_{A(t,x)} g_x(t-s,\xi)\sigma(s,\xi)W(\mathrm{d}s,\mathrm{d}\xi)$$

is the *core* of Y. Moreover σ is referred to as the *intermittency, volatility* or *modulating* field or process. It is difficult to say interesting statements for such general objects. To obtain something remarkable about, for instance, how the trajectories are or if the ambit process is a semimartingale or not, we need specific kernels, volatilities and noises.

Consider just the particular case $(X_t)_{t\in\mathbf{R}}$ of the form

$$X_t = \int_{-\infty}^{t} g(t-s)W(\mathrm{d}s),$$

where W is a Gaussian white noise in $\mathbf{R}$, σ an adapted càdlag process and $g \in L^2(\mathbf{R}_+)$.

The path properties of the process $(X_t)_{t\in\mathbf{R}}$ crucially depend on the behaviour of the weight function g near 0. When $g(x) = x^\alpha L_g(x)$ (where $L_g(x)$ is a slowly varying function at 0) with $\alpha \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$, X has r-Hölder continuous paths for any $r < \alpha + \frac{1}{2}$. The analysis of the regularity of the sample paths follows the same routes that in the case of Volterra processes; see [34]. In fact X is a Volterra process though starting at $-\infty$.

Another important fact is that X is not a semimartingale, because g' is not square integrable in the neighbourhood of 0. In fact, observing the decomposition

$$X_{t+\Delta} - X_t = \int_t^{t+\Delta} g(t+\Delta-s)W(\mathrm{d}s) + \int_{-\infty}^{t} \{g(t+\Delta-s) - g(t-s)\}W(\mathrm{d}s),$$

we obtain by formal differentiation that

$$\mathrm{d}X_t = g(0+)\mathrm{d}W(t) + \left(\int_{-\infty}^{t} g'(t-s)W(\mathrm{d}s)\right)\mathrm{d}t,$$

Then, the Gaussian process X is an Itô semimartingale when $g(0+) < \infty$ and $g' \in L^2(\mathbf{R}_+)$ and this property also transfers to the $\mathcal{BSS}$ process Y under mild assumptions. It can be shown (see [14]) that the conditions $g(0+) < \infty$ and $g' \in L^2(\mathbf{R}^+)$ are also necessary conditions for X to be a semimartingale. So, if we assume that $g(x) = x^\alpha L_g(x)$, with $\alpha \in (-\frac{1}{2}, 0) \cup (0, \frac{1}{2})$, we have that $g' \notin L^2(\mathbf{R}^+)$ and the process X, and so the process Y (unless $\sigma = 0$), is not a semimartingale.

A similar analysis can be done to see if a $\mathcal{LSS}$ is a semimartingale. See, for instance, [9].

Moreover ambit processes can be used as leading noises of stochastic differential equations and we can construct a stochastic calculus with respect to this processes; see Sect. 4.1 in [23].

3 Models in Turbulence

In the framework of *stochastic modelling in turbulence* (see [26] for a description of this approach), Barndorff-Nielsen and Schmiegel [12, 13] propose to model the main component of the velocity by a process of the form

$$Y_t = \mu + \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s) + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s, \tag{3}$$

where μ is a constant, W is a Gaussian white noise on $\mathbf{R}$, g and q are nonnegative deterministic functions on $\mathbf{R}$, with $g(t) = q(t) = 0$ for $t \leq 0$, and σ and a are adapted càdlàg processes.

Other approaches, out of the scope of this paper, combine the classical Navier–Stokes equation for a fluid and randomness. The results in this framework are, however, quite implicit; see, for instance, [15, 32] or the more oriented toward applications [18].

3.1 Volatility Determination

One crucial *quantity* in the model (3) is the volatility and some effort has been done to estimate σ. It is apparent, from [3, 4, 8, 10, 21, 22], that a key tool to estimate σ is the *realized multipower variation (*RMV*)* of the process Y. It is an object of the type

$$\sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Delta_{i+j-1}^n Y|^{p_j} , \qquad \Delta_i^n Y = Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}} , \qquad p_1, \ldots, p_k \geq 0 ,$$

for some fixed number $k \geq 1$.

For simplicity of the exposition, we shall consider the *core* of (3)

$$Y_t = \int_{-\infty}^{t} g(t-s)\sigma_s W(\mathrm{d}s), \tag{4}$$

where we assume that

$$\int_{-\infty}^{t} g^2(t-s)\sigma_s^2 \mathrm{d}s < \infty, \quad \text{a.s..}$$

and also that the function g is continuously differentiable on $(0, \infty)$, $|g'|$ is nonincreasing on (b, ∞) for some $b > 0$, and $g' \in L^2((\varepsilon, \infty))$ for any $\varepsilon > 0$. Moreover, we assume that for any $t > 0$,

$$F_t = \int_{1}^{\infty} (g'(s))^2 \sigma_{t-s}^2 \mathrm{d}s < \infty, \quad \text{a.s..}$$

See [8] for a discussion of this latter conditions.

The process Y is supposed to be observed at time points $t_i = i/n$, $i = 1, \ldots, [nt]$. Now, let G be the stationary Gaussian process defined as

$$G_t = \int_{-\infty}^{t} g(t-s) W(\mathrm{d}s).$$

We are interested in the asymptotic behaviour of the functionals

$$V(Y, p_1, \ldots, p_k)_t^n = \frac{1}{n\tau_n^{p+}} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |\Delta_{i+j-1}^n Y|^{p_j} , \qquad p_1, \ldots, p_k \geq 0 ,$$

where $\Delta_i^n Y = Y_{\frac{i}{n}} - Y_{\frac{i-1}{n}}$ and $\tau_n^2 = \bar{R}(1/n)$ with $\bar{R}(t) = \mathbf{E}[|G_{s+t} - G_s|^2]$, $t \geq 0$, and when n goes to infinity, in such a way that we are in the context of infill asymptotics. We define the correlation function of the increments of G:

$$r_n(j) = \operatorname{cov}\left(\frac{\Delta_1^n G}{\tau_n}, \frac{\Delta_{1+j}^n G}{\tau_n} \right) = \frac{\bar{R}(\frac{j+1}{n}) + \bar{R}(\frac{j-1}{n}) - 2\bar{R}(\frac{j}{n})}{2\tau_n^2} , \qquad j \geq 0.$$

Next, we introduce a class of measures:

$$\pi^n(A) = \frac{\int_A (g(x-\frac{1}{n}) - g(x))^2 \mathrm{d}x}{\int_0^\infty (g(x-\frac{1}{n}) - g(x))^2 \mathrm{d}x}, \qquad A \in \mathcal{B}(\mathbf{R}).$$

Finally, we define

$$\rho^{(n)}_{p_1,\ldots,p_k} = \mathbf{E}\left[\left| \frac{\Delta_1^n G}{\tau_n} \right|^{p_1} \cdots \left| \frac{\Delta_k^n G}{\tau_n} \right|^{p_k} \right].$$

To have a weak law of large numbers, we require the following assumptions: **(LLN)**: There exists a sequence $r(j)$ with

$$r_n^2(j) \le r(j), \qquad \frac{1}{n} \sum_{j=1}^{n-1} r(j) \to 0.$$

Moreover, it holds that

$$\lim_{n\to\infty} \pi^n((\varepsilon, \infty)) = 0$$

for any $\varepsilon > 0$.

For the **CLT** we need to introduce another Gaussian process. Let $(Q_i)_{i\ge 1}$ be a non-degenerate stationary centred (discrete time) Gaussian process with variance 1 and correlation function

$$\rho(j) = \mathrm{cor}\left(Q_1, Q_{1+j}\right), \qquad j \ge 1.$$

Define

$$V_Q(p_1,\ldots,p_k)_t^n = \frac{1}{n} \sum_{i=1}^{[nt]-k+1} \prod_{j=1}^{k} |Q_{i+j-1}|^{p_j}$$

and let $\rho_{p_1,\ldots,p_k} = \mathbf{E}(|Q_1|^{p_1} \cdots |Q_k|^{p_k})$.

Now we can specify the condition (**CLT**): Assume **(LLN)** holds, and

$$r_n(j) \to \rho(j), \qquad j \ge 0,$$

where $\rho(j)$ is the correlation function of (Q_i). Furthermore, there exists a sequence $r(j)$ such that, for any $j, n \ge 1$,

$$r_n^2(j) \le r(j), \qquad \sum_{j=1}^{\infty} r(j) < \infty,$$

and we have

$$\mathbf{E}[|\sigma_t - \sigma_s|^A] \le C|t-s|^{A\gamma},$$

for any $A > 0$, with $\gamma(p \wedge 1) > \frac{1}{2}$, and $p = \min_{1\le i\le k, 1\le j\le d}(p_i^j)$. Finally we assume that there exists a constant $\lambda < -\frac{1}{p\wedge 1}$ such that for any $\varepsilon_n = O(n^{-\kappa})$, $\kappa \in (0,1)$, we have

$$\pi^n((\varepsilon_n,\infty)) = O(n^{\lambda(1-\kappa)}).$$

Set $p^+ = \sum_{l=1}^k p_l$. We have the following main theorem; see [8].

Theorem 1. *Consider the process Y given by (4). Assume that the condition* ***(CLT)*** *holds; then, we obtain the stable convergence*

$$\sqrt{n}\left(V(Y,p_1^j,\ldots,p_k^j)_t^n - \rho^{(n)}_{p_1^j,\ldots,p_k^j}\int_0^t |\sigma_s|^{p_+^j}\mathrm{d}s\right)_{1\le j\le d} \xrightarrow{st} \int_0^t A_s^{1/2}\mathrm{d}B_s,$$

where B is a d-dimensional Brownian motion independent of $\mathcal{F}$, and A is a $d\times d$-dimensional process given by

$$A_s^{ij} = \beta_{ij}|\sigma_s|^{p_+^i+p_+^j}, \qquad 1\le i,j\le d\ ,$$

with β the $d\times d$ matrix given by

$$\beta_{ij} = \lim_{n\to\infty} n\,\mathrm{cov}\Big(V_Q(p_1^i,\ldots,p_k^i)_1^n, V_Q(p_1^j,\ldots,p_k^j)_1^n\Big)\ , \qquad 1\le i,j\le d\ .$$

In [8] we worked with the function g

$$g(t) = t^{\nu-1}e^{-\lambda t}1_{(0,\infty)}(t)$$

for $\lambda > 0$ and with $\nu > \frac{1}{2}$. For t near 0, $g(t)$ behaves as t^δ with $\delta = \nu - 1$. If we check the conditions for the CTL, we have the restriction $1/2 < \nu < 1$. This forced us to consider higher-order differences:

$$\diamondsuit_i^n X = X_{i\Delta_n} - 2X_{(i-1)\Delta_n} + X_{(i-2)\Delta_n}.$$

and to study the multipower variation of the second-order differences of the $\mathcal{BSS}$ process X, i.e.

$$\mathrm{MPV}^{\diamondsuit}(X,p_1,\ldots,p_k)_t^n = \Delta_n(\tau_n^{\diamondsuit})^{-p^+}\sum_{i=2}^{[t/\Delta_n]-2k+2}\prod_{l=0}^{k-1}|\diamondsuit_{i+2l}^n X|^{p_l},$$

where $(\tau_n^{\diamondsuit})^2 = \mathbf{E}(|\diamondsuit_i^n G|^2)$ and $p^+ = \sum_{l=1}^k p_l$.

See [10, 24] for the development and application to real turbulence data of the high-order multipower variation.

It is worthwhile to comment that the limit theory for multipower variation of Lévy semistationary processes does not yet exist.

3.1.1 Volatility Determination in an Ambit Field Setting

Now we try to show a relation between the realized quadratic variation (RQV) along a curve and the volatility of the underlying random field. We refer to [11] for more details.

Consider a random field

$$Y(x) = \int_{A(x)} g(x-\xi)\sigma(\xi)W(\mathrm{d}\xi),$$

where $x \in \mathbf{R}^n$, W is the Gaussian white noise in $\mathbf{R}^n$, $g : \mathbf{R}^n \to \mathbf{R}$, with $g(x_1, .., x_n) = 0$ if $x_1 < 0$ (the first coordinate indicates time), and σ is either deterministic or independent of W. Then, assume that $A(x) = A + x$,

$$Y(x) = \int_{A(x)} g(x-\xi)\sigma(\xi)W(\mathrm{d}\xi) = \int_{\mathbf{R}^n} g\mathbf{1}_{-A}(v)\sigma(x-v)W(x-\mathrm{d}v),$$

in such a way that

$$Y(x+\Delta x) - Y(x) = \int_{\mathbf{R}^n} (g\mathbf{1}_{-A}(v+\Delta x) - g\mathbf{1}_{-A}(v))\,\sigma(x-v)W(x-\mathrm{d}v),$$

and

$$\mathbf{E}\left[(Y(x+\Delta x) - Y(x))^2 \middle| \sigma \right] = \int_{\mathbf{R}^n} (g\mathbf{1}_{-A}(v+\Delta x) - g\mathbf{1}_{-A}(v))^2\,\sigma^2(x-v)\mathrm{d}v.$$

Then

$$\sum_{i=1}^{n} \mathbf{E}\left[(Y(x_{i-1}+\Delta x_i) - Y(x_i))^2 \middle| \sigma \right]$$
$$= \int_{\mathbf{R}^n} \sum_{i=1}^{n} (g\mathbf{1}_{-A}(v+\Delta x_i) - g\mathbf{1}_{-A}(v))^2\,\sigma^2(x_i - v)\mathrm{d}v.$$

Assume now that $\Delta x_i = \Delta x(\delta) = (\tau_1(\delta), \tau_2(\delta), \ldots, \tau_n(\delta))$ for all $i = 1, \ldots, n$, with $\tau_1(\delta) = \delta$ (in particular this happens if we are moving along a straight line). We take $n = [t/\delta]$. Then if we define

$$\pi_\delta(\mathrm{d}v) := \frac{(g\mathbf{1}_{-A}(v+\Delta x(\delta)) - g\mathbf{1}_{-A}(v))^2}{c(\delta)}\mathrm{d}v,$$

where $c(\delta) = \int_{\mathbf{R}^n} (g\mathbf{1}_{-A}(u+\Delta x_i) - g\mathbf{1}_{-A}(u))^2\,\mathrm{d}u$, we have that

$$\frac{\delta}{c(\delta)}\sum_{i=1}^{[t/\delta]}\mathbf{E}\left[\left.(Y(x_{i-1}+\Delta x_i)-Y(x_i))^2\right|\sigma\right]=\int_{\mathbf{R}^n}\delta\sum_{i=1}^{[t/\delta]}\sigma^2(x_i(\delta)-v)\pi_\delta(\mathrm{d}v)$$
$$\overset{\delta\to 0}{\longrightarrow}\int_{\mathbf{R}^n}\left(\int_0^t\sigma^2(x(s)-v)\mathrm{d}s\right)\pi_0(\mathrm{d}v),$$

provided that

$$\pi_\delta \overset{\delta\to 0}{\longrightarrow} \pi_0$$

and σ is continuous. We have also the following result; see [11].

Proposition 1. *If* π_0 *is concentrated on* $-\partial A$*, then*

$$\operatorname{var}\left(\left.\frac{\delta}{c(\delta)}\sum_{i=1}^{[t/\delta]}(Y(x_{i-1}+\Delta x_i)-Y(x_i))^2\right|\sigma\right)\overset{\delta\to 0}{\to}0.$$

As a corollary, we have the convergence in probability

$$\frac{\delta}{c(\delta)}\sum_{i=1}^{[t/\delta]}(Y(x_{i-1}+\Delta x_i)-Y(x_i))^2\overset{\delta\to 0}{\longrightarrow}\int_{\mathbf{R}^n}\left(\int_0^t\sigma^2(x(s)-v)\mathrm{d}s\right)\pi_0(\mathrm{d}v).$$

But when is π_0 concentrated on $-\partial A$? In [11] authors give some sufficient conditions for A (bounded, closed, convex with nonempty interior and piecewise smooth boundary) and g, but they are quite restrictive.

The behaviour of the RQV along smooth curves and for some particular shapes of A, for instance, $A = (\mathbf{R}_+)^n$, and memory functions of the kind $g(x) = ||x||^\alpha L_g(||x||)$ is a topic of present research. The purpose is to relate σ or some integral of it, with the limit of the RQV along lines, or surfaces.

To remark that the asymptotic behaviour of the multipower variation of general tempo-spatial ambit fields is an open problem.

4 Models in Finance

4.1 A Short-Rate Model

4.1.1 The Model

Let $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$ be a complete probability space with a filtration $\mathbf{F} = (\mathcal{F}_t)_{t\in\mathbf{R}_+}$. Assume that in this probability space

$$r_t = \int_{-\infty}^t g(t-s)\sigma_s W(\mathrm{d}s) + \mu_t \tag{5}$$

where W is an $(\mathcal{F}_t)$-Gaussian noise in $\mathbf{R}$ under the risk-neutral probability, $\mathbf{P}^* \sim \mathbf{P}$, g is a deterministic function on $\mathbf{R}_+$, $g \in L^2((0,\infty))$, and $\sigma \geq 0$ and μ are also deterministic. Notice that the process r is not a semimartingale if $g' \notin L^2((0,\infty))$. Furthermore, we also assume that

$$\int_{-\infty}^{t} g^2(t-s)\sigma_s^2 \mathrm{d}s < \infty \quad \text{a.s.}$$

which ensures that r_t is well defined. Then, we consider a financial bond market with short rate r. Here we follow [23].

4.1.2 Bond Prices

Assume that $\exp\left\{-\int_0^T r_s \mathrm{d}s\right\} \in L^1(\mathbf{P}^*)$ and denote $P(t,T)$ and $\tilde{P}(t,T)$ the price and the discounted price at t of the zero coupon bond with maturity time T:

$$P(t,T) = \mathbf{E}_{\mathbf{P}^*}\left[\exp\left\{-\int_t^T r_s \mathrm{d}s\right\} \middle| \mathcal{F}_t\right]$$

$$\tilde{P}(t,T) = P(t,T)\exp\left\{-\int_0^t r_s \mathrm{d}s\right\},$$

where $\tilde{P}(t,T)$ is a $\mathbf{P}^*$-martingale. Then, writing $c(u;t,T) := \int_t^T g(s-u)\mathrm{d}s$ for $t \geq u$, and by using Fubini's theorem, we have

$$\int_t^T r_s \mathrm{d}s = \int_{-\infty}^{t} \sigma_u c(u;t,T) W(\mathrm{d}u) + \int_t^T \sigma_u c(u;u,T) W(\mathrm{d}u) + \int_t^T \mu_s \mathrm{d}s.$$

Then

$$P(t,T) = \exp\left\{A(t,T) - \int_{-\infty}^{t} \sigma_u c(u;t,T) W(\mathrm{d}u)\right\},$$

where

$$A(t,T) = \log \mathbf{E}_{\mathbf{P}^*}\left[\exp\left\{-\int_t^T \sigma_u c(u;u,T) W(\mathrm{d}u) - \int_t^T \mu_s \mathrm{d}s\right\} \middle| \mathcal{F}_t\right]$$
$$= \frac{1}{2}\int_t^T \sigma_u^2 c^2(u;u,T)\mathrm{d}u - \int_t^T \mu_s \mathrm{d}s.$$

and the variance of the yield $-\frac{1}{T-t}\log P(t,T)$ is given by

$$\operatorname{var}\left(-\frac{1}{T-t}\log P(t,T)\right) = \frac{1}{(T-t)^2}\int_{-\infty}^{t} \sigma_u^2 c^2(u;t,T)\mathrm{d}u.$$

The corresponding forward rates are given by

$$
\begin{aligned}
f(t,T) &= -\partial_T \log P(t,T) \\
&= -\int_t^T \sigma_u^2 g(T-u)c(u;u,T)\mathrm{d}u + \int_{-\infty}^t \sigma_u g(T-u)W(\mathrm{d}u) + \mu_T
\end{aligned}
$$

and

$$
\operatorname{var}(f(t,T)) = \int_{-\infty}^t \sigma_u^2 g^2(T-u)\mathrm{d}u.
$$

Note that

$$
\mathrm{d}_t f(t,T) = \alpha(t,T)\mathrm{d}t + \sigma(t,T)W(\mathrm{d}t),
$$

with

$$
\begin{aligned}
\sigma(t,T) &= \sigma_t g(T-t), \\
\alpha(t,T) &= \sigma_t^2 g(T-t)c(t;t,T).
\end{aligned}
$$

4.1.3 Completeness of the Market

It is easy to see that

$$
\begin{aligned}
\tilde{P}(t,T) := \frac{P(t,T)}{\exp\left\{\int_0^t r_s \mathrm{d}s\right\}} = P(0,T)\exp\Big\{ &-\int_0^t \sigma_u c(u;u,T)W(\mathrm{d}u) \\
&-\frac{1}{2}\int_0^t \sigma_u^2 c(u;u,T)^2 \mathrm{d}u \Big\},
\end{aligned}
$$

so we have

$$
\begin{aligned}
P(t,T) = P(0,T)&\exp\left\{-\frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)\mathrm{d}u + \int_0^t \mu_s \mathrm{d}s\right\} \\
&\times \exp\left\{\int_{-\infty}^0 \sigma_u c(u;0,t)W(\mathrm{d}u) - \int_0^t \sigma_u c(u;t,T)W(\mathrm{d}u)\right\}
\end{aligned}
$$

and

$$
\tilde{P}(t,T) = P(0,T)\exp\left\{-\int_0^t \sigma_u c(u;u,T)W(\mathrm{d}u) - \frac{1}{2}\int_0^t \sigma_u^2 c^2(u;u,T)\mathrm{d}u\right\}.
$$

Therefore,

$$\mathrm{d}\tilde{P}(t,T) = -\tilde{P}(t,T)\sigma_t c(t;t,T)W(\mathrm{d}t), t \geq 0,$$

Let X be a $\mathbf{P}^*$-square integrable, $\mathcal{F}_T$-measurable payoff. Consider the $(\mathcal{F}_t)$-martingale

$$M_t := \mathbf{E}_{\mathbf{P}^*}\left[X \mid \mathcal{F}_t\right], t \geq 0,$$

then, by an extension of Brownian martingale representation theorem, we can write

$$\mathrm{d}M_t = H_t W(\mathrm{d}t),$$

where H is an adapted square integrable process.

Let $\left(\phi_t^0, \phi_t^1\right)$ be a self-financing portfolio built with a bank account and a bond with maturity T; its value process is given by

$$V_t = \phi_t^0 e^{\int_0^t r_s \mathrm{d}s} + \phi_t^1 P(t,T),$$

and, by the self-financing condition, the discounted value process $\tilde{V}$ satisfies

$$\mathrm{d}\tilde{V}_t = \phi_t^1 \mathrm{d}\tilde{P}(t,T).$$

So, if we take

$$\phi_t^1 = -\frac{H_t}{\tilde{P}(t,T)\sigma_t c(t;t,T)}$$

we can replicate X. In particular the bond with maturity T^* can be replicated by taking

$$\frac{P(t,T^*)c(t;t,T^*)}{P(t,T)c(t;t,T)}$$

bonds with maturity time $T \geq T^*$.

4.1.4 Examples

Example 1. With $g(t) = e^{-bt}$, $\sigma_u = \sigma$ and $\mu = a$, we have

$$r_t = r_0 e^{-bt} + a(1 - e^{-bt}) + e^{-bt}\int_0^t e^{bs}\sigma W(\mathrm{d}s),$$

$$P(t,T) = \exp\left(A(t,T) + aB(t,T) - r_t B(t,T)\right),$$

with

$$B(t,T) = \frac{1}{b}(1 - e^{-b(T-t)})$$

and

$$A(t,T) = \frac{\sigma^2}{2}\int_t^T B(u,T)^2 \mathrm{d}u - a(T-t).$$

Then,

$$\mathrm{var}\left(-\frac{1}{T-t}\log P(t,T)\right) = \frac{\sigma^2}{2b^3}\frac{(1-e^{-b(T-t)})^2}{(T-t)^2} \sim T^{-2},$$

when $T \to \infty$, and the corresponding instantaneous forward rates and their variance are given by

$$f(t,T) = -\frac{\sigma^2}{2b^2}\left(1 - e^{-b(T-t)}\right)^2 + \sigma e^{-b(T-t)}(r_t - a) + a.$$
$$\mathrm{var}\,(f(t,T)) = \frac{\sigma^2}{2b}e^{-2b(T-t)} \sim e^{-2bT},$$

when $T \to \infty$. Moreover the volatility of the forward rates is given by $\sigma(t,T) = \sigma e^{-b(T-t)}$ and this is not too realistic.

Example 2. Assume that $\sigma_t = \sigma \mathbf{1}_{\{t \geq 0\}}$ and

$$g(t) = e^{-b(t)}\int_0^t e^{bs}\beta s^{\beta-1}\mathrm{d}s,$$

for $\beta \in (0, 1/2)$. Then

$$\mathrm{var}\,(f(t,T)) = \int_{-\infty}^t \sigma_u^2 g^2(T-u)\mathrm{d}u \sim T^{2\beta-2}.$$

And the volatility of the forward rates are given by

$$\sigma(t,T) = \sigma^2 g(T-t) \sim T^{\beta-1},$$

when $T \to \infty$, that is more realistic (see [20, Sect. 4.1] and also [2]) than the exponential decay in the Vasicek model. For $\beta \in (-1/2, 0)$ consider the memory function

$$g(t) = e^{-bt}t^\beta + \beta\int_0^t \left(e^{-b(t-u)} - e^{-bt}\right)u^{\beta-1}\mathrm{d}u,$$

and then

$$g(t) \sim t^{\beta-1}$$

when $x \to \infty$, in such a way that we obtain analogous asymptotic results to the previous case.

4.1.5 The Analog of a CIR Model

One of the drawbacks of the previous model is that it allows for negative short rates. An obvious way of avoiding this is to take

$$r_t = \sum_{i=1}^{d} \left(\int_0^t g(t-s)\sigma_s \mathrm{d}W_i(s) \right)^2 + r_0, \quad t \geq 0, r_0 > 0.$$

where $((W_i)_{1 \leq i \leq d})$ is a Brownian motion in $\mathbf{R}^d$.

Bond Prices

Given

$$r_t = \sum_{i=1}^{d} \int_0^t \int_0^t g(t-u)g(t-v)\sigma_s\sigma_u \mathrm{d}W_i(u)\mathrm{d}W_i(v),$$

(where by simplicity we take $r_0 = 0$), we have

$$\begin{aligned} \int_t^T r_s \mathrm{d}s = & \sum_{i=1}^{d} \int_0^t \int_0^t \sigma_u\sigma_v c_2(u, v; t, T)\mathrm{d}W_i(u)\mathrm{d}W_i(v) \\ & +2\sum_{i=1}^{d} \int_0^t \int_t^T \sigma_u\sigma_v c_2(u, v; u, T)\mathrm{d}W_i(u)\mathrm{d}W_i(v) \\ & +\sum_{i=1}^{d} \int_t^T \int_t^T \sigma_u\sigma_v c_2(u, v; u \vee v, T)\mathrm{d}W_i(u)\mathrm{d}W_i(v), \end{aligned}$$

with $c_2(u, v; t, T) := \int_t^T g(s-u)g(s-v)\mathrm{d}s$. Then, using this, we have

$$
\begin{aligned}
P(0,T) &= \mathbf{E}\left[\exp\left\{-\int_0^T r_s \mathrm{d}s\right\}\right] \\
&= \prod_{i=1}^{d} \mathbf{E}\left[\exp\left\{-T\int_0^1\int_0^1 \sigma_{Tu}\sigma_{Tv}c_2(Tu,Tv;T(u\vee v),T)\mathrm{d}W_i(u)\mathrm{d}W_i(v)\right\}\right] \\
&= d\,(2T)^{-d/2},
\end{aligned}
$$

where $d\,(\lambda)$ is the Fredholm determinant

$$
d(\lambda) = 1 + \sum_{n=1}^{\infty}\frac{\lambda^n}{n!}\int_0^1\cdots\int_0^1 \begin{vmatrix} R\,(s_1,s_1) & \cdots & R\,(s_1,s_n) \\ \vdots & & \vdots \\ R\,(s_n,s_1) & \cdots & R\,(s_n,s_n)\end{vmatrix} \mathrm{d}s_1\cdots\mathrm{d}s_n
$$

where

$$
R\,(u,v) = \sigma_{Tu}\sigma_{Tv}c_2(Tu,Tv;T(u\vee v),T).
$$

Example 3. Assume that $g(t) = \mathbf{1}_{\{t\geq 0\}}$ and $\sigma_t = \sigma$. Then r_t is a squared Bessel process of dimension d (see, for instance, [28]), and

$$
R\,(u,v) = \sigma^2 T(1-(u\vee v)),
$$

consequently

$$
P(0,T) = \left(\cosh(\sqrt{2}\sigma T)\right)^{-\frac{d}{2}} = \frac{2^{\frac{d}{2}}}{\left(e^{\sqrt{2}\sigma T}+e^{-\sqrt{2}\sigma T}\right)^{\frac{d}{2}}},
$$

see [37] for the calculations of the Fredholm determinant. Another procedure to calculate the Fredholm determinants is given in [29], where it is shown that provided the kernel $R\,(u,v)$ is of the form

$$
\begin{aligned}
R\,(u,v) &= M(u\vee v)N(u\wedge v) \\
d(\lambda) &= B_\lambda(1),
\end{aligned}
$$

and therefore

$$
P(0,T) = (B_{2T}(1))^{-\frac{d}{2}},
$$

where, in our case of having $M(t) = \sigma^2 T(1-t)$ and $N(t) = 1$, we obtain

$$
B_\lambda(t) = \sigma^2 T^2\left((1-t)\frac{e^{\sigma\sqrt{\lambda T}t}-e^{-\sigma\sqrt{\lambda T}t}}{\sigma\sqrt{\lambda T}} + \frac{e^{\sigma\sqrt{\lambda T}t}+e^{-\sigma\sqrt{\lambda T}t}}{\left(\sigma\sqrt{\lambda T}\right)^2}\right).
$$

Note that we can consider squared Bessel processes of dimension $d \geq 0$, where d is not necessarily an integer; see [28] and Corollary 6.2.5.5 therein. Due to the fact that discount values are in close form under the model, a calibration performs very fast.

Example 4. Another interesting example is the classical CIR model. In such a case

$$\begin{aligned}R(u, v) &= \sigma^2 \int_{T(u \vee v)}^{T} e^{-b(s-u)} e^{-b(s-v)} \mathrm{d}s = \frac{\sigma^2}{2b} e^{bT((u \wedge v)-1)} (e^{-bT((u \vee v)-1)} - e^{bT((u \vee v)-1)}) \\ &= M(u \vee v) N(u \wedge v),\end{aligned}$$

where

$$M(t) = \frac{\sigma}{\sqrt{2b}} \left(e^{-bT(t-1)} - e^{bT(t-1)} \right), \qquad \text{and} \qquad N(t) = \frac{\sigma}{\sqrt{2b}} e^{bT(t-1)}.$$

We obtain

$$\begin{aligned}B_{2T}(1) = \frac{1}{2\sqrt{b^2 + 2\sigma^2}} \Big(&(b + \sqrt{b^2 + 2\sigma^2}) e^{T(-b+\sqrt{b^2+2\sigma^2})} \\ &+ (-b + \sqrt{b^2 + 2\sigma^2}) e^{-T(b+\sqrt{b^2+2\sigma^2})} \Big).\end{aligned}$$

4.2 Models in Energy Markets

Like in other traditional commodities or stock markets, in the electricity market one finds trade in spot, forward/futures contracts as well as European options written on these (see [33, Capt. 1] for the definition and terminology of these contracts). Despite this parallelism, the distinctive features of the electricity market lead to specific problems of pricing and hedging. Let us mention two examples of such features. On the one hand, power market trades in contracts which deliver power over a delivery period. This adds an extra dimension to the models for forward dynamics which generally depend only on the current time and the maturity of the contract. On the other hand, the electricity spot cannot be stored directly except via reservoirs for hydro-generated power, or large and expensive batteries. This implies that prices may vary significatively when demand increases, for instance, due to a temperature drop. Moreover, due to the non-storability issue, the electricity spot cannot be held in a portfolio. Hence, the usual *buy-and-hold* hedging arguments break down, and the requirement of being a martingale under an *equivalent martingale measure* (EMM) is not necessary. Similarly, from a liquidity point of view, it would be possible to use non-martingales for modelling forward prices since in many emerging electricity markets, one may not be able to find any buyer to get rid of a forward contract, nor a seller when one wants to enter into one. Thus the illiquidity prevents possible arbitrage opportunities from being exercised.

These features, along with empirical evidence (see [16, 27, 36]) and statistical studies (see [31]), point to random field models in time and space which, in addition, allow for stochastic volatility. We present below two examples of modelling spot and forward prices via ambit processes; these models grant rich flexibility and account for some of the stylized features in the context of energy markets. We note here that since spot prices are determined by supply and demand, strong mean reversion can be observed; the spot prices have clear deterministic patterns over the year, week and intra-day.

4.2.1 Modelling Spot Prices

In [5] the log-spot price $Y.$ is modelled by means of the *Lévy semistationary processes* ($\mathcal{LSS}$) presented in Sect. 2, *i.e.* processes of the form

$$Y_t := \mu + \int_{-\infty}^{t} g(t-s)\sigma_s \mathrm{d}L_s + \int_{-\infty}^{t} q(t-s)a_s \mathrm{d}s, \tag{6}$$

where μ is a constant; $(L_t)_{t\in\mathbf{R}}$ is a two-sided Lévy process; g and q are nonnegative deterministic functions on $\mathbf{R}$, with $g(t) = 0 = q(t)$ for $t \leq 0$; and $\sigma.$ and $a.$ are two càdlàg processes. The $\mathcal{LSS}$ are analytically tractable and encompasses some classical models, as that of Schwartz [36], along with a wider class of continuous-time autoregressive moving-average (*CARMA*) processes. Note that in (6) the log-spot price is modelled directly, as opposed to traditional approaches that focus on modelling the dynamics of the spot price.

Consider a forward contract stating the agreement to deliver electricity at time T, for a predetermined price $F_t(T)$, fixed today but payable at T with no other cash flow at $t < T$. This price is referred to as *forward price*, and it is fixed in such a way that the price of the contract, at the issue time t, is zero. Then by definition

$$0 = \mathbf{E}_{\mathbf{P}^*}\left[\exp\left\{-\int_t^T r_u \mathrm{d}u\right\}(\exp\{Y_T\} - F_t(T))\Big|\,\mathcal{F}_t\right].$$

From this equation and the abstract Bayes' rule (see [33, Lemma A.1.4]), which links the risk-neutral measure $\mathbf{P}^*$ with the T-forward measure

$\mathbf{P}^T$, we get, provided integrability conditions on $\exp\{Y_T\}$,

$$F_t(T) = \mathbf{E}_{\mathbf{P}^T}[\exp\{Y_T\}|\,\mathcal{F}_t]. \tag{7}$$

As mentioned before, due to the lack of an underlying, any measure $\mathbf{P}^T$ equivalent to $\mathbf{P}$ can be chosen as pricing measure. If we assume that under $\mathbf{P}^T$, the dynamics of the log-spot price is given by 6 with $(L_t)_{t\in\mathbf{R}} = (W_t)_{t\in\mathbf{R}}$ being a two-sided Brownian motion, then for a constant volatility $\sigma_s \equiv 1$, we have the simple expression for the forward price

$$F_t(T) = C(T) \exp\left\{ \int_{-\infty}^{t} g(T-s)\mathrm{d}W_s - \frac{1}{2}\int_{-\infty}^{t} g^2(T-s)\mathrm{d}s \right\}. \tag{8}$$

We refer to [38] for a multivariate version of (6) and a detailed empirical study using data from the European Energy Exchange.

4.2.2 Modelling Forward Prices

In [7] forward prices are modelled directly, rather than modelling the spot price and deducing the forward price from the conditional expectation of the spot at delivery (*cf.* 7). Moreover, as opposed to existing literature, the dynamics of the forward price are not specified; instead, the authors specify an ambit field which explicitly describes the forward price. More precisely, for each maturity T, the deseasonalized log-forward price at time t is modelled by

$$\log F_t(x) := \int_{A(t,x)} g(\xi, t-s, x)\sigma_s(\xi)L(\mathrm{d}\xi, \mathrm{d}s), \tag{9}$$

where the spatial component in (9) models the time to maturity, *i.e.* $x := T - t$, the ambit set is given by $A(t,x) := A_t := \{(\xi, s) : \xi > 0, s \leq t\}$, and the kernel g may be chosen in order to capture the so-called Samuelson effect (see [35]). In addition, the fact that forward contracts close in maturity dates are strongly correlated may be captured by assuming that the volatility is another ambit field, independent of L, and with a kernel warranting that $\mathrm{Cor}(\sigma_t^2(x), \sigma_t^2(\bar{x}))$ is high for values of x and $\bar{x}$ close to 0.

Traditionally, the forward price is modelled as a semimartingale such that there is an E(L)MM under which the price dynamics becomes a (local) martingale. According to [7, Corollary 1], $(F_t(T))_{t\in\mathbf{R}}$ is an $\mathbf{F}^L$-martingale if and only if the kernel g in (9) is deterministic and does not depend on t. For instance, one can consider

$$\log F_t(T-t) = \int_{A_t} \exp\{-\alpha(\xi + T - s)\}\sigma_s(\xi)W(\mathrm{d}\xi, \mathrm{d}s), \tag{10}$$

where $\alpha > 0$ and W a homogeneous Gaussian Lévy basis. Such rather strong condition rules out many interesting more general ambit fields; however, it still includes some *CARMA* and standard models as those of Heath et al. and Audet et al. (see [1, 25], respectively). Nevertheless, it would be possible to use non-martingales for modelling forward prices without given place to arbitrage opportunities, due to the specific features of electricity markets mentioned above.

Finally, let us mention that (9) induces a model for the log-spot price Y. which is consistent with that in (6). In particular (see [7, Example 2]) the example in (10) leads to

$$Y_t = \int_{-\infty}^{t} \exp\{-\alpha(t-s)\}\mathrm{d}W_s.$$

References

1. Audet, N., Heiskanen, P., Keppo, J., Vehviläinen, I.: Modelling electricity forward curve a dynamics in the Nordic Market. In: Bunn, D.W. (eds.) Modelling Prices in Competitive Electricity Markets, pp. 251–265. Wiley, Chichester (2004)
2. Backus, D.K., Zin S.E.: Long-memory inflation uncertainty: evidence from the term structure of interest rates. J. Money Credit Bank **25**, 681–700 (1995)
3. Barndorff-Nielsen, O.E., Corcuera, J.M., Podolskij, M.: Power variation for Gaussian processes with stationary increments. Stoch. Proc. Appl. **119**, 1845–1865 (2009)
4. Barndorff-Nielsen, O.E., Corcuera, J.M., Podolskij, M., Woerner, J.H.C.: Bipower variation for Gaussian processes with stationary increments. J. Appl. Probab **46**, 132–150 (2009)
5. Barndorff-Nielsen, O.E., Benth, F.E., Veraart, A.: Modelling Energy Spot Prices by Lévy Semistationary Processes (2010) Available at http://ssrn.com/abstract=1597700
6. Barndorff-Nielsen, O.E., Benth, F.E., Veraart, A.: Ambit Processes and Stochastic Partial Differential Equation (2011) Available at http://ssrn.com/abstract=1597697
7. Barndorff-Nielsen, O.E., Benth, F.E., Veraart, A.: Modelling electricity forward markets by ambit fields (2011) Available at http://ssrn.com/abstract=1938704
8. Barndorff-Nielsen, O.E., Corcuera J.M. and Podolskij M.: Multipower variation for Brownian semistationary processes. Bernoulli **17**(4), 1159–1194 (2011)
9. Barndorff-Nielsen, O.E., Benth, F.E., Veraart, A.: Recent Advances in Ambit Stochastics with a View Towards Tempo-Spatial Stocastic Volatility/Intermittency (2013). Available at arXiv: 1210.1354v1
10. Barndorff-Nielsen, O.E., Corcuera, J.M. and Podolskij, M.: Limit theorems for functionals of higher order differences of Brownian semi-stationary processes. In: Shiryaev, A., Varadhan, N., Presman, S.R.S., Ernst L. (eds.) Prokhorov and Contemporary Probability Theory, pp 69–96. Springer, Berlin (2013)
11. Barndorff-Nielsen, O.E. Graversen, S.E.: Volatility determination in an ambit process setting. J Appl. Probab. **48**(A), 263–275 (2011)
12. Barndorff-Nielsen, O.E., Schmiegel, J.: Ambit processes with applications to turbulence and cancer growth. In: Benth, F.E., Nunno, G.D., Linstrøm, T., Øksendal, B., Zhang, T. (eds.) Stochastic Analysis and Applications: The Abel Symposium 2005, pp. 93–124. Springer, Heidelberg (2007)
13. Barndorff-Nielsen, O.E. Schmiegel, J.: Brownian semistationary processes and volatility/intermittency. In: Albrecher, H., Runggaldier, W., Schachermayer, W. (eds.) Advanced Financial Modelling, volume 8 of Radon Series on Computational and Applied Mathematics, pp. 1–25. Walter de Gruyter, Berlin (2009)
14. Basse, A.: Gaussian moving averages and semimartingales. Electron. J. Probab. **13**(39), 1140–1165 (2008)
15. Bensoussan, A.: Stochastic Navier-Stokes equations. Acta Appl. Math. **38**, 267–304 (1995)
16. Benth, F.E., Cartea, A., Kiesel, R.: Pricing forward contracts in power markets by the certainty equivalence principle: explaining the sign of the market risk premium. J. Bank. Financ. **32**(10), 2006–2021 (2008)
17. Benth, F.E., Suess, A.: Integration Theory for Infinite Dimensional Volatility Modulated Volterra Processes (2013). Available at arXiv: 1303.7143v1
18. Birnir, B.: Existence, uniqueness and statistical theory of turbulents solutions of the stochastic Navier-Stokes equation, in three dimensions an overview. Banach J. Math. Anal. **4**(1), 53–86 (2010)
19. Chong, C., Klüppelberg, C.: Integrability Conditions for Space-time Stochastic Integrals: Theory and Applications (2013). Available at arXiv: 1303.2468v1
20. Comte, F., Renault, E.: Long memory continuous time models. J. Econom. **73**(1), 101–149 (1996)

21. Corcuera, J.M.: Power variation analysis of some long-memory processes. In: Benth, F.E., Nunno, G.D., Linstrøm, T., Øksendal, B., Zhang, T. (eds.) Stochastic Analysis and Applications: The Abel Symposium 2005, pp. 219–234. Springer, Heidelberg (2007)
22. Corcuera, J.M., Nualart, D., Woerner, J.H.C.: Power variation of some integral fractional processes. Bernoulli **12**, 713–735 (2006)
23. Corcuera, J.M., Farkas G., Schoutens, W., Valkeyla, E.: A short rate model using ambit processes. In: Viens, F., Feng, J., Hu, Y., Nualart, E. (eds.) Malliavin Calculus and Stochastic Analysis, A Festschrift in Honor of David Nualart, pp. 525–553. Springer, New York (2013)
24. Corcuera, J.M.; Hedevang, E., Pakkanen, M. and Podolskij, M.: Asymptotic theory for Brownian semi-stationary processes with application to turbulence (2013). Stoch. Proc. Appl. **123**(7), 2552-2574 (2013).
25. Heath, D., Jarrow, R., Morton, A.: Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. Econometrica **60**(1), 77–105 (1992)
26. Hedevang, E.: Stochastic modelling of turbulence with applications to wind energy. Ph.D Thesis (2012). Available at http://pure.au.dk/portal/files/51621098/math_phd_2012_eh.pdf.
27. Hikspoors, S., Jaimungal, S.: Asymptotic pricing of commodity derivatives for stochastic volatility spot models. Appl. Math. Finance **15**(5&6), 449–467 (2008)
28. Jeanblanc, M., Yor, M., Chesney, M.: Mathematical Methods for Financial Markets. Springer Finance, London (2009)
29. Kailath, T.: Fredholm resolvents, Wiener-Hopf equations, and Riccati differential equations. IEEE Trans. Inf. Theory IT-**15**(6), 665–672 (1969)
30. Khoshnevisan, D.: A primer on Stochastic Partial Differential Equations. Lecture Notes in Mathematics, vol. 1962, pp. 1–36. Springer, Berlin (2009)
31. Koekebakker, S., Ollmar, F.: Forward curve dynamics in the Nordic electricity market. Manage. Financ. **31**(6), 73–94 (2005)
32. Mikulevicius, R., Rozoskii, L.B.: Stochastic Navier-Stokes equations for turbulent flows. SIAM J. Math. Anal. **35**(5), 1250–1310 (2004)
33. Musiela, M., Rutkowski, M.: Martingale Methods in Financial Modelling. Stochastic Modelling and Applied Probability, vol. 36. Springer, Heildeberg (2006)
34. Mytnik, L., Neuman E.: Sample Path Properties of Volterra Processes (2011). Available at Arxiv: 1210.1354v1
35. Samuelson, P.: Proof that properly anticipated prices fluctuate randomly. Ind. Manage. Rev. **6**, 41–44 (1965)
36. Schwartz, E.: The stochastic behavior of commodity prices: Implications for valuation and hedging. J. Financ. **52**(3), 923–973 (1997)
37. Varberg, D.E.: Convergence of quadratic forms in independent random variables. Ann. Math. Statist. **37**, 567–576 (1966)
38. Veraart, A.E.D., Veraart, L.A.M.: Modelling electricity day-ahead prices by multivariate Levy semistationary processes. In: Benth, F.E., Kholodnyi, V., Laurence, P. (eds.) Quantitative Energy Finance. Springer, Heidelberg (2013)
39. Walsh, J.B.: An Introduction to Stochastic Partial Differential Equations. Lecture Notes in Mathematics vol. 1180, pp. 265–439. Springer, Berlin (1986)

Some Functional Analytic Tools for Utility Maximization

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Abstract The aim of the chapter is to extend the application of convex duality methods to the problem of maximizing expected utility from terminal wealth. More precisely, we restrict attention to a dual characterization of the value function of this problem and to a static setting. A general scheme to solve this problem is proposed. In the case where the utility function is finite on $\mathbb{R}$, we use the approach, suggested by Biagini and Frittelli, based on using an Orlicz space constructed from an investor's utility function. We reduce the original problem to an optimization problem in this space in a nontrivial way, which allows us to weaken essentially assumptions on the model. We also study the problem of utility maximization with random endowment considered by Cvitanić, Schachermayer, and Wang. Using the space ψL^∞ with a weight function ψ constructed from a random endowment permits us to consider unbounded random endowments. Another important contribution is that in both problems under consideration, we provide versions of the dual problem that are free of singular functionals.

1 Introduction

Convex duality is widely used in expected utility maximization; see [3] for a detailed survey. Here we deal only with the problem of maximizing expected utility from (discounted) terminal wealth in a static setting, i.e., we consider the problem of maximizing the functional

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$$\xi \rightsquigarrow \mathsf{E}\, U(x + \xi + B), \quad \xi \in \mathscr{A}, \tag{1}$$

over a set $\mathscr{A}$ of random variables defined on a probability space $(\Omega, \mathscr{F}, \mathsf{P})$, where $x \in \mathbb{R}$ is a real number, B is a random variable on $(\Omega, \mathscr{F}, \mathsf{P})$, and U is a concave increasing function on $\mathbb{R}$ with values in $\mathbb{R} \cup \{-\infty\}$. Expectation E with respect to P is assumed to be equal to $-\infty$ if it is not defined, i.e., $\mathsf{E}\eta = \lim_{n\to+\infty} \mathsf{E}(\eta \wedge n)$. These objects are interpreted as follows. Consider an agent investing in a financial market. Then

- x is the initial wealth of the agent;
- $\mathscr{A}$ is the set of all possible (discounted) incomes (profits) of the agent. An income comes as a result of applying a certain (admissible) trading strategy. Since a dynamic structure is not involved, we identify elements of $\mathscr{A}$ and strategies corresponding to them;
- B is a random endowment, i.e., an additional cashflow, received by the agent at the terminal time;
- U is a utility function describing preferences of the agent.

Thus, the terminal wealth of the agent resulting from a strategy $\xi \in \mathscr{A}$ is $x + \xi + B$, and the aim of the agent is to maximize the expected utility of her terminal wealth over the set of her admissible strategies, i.e., the problem (1).

A dynamic model may be specified, e.g., by introducing a discounted price process S of the traded assets and a set $\mathscr{H}$ of admissible self-financing trading strategies H. Then

$$\mathscr{A} = \left\{ \xi = \int_0^T H_t \,\mathrm{d}S_t \colon H \in \mathscr{H} \right\}, \tag{2}$$

where $[0, T]$ is a trading period. The idea of reduction from a dynamic model to a static model of type (1) goes back to Pliska [18] who first used duality methods in utility maximization. Advantages of this reduction were fully demonstrated in the seminal paper by Kramkov and Schachermayer [16], who solved the problem of maximizing the expected utility of terminal wealth in the framework of a general incomplete semimartingale model of a financial market deducing their main results from corresponding results in a static model.

In this paper we consider only the question of a dual characterization of the value function

$$u(x) = \sup_{\xi\in\mathscr{A}} \mathsf{E}\, U(x + \xi + B)$$

of the problem (1) in terms of the value function of a certain dual problem. More precisely, the value function u, being concave (if $\mathscr{A}$ is convex) and increasing, admits the representation

$$u(x) = \inf_{y\geq 0} [v(y) + xy] \tag{3}$$

with a convex function v. The question is to find a dual problem whose value function is v. A standard way of doing this includes two steps. The original optimization problem (1) is defined on the space L^0 of all (equivalence classes of) random variables, which, typically, is not a locally convex space. So the first step is

[A] To embed the problem into an appropriate space, more precisely, to show that the value function $u(x)$ does not change if, in its definition, the set $\mathscr{A}$ is replaced by $(\mathscr{A} - L^0_+) \cap L$, where L is an appropriate space (say, a Banach space)

Then the second step is

[B] To apply the Fenchel-type duality theorem which allows us to state a dual optimization problem in the dual space L' and to establish dual relations between the value functions in the primal and dual problems

However, we would like to mention an additional step which is desirable. It often happens that the dual space L' contains singular functionals (e.g., finitely additive measures if $L = L^\infty$) entering the dual problem. So, the third step is

[C] To get rid of singular functionals in the dual problem

The main purpose of the paper is to represent the authors' contribution to these issues. In our papers [11, 14, 17], we considered the utility maximization problem in different settings and developed some new tools, mainly from functional analysis, which allowed us to strengthen existing results. See also [10] for a special result connected to Step C. Here we consider two problems of utility maximization corresponding to [14, 17] and present some of these tools in a refined form:

- Concerning Step A, we provide, following [17], nontrivial arguments allowing us to embed the original problem (1) into an appropriate functional space.
- Concerning Step B, we propose to add an extra variable to the primal problem. This idea was firstly used in [11] and then in [17]. In these two papers a *robust* setting of utility maximization is considered. The word 'robust' means that instead of probability P, we have a family $\mathscr{Q}$ of (subjective) probability measures, and the functional to be maximized is, e.g., of the form

$$\xi \rightsquigarrow \inf_{\mathsf{Q} \in \mathscr{Q}} \big[\mathsf{E}_{\mathsf{Q}}\, U(x + \xi) + \gamma(\mathsf{Q}) \big], \quad \xi \in \mathscr{A}.$$

  For such a problem, an additional variable appears naturally in the dual problem. This justifies the appearance of an extra variable in the primal problem. For the problem (1), this trick seems to be artificial and complicating the arguments. However, we shall see that it has some advantages in our setting as well.
- Concerning Step C, we demonstrate, following [10, 14], how to get rid of singular functionals in the dual problem. For the models under consideration, this difficulty was not overcome in [7, 8]. As a certain drawback of our approach, we should mention that the dual problem after its transformation may include, instead of singular functionals, a new optimization problem in primal variables. This does happen in the models under consideration but does not occur in the setting considered in [11].

We shall study the problem (1) separately in the following two cases:

- U is finite everywhere on $\mathbb{R}$, $B = 0$.
- U is finite on the half-line $(0, +\infty)$ and equals $-\infty$ on $(-\infty, 0)$.

In both cases the convex conjugate V of the utility function U is defined by

$$V(y) = \sup_{x \in \mathbb{R}} [U(x) - xy], \quad y \in \mathbb{R}. \tag{4}$$

In other words, $V(y)$ is the Fenchel conjugate of $-U(-x)$. Then $V : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous convex function with $\operatorname{dom} V \subseteq \mathbb{R}_+$ (under Assumptions 1 or 3 on U), where $\operatorname{dom} f = \{x \in X : f(x) < +\infty\}$ for a function $f : X \to \mathbb{R} \cup \{+\infty\}$. Let us also introduce another notation used throughout the paper. δ_A is the indicator function of a set A in the sense of convex analysis: $\delta_A(x) = 0$ if $x \in A$ and $\delta_A(x) = +\infty$ otherwise. It is always assumed that a probability space $(\Omega, \mathscr{F}, \mathsf{P})$ is given. L^0 is the space of equivalence classes (with respect to equality P-a.s.) of real-valued random variables. $\mathfrak{ba}$ denotes the space of bounded finitely additive measures that vanish on P-null sets, with the total variation norm, i.e., $\mathfrak{ba}$ is the dual of L^∞. Correspondingly, we shall write $\mu(\xi)$ for $\mu \in \mathfrak{ba}$, $\xi \in L^\infty$, instead of $\int \xi \, d\mu$. $\mathfrak{ca}$ stands for the subspace of $\mathfrak{ba}$ consisting of countably additive measures; $\mathfrak{ba}_+$ and $\mathfrak{ca}_+$ are the corresponding cones of measures with nonnegative values. f^* means the Fenchel conjugate of f. Bar over a set means its closure. $\mathbb{1}$ stands for the function that is equal to 1 identically.

2 The First Problem: U Is Finite on $\mathbb{R}$, $B = 0$

In this section we impose the following assumptions on the model.

Assumption 1. A utility function U is an increasing concave function on $\mathbb{R}$ with finite values which is not identically a constant.

Assumption 2. A set $\mathscr{A}$ of possible incomes is a convex cone.

Now the value function u of the problem (1) is of the form

$$u(x) = \sup_{\xi \in \mathscr{A}} \mathsf{E}\, U(x + \xi), \quad x \in \mathbb{R}. \tag{5}$$

It is well known that an appropriate choice of the class of strategies of an agent is a difficult task in this setting. In the semimartingale market model, when $\mathscr{A}$ is given by (2), the class $\mathscr{H} = L(S)$, where $L(S)$ is the set of all predictable H that are integrable with respect to a semimartingale S, is too rich as it typically leads to arbitrage opportunities and hence to a degenerate utility maximization problem in the sense $u(x) \equiv U(+\infty)$. On the other hand, the class $\mathscr{H}^b \subset L(S)$ of strategies H such that the integral process $H \cdot S$ is bounded from below by a constant, may be not sufficiently rich. First, it typically fails to contain the optimal solution. Second, if S is not locally bounded, it may even happen that $\mathscr{H}^b = \{0\}$ and $u(x) \equiv U(x)$, which can be considered as another degenerate case. There are examples (see, e.g., Example 3.2 in [5]) showing that in the last case it is possible to choose the class of strategies in such a way that $U(x) < u(x) < U(+\infty)$. We refer the reader to [4–7, 24, 25] and to the references therein for different approaches to the problem of a "good" choice of $\mathscr{H}$.

Our aim is different. We just want to obtain the dual characterization (3) of the value function using only Assumption 2. Of course, this does not exclude degenerate cases. Realizing Step A, we shall find an equivalent maximization problem in the Orlicz space L^Φ with a Young function Φ constructed from the utility function U similarly to as it was done in [2, 7]. In contrast to these papers, we do not assume that wealth processes (or terminal wealths) are bounded from below by $-cW$, where $W \in L^\Phi$ is a fixed random variable and $c > 0$.

A more general robust utility maximization problem was considered in [17], where an appropriate functional space is an Orlicz space with respect to a family of measures.

We start with necessary facts concerning Orlicz spaces. More details can be found in [19].

Let $(\Omega, \mathscr{F}, \mathsf{P})$ be a probability space and $\Phi\colon \mathbb{R} \to \mathbb{R}_+ \cup \{+\infty\}$ a nonzero even lower semicontinuous convex function with $\Phi(0) = 0$ (a Young function). Then the Orlicz space $L^\Phi = L^\Phi(\mathsf{P})$ on $(\Omega, \mathscr{F}, \mathsf{P})$, associated with Φ, is defined by

$$L^\Phi = \{\xi \in L^0(\mathsf{P})\colon \mathsf{E}\,\Phi(\varepsilon\xi) < +\infty \text{ for some } \varepsilon > 0\}.$$

L^Φ is a Banach space (and even a Banach lattice) with respect to the Luxemburg norm N_Φ given by

$$N_\Phi(\xi) = \inf\left\{K > 0\colon \mathsf{E}\,\Phi\left(\frac{\xi}{K}\right) \le 1\right\}, \quad \xi \in L^\Phi.$$

Let Ψ be the Fenchel conjugate of Φ:

$$\Psi(y) = \sup_{x\in\mathbb{R}} [xy - \Phi(x)],$$

then Ψ is also a Young function. Note that the Fenchel conjugate of Ψ is Φ. Another norm on L^Φ, equivalent to N_Φ, is the Orlicz norm $\|\cdot\|_\Phi$:

$$\|\xi\|_\Phi = \sup_{\mathsf{E}\,\Psi(\eta)\le 1} |\mathsf{E}\,\xi\eta|.$$

As an example, let $\Phi(x) = |x|$, then $\Psi(x) = \delta_{[-1.1]}(x)$. Then $L^\Phi = L^1$ and $L^\Psi = L^\infty$; the Luxemburg and Orlicz norms equal each other in both spaces and coincide with the usual norms in L^1 and L^∞, respectively. In general, if Φ takes value $+\infty$, then L^Φ is just the space L^∞, the Luxemburg and Orlicz norms being equivalent to the usual norm in L^∞.

It is assumed below that L^Φ is equipped with the Luxemburg norm. It follows from the general theory of Banach lattices that the norm dual space $(L^\Phi)'$ admits a decomposition into the direct sum

$$(L^\Phi)' = (L^\Phi)'_r \oplus (L^\Phi)'_s,$$

where $(L^\Phi)'_r$ is the band of regular, i.e., order continuous functionals, and $(L^\Phi)'_s$ is the band of singular functionals that are disjoint to every regular functional. $(L^\Phi)'_r$ can be identified with L^Ψ: to every $\eta \in L^\Psi$, there corresponds a functional $\xi \rightsquigarrow \mathsf{E}\,\xi\eta$ on L^Φ, which belongs to $(L^\Phi)'_r$; this correspondence is a linear one-to-one isometry between L^Ψ with the Orlicz norm and $(L^\Phi)'_r$ with the dual norm. In what follows it is convenient to consider the above functional associated with η as the measure with the density η with respect to P. If Φ takes value $+\infty$, then elements of $(L^\Phi)'_s$ are purely finitely additive measures that vanish on P-null sets. We shall deal with the case where Φ takes only finite values. Then elements of $(L^\Phi)'_s$ are characterized by the following property: $\mu \in (L^\Phi)'_s$ if and only if $\mu(\xi) = 0$ for all $\xi \in L^\infty$. A sufficient condition for $(L^\Phi)'_s = \{0\}$ is the Δ_2-condition: there are $x_0 > 0$ and $K > 0$ such that

$$\Phi(2x) \le K\Phi(x), \quad x \ge x_0.$$

Let us return to the utility maximization problem. Define a Young function Φ by

$$\Phi(x) = -U(-|x|) + U(0), \quad x \in \mathbb{R}. \tag{6}$$

Put $\mathscr{C} = (\mathscr{A} - L^0_+) \cap L^\Phi$ and introduce the set

$$\mathscr{R} = \{\mu \in (L^\Phi)' \colon \mu(\mathbb{1}) = 1 \text{ and } \mu(\xi) \le 0 \text{ for every } \xi \in \mathscr{C}\}$$

of "separating" functionals. Note that functionals in $\mathscr{R}$ are positive since any negative random variable from L^Φ belongs to $\mathscr{C}$. In particular, if $\mu \in \mathscr{R}$, then μ^r and μ^s are positive, where $\mu = \mu^r + \mu^s$ is the decomposition of μ into the sum of regular and singular components.

Here is the main result of this section. As usual, min stands for the infimum that is attained and $\min\varnothing = +\infty$. Recall that V is defined in (4).

Theorem 1. *Let Assumptions 1 and 2 be satisfied. Then*

$$u(x) = \min_{y\ge 0}\,[v(y) + xy], \quad x \in \mathbb{R}, \tag{7}$$

where $v(0) = V(0)$ *and*

$$v(y) = \min_{\mu\in\mathscr{R}}\left[y\|\mu^s\| + \mathsf{E}\,V\left(y\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathsf{P}}\right)\right] \tag{8}$$

$$= \min_{\mathsf{Q}\in\mathscr{Q}}\left[ya(\mathsf{Q}) + \mathsf{E}\,V\left(y\frac{\mathrm{d}\mathsf{Q}}{\mathrm{d}\mathsf{P}}\right)\right], \quad y > 0, \tag{9}$$

where

$$\mathcal{Q} = \{\mathsf{Q} \text{ is a probability measure}\colon \mathsf{Q} \ll \mathsf{P},\ \mathrm{d}\mathsf{Q}/\mathrm{d}\mathsf{P} \in L^{\Psi},\ a(\mathsf{Q}) < \infty\},$$

$$a(\mathsf{Q}) = \sup_{\xi \in \mathscr{C}:\mathsf{E}\, U(-\xi^-) > -\infty} \mathsf{E}_{\mathsf{Q}}\xi. \tag{10}$$

It is easy to express Ψ via V and, thus, to rewrite the condition $\mathrm{d}\mathsf{Q}/\mathrm{d}\mathsf{P} \in L^{\Psi}$ in the definition of $\mathcal{Q}$ in terms of V.

The proof of the theorem uses several lemmas. The first one provides a reduction of our optimization problem to a similar one in the space L^{Φ}.

Lemma 1. *We have*

$$u(x) = \sup_{\xi \in \overline{\mathscr{C}},\ U(x+\xi) \in L^1} \mathsf{E}\, U(x+\xi),$$

where $\overline{\mathscr{C}}$ is the closure of $\mathscr{C}$ in the norm of L^{Φ}.

Proof. Define

$$\mathscr{A}_x = \{\xi \in \mathscr{A}\colon \mathsf{E}\, U(x - \xi^-) > -\infty\}.$$

Obviously, the supremum in (5) can be taken over $\mathscr{A}_x$. Put also

$$\mathscr{C}_x = (\mathscr{A}_x - L^0_+) \cap L^{\Phi},$$

so we have

$$\begin{aligned} u(x) &= \sup_{\xi \in \mathscr{A}_x} \mathsf{E}\, U(x+\xi) = \sup_{\xi \in \mathscr{A}_x,\ n \in \mathbb{N}} \mathsf{E}\, U(x + \xi \wedge n) \\ &= \sup_{\xi \in \mathscr{C}_x,\ U(x+\xi) \in L^1} \mathsf{E}\, U(x+\xi). \end{aligned} \tag{11}$$

Next, we show that the supremum in the last expression does not change if $\mathscr{C}_x$ is replaced by $\overline{\mathscr{C}_x}$. To this end, let $F(\xi) = -\mathsf{E}\ U(x+\xi)$, $\xi \in L^{\Phi}$, be a convex function on L^{Φ}. Let also B be the open ball of radius $1/2$ (in the Luxemburg norm) centered at $0 \in \mathscr{C}_x$ in L^{Φ}. By the definition of the Luxemburg norm, $\mathsf{E}\, \Phi(2\xi) \le 1$, $\xi \in B$. Hence, for $\xi \in B$,

$$\begin{aligned} F(\xi) = -\int U(x+\xi)\, \mathrm{d}\mathsf{P} &= \int_{\{\xi \ge -|x|\}} + \int_{\{\xi < -|x|\}} \\ &\le |U(x - |x|)| + \int_{\{\xi < -|x|\}} \big(\Phi(2\xi) - U(0)\big)\, \mathrm{d}\mathsf{P} \\ &\le |U(x - |x|)| + |U(0)| + 1, \end{aligned}$$

i.e., F is bounded from above on B. Applying Lemma 2 below, we obtain that $\mathscr{C}_x$ can be replaced by $\overline{\mathscr{C}_x}$ in (11), i.e.,

$$u(x) = \sup_{\xi\in\overline{\mathscr{C}_x},\ U(x+\xi)\in L^1} \mathsf{E}\, U(x+\xi).$$

Finally, we shall prove that, for every $x \in \mathbb{R}$,

$$\{\xi \in \overline{\mathscr{C}_x},\ U(x+\xi) \in L^1\} = \{\xi \in \overline{\mathscr{C}},\ U(x+\xi) \in L^1\}.$$

The inclusion $\subseteq$ is trivial. Let $\xi \in \overline{\mathscr{C}}$ and $U(x+\xi) \in L^1$. Our goal is to show that $\xi \in \overline{\mathscr{C}_x}$. Choose any $\alpha \in (0,1)$. Since $\mathscr{C}$ is a cone, its closure $\overline{\mathscr{C}}$ is also a cone. Therefore, $\alpha\xi \in \overline{\mathscr{C}}$ and there are $\xi_n \in \mathscr{C}$ such that $N_\Phi(\xi_n - \alpha\xi) \underset{n\to\infty}{\longrightarrow} 0$ and, hence, $N_\Phi(\xi_n^- - \alpha\xi^-) \underset{n\to\infty}{\longrightarrow} 0$. We have

$$\xi_n^- - x = \alpha(\xi^- - x) + (1-\alpha)\left(\frac{\xi_n^- - \alpha\xi^-}{1-\alpha} - x\right),$$

hence, due to the convexity of Φ,

$$\Phi(\xi_n^- - x) \le \alpha\Phi(\xi^- - x) + (1-\alpha)\Phi\left(\frac{\xi_n^- - \alpha\xi^-}{1-\alpha} - x\right).$$

The assumption $U(x+\xi) \in L^1$ implies $\Phi(\xi^- - x) \in L^1$. It is also easy to show that, for n large enough, $\Phi\left(\frac{\xi_n^- - \alpha\xi^-}{1-\alpha} - x\right) \in L^1$. Thus, $\Phi(\xi_n^- - x) \in L^1$ and, hence, $\xi_n \in \mathscr{C}_x$. This implies $\alpha\xi \in \overline{\mathscr{C}_x}$. Passing to the limit as $\alpha \uparrow 1$, we get $\xi \in \overline{\mathscr{C}_x}$. □

Lemma 2. *Let X be a topological vector space, A a convex subset of X, and let $F\colon X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. If F is bounded from above in a neighborhood of a point $a \in \bar{A}$, then*

$$\inf_{x\in A} F(x) = \inf_{x\in\bar{A}} F(x).$$

The proof is based on standard arguments and is omitted.

The next lemma will allow us to get rid of singular functionals. The connection between the existence of a positive element μ such that $\vartheta + \mu$ belongs to the polar cone K° and the boundedness of ϑ on a part of K was firstly established in Rokhlin [23, Lemma 2.5] for the space L^∞. A quantitative version of Rokhlin's result was proved in the first version of [11]; however, it did not appear in the final version and has been published only recently in [12, Lemma 1]. Lemma 8 below taken from [14] is its generalization. The case of Orlicz spaces needs different arguments comparing to that of L^∞. Lemma 3 firstly appeared in [10] in connection with Theorem 21 in [7], where the dual problem has a form similar to (8). In the lemma Φ is an arbitrary Young function with finite values.

Lemma 3. *Let K be a convex cone in L^Φ and $K^\circ = \{\mu \in (L^\Phi)' \colon \mu(\xi) \leq 0$ for every $\xi \in K\}$. Fix $\vartheta \in (L^\Phi)'$. Then*

$$\min\{\|\mu\| \colon \mu \in (L^\Phi)'_s,\ \mu \geq 0,\ \vartheta + \mu \in K^\circ\} = \sup_{\xi \in K,\ \mathsf{E}\,\Phi(\xi^-) < \infty} \vartheta(\xi).$$

Remark 1. $\|\mu\|$ is the dual norm of $\mu \in (L^\Phi)'$. Though we consider the Luxemburg norm on L^Φ, the dual norm of a positive singular functional μ is the same if L^Φ is equipped with the Orlicz norm and satisfies

$$\|\mu\| = \sup\{\mu(\xi) \colon \xi \geq 0,\ \xi \in L^\Phi,\ \mathsf{E}\,\Phi(\xi) < +\infty\}, \tag{12}$$

see [19, Proposition IV.3.4], [7, Proposition 11].

Proof. Put $\chi(\xi) = \inf\{\alpha > 0 \colon \mathsf{E}\Phi(\xi^-/\alpha) < \infty\}$, $\xi \in L^\Phi$. It is easy to see that χ is a convex continuous function, and its Fenchel conjugate $\chi^*(\mu) = \sup_{\xi \in L^\Phi}[\mu(\xi) - \chi(\xi)]$, $\mu \in (L^\Phi)'$, satisfies $\chi^* = \delta_{-B}$, where $B = \{\mu \in (L^\Phi)'_s \colon \mu \geq 0,\ \|\mu\| \leq 1\}$. Applying Rockafellar's version [20] of the Fenchel duality theorem, we get $(\delta_K + r\chi)^* = \delta_{K^\circ - rB}$ for every $r > 0$. The latter is equivalent to the statement of the lemma. □

In the proof of Theorem 1 we shall need to compute the Fenchel conjugate of a convex integral functional of a special form on the product of Orlicz spaces. This can be done directly using Theorem 2.6 in [15]; the idea goes back to [22] for the case of L^∞. We prefer to discuss this question in a greater generality to establish a relationship with the notion of f-divergence. In our opinion, an information theoretical spirit of this notion sheds more light on the subject.

Let $f \colon \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function with $\operatorname{dom} f \subseteq \mathbb{R}_+$. Let g be the Fenchel conjugate of f:

$$g(x) = \sup_{y \in \mathbb{R}} [xy - f(y)], \quad x \in \mathbb{R},$$

and F the Fenchel conjugate of the convex indicator function δ_D, where $D = \{(x, y) \in \mathbb{R}^2 \colon y + g(x) \leq 0\}$:

$$F(s,t) = \sup_{(x,y) \in D} (sx + ty), \quad (s,t) \in \mathbb{R}^2. \tag{13}$$

It can be easily checked (see [9, Lemma 2]) that

$$F(s,t) = \begin{cases} tf\left(\frac{s}{t}\right), & \text{if } s \geq 0 \text{ and } t > 0, \\ s\frac{f(\infty)}{\infty}, & \text{if } s > 0 \text{ and } t = 0, \\ 0, & \text{if } s = 0 \text{ and } t = 0, \\ +\infty, & \text{if } s < 0 \text{ or } t < 0, \end{cases}$$

where

$$\frac{f(\infty)}{\infty} = \lim_{y\uparrow+\infty} \frac{f(y)}{y} = \sup\{x\colon x \in \operatorname{dom} g\}.$$

It follows from the formula for the conjugate of a convex integral functional (see Theorem 2 in [21]) that if μ and ν are finite positive (countably additive) measures, then

$$\sup_{\xi,\eta\in L^\infty:\eta+g(\xi)\le 0} [\mu(\xi) + \nu(\eta)] = \mathsf{E}\, F\left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathsf{P}}, \frac{\mathrm{d}\nu}{\mathrm{d}\mathsf{P}}\right). \tag{14}$$

If $\operatorname{dom} f \supseteq (0, +\infty)$, the right-hand side of the above equality is Csiszár's f-divergence of μ and ν. This idea was used in [9] to define the f-divergence $\mathscr{I}_f(\mu, \nu)$ of finitely additive $\mu,\ \nu \in \mathfrak{ba}$ as the left-hand side of (14). More generally, if Φ_1 and Φ_2 are Young functions, then one can define the f-divergence of continuous linear functionals $\mu \in (L^{\Phi_1})'$, $\nu \in (L^{\Phi_2})'$ by

$$\mathscr{I}_f(\mu, \nu) = \sup_{\xi\in L^{\Phi_1},\ \eta\in L^{\Phi_2}:\eta+g(\xi)\le 0} [\mu(\xi) + \nu(\eta)]. \tag{15}$$

Being the Fenchel conjugate of the indicator function of the set $\{(\xi, \eta) \in L^{\Phi_1} \times L^{\Phi_2}\colon \eta + g(\xi) \le 0\}$, the f-divergence is a convex $\sigma((L^{\Phi_1})' \times (L^{\Phi_2})', L^{\Phi_1} \times L^{\Phi_2})$-lower semicontinuous function on $(L^{\Phi_1})' \times (L^{\Phi_2})'$. The properties of the f-divergence that we need are gathered in the following lemma.

Lemma 4. *Let Φ_1 and Φ_2 be arbitrary Young functions.*

- *If μ and ν are regular functionals, then*

$$\mathscr{I}_f(\mu, \nu) = \mathsf{E}\, F\left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathsf{P}}, \frac{\mathrm{d}\nu}{\mathrm{d}\mathsf{P}}\right).$$

- $\mathscr{I}_f(\mu, \nu) = \mathscr{I}_f(\mu^r, \nu^r) + \mathscr{I}_f(\mu^s, \nu^s)$.

Assume that $\Phi_1 = \Phi$, where Φ is given by (6) *and U satisfies Assumption 1, $\Phi_2(x) \equiv |x|$, $f = V$, where V is defined in* (4).

- *If μ is a positive singular functional on L^Φ, then $\mathscr{I}_f(\mu, 0) = \|\mu\|$.*
- *The level sets of the function $\mu \rightsquigarrow \mathscr{I}_f(\mu, \mathsf{P})$ are weakly* compact, i.e., for every $c \in \mathbb{R}$, the set $\{\mu \in (L^\Phi)'\colon \mathscr{I}_f(\mu, \mathsf{P}) \le c\}$ is a compact in the topology $\sigma((L^\Phi)', L^\Phi)$.*

A more general case of Orlicz spaces with respect to a family of probability measures is considered in [17].

Remark 2. A measure $\mu \in \mathfrak{ca}_+$ can be identified with a regular functional on L^{Φ_1} if $\mathrm{d}\mu/\mathrm{d}\mathsf{P} \in L^{\Psi_1}$; similarly, $\nu \in \mathfrak{ca}_+$, if $\mathrm{d}\nu/\mathrm{d}\mathsf{P} \in L^{\Psi_2}$, can be identified with a regular functional on L^{Φ_2}. The first statement of Lemma 4 shows that the quantity

$\mathscr{I}_f(\mu,\nu)$ does not depend on the choice of Orlicz spaces L^{Φ_1} and L^{Φ_2}. In general, the notation $\mathscr{I}_f(\mu,\nu)$ may be ambiguous if underlying spaces L^{Φ_1} and L^{Φ_2} are not specified. For example, the expressions $\mathscr{I}_f(\mu,0)$ and $\mathscr{I}_f(\mu,\mathsf{P})$, given $\mu \in (L^{\Phi_1})'$, may differ if 0 or P are interpreted as functionals on L^∞ or on L^1 (of course, if μ is not regular).

Proof. For regular $\mu \in (L^{\Phi_1})'_r$, $\nu \in (L^{\Phi_2})'_r$, we have

$$\begin{aligned}\mathscr{I}_f(\mu,\nu) &= \sup_{\xi\in L^{\Phi_1},\ \eta\in L^{\Phi_2}:\eta+g(\xi)\le 0} [\mu(\xi)+\nu(\eta)] \\ &\ge \sup_{\xi\in L^{\infty},\ \eta\in L^{\infty}:\eta+g(\xi)\le 0} [\mu(\xi)+\nu(\eta)] = \mathsf{E}\,F\left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathsf{P}},\frac{\mathrm{d}\nu}{\mathrm{d}\mathsf{P}}\right),\end{aligned}$$

where the last equality follows from (14). On the other hand, (13) implies that for $\xi \in L^{\Phi_1}$, $\eta \in L^{\Phi_2}$ such that $\eta + g(\xi) \le 0$,

$$F\left(\frac{\mathrm{d}\mu}{\mathrm{d}\mathsf{P}},\frac{\mathrm{d}\nu}{\mathrm{d}\mathsf{P}}\right) \ge \frac{\mathrm{d}\mu}{\mathrm{d}\mathsf{P}}\xi + \frac{\mathrm{d}\nu}{\mathrm{d}\mathsf{P}}\eta.$$

Since $\frac{\mathrm{d}\mu}{\mathrm{d}\mathsf{P}} \in L^{\Psi_1}$, $\frac{\mathrm{d}\nu}{\mathrm{d}\mathsf{P}} \in L^{\Psi_2}$, where Ψ_1 and Ψ_2 are the corresponding conjugate functions, the expression on the right in the previous inequality is integrable, and its expectation is equal to $\mu(\xi)+\nu(\eta)$. The first statement follows.

The proof of the second statement goes along the same lines as the proofs of similar results in [9, 15, 17, 22].

It follows from the definition that

$$\mathscr{I}_f(\mu,0) = \sup_{\xi\in L^{\Phi},\ \eta\in L^1:\eta\le U(-\xi)} \mu(\xi) = \sup_{\xi\in L^{\Phi}:\Phi(\xi^+)\in L^1} \mu(\xi),$$

and the expression on the right is equal to $\|\mu\|$ because of (12).

It remains to prove the last statement. Let ξ be from the unit ball in L^Φ. Then $\mathsf{E}\,\Phi(\xi) \le 1$, hence $U(-|\xi|) \in L^1$. By Jensen's inequality, $\mathsf{E}U(\xi^-)$ is finite as well. Thus, $U(-\xi) \in L^1$. From the definition of the f-divergence, $\mathscr{I}_f(\mu,\mathsf{P}) \ge \mu(\xi) + \mathsf{E}\,U(-\xi) \ge \mu(\xi) + \mathsf{E}\,U(-|\xi|) = \mu(\xi) + U(0) - \mathsf{E}\,\Phi(|\xi|) \ge \mu(\xi) + U(0) - 1$. This implies that the level sets are norm bounded. The claim follows from the Banach–Alaoglu theorem and the lower semicontinuity of the f-divergence. □

Proof (of Theorem 1). By Lemma 1 and elementary calculations,

$$\begin{aligned}u(x) &= \sup_{\xi\in\overline{\mathscr{C}},\ U(x+\xi)\in L^1} \mathsf{E}\,U(x+\xi) = -\inf_{\xi\in\overline{\mathscr{C}},\ U(x+\xi)\in L^1,\ \eta\ge -U(x+\xi)} \mathsf{E}\,\eta \\ &= -\inf_{(\xi,\eta)\in L^{\Phi}\times L^1} \left[\mathsf{E}\,\delta_{\{\eta\ge -U(\xi)\}} + \delta_{x+\overline{\mathscr{C}}}(\xi) + \mathsf{E}\,\eta\right] \\ &= -\inf_{(\xi,\eta)\in L^{\Phi}\times L^1} [\varphi_1(\xi,\eta)+\varphi_2(\xi,\eta)],\end{aligned}$$

where

$$\varphi_1(\xi,\eta) = \mathsf{E}\,\delta_{\{\eta \geq -U(\xi)\}}, \quad \varphi_2(\xi,\eta) = \delta_{x+\overline{\mathscr{C}}}(\xi) + \mathsf{E}\,\eta$$

We assert that the assumptions of the Attouch–Brezis version [1] of the Fenchel duality theorem are satisfied for the pair (φ_1, φ_2). Indeed, it is enough to check that $\{(\xi,\eta) \in L^\Phi \times L^1 : N_\Phi(\xi) < 1/2,\ \mathsf{E}\,|\eta| < 1/2\} \subseteq \operatorname{dom}\varphi_1 - \operatorname{dom}\varphi_2$. Put $\xi_1 = x+\xi$, $\xi_2 = x$, $\eta_1 = -U(\xi_1)$, $\eta_2 = \eta_1 - \eta$. It was proved in the proof of Lemma 1 that $\eta_1 \in L^1$. Hence, $(\xi_1,\eta_1) \in \operatorname{dom}\varphi_1$, $(\xi_2,\eta_2) \in \operatorname{dom}\varphi_2$.

By the theorem mentioned above,

$$u(x) = \min_{\mu \in (L^\Phi)',\ \zeta \in L^\infty} \left[\varphi_1^*(-\mu,-\zeta) + \varphi_2^*(\mu,\zeta)\right].$$

Let us calculate the expression on the right. For $(\mu,\zeta) \in (L^\Phi)' \times L^\infty$,

$$\varphi_2^*(\mu,\zeta) = x\mu(\mathbb{1}) + \delta^*_{\overline{\mathscr{C}}}(\mu) + \delta_{\{\mathbb{1}\}}(\zeta).$$

Next, taking $\Phi_1 = \Phi$, $\Phi_2(x) \equiv |x|$, $f = V$ (and, hence, $g(x) \equiv -U(-x)$), we get from (15) that $\varphi_1^*(-\mu,-\mathbb{1}) = \mathscr{J}_V(\mu,\mathsf{P})$, hence,

$$u(x) = \min_{\mu \in (L^\Phi)'} \left[\mathscr{J}_V(\mu,\mathsf{P}) + x\mu(\mathbb{1}) + \delta^*_{\overline{\mathscr{C}}}(\mu)\right]. \tag{16}$$

Since $\delta^*_{\overline{\mathscr{C}}} = \delta^*_{\mathscr{C}}$, we have $\delta^*_{\overline{\mathscr{C}}} = \delta_{\mathscr{K}}$, where $\mathscr{K} = \bigcup_{y \geq 0} \mathscr{K}_y$ consists of positive functionals,

$$\mathscr{K}_y = \{\mu \in (L^\Phi)' : \mu(\mathbb{1}) = y \text{ and } \mu(\xi) \leq 0 \text{ for every } \xi \in \mathscr{C}\}.$$

Thus, (16) implies (7) with

$$v(y) = \inf_{\mu \in \mathscr{K}_y} \mathscr{J}_V(\mu,\mathsf{P}), \quad y \geq 0.$$

Since the sets $\mathscr{K}_y$ are weakly* closed, the infimum in the previous formula is attained by the concluding statement of Lemma 4. By the same lemma,

$$\mathscr{J}_V(\mu,\mathsf{P}) = \|\mu^s\| + \mathsf{E}\,V\left(\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathsf{P}}\right).$$

Moreover, $\mathscr{K}_y = y\mathscr{R}$ if $y > 0$, $\mathscr{K}_0$ is a cone and contains only singular functionals. Thus, $v(0) = V(0)$ and we have (8) (consider separately the cases $\mathscr{R} = \varnothing$ and $\mathscr{R} \neq \varnothing$). Finally, since $\mu^r(\mathbb{1}) = \mu(\mathbb{1}) = 1$ for every $\mu \in \mathscr{R}$, μ^r is a probability measure, and (9) follows from Lemma 3. □

Remark 3 (cf. [5]). Assume that every $\xi \in \mathscr{C}$ satisfies

$$\mathsf{E}\,\Phi(\lambda\xi^-) < \infty \quad \text{for every } \lambda > 0,$$

i.e., ξ^- belongs to the Orlicz heart M^Φ (this happens, in particular, if ξ is bounded from below). Then

$$\mathscr{Q} = \{\mathsf{Q} \text{ is a probability measure: } \mathsf{Q} \ll \mathsf{P},\ d\mathsf{Q}/d\mathsf{P} \in L^\Psi,\ \mathsf{E}_\mathsf{Q}\xi \leq 0 \text{ for every } \xi \in \mathscr{C}\},$$

$a(\mathsf{Q}) = 0$ for $\mathsf{Q} \in \mathscr{Q}$, and (9) takes the form

$$v(y) = \min_{\mathsf{Q}\in\mathscr{Q}} \left[\mathsf{E}\,V\left(y\frac{d\mathsf{Q}}{d\mathsf{P}}\right)\right], \quad y > 0.$$

The analysis of the whole proof leads to the following conclusions:

- Since our only aim is to characterize the function v satisfying (7), there is no need to impose further assumptions on U such as differentiability and strict convexity. Also, we do not use such assumptions as finiteness of u or v at some point, etc.
- Due to Lemma 1, we succeed to realize Step A with no assumptions on $\mathscr{A}$ except those imposed in Assumption 2. Note that it is not satisfactory to reduce the original problem to the maximization problem over the set $\mathscr{C}_x \subseteq L^\Phi$ (see (11)) or over its closure because these sets are not cones, which is important at Step B.
- At Step B, we introduced an additional variable η and applied a version of the Fenchel duality theorem to obtain the dual characterization of the value function in (7) and (8). Comparing this part of the proof with a similar proof in [7], we must admit that the only advantage of introducing this extra variable is a short proof of the attainment of the minimum in (8).
- Due to Lemma 3, we succeeded to realize Step C, which was not done in [7]. However, instead of singular functionals we obtain the optimization problem (10) inside the dual problem (9).

3 The Second Problem: U Is Finite on $(0, +\infty)$

In this section we impose the following assumptions on the model.

Assumption 3. A utility function $U\colon \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is an increasing concave function, $U(x)$ is finite on $(0, \infty)$ and right-continuous at 0, $U(x) = -\infty$ if $x < 0$.

Assumption 4. A set $\mathscr{A}$ of possible incomes is a convex cone.

Assumption 5. A contingent claim $|B|$ can be superreplicated, that is, there exist $x \in \mathbb{R}$ and $\xi \in \mathscr{A}$ such that $x + \xi \geq |B|$; moreover, $\mathsf{E}\,U(a\psi) < +\infty$ for some $a > 0$, where $\psi = 1 + |B|$.

It is useful to note that, since U is increasing and concave, Assumption 5 implies $\mathsf{E}\,U(a\psi) < +\infty$ for all $a > 0$.

Now the value function u has the form

$$u(x) = \sup_{\xi \in \mathscr{A}} \mathsf{E}\, U(x + \xi + B), \quad x \in \mathbb{R}.$$

This problem was studied by Cvitanić, Schachermayer, and Wang [8] in the semimartingale market model with strategies whose wealth process is bounded from below. A contingent claim B was assumed to be bounded. In this paper we do not require boundedness assumptions on variables in $\mathscr{A}$ and the contingent claim B. However, we use Assumption 5 which is automatically satisfied if B is bounded. The idea is to reduce the original problem to a problem in the space L^∞ with the weight function ψ. The use of this space, ψL^∞, which is linearly isometric to L^∞, keeps the possibility to use duality approach and allows us to weaken essentially conditions on the model.

It should be mentioned that Hugonnier and Kramkov [13] suggested another approach that allowed them to remove the boundedness assumption on B and to avoid finitely additive measures in the corresponding dual problem as well. Namely, they extended the original optimization problem considering not only the initial capital but also the number of units of random endowments as parameters of the problem.

Let $\mathscr{C}^\psi = (\mathscr{A} - L^0_+) \cap (\psi L^\infty)$. The set of separating functionals is defined by

$$\mathscr{R} = \left\{ \mu \in \mathfrak{ba}_+ \colon \mu\left(\frac{1}{\psi}\right) = 1,\ \mu(\xi) \le 0 \text{ for all } \xi \in \frac{\mathscr{C}^\psi}{\psi} \right\}.$$

Here is the main result of this section. The function V is defined in (4).

Theorem 2. *Let Assumptions 3, 4, and 5 be satisfied and $\mathscr{R} \neq \varnothing$. Then*

$$u(x) = \min_{y \ge 0} [v(y) + xy], \quad x \in \mathscr{M}, \tag{17}$$

where

$$\mathscr{M} = \{x \in \mathbb{R} \colon \textit{there are } \xi \in \mathscr{A} \textit{ and } \varepsilon > 0 \textit{ such that } x + \xi + B \ge \varepsilon\psi\},$$

$v(0) = V(0)$ *and, for* $y > 0$,

$$v(y) = \min_{\mu \in \mathscr{R}} \left\{ \mathsf{E}\left[V\left(\frac{y}{\psi} \frac{\mathrm{d}\mu^r}{\mathrm{d}\mathsf{P}} \right) \right] + y\mu\left(\frac{B}{\psi}\right) \right\} \tag{18}$$

$$= \min_{\mu \in (\mathscr{R} - \mathfrak{ba}_+) \cap \mathfrak{ca}_+} \left\{ \mathsf{E}\left[V\left(\frac{y}{\psi} \frac{\mathrm{d}\mu}{\mathrm{d}\mathsf{P}} \right) \right] + y\mu\left(\frac{B}{\psi}\right) + y g(\mu, B) \right\}, \tag{19}$$

$$g(\mu, B) = \sup_{\eta + \frac{B}{\psi} \in L^\infty_+} \left\{ \mu(\eta) - \inf_{\xi \in \frac{\mathscr{C}^\psi}{\psi}} \operatorname{ess\,sup}\, [(\eta - \xi)\psi] \right\}. \tag{20}$$

Moreover,

$$(\mathscr{R} - \mathfrak{ba}_+) \cap \mathfrak{ca}_+ = \left\{ \mu \in \mathfrak{ca}_+ : \mu(\xi) \le 1 \text{ for every } \xi \in \left(\frac{1}{\psi} + \overline{\left(\frac{\mathscr{C}^\psi}{\psi} \right)} \right) \cap L^\infty_+ \right\},$$

where the closure of $\frac{\mathscr{C}^\psi}{\psi}$ *is taken with respect to the norm of* L^∞.

In the proof of the theorem the following lemmas are used. Their proofs are given in [14].

Lemma 5. *Let Assumptions 4 and 5 be satisfied. Then*

- $\mu(\frac{1}{\psi}) > 0$ *for every* $0 \neq \mu \in \mathscr{K}$, *where*

$$\mathscr{K} = \left\{ \mu \in \mathfrak{ba}_+ : \mu(\xi) \le 0 \text{ for all } \xi \in \frac{\mathscr{C}^\psi}{\psi} \right\}.$$

- $\mathscr{R}$ *is a convex compact in the topology* $\sigma(\mathfrak{ba}, L^\infty)$;
- $\mathscr{R} = \varnothing$ *if and only if* $\frac{\mathscr{C}^\psi}{\psi} = L^\infty$.

Lemma 6. *Let Assumptions 4 and 5 be satisfied and* $\mathscr{R} \neq \varnothing$. *Then, for* $\mu \in \mathscr{R} - \mathfrak{ba}_+$,

$$\min_{\nu \in (\mathscr{R} - \mu) \cap \mathfrak{ba}_+} \nu\left(\frac{B}{\psi}\right) = \sup_{\eta + \frac{B}{\psi} \in L^\infty_+} \{\mu(\eta) - \delta^*_{\mathscr{R}}(\eta)\}.$$

Lemma 7. *Let Assumptions 4 and 5 be satisfied and* $\mathscr{R} \neq \varnothing$. *Then, for* $\eta \in L^\infty$,

$$\inf_{\xi \in \frac{\mathscr{C}^\psi}{\psi}} \operatorname{ess\,sup} (\eta - \xi)\psi = \delta^*_{\mathscr{R}}(\eta).$$

Lemma 8. *Let Assumptions 4 and 5 be satisfied and* $\mathscr{R} \neq \varnothing$. *Let* $\mu \in \mathfrak{ba}$. *The following statements are equivalent:*

- $\mu \in \mathscr{R} - \mathfrak{ba}_+$;
- $\mu(\xi) \le 1$ *for every* $\xi \in \left(\frac{1}{\psi} + \overline{\left(\frac{\mathscr{C}^\psi}{\psi} \right)} \right) \cap L^\infty_+$.

Proof (of Theorem 2). Let $x \in \mathscr{M}$. Then

$$\begin{aligned} u(x) &= \sup_{\xi \in \frac{\mathscr{C}^\psi}{\psi}} \mathsf{E}\, U(x + \xi\psi + B) = \sup_{\xi \in \overline{\left(\frac{\mathscr{C}^\psi}{\psi}\right)}} \mathsf{E}\, U(x + \xi\psi + B) \\ &= \sup_{\xi \in \overline{\left(\frac{\mathscr{C}^\psi}{\psi}\right)},\ U(x+\xi\psi+B) \in L^1} \mathsf{E}\, U(x + \xi\psi + B). \end{aligned} \tag{21}$$

Indeed, the third equality is trivial, and the second one follows from Lemma 2 and the definition of $\mathscr{M}$. As for the first equality, let $\xi \in \mathscr{A}$ be such that $\mathsf{E}\, U(x + \xi + B) > -\infty$. Then $x + \xi + B \geq 0$ and, hence, $\xi \wedge (n\psi) \in \mathscr{C}^\psi$. Now $\lim_n \mathsf{E}\, U(x + \xi \wedge (n\psi) + B) = \mathsf{E}\, U(x + \xi + B)$ by the monotone convergence theorem.

Similar to the proof of Theorem 1, using (21), we represent u in the form

$$u(x) = -\inf_{(\xi,\eta)\in L^\infty\times L^1} \{\varphi_1(\xi,\eta) + \varphi_2(\xi,\eta)\},$$

where $\varphi_1, \varphi_2 \colon L^\infty \times L^1 \to \mathbb{R} \cup \{+\infty\}$ are defined by

$$\varphi_1(\xi,\eta) = \mathsf{E}\, \delta_{\{\eta\geq -U(\xi\psi)\}}(\xi,\eta), \quad \varphi_2(\xi,\eta) = \delta_{\frac{x}{\psi}+\overline{\left(\frac{\mathscr{C}^\psi}{\psi}\right)}+\frac{B}{\psi}}(\xi) + \mathsf{E}\, \eta.$$

For these φ_1, φ_2, and for $x \in \mathscr{M}$, the Attouch–Brezis version [1] of the Fenchel duality theorem can be applied again, which gives

$$u(x) = \min_{(\mu,\zeta)\in \mathfrak{ba}\times L^\infty} \{\varphi_1^*(-\mu,-\zeta) + \varphi_2^*(\mu,\zeta)\}. \tag{22}$$

It is easy to see that

$$\varphi_2^*(\mu,\zeta) = \delta_{\mathscr{K}}(\mu) + x\mu\left(\frac{1}{\psi}\right) + \mu\left(\frac{B}{\psi}\right) + \delta_{\{\mathbb{1}\}}(\zeta),$$

where $\mathscr{K}$ is defined in Lemma 5. One can give a general formula for φ_1^*; however, to calculate the right-hand side of (22), it is enough to find $\varphi_1^*(-\mu,-\zeta)$ only for $\mu \in \mathscr{K} \subseteq \mathfrak{ba}_+$ and $\zeta = 1$. For such μ and ζ, taking $f(y) = V(y)$ and $g(x) = -U(-x)$,

$$\begin{aligned} \varphi_1^*(-\mu,-\zeta) &= \sup_{(\xi,\eta)\in L^\infty\times L^1,\ \eta+g(\xi\psi)\leq 0} [\mu(\xi) + \mathsf{E}\, \eta] \\ &\geq \sup_{(\xi,\eta)\in L^\infty\times L^\infty,\ \eta+g(\xi\psi)\leq 0} [\mu(\xi) + \mathsf{E}\, \eta] = \mathsf{E} F\left(\frac{1}{\psi}\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathsf{P}}, 1\right), \end{aligned} \tag{23}$$

where F is defined in (13) and the last equality follows from Theorem 1 in [22]. On the other hand, let $\xi \in L^\infty$, $\eta \in L^1$, $\eta + g(\xi\psi) \leq 0$. Then, in particular, $\xi \leq 0$ and, by the definition of F,

$$F\left(\frac{1}{\psi}\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathsf{P}}, 1\right) \geq \frac{\mathrm{d}\mu^r}{\mathrm{d}\mathsf{P}}\xi + \eta,$$

therefore,

$$\mathsf{E} F\left(\frac{1}{\psi}\frac{\mathrm{d}\mu^r}{\mathrm{d}\mathsf{P}}, 1\right) \geq \mu^r(\xi) + \mathsf{E}\, \eta \geq \mu(\xi) + \mathsf{E}\, \eta. \tag{24}$$

Combining (23) and (24), we get

$$\varphi_1^*(-\mu, -\zeta) = \mathsf{E} V\left(\frac{1}{\psi}\frac{\mathrm{d}\mu^r}{\mathrm{dP}}\right).$$

Therefore,

$$u(x) = \min_{\mu \in \mathscr{K}} \left\{ \mathsf{E}\left[V\left(\frac{1}{\psi}\frac{\mathrm{d}\mu^r}{\mathrm{dP}}\right)\right] + x\mu\left(\frac{1}{\psi}\right) + \mu\left(\frac{B}{\psi}\right)\right\}.$$

Now (17) with v given by (18) follows from Lemma 5; in particular, the lower bound in (18) is attained because $\mathscr{R}$ is a weak* compact and the function $\mu \rightsquigarrow \mathsf{E}\, V\left(\frac{1}{\psi}\frac{\mathrm{d}\mu^r}{\mathrm{dP}}\right)$ is weak* lower semicontinuous on $\mathfrak{ba}$. An easy way to check the last statement is to use the facts that $\mathsf{E}\, V(\mathrm{d}\mu^r/\mathrm{dP}) = \mathscr{I}_V(\mu, \mathsf{P})$ (where the V-divergence corresponds to $L^{\Phi_1} = L^{\Phi_2} = L^\infty$) and that the V-divergence is weak* lower semicontinuous in both variables; see [9].

Using the fact that V is decreasing (since $U(x) = -\infty$ for negative x), (18) can be rewritten in the form

$$v(y) = \min_{\substack{\mu+\nu \in \mathscr{R},\\ \mu \in \mathfrak{ca}_+,\\ \nu \in \mathfrak{ba}_+}} \left\{ \mathsf{E}\left[V\left(\frac{y}{\psi}\frac{\mathrm{d}\mu}{\mathrm{dP}}\right)\right] + y\mu\left(\frac{B}{\psi}\right) + y\nu\left(\frac{B}{\psi}\right)\right\}.$$

Hence, to prove (19) it remains to note that, by Lemmas 6 and 7,

$$\min_{\nu \in (\mathscr{R}-\mu)\cap \mathfrak{ba}_+} \nu\left(\frac{B}{\psi}\right) = g(\mu, B).$$

The final statement of the theorem is stated in Lemma 8. □

Finally, we make the following conclusions from the results of this section:

- As in the previous section, we do not need any additional assumptions on the function U such as differentiability and strict convexity and also on the finiteness of u or v at some point.
- At Step A, we reduced our problem to an appropriate space (essentially, ψL^∞) only for $x \in \mathscr{M}$, which was used when passing to the closure of $\frac{\mathscr{C}^\psi}{\psi}$; see (21). In [14] the proof is different and the closure is not taken; however, the restriction $x \in \mathscr{M}$ is also needed there to apply a duality theorem.
- At Step B, we introduced an additional variable η and applied a version of the Fenchel duality theorem to obtain the dual characterization of the value function in (17) and (18). Comparing to the proof in [14], more precisely, comparing the assumptions that are needed to apply Rockafellar's result [22] on the conjugate of a convex integral functional, one can see that this trick permits us to drop an (rather mild) additional assumption. Again, the proof of the attainment of the lower bound in (18) is more simple here.

- As in the previous section, we succeeded to realize Step C, which was not done in [8]. However, instead of singular functionals we again obtain the optimization problem (20) inside the dual problem (19).

References

1. Attouch, H., Brezis, H.: Duality for the sum of convex functions in general Banach spaces. In: Barroso, J.A. (ed.) Aspects of Mathematics and its Applications, pp. 125–133. North-Holland, Amsterdam (1986)
2. Biagini, S.: An Orlicz spaces duality for utility maximization in incomplete markets. In: Seminar on Stochastic Analysis, Random Fields and Applications V, Progress Probabability vol. 59, pp. 445–455. Birkhäuser, Basel (2008)
3. Biagini, S.: Expected utility maximization: Duality methods. In: Cont, R. (ed.) Encyclopedia of Quantitative Finance, pp. 638–645. Wiley, Chichester (2010)
4. Biagini, S., Černý, A.: Admissible strategies in semimartingale portfolio selection. SIAM J. Control Optim. **49**, 42–72 (2011)
5. Biagini, S., Frittelli, M.: Utility maximization in incomplete markets for unbounded processes. Financ. Stoch. **9**, 493–517 (2005)
6. Biagini, S., Frittelli, M.: The supermartingale property of the optimal wealth process for general semimartingales. Financ. Stoch. **11**, 253–266 (2007)
7. Biagini, S., Frittelli, M.: A unified framework for utility maximization problems: an Orlicz space approach. Ann. Appl. Probab. **18**, 929–966 (2008)
8. Cvitanić, J., Schachermayer, W., Wang, H.: Utility maximization in incomplete markets with random endowment. Financ. Stoch. **5**, 259–272 (2001)
9. Gushchin, A.A.: On extension of f-divergence. Theory Probab. Appl. **52**, 439–455 (2008)
10. Gushchin, A.A.: A lemma from the theory of Orlicz spaces. Surv. Appl. Ind. Math. **16**, 1056–1057 (2009) (in Russian)
11. Gushchin, A.A.: Dual characterization of the value function in the robust utility maximization problem. Theory Probab. Appl. **55**, 611–630 (2011)
12. Gushchin, A.A.: On the upper hedging price of nonnegative contingent claims. In: Shiryaev, A.N., Lebedev, A.V. (eds.) Contemporary Problems of Mathematics and Mechanics **VIII**, Mathematics, Issue 3, Probability Theory and Mathematical Statistics, pp. 60–72. MSU, Moscow (2013) (in Russian)
13. Hugonnier, J., Kramkov, D.: Optimal investment with random endowments in incomplete markets. Ann. Appl. Probab. **14**, 845–864 (2004)
14. Khasanov R.V.: Utility maximization problem in the case of unbounded endowment. Moscow Univ. Math. Bull. **68**, 138–147 (2013)
15. Kozek, A.: Convex integral functionals on Orlicz spaces. Ann. Soc. Math. Polonae Ser. 1. Comm. Math. **XXI**, 109–134 (1979)
16. Kramkov, D., Schachermayer, W.: The asymptotic elasticity of utility functions and optimal investment in incomplete markets. Ann. Appl. Prob. **9**, 904–950 (1999)
17. Morozov, I.S.: An extension of the class of admissible strategies in the robust utility maximization problem with a utility function finite on **R**. Surv. Appl. Ind. Math. **17**, 617–634 (2010) (in Russian)
18. Pliska, S.R.: A stochastic calculus model of continuous trading: optimal portfolios. Math. Oper. Res. **11**, 371–382 (1986)
19. Rao, M.M., Ren, Z.D.: Theory of Orlicz Spaces. Marcel Dekker, New-York (1991)
20. Rockafellar, R.T.: Extension of Fenchel's duality theorem for convex functions. Duke Math. J. **33**, 81–89 (1966)
21. Rockafellar, R.T.: Integrals which are convex functionals. Pacific J. Math. **24**, 525–539 (1968)

22. Rockafellar, R.T.: Integrals which are convex functionals, II. Pacific J. Math. **39**, 439–469 (1971)
23. Rokhlin, D.B.: The Kreps–Yan theorem for L^∞. Int. J. Math. Math. Sci. **2005**, 2749–2756 (2005)
24. Schachermayer, W.: Optimal investment in incomplete markets when wealth may become negative. Ann. Appl. Probab. **11**, 694–734 (2001)
25. Schachermayer, W.: A super-martingale property of the optimal portfolio process. Financ. Stoch., **4**, 433–457 (2003)

Maximization of the Survival Probability by Franchise and Deductible Amounts in the Classical Risk Model

Olena Ragulina

Abstract We consider the classical risk model when an insurance company has the opportunity to adjust franchise amount continuously. The problem of optimal control by franchise amount is solved from viewpoint of survival probability maximization. We derive the Hamilton–Jacobi–Bellman equation for the optimal survival probability and prove the existence of a solution of this equation with certain properties. The verification theorem gives the connection between this solution and the optimal survival probability. Then we concentrate on the case of exponentially distributed claim sizes. Finally, we extend the obtained results to the problem of optimal control by deductible amount.

1 Introduction

This contribution deals with the problem of survival probability maximization in the classical risk model when an insurance company has the opportunity to choose franchise and deductible amounts continuously. A franchise is a provision in the insurance policy whereby the insurer does not pay unless damage exceeds the franchise amount, whereas, deductible is a provision whereby the insurer pays any amounts of damage that exceed the deductible amount. As a rule, these provisions are applied when the insured's losses are relatively small to deter the large number of trivial claims. Moreover, a deductible stimulates the insured to take more care of the insured property.

Normally, a franchise and a deductible imply also reduction of insurance premiums. Thus, changes in claim and premium sizes have an influence on the survival probability of an insurance company. Our problem is to maximize the survival

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probability adjusting franchise or deductible amounts. We apply stochastic control theory to solve this problem. Similar approaches were used to solve optimal control problems by investment [1, 3, 6, 7, 10], reinsurance [9, 14], or investment and reinsurance simultaneously [8, 12, 15, 16]. The problems of optimal control by franchise and deductible amounts are considered in Sects. 2 and 3, respectively.

2 Optimal Control by Franchise Amount

In this section, we solve a problem of optimal control by franchise amount from viewpoint of survival probability maximization.

2.1 Description of the Model

Let $(\Omega, \mathbf{F}, (\mathbf{F}_t)_{t\geq 0}, \mathbf{P})$ be a stochastic basis and all objects be defined on it. In the classical risk model (see [2, 4, 5, 13]) the number of claims in the time interval $[0, t]$ is a Poisson process $N(t)$ with intensity $\lambda > 0$, and the claim sizes are nonnegative i.i.d. random variables Y_i, $i \geq 1$, independent of $N(t)$, with a distribution function $F(y) = \mathbf{P}\{Y_i \leq y\}$, $\mathbf{E}Y_i = \mu < \infty$. Let τ_i be the occurrence time of the ith claim. The initial surplus equals $x \geq 0$, and $c > 0$ is a constant premium intensity. We assume that $c > \lambda\mu$. If the insurance company uses the expected value principle for premium calculation, then the safety loading is defined as $\theta = c/\lambda\mu - 1$.

In addition, we assume that the insurance company uses a franchise and adjusts its amount d_t at every time $t \geq 0$ based on the information available just before time t, i.e. every admissible strategy (d_t) of franchise amount choice is a predictable process with respect to the natural filtration generated by $N(t)$ and Y_i, $1 \leq i \leq N(t)$. Moreover, we assume that $0 \leq d_t \leq d_{\max}$. Here $d_{\max}$ is a maximum allowed amount, and $0 < F(d_{\max}) < 1$. In particular, if $d_t = 0$, then a franchise is not used at time t. The safety loading $\theta > 0$ is constant. In this case the premium intensity is given by

$$c(d_t) = \lambda(1+\theta)\int_{d_t}^{+\infty} y\, \mathrm{d}F(y).$$

Let $X_x^{(d_t)}(t)$ be a surplus of the insurance company at time t if its initial surplus is x and the strategy (d_t) is used. Then

$$X_x^{(d_t)}(t) = x + \int_0^t c(d_s)\,\mathrm{d}s - \sum\nolimits_{i=1}^{N(t)} Y_i \;\; I\left\{Y_i > d_{\tau_i}\right\}, \tag{1}$$

where $I\{\cdot\}$ is an indicator of event. The ruin time under the admissible strategy (d_t) is defined as $\tau_x^{(d_t)} = \inf\left\{t \geq 0 :\ X_x^{(d_t)}(t) < 0\right\}$, and the corresponding infinite-horizon survival probability is given by $\varphi^{(d_t)}(x) = \mathbf{P}\left\{\tau_x^{(d_t)} = \infty\right\}$.

Our aim is to maximize the survival probability over all admissible strategies (d_t), i.e. to find

$$\varphi^*(x) = \sup_{(d_t)} \varphi^{(d_t)}(x),$$

and show that there exists an optimal strategy (d_t^*) such that $\varphi^*(x) = \varphi^{(d_t^*)}(x)$ for all $x \geq 0$. We will show later that the optimal strategy is a function of the initial surplus only.

We apply approaches of [1, 6, 7, 9, 14, 15] for solving this problem. First of all, we derive the Hamilton–Jacobi–Bellman equation for $\varphi^*(x)$, provided that this function is differentiable. Next, we prove the existence of a solution of this equation that satisfies certain conditions. Finally, we establish the connection between this solution and $\varphi^*(x)$ that gives us a way of finding the optimal survival probability.

2.2 Hamilton–Jacobi–Bellman Equation

For arbitrary $h > 0$ consider the strategy (d_t) such that franchise amount at time t equals

$$d_t = \begin{cases} d & \text{if } 0 \leq t \leq h \wedge \tau_1, \\ \tilde{d}_{t-(h\wedge\tau_1)}\left(X_x^{(d)}(h \wedge \tau_1)\right) & \text{if } t > h \wedge \tau_1 \text{ and } h \wedge \tau_1 < \tau_x^{(d)}, \end{cases}$$

where (d) is an arbitrary admissible constant strategy, $\left(\tilde{d}_t(x)\right)$ is an admissible strategy such that $\varphi^{(\tilde{d}_t)}(x) > \varphi^*(x) - \varepsilon$, here $\varepsilon > 0$ is an arbitrary fixed number.

For this strategy (d_t) by the law of total probability we have

$$\begin{aligned} \varphi^*(x) \geq \varphi^{(d_t)}(x) &= \mathrm{e}^{-\lambda h}\varphi^{(d_t)}\big(x + c(d)h\big) \\ &\quad + \int_0^h \lambda \mathrm{e}^{-\lambda s}\Bigg(F(d)\,\varphi^{(d_t)}\big(x + c(d)s\big) \\ &\quad + \int_d^{d\vee(x+c(d)s)} \varphi^{(d_t)}\big(x + c(d)s - y\big)\,\mathrm{d}F(y)\Bigg)\mathrm{d}s \\ &\geq \mathrm{e}^{-\lambda h}\varphi^*\big(x + c(d)h\big) + \int_0^h \lambda \mathrm{e}^{-\lambda s}\Bigg(F(d)\,\varphi^*\big(x + c(d)s\big) \\ &\quad + \int_d^{d\vee(x+c(d)s)} \varphi^*\big(x + c(d)s - y\big)\,\mathrm{d}F(y)\Bigg)\mathrm{d}s - \varepsilon. \end{aligned} \tag{2}$$

Letting $\varepsilon \to 0$ and doing elementary calculations in (2) we arrive at

$$\frac{\varphi^*\big(x + c(d)h\big) - \varphi^*(x)}{h}\mathrm{e}^{-\lambda h} - \frac{1 - \mathrm{e}^{-\lambda h}}{h}\varphi^*(x)$$
$$+\frac{1}{h}\int\limits_0^h \lambda \mathrm{e}^{-\lambda s}\Bigg(F(d)\varphi^*\big(x + c(d)s\big)$$
$$+\int_d^{d\vee(x+c(d)s)} \varphi^*\big(x + c(d)s - y\big)\,\mathrm{d}F(y)\Bigg)\mathrm{d}s \le 0. \tag{3}$$

Assuming that $\varphi^*(x)$ is differentiable on $[0, +\infty)$ and letting $h \to 0$ in (3) we obtain

$$(1+\theta)\int_d^{+\infty} y\,\mathrm{d}F(y)\,\big(\varphi^*(x)\big)' + \big(F(d) - 1\big)\varphi^*(x)$$
$$+\int_d^{d\vee x} \varphi^*(x - y)\,\mathrm{d}F(y) \le 0. \tag{4}$$

Inequality (4) is true for all $d \in [0, d_{\max}]$, and equality in (4) is attained when d is optimal at initial time. This yields the Hamilton–Jacobi–Bellman equation for $\varphi^*(x)$

$$\sup_{d\in[0,\,d_{\max}]}\Bigg((1+\theta)\int_d^{+\infty} y\,\mathrm{d}F(y)\,\big(\varphi^*(x)\big)'$$
$$+\big(F(d) - 1\big)\varphi^*(x) + \int_d^{d\vee x} \varphi^*(x - y)\,\mathrm{d}F(y)\Bigg) = 0. \tag{5}$$

Note that (5) can be rewritten in the following way:

$$\sup_{d\in[0,\,d_{\max}]}\big(A^d\varphi^*(x)\big) = 0,$$

where A^d is an infinitesimal generator of $X_x^{(d_t)}(t)$ as $d_t \equiv d$.

Rewrite (5) in a more convenient form. Since $0 < F(d_{\max}) < 1$, then expressing $\big(\varphi^*(x)\big)'$ from (4) gives

$$\big(\varphi^*(x)\big)' = \inf_{d\in[0,\,d_{\max}]}\left(\frac{\big(1 - F(d)\big)\varphi^*(x) - \int_d^{d\vee x} \varphi^*(x - y)\,\mathrm{d}F(y)}{(1+\theta)\int_d^{+\infty} y\,\mathrm{d}F(y)}\right). \tag{6}$$

Remark 1. Note that if there exists one solution of (5) or (6), then there exist infinitely many solutions of these equations which differ one from other with a multiplicative constant.

2.3 Existence Theorem

Now we prove the existence of a solution to (6).

Theorem 1. *If* Y_i, $i \geq 1$, *have the density function* $f(y)$, *then there exists a solution* $V(x)$ *of (6) which is nondecreasing and continuously differentiable on* $[0, +\infty)$, *with* $V(0) = \theta/(1+\theta)$, *and* $\theta/(1+\theta) \leq \lim_{x\to+\infty} V(x) \leq 1$.

Proof. Define a sequence of functions $V_n(x)$, $n \geq 0$, on $[0, +\infty)$ in the following way. Let $V_0(x) = \varphi^{(0)}(x)$, where $\varphi^{(0)}(x)$ is the survival probability under the strategy $d_t \equiv 0$, and

$$V_n'(x) = \inf_{d\in[0,\,d_{\max}]} \left(\frac{\big(1 - F(d)\big)V_{n-1}(x) - \int_d^{d\vee x} V_{n-1}(x-y)\,\mathrm{d}F(y)}{(1+\theta)\int_d^{+\infty} y\,\mathrm{d}F(y)} \right),$$
$$V_n(0) = \theta/(1+\theta), \tag{7}$$

for $n \geq 1$. This gives

$$V_n(x) = \frac{\theta}{1+\theta} + \int_0^x V_n'(u)\,\mathrm{d}u. \tag{8}$$

Since Y_i, $i \geq 1$, have the density function $f(y)$ and $\mathbf{E}Y_i < \infty$, then $V_0(x)$ is continuously differentiable on $[0, +\infty)$ and satisfies the equation

$$V_0'(x) = \frac{V_0(x) - \int_0^x V_0(x-y)f(y)\,\mathrm{d}y}{(1+\theta)\int_0^{+\infty} yf(y)\,\mathrm{d}y}, \quad V_0(0) = \frac{\theta}{(1+\theta)}. \tag{9}$$

Since $\theta > 0$, there exists a unique solution of (9) that is nondecreasing function, and $\lim_{x\to+\infty} V_0(x) = 1$; see [13].

Equation (9) and equality (7) for $n = 1$ yield $V_1'(x) \leq V_0'(x)$ for all $x \geq 0$. Therefore, by (8) we arrive at $V_1(x) \leq V_0(x)$. Besides, (7) for $n = 1$ and properties of $V_0(x)$ give that $V_1'(x)$ is nonnegative and continuous on $[0, +\infty)$.

Let $V_n'(x) \leq V_{n-1}'(x)$ for all $x \geq 0$ (this implies $V_n(x) \leq V_{n-1}(x)$), and $V_n'(x)$ is nonnegative and continuous on $[0, +\infty)$. Then for all $d \in [0, d_{\max}]$ and $x \geq 0$, we have

$$\begin{aligned}
&\big(1 - F(d)\big)V_{n-1}(x) - \int_d^{d\vee x} V_{n-1}(x-y)f(y)\,\mathrm{d}y \\
&= \int_{d\vee x}^{+\infty} V_{n-1}(x)f(y)\,\mathrm{d}y + \int_d^{d\vee x} \left(\int_{x-y}^{x} V_{n-1}'(u)\,\mathrm{d}u\right) f(y)\,\mathrm{d}y \\
&\geq \int_{d\vee x}^{+\infty} V_n(x)f(y)\,\mathrm{d}y + \int_d^{d\vee x} \left(\int_{x-y}^{x} V_n'(u)\,\mathrm{d}u\right) f(y)\,\mathrm{d}y \\
&= \big(1 - F(d)\big)V_n(x) - \int_d^{d\vee x} V_n(x-y)f(y)\,\mathrm{d}y.
\end{aligned}$$

This together with (7) yields that $V'_{n+1}(x) \le V'_n(x)$ for all $x \ge 0$ (therefore, $V_{n+1}(x) \le V_n(x)$), and $V'_{n+1}(x)$ is nonnegative and continuous on $[0, +\infty)$.

Thus, by induction $V'_n(x)$, $n \ge 0$, is a nonincreasing sequence of functions, which are nonnegative and continuous on $[0, +\infty)$. Hence, there exists a pointwise limit of these functions. We denote it by $v(x)$.

By (8) we obtain that $V_n(x)$, $n \ge 0$, is a nonincreasing sequence of nondecreasing functions, and this sequence is bounded below by the value of these functions at $x = 0$, i.e. $\theta/(1+\theta)$. Hence, there exists a pointwise limit of these functions. By Cauchy criterion this gives that the number sequence $V_n(z)$, $n \ge 0$, is fundamental for every fixed $z \ge 0$.

Now we show by Cauchy criterion that $V'_n(x)$, $n \ge 0$, converges uniformly over every $[0, z]$, where $0 < z < \infty$. For all $m \ge 1$ and $n \ge 1$ we have that

$$\begin{aligned}
&\sup_{x\in[0,z]} \left|V'_m(x) - V'_n(x)\right| \\
&\quad\le \sup_{\substack{x\in[0,z],\\ d\in[0,d_{\max}]}} \left| \frac{\big(1-F(d)\big)\big(V_{m-1}(x)-V_{n-1}(x)\big)}{(1+\theta)\int_d^{+\infty} yf(y)\,\mathrm{d}y} \right. \\
&\qquad \left. - \frac{\int_d^{d\vee x}\big(V_{m-1}(x-y) - V_{n-1}(x-y)\big) f(y)\,\mathrm{d}y}{(1+\theta)\int_d^{+\infty} yf(y)\,\mathrm{d}y} \right| \\
&\quad\le \sup_{d\in[0,d_{\max}]} \left(\frac{1-F(d)}{(1+\theta)\int_d^{+\infty} yf(y)\,\mathrm{d}y} \right) \sup_{x\in[0,z]} \left|V_{m-1}(x) - V_{n-1}(x)\right| \\
&\quad= \frac{\left|V_{m-1}(z) - V_{n-1}(z)\right|}{(1+\theta)\mu}.
\end{aligned} \tag{10}$$

Here we used the facts that $\sup_{d\in[0,d_{\max}]} \Big(\big(1 - F(d)\big) \Big/ \big((1+\theta)\int_d^{+\infty} yf(y)\,\mathrm{d}y\big)\Big)$ is attained at $d = 0$, and moreover

$$\begin{aligned}
\sup_{x\in[0,z]} \left|V_{m-1}(x) - V_{n-1}(x)\right| &= \sup_{x\in[0,z]} \left|\int_0^x \big(V'_{m-1}(u) - V'_{n-1}(u)\big)\,\mathrm{d}u\right| \\
&= \left|\int_0^z \big(V'_{m-1}(u) - V'_{n-1}(u)\big)\,\mathrm{d}u\right| = \left|V_{m-1}(z) - V_{n-1}(z)\right|.
\end{aligned}$$

By (10) $V'_n(x)$, $n \ge 0$, converges uniformly on any $[0, z]$. Thus, $v(x)$ is continuous on any interval; hence, it is continuous on $[0, +\infty)$. Denote

$$V(x) = \frac{\theta}{1+\theta} + \int_0^x v(u)\,\mathrm{d}u. \tag{11}$$

This gives that $V(x)$ is continuously differentiable on $[0, +\infty)$ and $V'(x) = v(x)$. Since $0 \le v(x) \le V'_0(x)$, then (11) yields the relations $\theta/(1+\theta) \le \lim_{x\to+\infty} V(x) \le 1$. Furthermore,

$$\Bigg| \inf_{d\in[0,\,d_{\max}]} \left(\frac{\big(1-F(d)\big)V_{n-1}(x) - \int_d^{d\vee x} V_{n-1}(x-y)\,\mathrm{d}F(y)}{(1+\theta)\int_d^{+\infty} y\,\mathrm{d}F(y)} \right)$$

$$- \inf_{d\in[0,\,d_{\max}]} \left(\frac{\big(1-F(d)\big)V(x) - \int_d^{d\vee x} V(x-y)\,\mathrm{d}F(y)}{(1+\theta)\int_d^{+\infty} y\,\mathrm{d}F(y)} \right) \Bigg|$$

$$\le \sup_{d\in[0,\,d_{\max}]} \Bigg| \frac{\big(1-F(d)\big)\big(V_{n-1}(x)-V(x)\big)}{(1+\theta)\int_d^{+\infty} y\,\mathrm{d}F(y)}$$

$$- \frac{\int_d^{d\vee x}\big(V_{n-1}(x-y)-V(x-y)\big)\,dF(y)}{(1+\theta)\int_d^{+\infty} y\,\mathrm{d}F(y)} \Bigg|$$

$$\le \frac{1}{(1+\theta)\mu}\int_0^x \big|V_{n-1}'(u)-V'(u)\big|\,\mathrm{d}u \le \frac{x}{(1+\theta)\mu}\sup_{u\in[0,\,x]}\big|V_{n-1}'(u)-V'(u)\big|,$$

and $\lim_{n\to\infty}\sup_{u\in[0,\,x]}\big|V_{n-1}'(u)-V'(u)\big| = 0$. Letting $n \to \infty$ in (7) yields that $V(x)$ is a solution of (6). Thus, the theorem is proved. □

2.4 Verification Theorem

In this subsection, we prove that $\varphi^*(x)$ coincides with $V(x)/\big(\lim_{x\to+\infty} V(x)\big)$.

Theorem 2. *Let the surplus process $X_x^{(d_t)}(t)$ be defined by (1) and $V(x)$ be the solution of (6) that satisfies conditions of Theorem 1. Then for any $x \ge 0$ and arbitrary admissible strategy (d_t), we have*

$$\varphi^{(d_t)}(x) \le V(x)/\big(\lim_{x\to+\infty} V(x)\big), \tag{12}$$

and equality in (12) is attained under the strategy $(d_t^*) = \big(d_t^*\big(X_x^{(d_t^*)}(t-)\big)\big)$, where $\big(d_t^*(x)\big)$ minimizes the right-hand side of (6), i.e.

$$\varphi^*(x) = \varphi^{(d_t^*)}(x) \le V(x)/\big(\lim_{x\to+\infty} V(x)\big).$$

Proof. Let $V(x)$ be the solution of (6) that satisfies conditions of Theorem 1 with the corresponding strategy $(d_t^*) = \big(d_t^*(x)\big)$, and (d_t) is an arbitrary admissible strategy. Moreover, we suppose that $V(x) = 0$ for all $x < 0$, and $V'(x)$ at $x = 0$ means right-sided derivative of $V(x)$ at this point. Applying Dynkin's formula (see [11], p. 11) to the process $V\big(X_x^{(d_t)}\big(t \wedge \tau_x^{(d_t)}\big)\big)$ we get

$$\begin{aligned}&\mathbf{E}\, V\Big(X_x^{(d_t)}\big(t\wedge\tau_x^{(d_t)}\big)\Big)\\&\quad= V(x)+\mathbf{E}\left(\int\limits_0^{t\wedge\tau_x^{(d_t)}}\Big((1+\theta)\int_{d_s}^{+\infty} y\,\mathrm{d}F(y)\,V'\big(X_x^{(d_t)}(s)\big)\Big)\,\mathrm{d}s\right.\\&\qquad\left.+\int\limits_0^{t\wedge\tau_x^{(d_t)}}\Big(\big(F_{\mathrm{cl}}(d_s)-1\big)\,V(X_x^{(d_t)})+\int_{d_s}^{d_s\vee X_x^{(d_t)}} V\big(X_x^{(d_t)}-y\big)\,\mathrm{d}F(y)\Big)\,\mathrm{d}s\right).\end{aligned}$$

Similar formula holds for $V\Big(X_x^{(d_t^*)}\big(t\wedge\tau_x^{(d_t^*)}\big)\Big)$ as well. Therefore, for all $x\geq 0$ and $t\geq 0$

$$\mathbf{E}\, V\Big(X_x^{(d_t)}\big(t\wedge\tau_x^{(d_t)}\big)\Big)\leq V(x)=\mathbf{E}\, V\Big(X_x^{(d_t^*)}\big(t\wedge\tau_x^{(d_t^*)}\big)\Big), \tag{13}$$

since $V(x)$ is a solution of (6), and hence it is a solution of (5).

Now we show that for any admissible strategy (d_t), the process $X_x^{(d_t)}(t)$ is unbounded with probability 1 provided that ruin does not occur, i.e. for all $X>0$

$$\mathbf{P}\Big\{X_x^{(d_t)}(t)\leq X\quad\forall t\geq 0,\ \tau_x^{(d_t)}=\infty\Big\}=0. \tag{14}$$

The probability of more than $(X+c)/d_{\max}$ claims of size larger than $d_{\max}$ within any unit time is positive, and the claim process has stationary independent increments. Hence, by the Borel–Cantelli lemma with probability 1, there exists integer $T>0$ such that there are more than $(X+c)/d_{\max}$ claims of this kind in the time interval $[T,T+1]$. If in addition $X_x^{(d_t)}(t)\leq X$ for all $t\in[0,T]$, then

$$X_x^{(d_t)}(T+1)<X+c-d_{\max}(X+c)/d_{\max}=0,$$

i.e. $\tau_x^{(d_t)}<\infty$. This yields (14).

Now we fix arbitrary admissible strategy (d_t) and small enough $\varepsilon>0$. Let $z>x$ be large enough number, such that $1-\varphi^{(0)}(z)<\varepsilon$. Denote

$$T_z=\inf\big\{t>0:\ X_x^{(d_t)}(t)=z\big\}.$$

By (14) $T_z<\infty$ with probability 1 provided that $\tau_x^{(d_t)}=\infty$ with probability 1. Define the strategy $\big(d_t^{T_z}\big)$ in the following way:

$$d_t^{T_z}=\begin{cases} d_t & \text{if } t\leq T_z,\\ 0 & \text{if } t>T_z.\end{cases}$$

Then

$$\mathbf{P}\Big\{\tau_x^{(d_t)} = \infty,\ \tau_x^{\left(d_t^{T_z}\right)} < \infty\Big\} \le 1 - \varphi^{(0)}(z) < \varepsilon. \tag{15}$$

Furthermore, $\mathbf{P}\left\{\lim_{t\to+\infty} X_x^{\left(d_t^{T_z}\right)}(t) = +\infty,\ \tau_x^{\left(d_t^{T_z}\right)} = \infty\right\} = 1$ since $\mathbf{P}\{T_z < \infty\} = 1$ and $\mathbf{P}\left\{\lim_{t\to+\infty} X_x^{(0)}(t) = +\infty,\ \tau_x^{(0)} = \infty\right\} = 1$.

Relation (13), applied to the strategy $\left(d_t^{T_z}\right)$, gives

$$\mathbf{E}\, V\left(X_x^{\left(d_t^{T_z}\right)}\left(t \wedge \tau_x^{\left(d_t^{T_z}\right)}\right)\right) \le V(x) = \mathbf{E}\, V\left(X_x^{(d_t^*)}\left(t \wedge \tau_x^{(d_t^*)}\right)\right). \tag{16}$$

Since $V\left(X_x^{\left(d_t^{T_z}\right)}\left(\tau_x^{\left(d_t^{T_z}\right)}\right)\right) = 0$ and $V\left(X_x^{(d_t^*)}\left(\tau_x^{(d_t^*)}\right)\right) = 0$, then by (15) letting $t \to \infty$ in (16) yields

$$\mathbf{P}\left\{\tau_x^{(d_t)} = \infty\right\} - \varepsilon \le \mathbf{P}\left\{\tau_x^{(d_t)} = \infty,\ \tau_x^{\left(d_t^{T_z}\right)} = \infty\right\}$$
$$\le \frac{V(x)}{\lim_{x\to+\infty} V(x)} \le \mathbf{P}\left\{\tau_x^{(d_t^*)} = \infty\right\}, \tag{17}$$

where $\theta/(1+\theta) \le \lim_{x\to+\infty} V(x) \le 1$.

Note that $\varepsilon > 0$ is arbitrary in (17), whence the theorem follows. □

Remark 2. In the proof of Theorem 2 we used arbitrary solution of (6) that satisfies conditions of Theorem 1. However, notice that Theorem 2 also implies uniqueness of such solution. The corresponding strategy (d_t^*) may not be unique in the general case.

2.5 *Exponentially Distributed Claim Sizes*

In this subsection, we consider a case of exponentially distributed claim sizes.

Theorem 3. *Let the surplus process $X_x^{(d_t)}(t)$ be defined by (1), claim sizes be exponentially distributed with mean μ, and $d_{\max} = \mu$. Then the strategy $d_t(x) \equiv 0$ is not optimal.*

Proof. If the strategy $d_t(x) \equiv 0$ is optimal, then $\varphi^{(0)}(x)$ is the solution of (6) which satisfies conditions of Theorem 1, i.e. $V_1'(x) = V_0'(x)$ for all $x \ge 0$. Now we show that this is not true.

Indeed, note that

$$
\begin{aligned}
V_0(x) =&\varphi^{(0)}(x) = 1 - \frac{1}{1+\theta} \exp\left(-\frac{\theta x}{(1+\theta)\mu}\right),\\
V_1'(x) =& \inf_{d\in[0,\mu]} \left(\frac{\exp\left(-\frac{d}{\mu}\right)\left(1-\frac{1}{1+\theta}\exp\left(-\frac{\theta x}{(1+\theta)\mu}\right)\right)}{(1+\theta)(d+\mu)\exp\left(-\frac{d}{\mu}\right)} \right.\\
&\left. - \frac{\int_d^{d\vee x}\left(1-\frac{1}{1+\theta}\exp\left(-\frac{\theta(x-y)}{(1+\theta)\mu}\right)\right)\frac{1}{\mu}\exp\left(-\frac{y}{\mu}\right)\mathrm{d}y}{(1+\theta)(d+\mu)\exp\left(-\frac{d}{\mu}\right)} \right).
\end{aligned}
$$

Let $x \geq 0$ be an arbitrary fixed number. Consider two cases.

(1) If $d \geq x$ (additionally, $x \in [0,\mu]$), then

$$
\inf_{d\in[x,\mu]} \left(\frac{1+\theta-\exp\left(-\frac{\theta x}{(1+\theta)\mu}\right)}{(1+\theta)^2(d+\mu)} \right) = \frac{1+\theta-\exp\left(-\frac{\theta x}{(1+\theta)\mu}\right)}{2\mu(1+\theta)^2}
$$

at $d = \mu$.

(2) If $d < x$, then

$$
\begin{aligned}
&\inf_{d\in[0,\mu\wedge x]} \left(\frac{\exp\left(-\frac{\theta x}{(1+\theta)\mu}\right)\left(\exp\left(\frac{\theta d}{(1+\theta)\mu}\right)-\frac{1}{1+\theta}\right)}{(1+\theta)(d+\mu)} \right)\\
&=\frac{\theta}{\mu(1+\theta)^2}\exp\left(-\frac{\theta x}{(1+\theta)\mu}\right)
\end{aligned}
$$

at $d = 0$ because for all $d > 0$

$$
\left(\frac{\exp\left(\frac{\theta d}{(1+\theta)\mu}\right)-\frac{1}{1+\theta}}{(d+\mu)} \right)'_d = \frac{\left(\frac{\theta d}{\mu}-1\right)\exp\left(\frac{\theta d}{(1+\theta)\mu}\right)+1}{\left((1+\theta)(d+\mu)^2\right)} > 0.
$$

Notice that

$$
\frac{1+\theta-\exp\left(-\frac{\theta x}{(1+\theta)\mu}\right)}{2\mu(1+\theta)^2} \leq \frac{\theta}{\mu(1+\theta)^2}\exp\left(-\frac{\theta x}{(1+\theta)\mu}\right)
$$

for $x \leq x_0$, where $x_0 = \frac{(1+\theta)\mu}{\theta}\ln\left(1+\frac{\theta}{1+\theta}\right)$ and

$$
x_0 = \frac{(1+\theta)\mu}{\theta}\left(\frac{\theta}{1+\theta}+\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\frac{\theta}{1+\theta}\right)^k\right) < \mu.
$$

Here we used the fact that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\frac{\theta}{1+\theta}\right)^k$ is a Leibniz series; therefore, its sum is negative.

Hence, we arrive at

$$V_1'(x) = \begin{cases} \dfrac{1+\theta-\exp\left(-\frac{\theta x}{(1+\theta)\mu}\right)}{2\mu(1+\theta)^2} & \text{if } x \in [0, x_0], \\ \dfrac{\theta}{\mu(1+\theta)^2}\exp\left(-\dfrac{\theta x}{(1+\theta)\mu}\right) & \text{if } x \in (x_0, +\infty). \end{cases}$$

This yields $V_1'(x) \neq V_0'(x)$ for all $x \in [0, x_0]$, which completes the proof. □

Remark 3. Theorem 3 implies that we can always increase the survival probability adjusting the franchise amount for exponentially distributed claim sizes.

Example 1. If claim sizes are exponentially distributed with mean $\mu = 10$, $d_{\max} = \mu$, and $\theta = 0.1$, then

$$\varphi^{(0)}(x) \approx 1 - 0.9090909\,\mathrm{e}^{-x/110}, \quad x \geq 0,$$

$$\varphi^*(x) \approx \begin{cases} 0.111048767\,\mathrm{e}^{x/22} & \text{if } x \leq 8.93258, \\ 1 - 0.90382792\,\mathrm{e}^{x/110} & \text{if } x > 8.93258, \end{cases}$$

$$d_t^*(x) = \begin{cases} 10 \text{ if } x \leq 8.93258, \\ 0 \text{ if } x > 8.93258. \end{cases}$$

3 Optimal Control by Deductible Amount

Now we assume that in the classical risk model the insurance company has the opportunity to use a deductible and adjust its amount $\bar{d}_t$ at every time $t \geq 0$ based on the information available just before time t. That is, every admissible strategy $\left(\bar{d}_t\right)$ of deductible amount choice is a predictable process with respect to the natural filtration generated by $N(t)$ and Y_i, $1 \leq i \leq N(t)$. Furthermore, we assume that $0 \leq \bar{d}_t \leq \bar{d}_{\max}$, where $\bar{d}_{\max}$ is a maximum allowed deductible amount, and $0 < F\left(\bar{d}_{\max}\right) < 1$. Let the safety loading $\theta > 0$ be constant, and then the premium intensity is given by

$$c\left(\bar{d}_t\right) = \lambda(1+\theta)\int_{\bar{d}_t}^{+\infty} \left(y - \bar{d}_t\right)\,\mathrm{d}F(y).$$

Let $X_x^{(\bar{d}_t)}(t)$ denote a surplus of the insurance company at time t if its initial surplus equals x and the strategy $\left(\bar{d}_t\right)$ is used. Then

$$X_x^{(\bar{d}_t)}(t) = x + \int_0^t c\left(\bar{d}_s\right) \mathrm{d}s - \sum\nolimits_{i=1}^{N(t)} \left(Y_i - \bar{d}_{\tau_i}\right)^+, \tag{18}$$

where $\left(Y_i - \bar{d}_{\tau_i}\right)^+ = \max\left\{Y_i - \bar{d}_{\tau_i}, 0\right\}$.

The ruin time and the infinite-horizon survival probability under the admissible strategy $\left(\bar{d}_t\right)$ are defined as $\tau_x^{(\bar{d}_t)} = \inf\left\{t \geq 0 : \ X_x^{(\bar{d}_t)}(t) < 0\right\}$ and $\varphi^{(\bar{d}_t)}(x) = \mathbf{P}\left\{\tau_x^{(\bar{d}_t)} = \infty\right\}$, respectively.

Our aim is to find

$$\varphi^*(x) = \sup_{(\bar{d}_t)} \varphi^{(\bar{d}_t)}(x),$$

and show that there exists an optimal strategy $\left(\bar{d}_t^*\right)$ such that $\varphi^*(x) = \varphi^{(\bar{d}_t^*)}(x)$ for all $x \geq 0$.

Applying the techniques of Sect. 2.2 we arrive at the Hamilton–Jacobi–Bellman equation for $\varphi^*(x)$

$$\sup_{\bar{d} \in [0,\, \bar{d}_{\max}]} \Bigg((1+\theta) \int_{\bar{d}}^{+\infty} \left(y - \bar{d}\right) \mathrm{d}F(y) \left(\varphi^*(x)\right)' + \left(F\left(\bar{d}\right) - 1\right)\varphi^*(x) + \int_{\bar{d}}^{x+\bar{d}} \varphi^*\left(x + \bar{d} - y\right) \mathrm{d}F(y) \Bigg) = 0$$

or, that is equivalent,

$$\left(\varphi^*(x)\right)' = \inf_{\bar{d} \in [0,\, \bar{d}_{\max}]} \left(\frac{\left(1 - F\left(\bar{d}\right)\right)\varphi^*(x) - \int_{\bar{d}}^{x+\bar{d}} \varphi^*\left(x + \bar{d} - y\right) \mathrm{d}F(y)}{(1+\theta) \int_{\bar{d}}^{+\infty} \left(y - \bar{d}\right) \mathrm{d}F(y)} \right) \tag{19}$$

provided that function φ^* is differentiable on $[0, +\infty)$.

We can formulate following additional results.

Theorem 4. *If* Y_i, $i \geq 1$, *have the density function* $f(y)$, *then there exists a solution* $V(x)$ *of (19) which is nondecreasing and continuously differentiable on* $[0, +\infty)$, *with* $V(0) = \theta/(1+\theta)$, *and* $\theta/(1+\theta) \leq \lim_{x\to+\infty} V(x) \leq 1$.

Theorem 5. *Let the surplus process* $X_x^{(\bar{d}_t)}(t)$ *be defined by (18) and* $V(x)$ *be the solution of (19) that satisfies conditions of Theorem 4. Then for any* $x \geq 0$ *and arbitrary admissible strategy* $\left(\bar{d}_t\right)$, *we have*

$$\varphi^{(\bar{d}_t)}(x) \leq V(x) \big/ \left(\lim_{x\to+\infty} V(x)\right), \tag{20}$$

and equality in (20) is attained under the strategy $\left(\bar{d}_t^*\right) = \left(\bar{d}_t^*\left(X_x^{(\bar{d}_t^*)}(t-)\right)\right)$, *where* $\left(\bar{d}_t^*(x)\right)$ *minimizes the right-hand side of (19), i.e.*

$$\varphi^*(x) = \varphi^{(\bar{d}_t^*)}(x) \leq V(x)/\left(\lim_{x\to+\infty} V(x)\right).$$

The proofs of Theorems 4 and 5 are similar to the proofs of Theorems 1 and 2, respectively.

Note that the solution $V(x)$ of (19) which satisfies conditions of Theorem 4 can be found as a limit of the sequence $V_n(x)$, $n \geq 0$, where $V_0(x) = \varphi^{(0)}(x)$ is the survival probability under the strategy $\bar{d}_t \equiv 0$, and

$$V_n'(x) = \inf_{\bar{d}\in[0,\bar{d}_{\max}]} \left(\frac{\left(1 - F\left(\bar{d}\right)\right) V_{n-1}(x) - \int_{\bar{d}}^{x+\bar{d}} V_{n-1}\left(x + \bar{d} - y\right)\, \mathrm{d}F(y)}{(1+\theta)\int_{\bar{d}}^{+\infty} \left(y - \bar{d}\right)\, \mathrm{d}F(y)} \right),$$
$$V_n(0) = \theta/(1+\theta),$$

for $n \geq 1$.

Theorem 6. *Let the surplus process* $X_x^{(\bar{d}_t)}(t)$ *be defined by (18), and claim sizes are exponentially distributed. Then* $\varphi^*(x) = \varphi^{(\bar{d}_t)}(x)$ *for every admissible strategy* $\left(\bar{d}_t\right)$, *i.e. every admissible strategy is optimal.*

Proof. If claim sizes are exponentially distributed with mean μ, then $V_0(x)$ is given in the proof of the Theorem 3, and

$$\begin{aligned} V_1'(x) = &\inf_{\bar{d}\in[0,\bar{d}_{\max}]} \left(\frac{\exp\left(-\frac{\bar{d}}{\mu}\right)\left(1 - \frac{1}{1+\theta}\exp\left(-\frac{\theta x}{(1+\theta)\mu}\right)\right)}{(1+\theta)\int_{\bar{d}}^{+\infty}\left(y-\bar{d}\right)\frac{1}{\mu}\exp\left(-\frac{y}{\mu}\right)\mathrm{d}y} \right. \\ &\left. - \frac{\int_{\bar{d}}^{x+\bar{d}}\left(1 - \frac{1}{1+\theta}\exp\left(-\frac{\theta(x+\bar{d}-y)}{(1+\theta)\mu}\right)\right)\frac{1}{\mu}\exp\left(-\frac{y}{\mu}\right)\mathrm{d}y}{(1+\theta)\int_{\bar{d}}^{+\infty}\left(y-\bar{d}\right)\frac{1}{\mu}\exp\left(-\frac{y}{\mu}\right)\mathrm{d}y} \right) \\ = &\inf_{\bar{d}\in[0,\bar{d}_{\max}]} \left(\frac{\frac{1}{1+\theta}\exp\left(-\frac{\bar{d}}{\mu}\right)\exp\left(-\frac{\theta x}{(1+\theta)\mu}\right)}{(1+\theta)\mu\exp\left(-\frac{\bar{d}}{\mu}\right)} \right) \\ = &\frac{\theta}{(1+\theta)^2\mu}\exp\left(-\frac{\theta x}{(1+\theta)\mu}\right) = V_0'(x), \end{aligned}$$

where the infimum is attained at arbitrary $\bar{d} \in [0, \bar{d}_{\max}]$. This yields $V(x) = \varphi^{(0)}(x)$, which completes the proof. □

Remark 4. Theorem 6 implies that we cannot increase the survival probability adjusting the deductible amount for exponentially distributed claim sizes.

References

1. Androshchuk, M., Mishura, Yu.: Optimal investment for an insurance company into risky and non-risky assets in the model with variable premium income process. J. Numer. Appl. Math. **1(96)**, 11–27 (2008)
2. Asmussen, S.: Ruin Probabilities. World Scientific, Singapore (2000)
3. Azcue, P., Muler, N.: Optimal investment strategy to minimize the ruin probability of an insurance company under borrowing constrains. Insur. Math. Econ. **44**, 26–34 (2009)
4. Beard, R.E., Pentikainen, T., Pesonen, E.: Risk Theory. Chapman and Hall, New York (1984)
5. Grandell, J.: Aspects of Risk Theory. Springer, New York (1991)
6. Hipp, C., Plum, M.: Optimal investment for insurers. Insur. Math. Econ. **27**, 215–228 (2000)
7. Hipp, C., Plum, M.: Optimal investment for investors with state dependent income, and for insurers. Financ. Stoch. **7**, 299–321 (2003)
8. Hipp, C., Taksar, M.: Optimal non-proportional reinsurance control. Insur. Math. Econ. **47**, 246–254 (2010)
9. Hipp, C., Vogt, M.: Optimal dynamic XL reinsurance. ASTIN BULL. **33**, 193–207 (2003)
10. Liu, C.S., Yang, H.: Optimal investment for an insurer to minimize its probability of ruin. N. Am. Actuar. J. **8**, 11–31 (2004)
11. Øksendal, B., Sulem, A.: Applied Stochastic Control of Jump Diffusions. Springer, Berlin (2005)
12. Promislow, S.D., Young, V.R.: Minimizing the probability of ruin when claims follow Brownian motion with drift. N. Am. Actuar. J. **9**, 109–128 (2005)
13. Rolski, T., Schmidli, H., Schmidt V., Teugels J.: Stochastic Processes for Insurance and Finance. Wiley, Chichester (1999)
14. Schmidli, H.: Optimal proportional reinsurance policies in a dynamic setting. Scand. Actuar. J., 55–68 (2001)
15. Schmidli, H.: On minimizing the ruin probability by investment and reinsurance. Ann. Appl. Probab. **12**, 890–907 (2002)
16. Taksar, M.I., Markussen, C.: Optimal dynamic reinsurance policies for large insurance portfolios. Financ. Stoch. **7**, 97–121 (2003)

Part V
Statistics

Asymptotic Properties of Drift Parameter Estimator Based on Discrete Observations of Stochastic Differential Equation Driven by Fractional Brownian Motion

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Abstract In this chapter, we consider a problem of statistical estimation of an unknown drift parameter for a stochastic differential equation driven by fractional Brownian motion. Two estimators based on discrete observations of solution to the stochastic differential equations are constructed. It is proved that the estimators converge almost surely to the parameter value, as the observation interval expands and the distance between observations vanishes. A bound for the rate of convergence is given and numerical simulations are presented. As an auxilliary result of independent interest we establish global estimates for fractional derivative of fractional Brownian motion.

1 Introduction

A fractional Brownian motion (fBm) with the Hurst parameter $H \in (0,1)$ is a centered Gaussian process $\{B_t^H, t \geq 0\}$ having the covariance $\mathsf{E}\left[B_t^H B_s^H \right] = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$. Stochastic differential equations driven by an fBm have been a subject of active research for the last two decades. Main reason is that such equations seem to be one of the most suitable tools to model long-range dependence in many applied areas, such as physics, finance, biology, and network studies.

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In modeling, the problems of statistical estimation of model parameters are of a particular importance, so the growing number of papers devoted to statistical methods for equations with fractional noise is not surprising. We will cite only few of them; further references can be found in [2, 12, 14]. In [6], the authors proposed and studied maximum likelihood estimators for fractional Ornstein–Uhlenbeck process. Related results were obtained in [13], where a more general model was considered. In [4] the authors proposed a least squares estimator for fractional Ornstein–Uhlenbeck process and proved its asymptotic normality. The estimators constructed in these papers have the same disadvantage: they are based on the whole trajectory of solution to stochastic differential equations, so are not directly implementable. In view of this, estimators based on discrete observations of solutions were proposed in [1, 5, 15, 16]. We note that the discretization of the maximum likelihood estimator is extremely involved in the fractional Brownian case; see discussion in [15].

It is worth to mention that papers [5, 15] deal with the whole range of Hurst parameter $H \in (0, 1)$, while other papers cited here investigate only the case $H > 1/2$ (which corresponds to long-range dependence); recall that in the case $H = 1/2$, we have a classical diffusion, and there is a huge literature devoted to it; we refer to books [8, 9] for the review of the topic. We also mention papers [7, 17], which deal with parameter estimation in so-called mixed models, involving standard Wiener process along with an fBm.

This chapter deals with statistical estimation of drift parameter for a stochastic differential equation driven by an fBm based on discrete observation of its solution. The model we consider is fully nonlinear, in contrast to [1, 5], which deal with a simple linear model, [16], devoted to the problem of estimating the parameters for fractional Ornstein–Uhlenbeck processes from discrete observations, and [15], which studies a model where the fractional Brownian motion enters linearly. We propose two new estimators and prove their strong consistency under the so-called "high-frequency data" assumption that the horizon of observations tends to infinity while the distance between them goes to zero. Moreover, we obtain almost sure upper bounds for the rate of convergence of the estimators. The estimators proposed go far away from being maximum likelihood estimators, and this is their crucial advantage, because they keep strong consistency but they are not complicated technically and are convenient for the simulations. This chapter is organized as follows. In Sect. 2, we give preliminaries on stochastic integration with respect to an fBm. In this section, we also give some auxiliary results, which are of independent interest: global estimates for the fractional derivative of an fBm and for increments of a solution to an fBm-driven stochastic differential equation. In Sect. 3, we construct estimators for the drift parameter, prove their strong consistency, and establish their rate of convergence. Section 4 illustrates our findings with simulation results.

2 Preliminaries

For a fixed real number $H \in (1/2, 1)$, let $\{B_t^H, t \geq 0\}$ be a fractional Brownian motion with the Hurst parameter H on a complete probability space $(\Omega, \mathcal{F}, \mathsf{P})$. The integral with respect to the fBm B^H will be understood in the generalized Lebesgue–Stieltjes sense (see [18]). Its construction uses the fractional derivatives, defined for $a < b$ and $\alpha \in (0, 1)$ as

$$\big(D_{a+}^{\alpha} f\big)(x) = \frac{1}{\Gamma(1-\alpha)}\left(\frac{f(x)}{(x-a)^{\alpha}} + \alpha \int_a^x \frac{f(x)-f(u)}{(x-u)^{1+\alpha}} du\right),$$

$$\big(D_{b-}^{1-\alpha} g\big)(x) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)}\left(\frac{g(x)}{(b-x)^{1-\alpha}} + (1-\alpha) \int_x^b \frac{g(x)-g(u)}{(u-x)^{2-\alpha}} du\right).$$

Provided that $D_{a+}^{\alpha} f \in L_1[a, b]$, $D_{b-}^{1-\alpha} g_{b-} \in L_\infty[a, b]$, where $g_{b-}(x) = g(x) - g(b)$, the generalized Lebesgue-Stieltjes integral $\int_a^b f(x)dg(x)$ is defined as

$$\int_a^b f(x)dg(x) = e^{i\pi\alpha} \int_a^b \big(D_{a+}^{\alpha} f\big)(x)\big(D_{b-}^{1-\alpha} g_{b-}\big)(x)dx. \tag{1}$$

It follows from the Hölder continuity of B^H that for $\alpha \in (1-H, 1)$, $D_{b-}^{1-\alpha} B_{b-}^H \in L_\infty[a, b]$ a.s. (we will prove this result in a stronger form further). Then for a function f with $D_{a+}^{\alpha} f \in L_1[a, b]$, we can define integral with respect to B^H through (1):

$$\int_a^b f(x)\, dB^H(x) = e^{i\pi\alpha} \int_a^b (D_{a+}^{\alpha} f)(x)(D_{b-}^{1-\alpha} B_{b-}^H)(x)\, dx. \tag{2}$$

Throughout the paper, the symbol C will denote a generic constant, whose value is not important and may change from one line to another. If a constant depends on some variable parameters, we will put them in subscripts.

2.1 Estimate of Derivative of Fractional Brownian Motion

In order to estimate integrals with respect to fractional Brownian motion, we need to estimate the fractional derivative of B^H. Let some $\alpha \in (1-H, 1/2)$ be fixed in the rest of this paper. Denote for $t_1 < t_2$

$$Z(t_1, t_2) = \big(D_{t_2-}^{1-\alpha} B_{t_2-}^H\big)(t_1) = \frac{e^{-i\pi\alpha}}{\Gamma(\alpha)}\left(\frac{B_{t_1}^H - B_{t_2}^H}{(t_2-t_1)^{1-\alpha}} + (1-\alpha)\int_{t_1}^{t_2} \frac{B_{t_1}^H - B_u^H}{(u-t_1)^{2-\alpha}} du\right).$$

The following proposition is a generalization of [7, Theorem 3].

Theorem 1. *For any $\gamma > 1/2$, the random variable*

$$\xi_{H,\alpha,\gamma} = \sup_{0\le t_1<t_2\le t_1+1} \frac{|Z(t_1,t_2)|}{(t_2-t_1)^{H+\alpha-1}\left(|\log(t_2-t_1)|^{1/2}+1\right)(\log(t_2+2))^{\gamma}} \tag{3}$$

is finite almost surely. Moreover, there exists $c_{H,\alpha,\gamma} > 0$ such that $\mathsf{E}\left[\exp\left\{x\xi^2_{H,\alpha,\gamma}\right\}\right] < \infty$ for $x < c_{H,\alpha,\gamma}$.

Proof. Let $h(s) = s^{H+\alpha-1}\left(|\log s|^{1/2}+1\right)$, $s > 0$. Define for $T > 0$

$$M_T = \sup_{0\le t_1<t_2\le t_1+1\le T} \frac{|Z(t_1,t_2)|}{h(t_2-t_1)}.$$

We will first prove that M_T is finite almost surely. Since $\mathsf{E}\left[\left(B^H_t - B^H_s\right)^2\right] = (t-s)^{2H}$, it follows from [11, Theorem 4] that there exists a random variable ξ_T such that almost surely for all t_1, t_2 with $0 \le t_1 < t_2 \le t_1 + 1$

$$\left|B^H_{t_1} - B^H_{t_2}\right| \le \xi_T (t_2-t_1)^H \left(|\log(t_2-t_1)|^{1/2}+1\right).$$

Then

$$|Z(t_1,t_2)| \le \frac{\xi_T}{\Gamma(\alpha)}(t_2-t_1)^{H+\alpha-1}\left(|\log(t_2-t_1)|^{1/2}+1\right) + I,$$

where

$$\begin{aligned}
I &= \left|\int_{t_1}^{t_2} \frac{B^H_u - B^H_{t_1}}{(u-t_1)^{2-\alpha}}du\right| \le \frac{\xi_T}{\Gamma(\alpha)}\int_{t_1}^{t_2}(u-t_1)^{H+\alpha-2}\left(|\log(u-t_1)|^{1/2}+1\right)du \\
&\le \frac{\xi_T}{\Gamma(\alpha)}(t_2-t_1)^{H+\alpha-1}\int_0^1 z^{H+\alpha-2}\left(|\log z|^{1/2}+|\log(t_2-t_1)|^{1/2}+1\right)dz \\
&\le C\xi_T (t_2-t_1)^{H+\alpha-1}\left(|\log(t_2-t_1)|^{1/2}+1\right),
\end{aligned}$$

whence finiteness of M_T follows. Since M_T is a supremum of Gaussian family, Fernique's theorem implies that $\mathsf{E}\left[e^{\varepsilon M_T^2}\right] < \infty$ for some $\varepsilon > 0$; in particular, all moments of M_T are finite.

Now observe that from H-self-similarity of B^H it follows that for any $a > 0$

$$\{Z(at_1,at_2), 0\le t_1<t_2\} \stackrel{d}{=} \left\{a^{H+\alpha-1}Z(t_1,t_2), 0\le t_1<t_2\right\}.$$

Therefore, for any $k \ge 1$

$$
\begin{aligned}
M_1 &\overset{d}{=} \sup_{0\le t_1<t_2\le 1} \frac{2^{-k(H+\alpha-1)}\left|Z(2^k t_1, 2^k t_2)\right|}{|t_2-t_1|^{H+\alpha-1}\left(|\log(t_2-t_1)|^{1/2}+1\right)} \\
&= \sup_{0\le s_1<s_2\le 2^k} \frac{|Z(s_1,s_2)|}{|s_2-s_1|^{H+\alpha-1}\left(|\log(s_2-s_1)-k\log 2|^{1/2}+1\right)} \\
&\ge \sup_{0\le s_1<s_2\le s_1+1\le 2^k} \frac{|Z(s_1,s_2)|}{|s_2-s_1|^{H+\alpha-1}\left(|\log(s_2-s_1)-k\log 2|^{1/2}+1\right)} \\
&\ge \sup_{0\le s_1<s_2\le s_1+1\le 2^k} \frac{|Z(s_1,s_2)|}{|s_2-s_1|^{H+\alpha-1}\left(|\log(s_2-s_1)|^{1/2}+(k\log 2)^{1/2}+1\right)} \\
&\ge \frac{M_{2^k}}{(k\log 2)^{1/2}+1}.
\end{aligned}
$$

Hence, for any $q \ge 1$

$$
\mathsf{E}\big[\, M_{2^k}^q \,\big] \le \mathsf{E}\left[\, M_1^q \,\right]\left((k\log 2)^{1/2}+1\right)^q .
$$

This implies that for any $p > q/2+1$

$$
\mathsf{E}\left[\sum_{k=1}^{\infty}\frac{M_{2^k}^q}{k^p}\right] = \sum_{k=1}^{\infty}\frac{\mathsf{E}\left[\, M_{2^k}\,\right]^q}{k^p} \le C\,\mathsf{E}\left[\, M_1^q \,\right]\sum_{k=1}^{\infty} k^{q/2-p} < \infty.
$$

In particular, the sum $\sum_{k=1}^{\infty}|M_{2^k}|^q k^{-p}$ is finite almost surely, so $M_{2^k} = o(k^{p/q})$, $k\to\infty$, a.s. If we choose some $q > (\gamma-1/2)^{-1}$, then $q/2+1 < \gamma q$. Hence, we can take some $p \in (q/2+1, \gamma q)$ and arrive at $M_{2^k} = o(k^{\gamma})$, $k\to\infty$, a.s. Thus, the random variable $\zeta = \sup_k M_{2^k} k^{-\gamma}$ is finite almost surely.

Obviously, for $t_2 \le 2$

$$
\frac{|Z(t_1,t_2)|}{h(t_2-t_1)\log(t_2+2)} \le \frac{M_2}{\log 2}.
$$

Now let $t_2 \in (2^{k-1}, 2^k]$ for some $k \ge 2$. Then we have for any $t_1 \in [t_2-1, t_2)$

$$
\begin{aligned}
|Z(t_1,t_2)| &\le M_{2^k} h(t_2-t_1) \le \zeta k^{\gamma} h(t_2-t_1) \le \zeta\left(\frac{\log t_2}{\log 2}+1\right)^{\gamma} h(t_2-t_1) \\
&\le 2^{\gamma}\zeta(\log t_2)^{\gamma} h(t_2-t_1) < 2^{\gamma}\zeta(\log(t_2+2))^{\gamma} h(t_2-t_1).
\end{aligned}
$$

Consequently, $\xi_{H,\alpha,\gamma} \le \max\{M_2/\log 2, 2^{\gamma}\zeta\} < \infty$ a.s.

The second statement follows from Fernique's theorem, since $\xi_{H,\alpha,\gamma}$ is a supremum of absolute values of a centered Gaussian family.

2.2 Estimates for Solution of SDE Driven by Fractional Brownian Motion

Consider a stochastic differential equation

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t b(X_s)dB_s^H, \tag{4}$$

where X_0 is nonrandom. In [10], it is shown that this equation has a unique solution under the following assumptions: there exist constants $\delta \in (1/H - 1, 1]$, $K > 0$, $L > 0$ and for every $N \geq 1$ there exists $R_N > 0$ such that

(A) $|a(x)| + |b(x)| \leq K \quad$ for all $x \in \mathbb{R}$,
(B) $|a(x) - a(y)| + |b(x) - b(y)| \leq L\,|x - y| \quad$ for all $x, y \in \mathbb{R}$,
(C) $|b'(x) - b'(y)| \leq R_N\,|x - y|^\delta \quad$ for all $x \in [-N, N]$, $y \in [-N, N]$.

Fix some $\beta \in (1/2, H)$. Denote for $t_1 < t_2$

$$\Lambda_\beta(t_1, t_2) = 1 \vee \sup_{t_1 \leq u < v \leq t_2} \frac{|Z(u, v)|}{(v - u)^{\beta+\alpha-1}}.$$

Define for $a < b$

$$\|f\|_{a,b,\beta} = \sup_{a \leq s < t \leq b} \frac{|f(t) - f(s)|}{|t - s|^\beta}.$$

Theorem 2. *There exists a constant $M_{\alpha,\beta}$ depending on α, β, K, and L such that for any $t_1 \geq 0, t_2 \in (t_1, t_1 + 1]$*

$$|X_{t_2} - X_{t_1}| \leq M_{\alpha,\beta}\left(\Lambda_\beta(t_1, t_2)(t_2 - t_1)^\beta + \Lambda_\beta(t_1, t_2)^{1/\beta}(t_2 - t_1)\right).$$

Proof. The proof follows the lines of [3, Theorem 2].

Fix $t_1 \geq 0$ and $t_2 \in (t_1, t_1 + 1]$. Abbreviate $\Lambda = \Lambda_\beta(t_1, t_2)$. Take any s, t such that $t_1 \leq s < t \leq t_2$. Write

$$|X_t - X_s| \leq \int_s^t |a(X_u)|\,du + \left|\int_s^t b(X_u)dB_u^H\right| \leq K(t - s) + \left|\int_s^t b(X_u)dB_u^H\right|.$$

Estimate

$$\begin{aligned}\left|\int_s^t b(X_u)dB_u^H\right| &\leq \int_s^t \left|\left(D_{s+}^\alpha b(X)\right)(u)\right| \left|\left(D_{t-}^{1-\alpha} B_{t-}^H\right)(u)\right| du \\ &\leq \Lambda \int_s^t \left|\left(D_{s+}^\alpha b(X)\right)(u)\right| (t - u)^{\beta+\alpha-1} du.\end{aligned} \tag{5}$$

Now

$$\left|\left(D_{s+}^{\alpha} b(X)\right)(u)\right| \leq \left(\frac{|b(X_u)|}{(u-s)^{\alpha}} + \int_s^u \frac{|b(X_u)-b(X_v)|}{(u-v)^{1+\alpha}} dv\right)$$
$$\leq K(u-s)^{-\alpha} + L \|X\|_{s,t,\beta} \int_s^u (u-v)^{\beta-\alpha-1} dv$$
$$\leq C_{\alpha,\beta} \left((u-s)^{-\alpha} + \|X\|_{s,t,\beta} (u-s)^{\beta-\alpha}\right).$$

Hence,

$$\left|\int_s^t b(X_s) dB_u^H\right| \leq C_{\alpha,\beta} \Lambda \left((t-s)^{\beta} + \|X\|_{s,t,\beta} (t-s)^{2\beta}\right)$$

and

$$\|X\|_{s,t,\beta} \leq K_{\alpha,\beta} \Lambda \left(1 + \|X\|_{s,t,\beta} (t-s)^{\beta}\right)$$

with a constant $K_{\alpha,\beta}$ depending only on α, β, K, and L. Setting $\Delta = (2K_{\alpha,\beta}\Lambda)^{-1/\beta}$, we obtain $\|X\|_{s,t,\beta} \leq 2K_{\alpha,\beta}\Lambda$ whenever $t-s < \Delta$.

Now, if $0 < t_2 - t_1 \leq \Delta$, then

$$|X_{t_2} - X_{t_1}| \leq \|X\|_{t_1,t_2,\beta} (t_2-t_1)^{\beta} \leq 2K_{\alpha,\beta}\Lambda(t_2-t_1)^{\beta}.$$

On the other hand, if $t_2 - t_1 > \Delta$, then, partitioning the interval $[t_1, t_2]$ into $k = [(t_2-t_1)/\Delta]$ parts of length Δ and, possibly, an extra smaller part, we obtain

$$|X_{t_2} - X_{t_1}| \leq |X_{t_1+\Delta} - X_{t_1}| + \cdots + \left|X_{t_1+k\Delta} - X_{t_1+(k-1)\Delta}\right| + |X_{t_2} - X_{t_1+k\Delta}|$$
$$\leq (k+1)\, 2K_{\alpha,\beta}\Lambda\Delta^{\beta} \leq 4kK_{\alpha,\beta}\Delta^{\beta} \leq 4K_{\alpha,\beta}\Lambda(t_2-t_1)\Delta^{\beta-1}$$
$$= 2(2K_{\alpha,\beta}\Lambda)^{1/\beta}(t_2-t_1).$$

The proof is now complete.

Corollary 1. *For any $\gamma > 1/2$, there exist random variables ξ and ζ such that for all $t_1 \geq 0$, $t_2 \in (t_1, t_1+1]$*

$$|X_{t_2} - X_{t_1}| \leq \zeta(t_2-t_1)^{\beta} (\log(t_2+2))^{\kappa}, \quad \Lambda_{\beta}(t_1,t_2) \leq \xi(\log(t_2+2))^{\kappa\beta},$$

where $\kappa = \gamma/\beta$. Moreover, there exists some $c > 0$ such that $\mathsf{E}\left[\exp\left\{x\xi^2\right\}\right] < \infty$ and $\mathsf{E}\left[\exp\left\{x\zeta^{2\beta}\right\}\right] < \infty$ for $x < c$. In particular, all moments of ξ and ζ are finite.

Proof. From Theorem 1 we have for all $u < v$

$$Z(u,v) \leq \xi_{H,\alpha,\gamma}(v-u)^{H+\alpha-1}\left(|\log(v-u)|^{1/2} + 1\right)(\log(v+2))^{\gamma}$$
$$\leq C_{H,\beta}\xi_{H,\alpha,\gamma}(v-u)^{\beta+\alpha-1}(\log(v+2))^{\gamma},$$

Dividing by $(v-u)^{\beta+\alpha-1}$ and taking supremum over u, v such that $t_1 \le u < v \le t_2$, we get

$$\Lambda_\beta(t_1,t_2) \le 1 \vee \left(C_{H,\beta}\xi_{H,\alpha,\gamma}\left(\log(t_2+2)\right)^\gamma\right) \le \left(1 \vee C_{H,\beta}\xi_{H,\alpha,\gamma}\right)\left(\log(t_2+2)\right)^\gamma .$$

Further, since $\Lambda_\beta(t_1,t_2) \ge 1$ and $t_2 - t_1 \le 1$, it follows from Theorem 2 that

$$|X_{t_2} - X_{t_1}| \le 2M_{\alpha,\beta}\Lambda_\beta(t_1,t_2)^{1/\beta}(t_2-t_1)^\beta.$$

Hence, the desired statement holds with $\xi = 1 \vee C_{H,\beta}\xi_{H,\alpha,\gamma}$ and $\zeta = 2M_{\alpha,\beta}\xi^{1/\beta}$.

The following lemma gives a particular case of Corollary 1, suitable for our needs. Let $\gamma > 1/2$ and $\kappa = \gamma/\beta$ be fixed; ξ and ζ will be the corresponding random variables from Corollary 1.

Lemma 1. *For any $n \ge 2$ and any $t_1, t_2 \in [0, 2^n]$ such that $t_1 < t_2 \le t_1 + 1$*

$$|X_{t_2} - X_{t_1}| \le \zeta n^\kappa (t_2-t_1)^\beta, \quad \Lambda_\beta(t_1,t_2) \le \xi n^\gamma.$$

Proof. In this case

$$\log(t_2+2) \le \log(2^n+2) \le \log 2^{n+1} = (n+1)\log 2 \le n,$$

whence the statement follows.

3 Drift Parameter Estimation

Now we turn to problem of drift parameter estimation in equations of type (4). Let $(\Omega, \mathcal{F})$ be a measurable space and $X : \Omega \to C[0,\infty)$ be a stochastic process. Consider a family of probability measures $\left\{\mathsf{P}^\theta, \theta \in \mathbb{R}\right\}$ on $(\Omega, \mathcal{F})$ such that for each $\theta \in \mathbb{R}$, $\mathcal{F}$ is P^θ-complete, and there is an fBm $\left\{B_t^{H,\theta}, t \ge 0\right\}$ on $(\Omega, \mathcal{F}, \mathsf{P}^\theta)$ such that X solves a parametrized version of (4):

$$X_t = X_0 + \theta \int_0^t a(X_s)ds + \int_0^t b(X_s)dB_s^{H,\theta}. \tag{6}$$

Our main problem is to construct an estimator for θ based on discrete observations of X. Specifically, we will assume that for some $n \ge 1$ we observe the values $X_{t_k^n}$ at the following uniform partition of $[0, 2^n]$: $t_k^n = k2^{-n}, k = 0, 1, \dots, 2^{2n}$.

To simplify the notation, in the following we will fix an arbitrary $\theta \in \mathbb{R}$ and denote simply $B^{H,\theta} = B^H$, $\mathsf{P}^\theta = \mathsf{P}$. We also fix the parameters $\alpha \in (1-H, 1/2)$, $\beta \in (1-\alpha, H)$, $\gamma > 1/2$, and $\kappa = \gamma/\beta$. Finally, with a slight abuse of notation, let ξ and ζ be the random variables from Corollary 1 applied to Eq. (6).

In order to construct a consistent estimator, we need a lemma concerning the discrete approximation of integrals in (6).

Lemma 2. *For all $n \geq 1$ and $k = 1, 2, \ldots, 2^{2n}$*

$$\left| \int_{t_{k-1}^n}^{t_k^n} \left(a(X_u) - a(X_{t_{k-1}^n}) \right) du \right| \leq C \zeta n^\kappa 2^{-n(\beta+1)}$$

and

$$\left| \int_{t_{k-1}^n}^{t_k^n} \left(b(X_u) - b(X_{t_{k-1}^n}) \right) dB_u^{H,\theta} \right| \leq C \xi \zeta n^{\gamma+\kappa} 2^{-2n\beta}.$$

Proof. Write

$$\left| \int_{t_{k-1}^n}^{t_k^n} \left(a(X_u) - a(X_{t_{k-1}^n}) \right) du \right| \leq \int_{t_{k-1}^n}^{t_k^n} \left| a(X_u) - a(X_{t_{k-1}^n}) \right| du$$
$$\leq K \zeta n^\kappa \int_{t_{k-1}^n}^{t_k^n} (u - t_{k-1}^n)^\beta du \leq C \zeta n^\kappa (t_k^n - t_{k-1}^n)^{\beta+1} = C \zeta n^\kappa 2^{-n(\beta+1)}.$$

Similarly to (5),

$$\left| \int_{t_{k-1}^n}^{t_k^n} \left(b(X_u) - b(X_{t_{k-1}^n}) \right) dB_u^H \right|$$
$$\leq \Lambda_\beta(t_{k-1}^n, t_k^n) \int_{t_{k-1}^n}^{t_k^n} \left| D_{t_{k-1}^n+}^\alpha (b(X) - b(X_{t_{k-1}^n}))(u) \right| (t_k^n - u)^{\beta+\alpha-1} du$$
$$\leq \xi n^\gamma \int_{t_{k-1}^n}^{t_k^n} \left| D_{t_{k-1}^n+}^\alpha (b(X) - b(X_{t_{k-1}^n}))(u) \right| (t_k^n - u)^{\beta+\alpha-1} du,$$

and

$$\left| D_{t_{k-1}^n+}^\alpha (b(X) - b(X_{t_{k-1}^n}))(u) \right| \leq \frac{\left| b(X_u) - b(X_{t_{k-1}^n}) \right|}{(u - t_{k-1}^n)^\alpha} + \int_{t_{k-1}^n}^u \frac{\left| b(X_u) - b(X_v) \right|}{(u-v)^{1+\alpha}} dv$$
$$\leq K \zeta n^\kappa \left(u - t_{k-1}^n \right)^{\beta-\alpha} + K \zeta n^\kappa \int_{t_{k-1}^n}^u (u-v)^{\beta-\alpha-1} dv \leq C \zeta n^\kappa \left(u - t_{k-1}^n \right)^{\beta-\alpha}.$$

Then we can write the estimate

$$\left|\int_{t_{k-1}^n}^{t_k^n} \left(b(X_u) - b(X_{t_{k-1}^n})\right) dB_u^H\right| \le C\xi\zeta n^{\gamma+\kappa} \int_{t_{k-1}^n}^{t_k^n} (u - t_{k-1}^n)^{2\beta-1} du$$

$$\le C\xi\zeta n^{\gamma+\kappa} \left(t_k^n - t_{k-1}^n\right)^{2\beta} = C\xi\zeta n^{\gamma+\kappa} 2^{-2n\beta},$$

which finishes the proof.

Now we are ready to construct consistent estimators for θ. In order to proceed, we need a technical assumption, in addition to conditions (A)–(C):

(D) There exists a constant $M > 0$ such that for all $x \in \mathbb{R}$

$$|a(x)| \ge M, \qquad |b(x)| \ge M.$$

Consider now the following estimator:

$$\hat{\theta}_n^{(1)} = \frac{\sum_{k=1}^{2^{2n}-1} \left(t_k^n\right)^{\lambda} \left(2^n - t_k^n\right)^{\lambda} b^{-1}\left(X_{t_{k-1}^n}\right)\left(X_{t_k^n} - X_{t_{k-1}^n}\right)}{\sum_{k=1}^{2^{2n}-1} \left(t_k^n\right)^{\lambda} \left(2^n - t_k^n\right)^{\lambda} b^{-1}\left(X_{t_{k-1}^n}\right) a\left(X_{t_{k-1}^n}\right) \frac{1}{2^n}},$$

where $\lambda = 1/2 - H$.

Theorem 3. *With probability one, $\hat{\theta}_n^{(1)} \to \theta$, $n \to \infty$. Moreover, there exists a random variable η with all finite moments such that $\left|\hat{\theta}_n^{(1)} - \theta\right| \le \eta n^{\kappa+\gamma} 2^{-\rho n}$, where $\rho = (1 - H) \wedge (2\beta - 1)$.*

Proof. It follows from (6) that

$$X_{t_k^n} - X_{t_{k-1}^n} = \theta \int_{t_{k-1}^n}^{t_k^n} a(X_v) dv + \int_{t_{k-1}^n}^{t_k^n} b(X_v) dB_v^H$$

$$= \theta \int_{t_{k-1}^n}^{t_k^n} a\left(X_{t_{k-1}^n}\right) dv + \theta \int_{t_{k-1}^n}^{t_k^n} \left(a(X_v) - a\left(X_{t_{k-1}^n}\right)\right) dv$$

$$+ \int_{t_{k-1}^n}^{t_k^n} b\left(X_{t_{k-1}^n}\right) dB_v^H + \int_{t_{k-1}^n}^{t_k^n} \left(b(X_v) - b\left(X_{t_{k-1}^n}\right)\right) dB_v^H.$$

Then

$$\hat{\theta}_n^{(1)} = \theta + \frac{B_n + E_n + D_n}{A_n},$$

where

$$A_n = 2^{n(2H-3)} \sum_{k=1}^{2^{2n}-1} \left(t_k^n\right)^{\lambda} \left(2^n - t_k^n\right)^{\lambda} a\left(X_{t_{k-1}^n}\right) b^{-1}\left(X_{t_{k-1}^n}\right),$$

$$B_n = 2^{2n(H-1)}\theta \sum_{k=1}^{2^{2n}-1} \left(t_k^n\right)^{\lambda} \left(2^n - t_k^n\right)^{\lambda} b^{-1}\left(X_{t_{k-1}^n}\right) \int_{t_{k-1}^n}^{t_k^n} \left(a(X_v) - a\left(X_{t_{k-1}^n}\right)\right) dv,$$

$$E_n = 2^{2n(H-1)} \sum_{k=1}^{2^{2n}-1} \left(t_k^n\right)^{\lambda} \left(2^n - t_k^n\right)^{\lambda} \left(B^H_{t_k^n} - B^H_{t_{k-1}^n}\right),$$

$$D_n = 2^{2n(H-1)} \sum_{k=1}^{2^{2n}-1} \left(t_k^n\right)^{\lambda} \left(2^n - t_k^n\right)^{\lambda} b^{-1}\left(X_{t_{k-1}^n}\right) \int_{t_{k-1}^n}^{t_k^n} \left(b(X_v) - b\left(X_{t_{k-1}^n}\right)\right) dB^H_v.$$

It is not hard to show that the sequence

$$\gamma_n = 2^{n(2H-3)} \sum_{k=1}^{2^{2n}-1} \left(t_k^n\right)^{\lambda} \left(2^n - t_k^n\right)^{\lambda} = \sum_{k=1}^{2^{2n}-1} \left(\frac{k}{2^{2n}}\right)^{\lambda} \left(1 - \frac{k}{2^{2n}}\right)^{\lambda} \frac{1}{2^{2n}}$$

converges to $\int_0^1 x^{\lambda}(1-x)^{\lambda}dx = B(1+\lambda, 1+\lambda)$, and, hence, is bounded and uniformly positive.

Indeed, $h(x) = x^{\lambda}(1-x)^{\lambda}$ increases for $x \in \left(0, \frac{1}{2}\right]$; then

$$\int_0^{\frac{1}{2}} h(x)dx = \sum_{k=0}^{2^{2n-1}-1} \int_{\frac{k}{2^{2n}}}^{\frac{k+1}{2^{2n}}} h(x)dx < \int_0^{\frac{1}{2^{2n}}} h(x)dx + \sum_{k=1}^{2^{2n-1}} h\left(\frac{k}{2^{2n}}\right) \frac{1}{2^{2n}}.$$

On the other hand,

$$\int_0^{\frac{1}{2}} h(x)dx = \sum_{k=1}^{2^{2n-1}} \int_{\frac{k-1}{2^{2n}}}^{\frac{k}{2^{2n}}} h(x)dx > \sum_{k=1}^{2^{2n-1}} h\left(\frac{k}{2^{2n}}\right) \frac{1}{2^{2n}}.$$

So

$$0 < \int_0^{\frac{1}{2}} h(x)dx - \sum_{k=1}^{2^{2n-1}} h\left(\frac{k}{2^{2n}}\right) \frac{1}{2^{2n}} < \int_0^{\frac{1}{2^{2n}}} h(x)dx \to 0,\ n \to \infty.$$

Hence,

$$\sum_{k=1}^{2^{2n-1}} h\left(\frac{k}{2^{2n}}\right) \frac{1}{2^{2n}} \to \int_0^{\frac{1}{2}} h(x)dx, n \to \infty.$$

Similarly one can prove that

$$\sum_{k=2^{2n-1}+1}^{2^{2n}-1} h\left(\frac{k}{2^{2n}}\right) \frac{1}{2^{2n}} \to \int_{\frac{1}{2}}^1 h(x)dx,\ n \to \infty.$$

By assumption (D), $a(x)b^{-1}(x)$ is bounded away from zero and keeps its sign. Therefore,

$$\liminf_{n\to\infty} |A_n| \geq MK^{-1} \lim_{n\to\infty} \gamma_n = MK^{-1} B(1+\lambda, 1+\lambda) > 0.$$

So it is sufficient to estimate B_n, E_n, and D_n.

By Lemma 2,

$$|B_n| \leq C\,|\theta|\,\zeta n^{\kappa} M^{-1} 2^{n(2H-\beta-3)} \sum_{k=1}^{2^{2n}-1} \left(t_k^n\right)^{\lambda} \left(2^n - t_k^n\right)^{\lambda} \leq C_\theta \zeta n^{\kappa} 2^{-n\beta};$$

$$|D_n| \leq C\xi\zeta n^{\gamma+\kappa} M^{-1} 2^{n(2H-2-2\beta)} \sum_{k=1}^{2^{2n}-1} \left(t_k^n\right)^{\lambda} \left(2^n - t_k^n\right)^{\lambda} \leq C\xi\zeta n^{\gamma+\kappa} 2^{n(1-2\beta)}.$$

Finally we estimate E_n. Start by writing

$$\mathsf{E}\left[\,E_n^2\,\right] = 2^{4n(H-1)} \mathsf{E}\left[\left(\sum_{k=1}^{2^{2n}-1} \int_{t_{k-1}^n}^{t_k^n} \left(t_k^n\right)^{\lambda} \left(2^n - t_k^n\right)^{\lambda} dB_s^H\right)^2\right].$$

According to [12, Corollary 1.9.4], for $f \in L_{1/H}[0,t]$ there exists a constant $C_H > 0$ such that

$$\mathsf{E}\left[\left(\int_0^t f(s)dB_s^H\right)^2\right] \leq C_H \left(\int_0^t |f(s)|^{1/H}\, ds\right)^{2H}.$$

Hence,

$$\mathsf{E}\left[\,E_n^2\,\right] \leq C2^{4n(H-1)} \left(\sum_{k=1}^{2^{2n}-1} \int_{t_{k-1}^n}^{t_k^n} \left(t_k^n\right)^{\lambda/H} \left(2^n - t_k^n\right)^{\lambda/H} ds\right)^{2H}$$

$$= C2^{2n(H-1)} \left(\sum_{k=1}^{2^{2n}-1} \left(\frac{k}{2^{2n}}\right)^{\lambda/H} \left(1 - \frac{k}{2^{2n}}\right)^{\lambda/H} \frac{1}{2^{2n}}\right)^{2H}.$$

As above,

$$\sum_{k=1}^{2^{2n}-1} \left(\frac{k}{2^{2n}}\right)^{\lambda/H} \left(1 - \frac{k}{2^{2n}}\right)^{\lambda/H} \frac{1}{2^{2n}} \to B\left(1+\lambda/H, 1+\lambda/H\right),\ n\to\infty,$$

which implies that $\mathsf{E}\left[E_n^2\right] \le C2^{2n(H-1)}$. Since E_n is Gaussian, $\mathsf{E}\left[|E_n|^p\right] \le C_p 2^{pn(H-1)}$ for any $p \ge 1$. Therefore, for any $\nu > 1$

$$\mathsf{E}\left[\sum_{n=1}^{\infty} \frac{|E_n|^p}{n^\nu 2^{pn(H-1)}}\right] = \sum_{n=1}^{\infty} \frac{\mathsf{E}\left[|E_n|^p\right]}{n^\nu 2^{pn(H-1)}} \le C_p \sum_{n=1}^{\infty} n^{-\nu} < \infty.$$

Consequently,

$$\xi' := \sup_{n\ge 1} \frac{|E_n|}{n^{\nu/p} 2^{n(H-1)}} < \infty$$

almost surely; moreover, by Fernique's theorem, all moments of ξ' are finite.

Let us summarize the estimates:

$$|B_n| \le C_\theta \zeta n^\kappa 2^{-n\beta},\ |D_n| \le C\xi\zeta n^{\gamma+\kappa} 2^{n(1-2\beta)},\ |E_n| \le \xi' n^\delta 2^{n(H-1)},$$

where $\delta > 0$ can be taken arbitrarily small. We have $-\beta < -1/2 < H-1$, $-\beta < 1-2\beta$, so $|B_n|$ is of the smallest order. Which of the remaining two estimates wins depends on values of β and H: for H close to $1/2$, $1-2\beta$ is close to 0, while $H-1$ is close to $-1/2$; for β close to 1, $1-2\beta$ is close to -1, while $H-1$ is close to 0. Thus, we arrive to

$$|B_n| + |E_n| + |D_n| \le \eta n^{\gamma+\kappa} 2^{-\rho n},$$

where $\eta \le C_\theta(\zeta + \xi\zeta + \xi')$, so all its moments are finite. The proof is now complete.

Consider a simpler estimator:

$$\hat{\theta}_n^{(2)} = \frac{\sum_{k=1}^{2^{2n}-1} b^{-1}\left(X_{t_{k-1}^n}\right)\left(X_{t_k^n} - X_{t_{k-1}^n}\right)}{\frac{1}{2^n}\sum_{k=1}^{2^{2n}-1} b^{-1}\left(X_{t_{k-1}^n}\right) a\left(X_{t_{k-1}^n}\right)}.$$

This is a discretized maximum likelihood estimator for θ in Eq. (4), where B^H is replaced by Wiener process. Nevertheless, this estimator is consistent as well. Namely, we have the following result, whose proof is similar to that of Theorem 3, but is much simpler, so we omit it.

Theorem 4. *With probability one, $\hat{\theta}_n^{(2)} \to \theta$, $n \to \infty$. Moreover, there exists a random variable η' with all finite moments such that $\left|\theta_n^{(2)} - \theta\right| \le \eta' n^{\kappa+\gamma} 2^{-\rho n}$.*

Remark 1. Using Theorem 1, it can be shown with some extra technical work that

$$\left|\theta_n^{(i)} - \theta\right| \le \eta_1 n^\mu 2^{-\tau n},\ i = 1, 2, \tag{7}$$

where $\mu = 1/2 + \gamma(1 + 1/H)$, $\tau = (2H - 1) \wedge (1 - H)$; η_1 is a random variable, for which there exists some $c_\theta > 0$ such that $\mathsf{E}\left[\exp\left\{x\eta_1^{1+1/H}\right\}\right] < \infty$ for $x < c_\theta$. Moreover, in order to estimate the estimators reliability, the constant c_θ can be computed explicitly in terms of H, K, L, θ. However, we will not undertake this tedious task.

4 Simulations

In this section we illustrate quality of the estimators with the help of simulation experiments. For each set of parameters, we simulate 20 trajectories of the solution to (6). Then for each of estimators $\theta_n^{(i)}$, $i = 1, 2$, we compute the average relative error $\delta_n^{(i)}$, i.e., the average of values $\left|\theta_n^{(i)} - \theta\right|/\theta$. We remind that for a particular value of n we take 2^{2n} equidistant observations of the process on the interval $[0, 2^n]$.

We start with a case of relatively "tame" coefficients $a(x) = 2\sin x + 3$, $b(x) = 2\cos x + 3$. We choose $\theta = 2$ (Table 1).

The first observation is that the estimators have similar performance. This means that $\theta_n^{(2)}$ is preferable to $\theta_n^{(1)}$, since it does not involve H (which might be unknown) and is computable faster (for $n = 6$, computation of $\theta_n^{(1)}$ takes 473 microseconds on Intel Core i5-3210M processor, while that of $\theta_n^{(2)}$ takes 32 microseconds).

The second observation is that the estimate (7) of the convergence rate is probably not optimal; it seems that the rate of convergence is around 2^{-n}; in particular, it is independent of H. Now take worse coefficients $a(x) = 2\sin x + 2.1$, $b(x) = 2\cos x + 2.1$; again $\theta = 2$ (Table 2).

The relative errors have increased two to three times due to the coefficients approaching zero closer. Also observe that in this case the convergence rate seems better than the estimate (7).

Further we show that, despite condition (D) might seem too restrictive, certain condition that the coefficients are nonzero is required.

To illustrate this, take first $a(x) = 2\cos x + 1$, $b(x) = 2\sin x + 3$, $\theta = 2$. From the first sight, it seems that the estimators should work fine here. Such intuition is based on the observation that the proof of Theorem 3 relies on sufficiently fast

Table 1 Relative errors of estimators $\theta_n^{(i)}$, $i = 1, 2$, for $a(x) = 2\sin x + 3$, $b(x) = 2\cos x + 3$, $\theta = 2$

	$H = 0.6$		$H = 0.7$		$H = 0.8$		$H = 0.9$	
n	$\delta_n^{(1)}$	$\delta_n^{(2)}$	$\delta_n^{(1)}$	$\delta_n^{(2)}$	$\delta_n^{(1)}$	$\delta_n^{(2)}$	$\delta_n^{(1)}$	$\delta_n^{(2)}$
3	0.093	0.093	0.097	0.094	0.098	0.096	0.091	0.092
4	0.043	0.044	0.047	0.047	0.046	0.046	0.048	0.047
5	0.025	0.024	0.027	0.027	0.029	0.029	0.028	0.028
6	0.011	0.011	0.012	0.012	0.016	0.016	0.016	0.016

Table 2 Relative errors of $\theta_n^{(i)}$, $i = 1, 2$, for $a(x) = 2\sin x + 2.1$, $b(x) = 2\cos x + 2.1$, $\theta = 2$

	$H = 0.6$		$H = 0.7$		$H = 0.8$		$H = 0.9$	
n	$\delta_n^{(1)}$	$\delta_n^{(2)}$	$\delta_n^{(1)}$	$\delta_n^{(2)}$	$\delta_n^{(1)}$	$\delta_n^{(2)}$	$\delta_n^{(1)}$	$\delta_n^{(2)}$
3	0.17	0.18	0.18	0.19	0.18	0.18	0.17	0.17
4	0.096	0.097	0.099	0.102	0.099	0.106	0.095	0.099
5	0.045	0.045	0.052	0.052	0.051	0.053	0.046	0.046
6	0.024	0.024	0.021	0.021	0.027	0.028	0.033	0.033

convergence of the denominator to $+\infty$, which somehow should follow from the fact that positive values of the ratio $a(x)/b(x)$ are overwhelming. Unfortunately, this intuition is wrong. Here are ten values of the estimator $\theta_n^{(1)}$ for $H = 0.7$, $n = 6$: 1.3152, 0.6402, 1.9676, 0.9600, 0.4627, 4.7017, 0.8386, 0.8425, 1.0247, 0.3902. Values of the estimator $\theta_n^{(2)}$ are also useless: 0.7499, 0.4081, 1.0179, 0.5725, 0.2668, −3.1605, 0.6556, 0.4413, 0.5586, 0.2115.

Now take $a(x) = 2\cos x + 3$, $b(x) = 2\sin x + 1$ and keep other parameters, i.e. $\theta = 2$, $H = 0.7$, $n = 6$. In this case, here are ten values of the estimator $\theta_n^{(1)}$: 1.5010, 1.9824, 2.0666, 2.0087, 1.6751, 1.8802, 2.1087, 2.3519, 2.0160, 2.0442; and ten values of $\theta_n^{(2)}$: 2.2076, 1.9853, 2.0975, 2.0109, 1.1202, 1.8768, 2.0964, 2.6175, 2.0176, 2.045. Although the performance of the estimators is mediocre, it is clear that it has improved significantly compared to the previous case. We can conclude that small values and sign changes of the coefficient a to zero affect the performance much stronger than those of the coefficient b.

References

1. Bertin, K., Torres, S., Tudor, C.A.: Drift parameter estimation in fractional diffusions driven by perturbed random walks. Statist. Probab. Lett. **81**(2), 243–249 (2011)
2. Bishwal, J.P.N.: Parameter estimation in stochastic differential equations. Lecture Notes in Mathematics, vol. 1923. Springer, Berlin (2008)
3. Hu, Y., Nualart, D.: Differential equations driven by Hölder continuous functions of order greater than 1/2. In: Stochastic analysis and applications, Abel Symposium, vol. 2, pp. 399–413. Springer, Berlin (2007)
4. Hu, Y., Nualart, D.: Parameter estimation for fractional Ornstein-Uhlenbeck processes. Statist. Probab. Lett. **80**(11–12), 1030–1038 (2010)
5. Hu, Y., Nualart, D., Xiao, W., Zhang, W.: Exact maximum likelihood estimator for drift fractional Brownian motion at discrete observation. Acta Math. Sci. Ser. B Engl. Ed. **31**(5), 1851–1859 (2011)
6. Kleptsyna, M.L., Le Breton, A.: Statistical analysis of the fractional Ornstein-Uhlenbeck type process. Stat. Inference Stoch. Process. **5**(3), 229–248 (2002)
7. Kozachenko, Y., Melnikov, A., Mishura, Y.: On drift parameter estimation in models with fractional Brownian motion (2011). ArXiv:1112.2330
8. Kutoyants, Y.A.: Parameter estimation for stochastic processes. Research and Exposition in Mathematics, vol. 6. Heldermann Verlag, Berlin (1984)

9. Liptser, R.S., Shiryaev, A.N.: Statistics of random processes II: Applications. Applications of Mathematics, vol. 6. Springer, Berlin (2001)
10. Lyons, T.J.: Differential equations driven by rough signals. Rev. Mat. Iberoamericana **14**(2), 215–310 (1998)
11. Marcus, M.B.: Hölder conditions for Gaussian processes with stationary increments. Trans. Amer. Math. Soc. **134**, 29–52 (1968)
12. Mishura, Y.S.: Stochastic calculus for fractional Brownian motion and related processes. Lecture Notes in Mathematics, vol. 1929. Springer, Berlin (2008)
13. Prakasa Rao, B.L.S.: Parametric estimation for linear stochastic differential equations driven by fractional Brownian motion. Random Oper. Stoch. Equ. **11**(3), 229–242 (2003)
14. Prakasa Rao, B.L.S.: Statistical Inference for Fractional Diffusion Processes. Wiley Series in Probability and Statistics. Wiley, Chichester (2010)
15. Tudor, C.A., Viens, F.G.: Statistical aspects of the fractional stochastic calculus. Ann. Statist. **35**(3), 1183–1212 (2007)
16. Xiao, W., Zhang, W., Xu, W.: Parameter estimation for fractional Ornstein-Uhlenbeck processes at discrete observation. Appl. Math. Model. **35**(9), 4196–4207 (2011)
17. Xiao, W.L., Zhang, W.G., Zhang, X.L.: Maximum-likelihood estimators in the mixed fractional Brownian motion. Statistics **45**(1), 73–85 (2011)
18. Zähle, M.: Integration with respect to fractal functions and stochastic calculus I. Probab. Theory Relat. Fields **111**(3), 333–374 (1998)

Minimum Contrast Method for Parameter Estimation in the Spectral Domain

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Abstract We provide a concise summary on the method of parameter estimation of random fields in the spectral domain developed in the papers [1–3], which is based on higher-order information and the minimum contrast principle. The exposition covers both continuous and discrete-time cases. Minimum contrast estimators are defined via minimization of a certain empirical spectral functional of kth order based on tapered data. Conditions for consistency and asymptotic normality of the estimators are stated.

1 Introduction: Main Assumptions and Definitions

This chapter is concerned with parameter estimation of random fields in the spectral domain based on higher-order information and minimum contrast method.

Let $X(t)$, $t \in I$, be a real-valued measurable strictly stationary zero-mean random field, where I is $\mathbb{R}^d$ or $\mathbb{Z}^d$ endowed with the measure $\mu(\cdot)$ which is the Lebesgue or the counting measure ($\mu(\{t\}) = 1$), respectively. We will assume throughout that all order cumulants of our field exist and also that the field $X(t)$ has spectral densities of all orders $k = 2, 3, \ldots$, that is, there exist the complex-valued functions $f_k(\lambda_1, \ldots, \lambda_{k-1}) \in L_1\left(\mathbb{S}^{k-1}\right)$ such that the cumulant function of kth order is given by

$$c_k(t_1, \ldots, t_{k-1}) = \int_{\mathbb{S}^{k-1}} f_k(\lambda_1, \ldots, \lambda_{k-1})\, e^{i\sum_{j=1}^{k-1}(\lambda_j, t_j)} d\lambda_1 \ldots d\lambda_{k-1},$$

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where $\mathbb{S} = \mathbb{R}^d$ or $(-\pi, \pi]^d$ for the continuous-parameter or discrete-parameter cases, respectively; (λ_j, t_j) is the inner product of the d-dimensional vectors λ_j and t_j.

Recall that the cumulant of kth order for a random field $X(t)$ is defined as follows:

$$\tilde{c}_k(t_1, \ldots, t_k) = (-i)^k \frac{\partial^k}{\partial x_1 \ldots \partial x_1} \ln E \exp\left\{ i \sum_{j=1}^{k} x_j X(t_j) \right\}\Bigg|_{x_1=\cdots=x_k=0}$$
$$= \sum (-1)^{q-1}(q-1)! m_{\nu_1}(t_{\nu_1}) \ldots m_{\nu_q}(t_{\nu_q}),$$

where the sum is taken over all unordered partitions $\{\nu_1, \ldots, \nu_q\}$ of $\{1, \ldots, k\}$, $m_{\nu_j}(t_{\nu_j}) = EX(t_{j_1}) \ldots X(t_{j_l})$ for $\nu_j = \{j_1, \ldots, j_l\}$. For the case under consideration, when the random field is strictly stationary, $\tilde{c}_k(t_1 + t, \ldots, t_k + t) = \tilde{c}_k(t_1, \ldots, t_k)$ $\forall t_i, t \in I$, and we have denoted above $c_k(t_1, \ldots, t_{k-1}) := \tilde{c}_k(t_1, \ldots, t_{k-1}, 0)$.

We will further assume that the spectral densities depend on an unknown parameter vector $\theta \in \Theta \subset \mathbb{R}^m$:

$$\begin{aligned} f_2(\lambda) &= f_2(\lambda; \theta), \\ f_k(\lambda_1, \ldots, \lambda_{k-1}) &= f_k(\lambda_1, \ldots, \lambda_{k-1}; \theta) \\ &= \mathrm{Re} f_k(\lambda_1, \ldots, \lambda_{k-1}; \theta) + i \mathrm{Im} f_k(\lambda_1, \ldots, \lambda_{k-1}; \theta) \\ &= f_k^{(1)}(\lambda_1, \ldots, \lambda_{k-1}; \theta) + i f_k^{(2)}(\lambda_1, \ldots, \lambda_{k-1}; \theta), \quad k \geq 3; \end{aligned}$$

the parametric set Θ is a compact and the true value of the parameter vector $\theta_0 \in$ *int* Θ, the interior of Θ. Suppose that $f_k(\lambda_1, \ldots, \lambda_{k-1}; \theta_1) \not\equiv f_k(\lambda_1, \ldots, \lambda_{k-1}; \theta_2)$ for $\theta_1 \neq \theta_2$ almost everywhere in $\mathbb{S}^{k-1}$ with respect to the Lebesgue measure.

Note that parameters of various models of stationary processes and fields often appear in the expressions for spectral densities in a simple form which makes very convenient and reasonable their estimation in the spectral domain.

Let the field $X(t)$ be observed over the domain $D_T = [-T, T]^d \subset I$. Extensive studies have been devoted to the estimation of an unknown parameter θ of the second-order spectral densities $f(\lambda) = f(\lambda, \theta)$ via the minimum contrast method. Generally, minimum contrast estimators are defined in the following way:

$$\hat{\theta}_T = \arg\min_{\theta \in \Theta} D(f(\cdot, \theta), \hat{f}_T(\cdot)),$$

where $D(f, g) = \int_{\mathbb{S}} K(f(\lambda), g(\lambda)) d\lambda$ is some criterion to measure the distance (or discrepancy) between f and g, with the corresponding function $K(\cdot, \cdot)$, and $\hat{f}_T(\cdot)$ is a nonparametric estimator of the spectral density (see, e.g., [12]).

Some examples of functions $K(f, g)$, which are of the form $K\left(\frac{f}{g}\right)$, can be found in [12]. In particular, the case when $K(x) = \log x + \frac{1}{x}$ leads to the criterion $D(f, \hat{f}_T) = \int_{\mathbb{S}} \left\{\log f(\lambda, \theta) - \log \hat{f}_T(\lambda) + \frac{\hat{f}_T(\lambda)}{f(\lambda,\theta)}\right\} d\lambda$; this criterion is equivalent to the classical Whittle functional

$$D(f, \hat{f}_T) = \int_{\mathbb{S}} \left\{\log f(\lambda, \theta) + \frac{\hat{f}_T(\lambda)}{f(\lambda, \theta)}\right\} d\lambda,$$

which is the most popular in applications. Note that the Whittle functional and corresponding estimates were originally introduced within maximum likelihood approach.

We will use more general approach to define minimum contrast estimators following Guyon ([10]). Within this approach one defines:

(1) A nonrandom real-valued function $K(\theta_0, \theta)$, $\theta \in \Theta$, to be called a contrast function, such that $K(\theta_0, \theta) \geq 0$ and $K(\theta_0, \theta)$ has its unique minimum at $\theta = \theta_0$;
(2) a contrast field for a contrast function $K(\theta_0, \theta)$, which is a random field $U_T(\theta)$, $\theta \in \Theta$, related to observations $\{X(t),\ t \in D_T\}$, and such that the following relation holds:

$$U_T(\theta) - U_T(\theta_0) \to U(\theta) - U(\theta_0) = K(\theta, \theta_0) \tag{1}$$

in P_0-probability (with $P_0 = P_{\theta_0}$).

The minimum contrast estimator $\hat{\theta}_T$ is defined as a minimum point of the functional $U_T\,(\theta)$:

$$\hat{\theta}_T = \arg\min_{\theta \in \Theta} U_T\,(\theta)\,. \tag{2}$$

With the purpose to use the higher-order information for statistical inference, the minimum contrast estimation procedure was developed in the series of papers [1–3], and here we survey the main results presenting them in the most general form and discussing the conditions used to state the results. The procedure is based on the minimization of the empirical spectral functional of kth order generalizing the approach of the paper [11]. We refer a reader to [1–3] for the proofs and all detail unexplained here.

A number of examples showing the need of higher-order information can be found in [2, 3]; we just point out here that, for some models of processes and fields, consideration of covariances and spectral densities of second order is not always sufficient to estimate all the parameters, or, in some situations, parameters may appear in spectral densities of higher orders in the form more convenient for their estimation.

We will base our analysis on the tapered data. Benefits of tapering the data have been widely reported in the literature; in particular, the use of tapers leads to the bias reduction, which is especially important when dealing with spatial data: tapers can help to fight the so-called edge effects (see, e.g., [10]).

Consider the tapered values

$$\{h_T\,(t)\,X\,(t)\,,\ t\in D_T\}\,,$$

where $h_T\,(t)=h\,(t/T)\,,\,t=(t^{(1)},\ldots,t^{(d)})\in R^d$, and the taper $h\,(t)$ factorizes as $h\,(t)=\prod_{i=1}^d\tilde{h}\left(t^{(i)}\right),t^{(i)}\in R^1$, with $\tilde{h}\,(\cdot)$ satisfying the assumption below.

Assumption H.I. $\tilde{h}\,(t)\,,\ t\in R^1$, is a positive even function of bounded variation with bounded support: $\tilde{h}\,(t)=0$ for $|t|>1$.

Note. All assumptions concerning the tapers will be enumerated separately, starting from the letter 'H'.

Denote

$$\tilde{H}_{k,T}(\lambda)=\int\tilde{h}_T(t)^k e^{-i\lambda t}\mu\,(dt)\,,\ \ H_{k,T}(\lambda)$$

$$=\int h_T(t)^k e^{-i(\lambda,t)}\mu\,(dt)=\prod_{i=1}^d\tilde{H}_{k,T}(\lambda^{(i)});$$

the integrals above are one-dimensional and d-dimensional with corresponding measure $\mu\,(\cdot)$ (for the discrete case, we deal actually with sums); $\tilde{h}_T\,(t)=\tilde{h}\,(t/T)$.

Note that evaluation of asymptotic behavior of spectral estimates is based on the properties of functions $\tilde{H}_{k,T}(\lambda)$, which, in its own turn, is based on properties of functions $\tilde{h}\,(t)$. For example, the assumption that $\tilde{h}\,(t)$ is of bounded variation allows to write down useful upper bounds for $\tilde{H}_{k,T}(\lambda)$. Some other assumptions on $\tilde{h}\,(t)$ are also of use in the literature, such as some kinds of Lipschitz condition (see, e.g., [7]).

Define the finite Fourier transform of tapered data $\{h_T\,(t)\,X\,(t)\,,t\in D_T\}$:

$$d_T^h(\lambda)=\int h_T(t)\,X(t)e^{-i(\lambda,t)}\mu(dt),\ \lambda\in\mathbb{S},\tag{3}$$

the tapered periodograms of the second and the third orders:

$$I_{2,T}^h(\lambda)=\frac{1}{(2\pi)^d\ \,H_{2,T}(0)}\ d_T^h(\lambda)d_T^h(-\lambda)$$

(provided that $H_{2,T}(0)\neq 0$),

$$I_{3,T}^h(\lambda_1,\lambda_2)=\frac{1}{(2\pi)^{2d}H_{3,T}(0)}\ d_T^h(\lambda_1)d_T^h(\lambda_2)d_T^h(-\lambda_1-\lambda_2)$$

(provided that $H_{3,T}(0) \neq 0$), and the tapered periodogram of kth order (provided that $H_{k,T}(0) \neq 0$):

$$I_{k,T}^{h}(\lambda_1, \ldots, \lambda_{k-1}) = \frac{1}{(2\pi)^{(k-1)d} H_{k,T}(0)} \prod_{i=1}^{k} d_T^h(\lambda_i), \ \lambda_i \in \mathbb{S}, \tag{4}$$

where $\sum_{i=1}^{k} \lambda_i = 0$. The statistic (4) is a natural generalization of the second-order periodogram and can be considered as an estimator for the spectral density of kth order at the frequencies $\lambda_1, \ldots, \lambda_k$ such that $\sum_{i=1}^{k} \lambda_i = 0$ but no proper subset of λ_i has sum 0. The hyperplane $\sum_{i=1}^{k} \lambda_i = 0$ sometimes is called the principal manifold; those sets where, moreover, some subset of $\{\lambda_i, i \in \nu\}$, $\nu = \{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$ has sum zero: $\sum_{i \in \nu} \lambda_i = 0$ are called submanifolds (see, e.g., [7, 8]).

To define the estimation procedure in the case when $k \geq 3$, we need the following assumptions. (The case $k = 2$ will be outlined in Sect. 4.)

Assumption I. Let the real-valued functions $w_k^{(i)}(\lambda)$, $i = 1, 2$, $w_{k,0}(\lambda)$, $\lambda \in \mathbb{S}^{k-1}$, and the spectral density of kth order satisfy the following conditions:

(i) $w_k^{(i)}(\lambda)$, $i = 1, 2$, and $w_{k,0}(\lambda)$ satisfy the same conditions of symmetry as the kth-order spectral density;
(ii) $w_{k,0}(\lambda)$ is nonnegative and $w_{k,0}(\lambda) \equiv 0$ on all hyperplanes of the form $\sum_{i \in \nu} \lambda_i = 0$, where $\nu = \{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$ and $1 \leq l < k$;
(iii) $f_k^{(i)}(\lambda; \theta) w_k^{(i)}(\lambda) w_{k,0}(\lambda) \in L_1(\mathbb{S}^{k-1})$, $i = 1, 2$, for all $\theta \in \Theta$;
(iv) $f_k^{(i)}(\lambda; \theta) w_k^{(i)}(\lambda) \geq 0$, $i = 1, 2$, $(\lambda; \theta) \in \mathbb{S}^{k-1} \times \Theta$.

For the case when $k = 3$ there is no need in the function $w_{k,0}(\lambda)$.
Under the Assumption I we set

$$\int_{\mathbb{S}^{k-1}} f_k^{(i)}(\lambda; \theta) w_k^{(i)}(\lambda) w_{k,0}(\lambda) d\lambda = \sigma_k^{(i)}(\theta), \ i = 1, 2 \tag{5}$$

and represent the real and imaginary parts of the spectral density $f_k(\lambda; \theta)$ in the form

$$f_k^{(i)}(\lambda; \theta) w_k^{(i)}(\lambda) = \sigma_k^{(i)}(\theta) \psi_k^{(i)}(\lambda; \theta), \ i = 1, 2, \ \theta \in \Theta, \tag{6}$$

where

$$\psi_k^{(i)}(\lambda; \theta) = \frac{f_k^{(i)}(\lambda; \theta) w_k^{(i)}(\lambda)}{\sigma_k^{(i)}(\theta)}, \quad \int_{\mathbb{S}^{k-1}} \psi_k^{(i)}(\lambda; \theta) w_{k,0}(\lambda) d\lambda = 1.$$

In what follows we will omit the subscript k in functions $w_k^{(i)}$ and $w_{k,0}$.

Introduce the contrast field constructed with the use of the periodogram of kth-order $I_{k,T}^{h}(\lambda)$:

$$U_T(\theta) = -\left(p \int_{\mathbb{S}^{k-1}} \mathrm{Re} I_{k,T}^{h}(\lambda)\, w^{(1)}(\lambda)\, w_0(\lambda) \log \psi_k^{(1)}(\lambda;\theta)\, d\lambda \right.$$
$$\left. + q \int_{\mathbb{S}^{k-1}} \mathrm{Im} I_{k,T}^{h}(\lambda)\, w^{(2)}(\lambda)\, w_0(\lambda) \log \psi_k^{(2)}(\lambda;\theta)\, d\lambda \right), \tag{7}$$

with nonnegative numbers p and q satisfying $p + q = 1$, and define the minimum contrast estimator:

$$\hat{\theta}_T = \arg\min_{\theta \in \Theta} U_T(\theta). \tag{8}$$

The corresponding contrast function is of the form

$$\mathcal{K}(\theta_0;\theta) = p \int_{\mathbb{S}^{k-1}} f_k^{(1)}(\lambda;\theta_0) \log \frac{\psi_k^{(1)}(\lambda;\theta_0)}{\psi_k^{(1)}(\lambda;\theta)} w^{(1)}(\lambda)\, w_0(\lambda)\, d\lambda$$
$$+ q \int_{\mathbb{S}^{k-1}} f_k^{(2)}(\lambda;\theta_0) \log \frac{\psi_k^{(2)}(\lambda;\theta_0)}{\psi_k^{(2)}(\lambda;\theta)} w^{(2)}(\lambda)\, w_0(\lambda)\, d\lambda. \tag{9}$$

Define also the function

$$U(\theta) = -\left(p \int_{\mathbb{S}^{k-1}} f_k^{(1)}(\lambda;\theta_0)\, w^{(1)}(\lambda)\, w_0(\lambda) \log \psi_k^{(1)}(\lambda;\theta)\, d\lambda \right.$$
$$\left. + q \int_{\mathbb{S}^{k-1}} f_k^{(2)}(\lambda;\theta_0)\, w^{(2)}(\lambda)\, w_0(\lambda) \log \psi_k^{(2)}(\lambda;\theta)\, d\lambda \right). \tag{10}$$

To establish the consistency of a minimum contrast estimator $\hat{\theta}_T$, which corresponds to a functional $U_T(\theta)$, one can use the following standard reasonings: to check that the convergence (1) holds in probability, and then, due to Theorem 3.4.1 [10], it is sufficient to prove that the convergence (1) holds uniformly with respect to θ.

The standard approach to state the asymptotic normality of the estimator $\hat{\theta}_T$ is to consider the relation

$$\nabla_\theta U_T^h\left(\hat{\theta}_T\right) = \nabla_\theta U_T^h(\theta_0) + \nabla_\theta \nabla_\theta' U_T^h\left(\theta_T^*\right)\left(\hat{\theta}_T - \theta_0\right), \quad |\theta_T^* - \theta_0| < \left|\hat{\theta}_T - \theta_0\right|,$$

and then evaluate the asymptotic behavior of $\nabla_\theta U_T^h(\theta_0)$ and $\nabla_\theta \nabla_\theta' U_T^h\left(\theta_T^*\right)$.

Therefore, one needs to study large sample properties of the empirical spectral functionals of the form

$$J_{k,T}^{h}(\varphi) = J_{k,T}^{h}(\varphi; w_0) = \int_{\mathbb{S}^{k-1}} I_{k,T}^{h}(\lambda)\, \varphi(\lambda)\, w_0(\lambda)\, d\lambda,$$

where $I^h_{k,T}(\lambda)$ is the periodogram based on tapered data, the function $w_0(\lambda)$ satisfies Assumption I (ii), and $\varphi(\lambda)$ is a weight function.

We consider asymptotic properties of functionals $J^h_{k,T}(\varphi)$ in Sect. 2 and then state the properties of minimum contrast estimators in Sect. 3. In Sect. 4 we give the details for the estimation procedure based on the second-order spectral densities.

2 Asymptotic Properties of Empirical Spectral Functionals

Let us first evaluate the asymptotic behavior of the cumulants of the spectral functionals. In order to calculate the expressions for cumulants of the functionals $J^h_{k,T}(\varphi)$, the following formula for the cumulants of the finite Fourier transform $d^h_T(\lambda)\,,\ \lambda \in \mathbb{S}$, is of use:

$$\begin{aligned}\operatorname{cum}\left(d^h_T(\alpha_1),\ldots,d^h_T(\alpha_k)\right) &= \int_{D^k_T}\prod_{j=1}^{k} h_T(t_j)e^{-i\sum_{j=1}^{k}(t_j,\alpha_j)} \\ &\quad\times \operatorname{cum}\left(X(t_1),\ldots,X(t_k)\right)\mu(dt_1)\ldots\mu(dt_k) \\ &= \int_{\mathbb{S}^{k-1}} f_k(\gamma_1,\ldots,\gamma_{k-1})\int_{D^k_T}\prod_{j=1}^{k} h_T(t_j)e^{-i\sum_{j=1}^{k-1}(t_j,\gamma_j-\alpha_j)} \\ &\quad\times e^{i\left(t_k,-\sum_{j=1}^{k-1}\gamma_j-\alpha_k\right)}\mu(dt_1)\ldots\mu(dt_k)\,d\gamma_1\ldots d\gamma_{k-1} \\ &= \int_{\mathbb{S}^{k-1}} f_k(\gamma_1,\ldots,\gamma_{k-1})\prod_{j=1}^{k-1} H_{1,T}\left(\gamma_j-\alpha_j\right)H_{1,T}\left(-\sum_{1}^{k-1}\gamma_j-\alpha_k\right)d\gamma_1\ldots d\gamma_{k-1},\end{aligned} \tag{11}$$

where

$$H_{1,T}(\lambda) = \int_{D_T} h_T(t)e^{-i(t,\lambda)}\mu(dt).$$

Introduce the next assumption concerning the tapering functions.

Assumption H.II. The functions

$$\Phi^h_{k,T}(\lambda_1,\ldots,\lambda_{k-1}) = \frac{1}{(2\pi)^{d(k-1)}H_{k,T}(0)}\prod_{j=1}^{k-1} H_{1,T}\left(\lambda_j\right)H_{1,T}\left(-\sum_{j=1}^{k-1}\lambda_j\right),$$

$$(\lambda_1,\ldots,\lambda_{k-1}) \in \mathbb{S}^{k-1},$$

have properties of δ-type kernels, that is,

$$\lim_{T\to\infty}\int_{S^{k-1}} G\left(u_1-v_1,\ldots,u_{k-1}-v_{k-1}\right)\Phi_{k,T}^{h}\left(u_1,\ldots,u_{k-1}\right)du_1\ldots du_{k-1}$$
$$=G\left(v_1,\ldots,v_{k-1}\right), \tag{12}$$

provided that the function $G\left(u_1,\ldots,u_{k-1}\right)$ is bounded and continuous at the point $\left(u_1,\ldots,u_{k-1}\right)=\left(v_1,\ldots,v_{k-1}\right)$.

Note that the above assumption holds in the case when $h(t)\equiv 1$ (no tapering), as shown, for example, in [6]. In the discrete case it holds for $\tilde{h}(t)$ of bounded variation (see, e.g., results in [9] for the case $d=1$, which straightforwardly extendable for the case $d>1$ for a taper which factorizes).

Using the formula (11) and the formulae giving expressions for the cumulants of products of random variables via products of cumulants of the individual variables, the cumulants of the functionals $J_{k,T}^{h}$ can be represented in the form of convolutions of kernels $\Phi_{k,T}^{h}$ and functions composed with the use of spectral densities of different orders as follows:

$$E J_{k,T}^{h}(\varphi)=\int_{\mathbb{S}^{k-1}}\Phi_{k,T}^{h}(u)\,G_k\left(u;\varphi,w_0\right)du,$$
$$\operatorname{cov}\left(J_{k,T}^{h}\left(\varphi_1\right),J_{k,T}^{h}\left(\varphi_2\right)\right)=(2\pi)^{n}H_{2k,T}(0)\left(H_{k,T}(0)\right)^{-2}$$
$$\times\int_{\mathbb{S}^{2k-1}}\Phi_{2k,T}^{h}(u)\,G_{2k}\left(u;\varphi_1,\varphi_2,w_0\right)du,$$
$$\operatorname{cum}\left(J_{k,T}^{h}\left(\varphi_1\right),\ldots,J_{k,T}^{h}\left(\varphi_m\right)\right)=(2\pi)^{n(m-1)}H_{km,T}(0)\left(H_{k,T}(0)\right)^{-m}$$
$$\times\int_{\mathbb{S}^{km-1}}\Phi_{km,T}^{h}(u)\,G_{km}\left(u;\varphi_1,\ldots,\varphi_m,w_0\right)du,$$

where

$$G_k(u)=G_k\left(u;\varphi,w_0\right)$$
$$=\sum_{\nu=\left(\nu_1,\ldots,\nu_p\right)}\int_{\mathbb{S}^{k-p}}\prod_{l=1}^{p}f_{|\nu_l|}\left(\lambda_j+u_j,\,j\in\tilde{\nu}_l\right)$$
$$\times\varphi(\lambda)\,w_0(\lambda)\prod_{l=1}^{p-1}\delta\left(\sum_{j\in\nu_l}\left(\lambda_j+u_j\right)\right)\delta\left(\sum_{i=1}^{k}\lambda_i\right)d\lambda', \tag{13}$$

where the sum is taken over all unordered partitions $\left(\nu_1,\ldots,\nu_p\right)$ of the set $\{1,\ldots,k\}$,

$$G_{km}(u) = G_{km}(u;\varphi_1,\dots,\varphi_m,w_0) = \sum_{\nu=(\nu_1,\dots,\nu_p)} \int_{\mathbb{S}^{(k-1)m-p+1}} \prod_{i=1}^{m} \varphi_i\left(\lambda_{(i-1)k+1},\dots,\lambda_{ik}\right)$$

$$\times \prod_{i=1}^{m} w_0\left(\lambda_{(i-1)k+1},\dots,\lambda_{ik}\right) f_{|\nu_1|}\left(u_j+\lambda_j, j\in\tilde{\nu}_1\right)\times\dots\times f_{|\nu_p|}\left(u_j+\lambda_j, j\in\tilde{\nu}_p\right)$$

$$\times \prod_{l=1}^{p-1} \delta\left(\sum_{j\in\nu_l}\left(u_j+\lambda_j\right)\right) \prod_{i=1}^{m} \delta\left(\lambda_{(i-1)k+1}+\dots+\lambda_{ik}\right) d\lambda', \quad (14)$$

where the summation is taken over all indecomposable partitions $\nu = \left(\nu_1,\dots,\nu_p\right)$ of the table $T_{m\times k}$ with the rows $r_1 = (1\dots k)$, $r_2 = (k+1\dots 2k)$, ..., $r_m = (m(k-1)+1\dots mk)$, that is, over those partitions ν of the elements of this table into disjoint sets in which there exist no sets $\nu_{i_1},\dots,\nu_{i_n}$ $(n<p)$ such that for some rows $r_{j_1},\dots,r_{j_m}$ $(m<k)$ of the table, the following equality holds: $r_{j_1}\cup\dots\cup r_{j_m} = \nu_{i_1}\cup\dots\cup\nu_{i_n}$.

Here and in similar formulae below, we use the following notations: having a set of natural numbers ν, we write $|\nu|$ to denote the number of elements in ν and $\tilde{\nu}$ to denote the subset of ν which contains all elements of ν except the last one. In the integrals of the form $\int_{\mathbb{S}^{k-p}} g(u)\prod_{l=1}^{p}\delta\left(\sum_{j\in\nu_l}u_j\right)du'$, where $(\nu_1,\dots,\nu_p)$ is a partition of the set $\{1,\dots,k\}$, integration is understood with respect to $(k-p)d$-dimensional vector u', obtained from the vector $u=(u_1,..,u_k)$ in view of p linear restrictions on k variables u_j.

Using the above formulae for cumulants and properties of kernels $\Phi_{k,T}^{h}$, one can obtain the following result.

Proposition 1. *Let Assumptions H.I, H.II hold:*

(1) If the function $G_k(u)$ given by (13) is bounded and continuous at $u=0$, then

$$EJ_{k,T}^{h}(\varphi) \to \int_{\mathbb{S}^{k-1}} f_k(\lambda)\,\varphi(\lambda)\,w_0(\lambda)\,d\lambda \text{ as } T\to\infty.$$

(2) If the function $G_{2k}(u;\varphi_1,\varphi_2,w_0)$ given by the formula (14) with $m=2$ is bounded and continuous at $u=0$, then as $T\to\infty$

$$\operatorname{cov}\left(T^{d/2}J_{k,T}^{h}(\varphi_1), T^{d/2}J_{k,T}^{h}(\varphi_2)\right) \to (2\pi)^d e_k(h) G_{2k}(0;\varphi_1,\varphi_2,w_0), \quad (15)$$

where

$$e_k(h) = \left(\int(\tilde{h}(t))^{2k}dt\left(\int(\tilde{h}(t))^{k}dt\right)^{-2}\right)^{d}. \quad (16)$$

(3) If the function $G_{km}(u;\varphi_1,\dots,\varphi_m,w_0)$ given by the formula (14) is bounded, then as $T\to\infty$,

$$\operatorname{cum}\left(J_{k,T}^{h}(\varphi_1),\dots,J_{k,T}^{h}(\varphi_m)\right) = O\left(T^{d(1-m)}\right).$$

Basing on the asymptotic behavior of cumulants stated above, one can derive the asymptotic normality result. Let us fix the weight functions $\varphi_1, \ldots, \varphi_m$ and set

$$J_{k,T}^h = \left\{J_{k,T}^h(\varphi_i)\right\}_{i=1,\ldots,m} = \left\{\int_{\mathbb{S}^{k-1}} I_{k,T}^h(\lambda)\,\varphi_i(\lambda)\,w_0(\lambda)\,d\lambda\right\}_{i=1,\ldots,m},$$

$$J_k = \left\{J_k(\varphi_i)\right\}_{i=1,\ldots,m} = \left\{\int_{\mathbb{S}^{k-1}} f_k(\lambda)\,\varphi_i(\lambda)\,w_0(\lambda)\,d\lambda\right\}_{i=1,\ldots,m}.$$

Let $\xi = \{\xi_i\}_{i=1,\ldots,m}$ be a complex-valued Gaussian random vector with mean zero and second-order moments

$$w_{ij} = E\xi_i\bar{\xi}_j = (2\pi)^d\,e_k(h)\,G_{2k}\left(0;\varphi_i,\varphi_j,w_0\right),\ i,j = 1,\ldots,m, \tag{17}$$

where the tapering factor $e_k(h)$ is given by (16).

Proposition 2. *Let the assumptions of Proposition 1 hold and the functions $G_{kl}\left(u;\varphi_{m_1},\ldots,\varphi_{m_l};w_0\right)$ defined by (14) are bounded for all $l = 2,3,\ldots$ and all choices $(m_1,\ldots,m_l)$ with $1 \le m_i \le m$, $i = 1,\ldots,l$. Then as $T \to \infty$,*

$$T^{d/2}\left(J_{k,T}^h - EJ_{k,T}^h\right) \xrightarrow{\mathcal{D}} \xi, \tag{18}$$

and, moreover, if

$$T^{d/2}\left(EJ_{k,T}^h(\varphi_i) - J_k(\varphi_i)\right) \to 0 \ \text{ as } \ T \to \infty, \quad i = 1,\ldots,m, \tag{19}$$

then as $T \to \infty$,

$$T^{d/2}\left(J_{k,T}^h - J_k\right) \xrightarrow{\mathcal{D}} \xi. \tag{20}$$

For the discrete-time case and $k = 2$, sufficient conditions for (19) were presented in Guyon [10] in the following form: $d = 1,2,3$ and $\varphi_i \in C(\mathbb{S})$, and the taper h and the spectral density f_2 belong to $C^2(\mathbb{S})$.

In view of the importance of the condition (19) for derivation of asymptotic normality results, we next consider the problem of bias of spectral estimates in more detail and present below the results from the paper [4], covering both continuous and discrete-parameter cases (i.e., random field over $\mathbb{R}^d$ as well as over $\mathbb{Z}^d$) and spectral functionals of the second, third, and higher orders.

Assumption H.III. The taper $\tilde{h}(t)$ is a Lipschitz-continuous function on $[-1,1]$.

Assumption H.IV. The function $\varkappa_2^h(u) = \left|\int \tilde{h}(t)\,e^{-itu}dt\right|^2$ satisfies $\int |u|^l \varkappa_2^h(u)\,du < \infty, l = 1,2.$

Theorem 1. *Let the taper $\tilde{h}(t)$ satisfy the Assumption H.III for the case of discrete-parameter fields, and for the case of continuous-parameter fields, let the Assumption H.IV hold.*

Suppose further that one of the following conditions holds:

(i) f_2 *is twice boundedly differentiable and* $\varphi_2 \in L_1(\mathbb{S})$;
(ii) φ_2 *is twice boundedly differentiable;*
(iii) the convolution $g_2(u) = \int_{\mathbb{S}} f_2(\lambda)\varphi(\lambda + u)\, d\lambda$ *is twice boundedly differentiable at zero.*

Then as $T \to \infty$,

$$E J_{2,T}^h(\varphi_2) - J_2(\varphi_2) = O\left(T^{-2}\right). \tag{21}$$

Assumption H.V. The function $\varkappa_3^h(u) = \int \tilde{h}(t)\, e^{-itu} dt \int \tilde{h}^2(t)\, e^{itu} dt$ satisfies $\int |u|^l \varkappa_3^h(u)\, du < \infty, l = 1, 2.$

Theorem 2. *Let the taper $\tilde{h}(t)$ satisfy the Assumption H.III for the case of discrete-parameter fields, and for the case of continuous-parameter fields, let the Assumption V hold.*

Suppose further that one of the following conditions holds:

(i) f_3 *is twice boundedly differentiable and* $\varphi_3 \in L_1(R^{2d})$;
(ii) φ_3 *is twice boundedly differentiable;*
(iii) the convolution $g_3(u_1, u_2) = \int_{\mathbb{S}^2} \varphi_3(\lambda_1, \lambda_2) f_3(\lambda_1 + u_1, \lambda_2 + u_2)\, d\lambda_1 d\lambda_2$ *is twice boundedly differentiable at zero.*

Then as $T \to \infty$,

$$E J_{3,T}^h(\varphi_3) - J_3(\varphi_3) = O\left(T^{-2}\right). \tag{22}$$

As we can see, if the standard normalizing factor $T^{d/2}$ is applied (under the conditions of Theorems 1 and 2), then the bias will be of order $T^{d/2-2}$, that is, we can handle dimensions $d = 1, 2, 3$ using the tapered periodogram in the estimator for $J_2(\varphi)$ and $J_3(\varphi)$. Also, when estimating the integrals of spectrum and bispectrum, one has a possibility for a trade-off between the smoothness properties of a spectral density f and that of weight function φ: as Theorems 1 and 2 show, one can relax conditions on f imposing at the same time stronger conditions on φ.

For the case $k > 3$, in order to exclude from consideration the points lying on submanifolds, as prescribed in the definition of the periodogram of order $k > 3$ (see formula (4)), we consider the empirical spectral functional

$$J_{k,T}^*(\varphi_k) = J_{k,T,\varepsilon}^*(\varphi_k) = \int_{\mathbb{S}_\varepsilon^{k-1}} \varphi_k(\lambda)\, I_{k,T}^h(\lambda)\, d\lambda, \tag{23}$$

as an estimator for

$$J_k^*(\varphi_k) = J_{k,\varepsilon}^*(\varphi_k) = \int_{\mathbb{S}_\varepsilon^{k-1}} \varphi_k(\lambda)\, f_k(\lambda)\, d\lambda, \tag{24}$$

for all $\varepsilon > 0$, where the integration is taken over $\mathbb{S}^{k-1}$, but avoiding the frequencies on and neighboring to the submanifolds $\{\sum_{i \in \nu} \lambda_i = 0$, where $\nu = \{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}\}$. More precisely, $\mathbb{S}_\varepsilon^{k-1} = \mathbb{S}^{k-1} \backslash \{\lambda : \left|\sum_{i \in \nu} \lambda_i\right| < \varepsilon$ for all $\nu \subset \{1, \ldots, k\}\}$, where $|y| = \max_{1 \le i \le d} |y^{(i)}|$.

Assumption H.VI. The function $\varkappa_k^h(u) = \int \tilde{h}(t) e^{-itu} dt \int \tilde{h}^{k-1}(t) e^{itu} dt$ satisfies
$\int |u|^l \varkappa_k^h(u)\, du < \infty, l = 1, 2.$

Theorem 3. *Let the taper $\tilde{h}(t)$ satisfy the Assumption H.III for the case of discrete-parameter fields, and for the case of continuous-parameter fields, let the Assumption H.IV hold. Suppose further that the spectral density of the kth-order $f_k(\lambda)$ is twice boundedly differentiable, $\varphi_k \in L_1(\mathbb{S}^{k-1})$, and the spectral densities $f_l(\lambda) \in L_{k-2}(\mathbb{S}^{l-1})$ for $l = 2, \ldots, k-2$.*

Then as $T \to \infty$,

$$E J_{k,T}^*(\varphi_k) - J_k^*(\varphi_k) = O\left(T^{-2}\right). \tag{25}$$

Note that in the discrete-time case the domain over which the field is observed has been traditionally taken to be $D_T = [1, T]^d$. The presented results remain valid for such a domain as well; one just needs to adjust the assumptions on a taper $\tilde{h}(t)$. Namely, Assumption H.I must be modified as follows: $\tilde{h}(t)$ is a positive function of bounded variation with support on $[0, 1]$ and $\tilde{h}(0) = 0$, $\tilde{h}(1 - v) = \tilde{h}(v)$ for $0 \le v \le \frac{1}{2}$.

An example of a taper $\tilde{h}(t)$ satisfying the assumptions introduced in the discrete case is $\tilde{h}(t) = \frac{1}{2}(1 + \cos(4\pi t)$, $t \in [-1, 1]$, which is a modification of the well-known cosine bell (or the Tukey-Hanning taper) $\tilde{h}(t) = \frac{1}{2}(1 - \cos(2\pi t)$, $t \in [0, 1]$, suitable for the domain $D_T = [1, T]^d$. For the continuous case, one can consider, for example, the taper $\tilde{h}(t) = 1 - |t|$ for $|t| \le 1$.

3 Asymptotic Properties of Minimum Contrast Estimators

We return now to consideration of the minimum contrast estimators (8) and formulate the results on their consistency and asymptotic normality. The derivation of these results is based on the results of the previous section.

We will need the following assumptions.

Assumption II. The derivatives $\nabla_\theta \psi_k^{(i)}(\lambda; \theta)$, $i = 1, 2$, exist and

$$\nabla_\theta \int_{\mathbb{S}^{k-1}} \psi_k^{(i)}(\lambda; \theta)\, w_{k,0}(\lambda)\, d\lambda' = \int_{\mathbb{S}^{k-1}} \nabla_\theta \psi_k^{(i)}(\lambda; \theta)\, w_{k,0}(\lambda). \tag{26}$$

Denote $\varphi_i(\lambda) = w^{(i)}(\lambda) \log \psi_k^{(i)}(\lambda)$.

Assumption III. For all $\theta \in \Theta$, the functions $G_k(u; \varphi_i, w_0), i = 1, 2$, are bounded and continuous at the point $u = 0$, where $G_k(u; \varphi, w)$ is defined by the formula (13).

Assumption IV. The functions $G_{2k}(u; \varphi_i, \varphi_i, w_0), i = 1, 2$,

(i) are bounded
(ii) are continuous at the point $u = 0$.

Assumption V. There exist nonnegative functions $v_1(\lambda)$ and $v_2(\lambda)$ such that

(i) the functions

$$a_k^{(i)}(\lambda; \theta) = v_i(\lambda) \log \psi_k^{(i)}(\lambda; \theta), \ i = 1, 2,$$

are uniformly continuous in $\mathbb{S}^{k-1} \times \Theta$.

(ii) the functions $G_k\left(u; \frac{w^{(i)}}{v_i}, w_0\right), i = 1, 2$, are bounded and continuous at $u = 0$ and the functions $G_{2k}\left(u; \frac{w^{(i)}}{v_i}, \frac{w^{(i)}}{v_i}, w_0\right), i = 1, 2$, are bounded.

Assumption VI. $w_k^{(1)}(\lambda) \,\mathrm{Re} I_k^T(\lambda) \geq 0$, $w_k^{(2)}(\lambda) \,\mathrm{Im} I_k^T(\lambda) \geq 0$.

Theorem 4. *Let the Assumptions I, III, IV(i), V, VI, and H.I, H.II hold. Then the function $\mathcal{K}(\theta_0; \theta)$ given by (10) is the contrast function for the contrast field $U_T^h(\theta)$ given by (8). The resulting minimum contrast estimator $\hat{\theta}_T$ is a consistent estimator of the parameter vector θ, that is, $\hat{\theta}_T \to \theta_0$ in P_0-probability as $T \to \infty$ and the estimators*

$$\hat{\sigma}_{k,T}^{(1)} = \int_{\mathbb{S}^{k-1}} \mathrm{Re} I_{k,T}^h(\lambda) \, w^{(1)}(\lambda) \, w_0(\lambda) \, d\lambda \tag{27}$$

and

$$\hat{\sigma}_{k,T}^{(2)} = \int_{\mathbb{S}^{k-1}} \mathrm{Im} I_{k,T}^h(\lambda) \, w^{(2)}(\lambda) \, w_0(\lambda) \, d\lambda \tag{28}$$

are consistent estimators of $\sigma_k^{(1)}(\theta)$ and $\sigma_k^{(2)}(\theta)$, respectively.

It should be noted that Theorem 4 actually holds with the use of the untapered periodogram ($h(t) = 1$) in the functional (8) for the multidimensional case as well as for the one-dimensional case. However, in order to state the result on asymptotic normality of the estimator (9), tapering is essential (or another adjustment of the periodogram is needed such as constructing the kth-order periodogram by means of unbiased estimators of the moments of second and higher orders).

We need some further assumptions to state the result on asymptotic normality of the estimator $\hat{\theta}_T$.

Assumption VII. The functions $\psi_k^{(i)}(\lambda; \theta), i = 1, 2$, are twice differentiable in the neighborhood of the point θ_0 and the functions

$$\varphi_l^{ij}(\lambda;\theta) = w^{(l)}(\lambda)\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log\psi_k^{(l)}(\lambda;\theta), \quad i,j=1,\ldots,m,\ l=1,2,\ \theta\in\Theta, \tag{29}$$

$$g_k^{(i)}(\lambda;\theta) = w^{(1)}(\lambda)\frac{\partial}{\partial\theta_i}\log\psi_k^{(1)}(\lambda;\theta), \quad i=1,\ldots,m,\ \theta\in\Theta, \tag{30}$$

$$g_k^{(i+m)}(\lambda;\theta) = w^{(2)}(\lambda)\frac{\partial}{\partial\theta_i}\log\psi_k^{(2)}(\lambda;\theta), \quad i=1,\ldots,m,\ \theta\in\Theta$$

are such that

(i) the functions $G_k(u;\varphi_l^{ij},w_0)$, $i,j=1,\ldots,m$, $l=1,2$, are bounded and continuous at $u=0$ for all $\theta\in\Theta$.
(ii) the functions $G_{2k}(u;\varphi_l^{ij},\varphi_l^{ij},w_0)$, $i,j=1,\ldots,m$, $l=1,2$, are bounded for all $\theta\in\Theta$;
(iii) the functions $G_{kl}(u;g_k^{(m_1)},\ldots,g_k^{(m_l)},w_0)$ are bounded for all $\theta\in\Theta$, $l=2,3,\ldots$ and all choices of $(m_1,\ldots,m_l)$, $1\le m_i\le 2m$, $i=1,\ldots,l$.
(iv) $T^{d/2}(E\int_{\mathbb{S}^{k-1}} I_{k,T}^h(\lambda)\,g_k^{(i)}(\lambda)\,w_0(\lambda)\,d\lambda - \int_{\mathbb{S}^{k-1}} f_k(\lambda)\,g_k^{(i)}(\lambda)\,w_0(\lambda)\,d\lambda)\to 0$, for $i=1,\ldots,2m$ and all $\theta\in\Theta$;
(v) the derivatives $\frac{\partial^2}{\partial\theta_i\partial\theta_j}\log\psi_k^{(l)}(\lambda;\theta)$, $i,j=1,\ldots,m$, $l=1,2$, are continuous in θ.

Assumption VIII. The matrices $S_k(\theta)=\{s_{ij}^{(k)}(\theta)\}_{i,j=\overline{1,m}}$ and $A_k(\theta)=\{a_{ij}^{(k)}(\theta)\}_{i,j=\overline{1,m}}$ are positive definite, where

$$\begin{aligned}
s_{ij}^{(k)}(\theta) &= p\int_{\mathbb{S}^{k-1}} f_k^{(1)}(\lambda;\theta)\,\varphi_1^{ij}(\lambda;\theta)\,w_0(\lambda)\,d\lambda' \\
&\quad + q\int_{\mathbb{S}^{k-1}} f_k^{(2)}(\lambda;\theta)\,\varphi_2^{ij}(\lambda;\theta)\,w_0(\lambda)\,d\lambda' \\
&= \sigma_k^{(1)}(\theta)\,p\int_{\mathbb{S}^{k-1}}\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\psi_k^{(1)} - \frac{1}{\psi_k^{(1)}}\,\frac{\partial}{\partial\theta_i}\psi_k^{(1)}\frac{\partial}{\partial\theta_j}\psi_k^{(1)}\right)d\lambda' \\
&\quad + \sigma_k^{(2)}(\theta)\,q\int_{\mathbb{S}^{k-1}}\left(\frac{\partial^2}{\partial\theta_i\partial\theta_j}\psi_k^{(2)} - \frac{1}{\psi_k^{(2)}}\,\frac{\partial}{\partial\theta_i}\psi_k^{(2)}\frac{\partial}{\partial\theta_j}\psi_k^{(2)}\right)d\lambda',
\end{aligned} \tag{31}$$

$$\begin{aligned}
a_{ij}^{(k)}(\theta) &= \frac{1}{2}\Big\{p^2\mathrm{Re}\,G_{2k}\left(0;g_k^{(i)},g_k^{(j)},w_0\right) + q^2\mathrm{Re}\,G_{2k}\left(0;g_k^{(i+m)},g_k^{(j+m)},w_0\right) \\
&\quad + pq\mathrm{Im}\,G_{2k}\left(0;g_k^{(i+m)},g_k^{(j)},w_0\right) - pq\mathrm{Im}\,G_{2k}\left(0;g_k^{(i)},g_k^{(j+m)},w_0\right)\Big\}.
\end{aligned} \tag{32}$$

Theorem 5. *Let the Assumptions I–VII, H.I, and H.II be satisfied. Then, as* $T\to\infty$,

$$T^{d/2}\left(\hat{\theta}_T-\theta_0\right)\xrightarrow{\mathcal{D}} N_m\left(0,e(h)S_k^{-1}\left(\theta_0\right)A_k\left(\theta_0\right)S_k^{-1}\left(\theta_0\right)\right), \tag{33}$$

where the matrices $S_k(\theta)$ and $A_k(\theta)$ are given by the formulae (33) and (34), respectively, and the tapering factor is given by the formula (16)

Note that the condition needed to control the bias is formulated here in the most general form in the Assumption VIII (iv). For $d=1,2,3$ and $k=2$ or $k=3$, this condition holds if the conditions of Theorems 1 and 2 are satisfied. For the case of general $k>3$ and $d=1,2,3$, one can consider the modified functional $\tilde{U}_T$, where the function w_0 is omitted but, instead, the domain of integration is taken to be $\mathbb{S}_{\varepsilon}^{k-1}$ (see (23)); then Assumption VIII (iv) holds under the conditions of Theorem 3. All the results hold true for such a modified functional.

4 Notes on the Estimation Procedure for the Second-Order Case

For the case of second-order spectral density, we introduce the following assumption.

Assumption IX. There exists a nonnegative even function $w(\lambda), \lambda\in\mathbb{S}$, such that $f(\lambda;\theta)w(\lambda)\in L_1(\mathbb{S})\ \forall\theta\in\Theta$.

We set

$$\sigma^2(\theta)=\int_{\mathbb{S}} f(\lambda;\theta)w(\lambda)\,d\lambda$$

and represent the spectral density in the form

$$f(\lambda;\theta)=\sigma^2(\theta)\psi(\lambda;\theta),\ \lambda\in\mathbb{S},\ \theta\in\Theta.$$

For the function $\psi(\lambda,\theta), \lambda\in\mathbb{S}, \theta\in\Theta$, we have

$$\int_{\mathbb{S}}\psi(\lambda;\theta)w(\lambda)\,d\lambda=1$$

and we additionally suppose the following:

Assumption X. The derivatives $\nabla_\theta\psi(\lambda;\theta)$ exist and

$$\nabla_\theta\int_{\mathbb{S}}\psi(\lambda;\theta)w(\lambda)\,d\lambda=\int_{\mathbb{S}}\nabla_\theta\psi(\lambda;\theta)w(\lambda)\,d\lambda=0.$$

The contrast field in this case is of the form

$$U_T(\theta)=-\int_{\mathbb{S}} I_{2,T}^h(\lambda)w(\lambda)\log\psi(\lambda;\theta)\,d\lambda,\quad \theta\in\Theta. \tag{34}$$

Conditions needed for consistency and asymptotic normality of the corresponding minimum contrast estimator can be written analogously to those presented in the previous section for the general kth-order case. These conditions become of much simpler form for Gaussian fields, when we deal only with second-order spectral density (higher-order (cumulant) spectral densities are zero in this case), and for linear fields.

We present below the results for linear processes, that is, we consider the stationary process $X(t)$ which admits the representation:

$$X(t) = \int_{u\in I} \hat{a}(t-u)\xi(du), \quad t \in I, \tag{35}$$

with a square-integrable kernel $\hat{a}(t)$, $t \in I$, with respect to, in continuous-time case, an independently scattered random measure with finite second moment, that is a homogeneous random measure $\xi(A)$, $A \subset \mathbb{R}^d$, with finite second moments and independent values over disjoint sets, and, in the discrete-time case, $\xi(u)$, $u \in Z$, are independent random variables such that $E\xi(0) = 0$, $E|\xi(0)|^k \le c_k < \infty$, $k = 1, 2, \ldots$ (see, e.g., [5]).

In this case we have the explicit representation of cumulants:

$$\tilde{c}_k(t_1, \ldots, t_k) = d_k \int_{s\in I} \prod_{j=1}^{k} \hat{a}(t_j - s)\, \nu(ds),$$

where d_k is the kth cumulant of $\xi(I_1)$ with I_1 being the unit rectangle in the continuous case and d_k is the kth cumulant of $\xi(0)$ in the discrete case, that is, in particular,

$$d_2 = \mathbb{E}\xi(I_1)^2, \quad d_4 = \mathbb{E}(\xi(I_1)^4) - 2[\mathbb{E}(\xi(I_1)^2)]^2$$

or

$$d_2 = \mathbb{E}\xi(0)^2, \quad d_4 = \mathbb{E}(\xi(0)^4) - 2[\mathbb{E}(\xi(0)^2)]^2$$

for the continuous and discrete cases, respectively.

For the spectral densities we have the representations:

$$f_k(\lambda_1, \ldots, \lambda_{k-1}) = d_k\, a\left(-\sum_{i=1}^{k-1} \lambda_i\right) \prod_{i=1}^{k-1} a(\lambda_i) = d_k \prod_{i=1}^{k} a(\lambda_i)\delta\left(\sum_{j=1}^{k} \lambda_j\right).$$

Assumption XI. $f(\lambda;\theta_0)\, w(\lambda) \log\psi(\lambda;\theta) \in L_1(\mathbb{S}) \cap L_2(\mathbb{S}), \quad \forall\theta \in \Theta.$

Assumption XII. There exists a function $v(\lambda)$, $\lambda \in \mathbb{S}$, such that

(i) the function $h(\lambda;\theta) = v(\lambda)\log\psi(\lambda;\theta)$ is uniformly continuous in $\mathbb{S} \times \Theta$;
(ii) $f(\lambda;\theta_0)\frac{w(\lambda)}{v(\lambda)} \in L_1(\mathbb{S}) \cap L_2(\mathbb{S})$.

Theorem 6. *Let Assumptions IX–XII be satisfied. Then the minimum contrast estimator $\hat{\theta}_T$ is a consistent estimator of the parameter θ, that is, $\hat{\theta}_T \to \theta_0$ in P_0-probability as $T \to \infty$, and the estimator $\hat{\sigma}_T^2 = \int_{\mathbb{S}} I_T(\lambda) w(\lambda) d\lambda$ is a consistent estimator of the parameter $\sigma^2(\theta)$, that is, $\hat{\sigma}_T^2 \to \sigma^2(\theta_0)$ in P_0-probability as $T \to \infty$.*

To formulate the result on the asymptotic normality of the minimum contrast estimator, we need some further assumptions.

Assumption XIII. The function $\psi(\lambda;\theta)$ is twice differentiable in a neighborhood of the point θ_0 and

(i) $f(\lambda;\theta) w(\lambda) \frac{\partial^2}{\partial\theta_i \partial\theta_j} \log \psi(\lambda,\theta) \in L_1(\mathbb{S}) \cap L_2(\mathbb{S}), i, j = 1,\ldots,m, \theta \in \Theta$;
(ii) $f(\lambda;\theta_0) \in \mathbf{L}_p(\mathbb{S}), \phi \in \mathbf{L}_q(\mathbb{S})$, for some p, q such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, $i = 1,\ldots,m$, $\theta \in \Theta$, where $\phi = w(\lambda) \frac{\partial}{\partial\theta_i} \log \psi(\lambda,\theta)$;
(iii) $T^{1/2} \int_{\mathbb{S}} (EI_T(\lambda) - f(\lambda;\theta_0)) w(\lambda) \frac{\partial}{\partial\theta_i} \log \psi(\lambda;\theta)\, d\lambda \to 0$ as $T \to \infty$, for all $i = 1,\ldots,m, \theta \in \Theta$;
(iv) the second-order derivatives $\frac{\partial^2}{\partial\theta_i \partial\theta_j} \log \psi(\lambda;\theta), i = 1,\ldots,m$, are continuous in θ.

Assumption XIV. The matrices $S(\theta) = \left(s_{ij}(\theta)\right)_{i,j=1,\ldots,m}$ and $A(\theta) = \left(a_{ij}(\theta)\right)_{i,j=1,\ldots,m}$ are positive definite where

$$
\begin{aligned}
s_{ij}(\theta) &= \int_{\mathbb{S}} f(\lambda;\theta) w(\lambda) \frac{\partial^2}{\partial\theta_i \partial\theta_j} \log \psi(\lambda;\theta) d\lambda \\
&= \sigma^2(\theta) \int_{\mathbb{S}} w(\lambda) \left[\frac{\partial^2}{\partial\theta_i \partial\theta_j} \psi(\lambda,\theta) - \frac{1}{\psi(\lambda,\theta)} \frac{\partial}{\partial\theta_i} \psi(\lambda,\theta) \frac{\partial}{\partial\theta_j} \psi(\lambda,\theta) \right] d\lambda, \\
a_{ij}(\theta) &= 4\pi \int_{\mathbb{S}} f^2(\lambda;\theta) w^2(\lambda) \frac{\partial}{\partial\theta_i} \log \psi(\lambda;\theta) \frac{\partial}{\partial\theta_j} \log \psi(\lambda;\theta) d\lambda \\
&\quad + 2\pi \frac{d_4}{d_2^2} \int_{\mathbb{S}} \frac{w(\lambda) f(\lambda;\theta)}{\psi(\lambda;\theta)} \frac{\partial}{\partial\theta_i} \psi(\lambda;\theta) d\lambda \int_{\mathbb{S}} \frac{w(\lambda) f(\lambda;\theta)}{\psi(\lambda;\theta)} \frac{\partial}{\partial\theta_j} \psi(\lambda;\theta) d\lambda \\
&= 4\pi \left(\sigma^2(\theta)\right)^2 \int_{\mathbb{S}} w^2(\lambda) \frac{\partial}{\partial\theta_i} \psi(\lambda;\theta) \frac{\partial}{\partial\theta_j} \psi(\lambda;\theta) d\lambda \\
&\quad + 2\pi \frac{d_4}{d_2^2} \left(\sigma^2(\theta)\right)^2 \int_{\mathbb{S}} w(\lambda) \frac{\partial}{\partial\theta_i} \psi(\lambda;\theta) d\lambda \int_{S} w(\lambda) \frac{\partial}{\partial\theta_j} \psi(\lambda;\theta) d\lambda
\end{aligned}
$$

Theorem 7. *Let the Assumptions IX–XIV be satisfied. Then as $T \longrightarrow \infty$*

$$
T^{1/2}\left(\widehat{\theta_T} - \theta_0\right) \xrightarrow{\mathfrak{D}} N_m\left(0, S^{-1}(\theta_0) A(\theta_0) S^{-1}(\theta_0)\right),
$$

where $N_m(\cdot,\cdot)$ denotes the m-dimensional Gaussian law.

The condition XIII (ii) guarantees that the limiting normal law holds for the normalized spectral functional of the second-order $T^{1/2}(J_T(\phi_i) - E J_T(\phi_i))$, where $J_T(\phi) = \int_{\mathbb{S}} I_T(\lambda)\,\phi(\lambda)\,d\lambda$ and $\phi_i(\lambda) = w(\lambda)\,\frac{\partial}{\partial\theta_i}\log\psi(\lambda,\theta)$ (see [5]).

Note that conditions for the asymptotic normal law for quadratic forms (or spectral functionals of the second order) $\int_{\mathbb{S}} I_T(\lambda)\phi(\lambda)\,d(\lambda)$, where I_T is the periodogram of the second order, have been intensively studied in the literature. For the processes all of whose moments exist, one can apply the methods of moments or cumulants to prove the convergence. Another approach is to reduce the problem of convergence of the spectral functionals $\int_{\mathbb{S}} I_T(\lambda)\phi(\lambda)\,d(\lambda)$ to the convergence of the empirical covariance function (and its integrals), which, in its own turn, can be reduced to the problem of validity of a central limit theorem for the process $Y_u = X_{t+u}X_t$, and at this point, one can apply the results on the central limit theorem for stationary processes. Thus, assuming the process to be weakly dependent (spectral density to be square integrable), we can demand instead of the Assumption XIII (ii) the asymptotic normality of empirical covariance function and its weighted integrals, and we note that sufficient conditions for this can be given, for example, for strongly mixing processes via conditions on mixing coefficients (demanding, e.g., the exponential decay for the Rosenblatt mixing coefficient α).

References

1. Anh, V.V., Leonenko, N.N., Sakhno, L.M.: Quasilikelihood-based higher-order spectral estimation of random processes and fields with possible long-range dependence. J. Appl. Probab. **41A**, 35–54 (2004)
2. Anh, V.V., Leonenko, N.N., Sakhno, L.M.: Minimum contrast estimation of random processes based on information of second and third orders. J. Statist. Plann. Inference. **137**, 1302–1331 (2007)
3. Anh, V.V., Leonenko, N.N., Sakhno, L.M.: Statistical inference using higher-order information. J. Multivariate Anal. **98**(4), 706–742 (2007)
4. Anh, V.V., Leonenko, N.N., Sakhno, L.M.: Evaluation of bias in higher-order spectral estimation. Probab. Theory and Math. Stat. **80**, 1–14 (2009)
5. Avram, F., Leonenko, N., Sakhno, L.: On a Szegö type limit theorem, the Hölder-Young-Brascamp-Lieb inequality, and the asymptotic theory of integrals and quadratic forms of stationary fields. ESAIM Probab. Stat. **14**, 210–255 (2010)
6. Bentkus, R.: On the error of the estimate of the spectral function of a stationary process. Liet. Mat. Rink. **12**, 55–71 (1972, in Russian)
7. Brillinger, D.R.: The frequency analysis of relations between stationary spatial series. In: Proceedings of the 12th Biennial Seminar of the Canadian Mathematical Congress, Montreal, pp. 39–81 (1970)
8. Brillinger, D.R., Rosenblatt, M.: Asymptotic theory of estimates of k-th order spectra. In: Harris, B. (ed.) Spectral Analysis Time Series, pp. 153–188. Wiley, New York (1967)
9. Dahlhaus, R.: Spectral analysis with tapered data. J. Time. Ser. Anal. **4**, 163–175 (1983)
10. Guyon, X.: Random Fields on a Network: Modelling, Statistics and Applications. Springer, New York (1995)
11. Ibragimov, I.A.: On maximum likelihood estimation of parameters of the spectral density of stationary time series. Theor. Prob. Appl. **12**, 115–119 (1967)
12. Taniguchi, M.: Minimum contrast estimation for spectral densities of stationary processes. J. R. Stat. Soc., Ser. B. **49**, 315–325 (1987)

Conditional Estimators in Exponential Regression with Errors in Covariates

Sergiy Shklyar

Abstract In this chapter we deal with a regression model in which there is Gaussian error in the regressor and the response variable has an exponential distribution. We consider three methods of estimation: Sufficiency estimator, Conditional Score estimators developed by Stefanski and Carroll (Biometrika 74, 703–716 1987), and Corrected Score estimator developed by Stefanski (Commun. Stat. Theory Methods 18, 4335–4358 1989) and Nakamura (Biometrika 77, 127–132 1990). We have written explicitly the estimating equations for these estimators. Sufficiency and Corrected Score estimators were compared numerically.

1 Introduction

Canonical generalized linear models are regression models, in which the distribution of response variable belongs to one-parameter exponential family of distributions, and the canonical parameter of the distribution is a linear function of regressor. In such models the log-likelihood function is concave.

We consider a model with Gaussian measurement error in regressor. For the case where the error-free model is canonical generalized linear, Stefanski and Carroll [8] developed Sufficiency estimator and a class of Conditional estimators.

The Corrected Score method proposed in Stefanski [7] and Nakamura [5] is more general. This method does not require that the error-free model is a canonical generalized linear model. Both Conditional Score and Corrected Score estimators are described in Chap. 7 of monograph by Carroll et al. [1].

We consider the exponential regression model with measurement errors. In this model, the estimating functions for Sufficiency, Conditional, and Corrected Score

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estimators can be written explicitly through the error function. We have written these estimating functions through the cdf of normal distribution.

We compare these estimators numerically. Similar comparison for Poisson regression model has been made by Stefanski [7].

The chapter is organized as follows. Section 2 includes the definition of one-parameter exponential family of distributions. The statistical model is set in Sect. 3. In Sect. 4 we obtained some formulas needed for Sufficiency and Conditional Score estimators. The estimating equations are obtained in Sects. 5 and 6. In Sect. 7 an estimating equation for Corrected Score estimator is obtained. In Sect. 8 we compare the estimators numerically. Section 9 concludes. In Appendix, we have proved a theorem that implies the uniqueness of solution to the deconvolution problem.

2 Exponential Family of Distributions

One-parameter exponential family of distributions is a family of distributions on $\mathcal{X} \subset \mathbb{R}$ with the following densities with respect to some σ-finite measure on a Borel σ-field on $\mathcal{X}$:

$$p(y, \eta) = \exp\left\{ \frac{y\eta - D(\eta)}{\phi} + c(y, \phi) \right\}, \tag{1}$$

where η is a canonical parameter, ϕ is a dispersion parameter, and $D(\eta)$ is a C^2-smooth convex function. Real-valued variable y denotes a point where the density is taken.

The domain for η (which is the parameter space for the exponential family) is assumed to be an open subset of $\mathbb{R}$. The dispersion parameter $\phi > 0$ can be fixed.

Let Y have a distribution from an exponential family with density (1). According to Lehmann and Casella [4], its expectation and variance equal

$$\mathsf{E}Y = D'(\eta), \qquad \mathsf{D}Y = \phi D''(\eta).$$

3 Exponential Regression

We consider a particular case of canonical generalized linear model with measurement error developed by Carroll et al. [1]. Let X_i be nonrandom true regressor. The response Y_i is exponentially distributed, $Y_i \sim \text{Exp}(\beta_0 + \beta_1 X_i)$, where the parameter $\beta_0 + \beta_1 X_i$ depends on X_i:

$$\mathsf{P}(Y_i > y) = \mathrm{e}^{-y(\beta_0 + \beta_1 X_i)}, \qquad y \geq 0.$$

Let us remark that the exponential distributions form a one-parameter exponential family. The parameter of exponential distribution should be positive. Therefore, we assume

$$\beta_0 + \beta_1 X_i > 0, \quad i=1,2,\ldots$$

The true regressor X_i is observed with measurement error U_i; thus, the observed surrogate data is $W_i = X_i + U_i$. The measurement error U_i is independent of Y_i and is Gaussian, $U_i \sim N(0, \sigma_u^2)$.

We assume that the measurement error variance σ_u^2 is known. For observations $\{(W_i, Y_i),\ i=1,2,\ldots,n\}$, we need to estimate the regression parameter $\boldsymbol{\beta} = (\beta_0, \beta_1)$.

4 Some Formulas Needed for Construction of the Estimators

Let us consider a single observation. For simplicity of notation, we omit the index i. The density of random response Y is

$$p_Y(y) = (\beta_0 + \beta_1 X)\mathrm{e}^{y\,(\beta_0+\beta_1 X)}, \quad y>0$$

(the distribution of Y is concentrated on half-line $(0,+\infty)$). The density of the surrogate regressor is

$$p_W(w) = \frac{1}{\sqrt{2\pi}\sigma_u} \exp\left\{-\frac{(w-X)^2}{2\sigma_u^2}\right\}, \quad w\in\mathbb{R}.$$

Since X is assumed to be nonrandom, independence between U and Y implies that W and Y are stochastically independent. The joint density of W and Y is

$$p_{W,Y}(w,y) = \frac{\beta_0 + \beta_1 X}{\sqrt{2\pi}\sigma_u} \exp\left\{-y\,(\beta_0 + \beta_1 X) - \frac{(w-X)^2}{2\sigma_u^2}\right\}. \tag{2}$$

Let us introduce a random variable

$$\Delta = W - Y\sigma_u^2\beta_1.$$

The joint density of Δ and Y is

$$p_{\Delta,Y}(\Delta,y) = \frac{\beta_0 + \beta_1 X}{\sqrt{2\pi}\sigma_u} \exp\left\{-y\,(\beta_0 + \beta_1 \Delta) - \frac{y^2\sigma_u^2\beta_1^2}{2} - \frac{(\Delta-X)^2}{2\sigma_u^2}\right\}.$$

The density of conditional distribution of Y given Δ is $p_{Y|\Delta}(y) = \frac{p_{\Delta,Y}(\Delta,y)}{p_\Delta(\Delta)}$, with

$$p_\Delta(\Delta) = \int_0^\infty p_{\Delta,Y}(\Delta, y)\,\mathrm{d}y$$
$$= \frac{\beta_0 + \beta_1 X}{\sqrt{2\pi}\sigma_u} \exp\left\{-\frac{(\Delta - X)^2}{2\sigma_u^2}\right\} \int_0^\infty \mathrm{e}^{-(\beta_0+\beta_1\Delta)-\frac{1}{2}y^2\sigma_u^2\beta_1^2}\mathrm{d}y.$$

If $\beta_1\sigma_u \neq 0$, then

$$p_\Delta(\Delta) = \frac{\beta_0 + \beta_1 X}{\sigma_u^2|\beta_1|} \exp\left\{-\frac{(\Delta - X)^2}{2\sigma_u^2}\right\} \exp\left\{\frac{(\beta_0 + \beta_1\Delta)^2}{2\sigma_u^2\beta_1^2}\right\} \Phi\left(-\frac{\beta_0 + \beta_1\Delta}{|\beta_1|\,\sigma_u}\right).$$

If $\beta_1\sigma_u = 0$, then $\Delta = X$ a.s.

Let us define a special function

$$\mathrm{mills}(t) := \int_0^\infty \exp\left\{-xt - \tfrac{1}{2}x^2\right\}\,\mathrm{d}x$$
$$= \sqrt{2\pi}\mathrm{e}^{t^2/2}\Phi(-t).$$

This function is called Mills ratio for standard normal distribution.

Using this notation, we obtain for $\beta_1\sigma_u \neq 0$ that

$$p_\Delta(\Delta) = \frac{\beta_0 + \beta_1 X}{\sqrt{2\pi}\sigma_u^2|\beta_1|} \exp\left\{-\frac{(\Delta - X)^2}{2\sigma_u^2}\right\} \mathrm{mills}\left(\frac{\beta_0 + \beta_1\Delta}{|\beta_1|\,\sigma_u}\right). \tag{3}$$

If $\beta_1\sigma_u \neq 0$, then

$$p_{Y|\Delta}(y) = \frac{|\beta_1|\sigma_u \exp\left\{-y\,(\beta_0 + \beta_1\Delta) - \frac{1}{2}y^2\sigma_u^2\beta_1^2\right\}}{\sqrt{2\pi}\ \exp\left\{\frac{(\beta_0+\beta_1\Delta)^2}{2\sigma_u^2\beta_1^2}\right\} \Phi\left(-\frac{\beta_0+\beta_1\Delta}{|\beta_1|\,\sigma_u}\right)}$$
$$= \frac{|\beta_1|\,\sigma_u \exp\left\{-y\,(\beta_0 + \beta_1\Delta) - \frac{1}{2}y^2\sigma_u^2\beta_1^2\right\}}{\mathrm{mills}\left(\frac{\beta_0+\beta_1\Delta}{|\beta_1|\,\sigma_u}\right)}.$$

Hence, the conditional distribution of Y given Δ is truncated normal distribution:

$$[Y \mid \Delta{=}d] \sim [\xi \mid \xi{>}0], \qquad \xi \sim N\left(-\frac{\beta_0 + \beta_1 d}{\beta_1^2\sigma_u^2}, \frac{1}{\beta_1^2\sigma_u^2}\right).$$

We can see that the conditional distribution of Y given Δ does not depend on X. This is consequence from the fact that for the fixed $\boldsymbol{\beta}$, the random variable Δ is a sufficient statistics for the parameter X.

If $\beta_1\sigma_u = 0$ and $\beta_0 + \beta_1\Delta > 0$, then

$$p_{Y|\Delta}(y) = p_Y(y) = (\beta_0 + \beta_1\Delta)\mathrm{e}^{-y\,(\beta_0+\beta_1\Delta)}.$$

It can be verified that for $\beta_0 > 0$, $p_{Y|\Delta}(y)$ is a continuous function of β_1.

Now let us evaluate conditional moments of Y given Δ. We can use formulas for moments of truncated normal distribution presented in Greene [2]. If $\beta_1\sigma_u \neq 0$, then

$$\mathsf{E}[Y|\Delta] = -\frac{\beta_0+\beta_1\Delta}{\beta_1^2\sigma_u^2} + \frac{1}{|\beta_1|\,\sigma_u\ \text{mills}\left(\frac{\beta_0+\beta_1\Delta}{|\beta_1|\,\sigma_u}\right)},$$

$$\mathsf{E}[Y^2|\Delta] = \frac{(\beta_0+\beta_1\Delta)^2+\beta_1^2\sigma_u^2}{\beta_1^4\sigma_u^4} - \frac{\beta_0+\beta_1\Delta}{|\beta_1|^3\sigma_u^3\ \text{mills}\left(\frac{\beta_0+\beta_1\Delta}{|\beta_1|\sigma_u}\right)}.$$

If $\beta_1\sigma_u = 0$, then $\beta_0+\beta_1\Delta > 0$ a.s. and

$$\mathsf{E}[Y|\Delta] = \frac{1}{\beta_0+\beta_1\Delta}, \qquad \mathsf{E}[Y^2|\Delta] = \frac{2}{(\beta_0+\beta_1\Delta)^2}.$$

Denoting $m(\Delta,\boldsymbol{\beta}) = \mathsf{E}[Y|\Delta]$ (i.e., $m(\delta,\mathbf{b}) = \mathsf{E}_{\boldsymbol{\beta}=\mathbf{b}}[Y \mid \Delta{=}\delta]$), we obtain

$$m(\Delta,\mathbf{b}) = -\frac{b_0+b_1\Delta}{b_1^2\sigma_u^2} + \frac{1}{|b_1|\,\sigma_u\ \text{mills}\left(\frac{b_0+b_1\Delta}{|b_1|\,\sigma_u}\right)} \quad \text{if } b_1\sigma_0 \neq 0;$$

$$m(\Delta,\mathbf{b}) = \frac{1}{b_0+b_1\Delta} \quad \text{if } b_1\sigma_u{=}0 \text{ and } b_0+b_1\Delta > 0.$$

With this notation,

$$\mathsf{E}[Y^2|\Delta] = \frac{1}{\beta_1^2\sigma_u^2} - \frac{(\beta_0+\beta_1\Delta)m(\Delta,\boldsymbol{\beta})}{\beta_1^2\sigma_u^2}.$$

Let us denote

$$v(\Delta,\boldsymbol{\beta}) = \mathsf{D}[Y \mid \Delta] = \mathsf{E}[Y^2 \mid \Delta] - m(\Delta,\boldsymbol{\beta})^2.$$

The notation $v(\Delta,\mathbf{b})$ will be used in Sect. 6 in the definition of Quasi-Score estimator.

The function mills(t) satisfies the following differential equation:

$$\text{mills}'(t) = t\,\text{mills}(t) - 1.$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\ln\text{mills}(t)) = t - \frac{1}{\text{mills}(t)}.$$

For $t = \dfrac{b_0+b_1\Delta}{|b_1|\sigma_u}$, we have $\dfrac{\mathrm{d}}{\mathrm{d}t}(\ln\text{mills}(t)) = -|b_1|\,\sigma_u m(\Delta,\mathbf{b})$.

5 Sufficiency Estimator

The Sufficiency estimator is defined by the following estimating equations:

$$\begin{cases} \sum_{i=1}^{n} \frac{\partial}{\partial \mathbf{b}} \left(\ln p_{[Y \mid \Delta=\Delta_i, \boldsymbol{\beta}=\mathbf{b}]}(Y_i) \right) = 0, \\ \Delta_i = W_i - Y_i \sigma_u^2 b_1. \end{cases}$$

(Here evaluating the derivative, we neglect the dependence of Δ_i on $\boldsymbol{\beta}$.) For

$$\ln p_{[Y \mid \Delta, \boldsymbol{\beta}=\mathbf{b}]}(y) \\ = \ln |b_1| + \ln \sigma_u - y(b_0 + b_1 \Delta) - \frac{1}{2} y^2 \sigma_u^2 b_1^2 - \ln \operatorname{mills} \left(\frac{b_0 + b_1 \Delta}{|\beta_1| \sigma_u} \right),$$

we can evaluate the derivatives

$$\frac{\partial}{\partial b_0} \left(\ln p_{[Y \mid \Delta, \boldsymbol{\beta}=\mathbf{b}]}(y) \right) = -y + m(\Delta, \mathbf{b});$$
$$\frac{\partial}{\partial b_1} \left(\ln p_{[Y \mid \Delta, \boldsymbol{\beta}=\mathbf{b}]}(y) \right) = \frac{1}{b_1} - y\Delta - y^2 \sigma_u^2 b_1 - \frac{b_0}{b_1} m(\Delta, \mathbf{b}).$$

Hence, the estimating equations become

$$\begin{cases} \sum_{i=1}^{n} m(\Delta_i, \mathbf{b}) = \sum_{i=1}^{n} Y_i, \\ \sum_{i=1}^{n} \left(-Y_i^2 b_1 \sigma_u^2 - Y_i \Delta_i + \frac{1 - b_0 m(\Delta_i, \mathbf{b})}{b_1} \right) = 0, \\ \Delta_i = W_i - Y_i \sigma_u^2 b_1. \end{cases} \tag{4}$$

(If $b_1 = 0$ and $b_0 > 0$, then the expression $\frac{1 - b_0 m(\Delta_i, \mathbf{b})}{b_1}$ should be replaced by $\frac{\Delta_i}{b_0}$.)

We add the first equation multiplied by b_0 to the second equation multiplied by b_1:

$$\sum_{i=1}^{n} \left(-Y_i^2 b_1^2 \sigma_u^2 - Y_i (b_0 + b_1 \Delta_i) + 1 \right) = 0.$$

We substitute $\Delta_i = W_i - Y_i \sigma_u^2 b_1$:

$$\sum_{i=1}^{n} (-Y_i (b_0 + b_1 W_i) + 1) = 0. \tag{5}$$

Combining the first equation in (4) and (5), we obtain that the Sufficiency estimator satisfies the relations

$$\begin{cases} \sum_{i=1}^{n} m(W_i - Y_i\sigma_u^2 b_1,\ \mathbf{b}) = \sum_{i=1}^{n} Y_i, \\ \sum_{i=1}^{n} Y_i\ (b_0 + b_1 W_i) = n. \end{cases} \tag{6}$$

Equation (6) always has a solution $b_1 = 0$, $b_0 = \frac{n}{\sum Y_i}$. This solution is superfluous in general case. Equation (6) may have no other solutions, or it may have many solutions. If the Eq. (6) has many solutions, we choose one as follows. If the sample correlation between W_i's and Y_i's $\operatorname{corr}(W_i, Y_i) < 0$, we search for a solution with smallest positive $b_1 > 0$. If $\operatorname{corr}(W_i, Y_i) > 0$, we search for a solution with $b_1 < 0$ closest to 0. If $\operatorname{corr}(W_i, Y_i) = 0$, we set $b_1 = 0$ and $b_0 = \frac{n}{\sum Y_i}$.

6 Conditional Score Estimators

In order to construct the Conditional Score estimator of $\boldsymbol{\beta}$, we regress Y on Δ. Consider a family of elementary estimating functions

$$(m(\Delta, \mathbf{b}) - Y)k(\Delta, \mathbf{b}),$$

with some vector-valued function $k(\Delta, \mathbf{b})$. Such estimating functions are unbiased. Choosing different functions $k(\Delta, \mathbf{b})$, we get different estimators.

Quasi-Score Estimator. For Quasi-Score estimator, the system of estimating equations is

$$\begin{cases} \sum_{i=1}^{n} \frac{m(\Delta_i, \mathbf{b}) - Y_i}{v(\Delta_i, \mathbf{b})} \frac{\partial m(\Delta_i, \mathbf{b})}{\partial \mathbf{b}} = 0, \\ \Delta_i = W_i - Y_i\sigma_u^2 b_1, \quad i=1,2,\ldots,n. \end{cases}$$

Conditional-Linear Estimator. The system of estimating equations for this estimator is

$$\begin{cases} \sum_{i=1}^{n} m(\Delta_i, \mathbf{b}) = \sum_{i=1}^{n} Y_i, \\ \sum_{i=1}^{n} m(\Delta_i, \mathbf{b})\Delta_i = \sum_{i=1}^{n} Y_i\Delta_i, \\ \Delta_i = W_i - Y_i\sigma_u^2 b_1, \quad i=1,2,\ldots,n. \end{cases}$$

This version of Conditional Score estimator is described in Carroll at al. ([1], Sect. 7.3) as the simplest approach to implement.

Optimal Estimator. The Optimal estimator is defined in the structural model, with random variable as the true regressor X. The joint density of X and Δ is

$$p_{X,\Delta}(x,\Delta) = p_X(x)\frac{\beta_0+\beta_1 x}{\sqrt{2\pi}\sigma_u^2|\beta_1|}\exp\left\{-\frac{(\Delta-x)^2}{2\sigma_u^2}\right\}\operatorname{mills}\left(\frac{\beta_0+\beta_1\Delta}{|\beta_1|\sigma_u}\right).$$

Let us evaluate the conditional expectation

$$t(\Delta) := \mathsf{E}[X \mid \Delta] = \frac{\int x p_X(x)(\beta_0+\beta_1 x)\exp\left\{-\frac{(\Delta-x)^2}{2\sigma_u^2}\right\}\mathrm{d}x}{\int p_X(x)(\beta_0+\beta_1 x)\exp\left\{-\frac{(\Delta-x)^2}{2\sigma_u^2}\right\}\mathrm{d}x},$$

$$t(d) = \mathsf{E}[X \mid \Delta{=}\delta] = \frac{\mathsf{E}[X\,(\beta_0+\beta_1 X) \mid W{=}\delta]}{\mathsf{E}[(\beta_0+\beta_1 X) \mid W{=}\delta]}.$$

We define the estimator as a measurable solution to the following system of estimating equations:

$$\begin{cases} \sum_{i=1}^n m(\Delta_i,\mathbf{b}) = \sum_{i=1}^n Y_i, \\ \sum_{i=1}^n m(\Delta_i,\mathbf{b})t(\Delta_i) = \sum_{i=1}^n Y_i t(\Delta_i), \\ \Delta_i = W_i - Y_i\sigma_u^2 b_1, \quad i{=}1,2,\ldots,n. \end{cases}$$

Stefanski and Carroll (1987) proved that the score function in the latter system of equations is optimal (in the sense of formal asymptotic covariance matrices) within certain class of estimating functions.

7 Corrected Score Estimator

The Corrected Score method is described in monograph by Carroll et al. ([1], Sect. 7.4). We are using some notations from Kukush and Schneeweiss ([3], Sect. 5), where Corrected Score estimator constructed for generalized linear model.

Here are Maximum Likelihood estimating equations in error-free model:

$$\begin{cases} \sum_{i=1}^n \frac{1}{b_0+b_1X_i} = \sum_{i=1}^n Y_i; \\ \sum_{i=1}^n \frac{X_i}{b_0+b_1X_i} = \sum_{i=1}^n X_iY_i. \end{cases}$$

In order to construct Corrected Score estimator, we have to solve the following deconvolution problems:

$$\mathsf{E} f_{c1}(W, \mathbf{b}) = \frac{1}{b_0 + b_1 x}, \qquad \mathsf{E} f_{c2}(W, \mathbf{b}) = \frac{x}{b_0 + b_1 x} \tag{7}$$

with $W \sim N(x, \sigma_u^2)$, i.e., for all x such that $b_0 + b_1 x > 0$,

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_u} f_{c1}(w, \mathbf{b}) \exp\left\{-\frac{(w-x)^2}{2\sigma_u^2}\right\} \mathrm{d}w = \frac{1}{b_0 + b_1 x}, \tag{8}$$
$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_u} f_{c2}(w, \mathbf{b}) \exp\left\{-\frac{(w-x)^2}{2\sigma_u^2}\right\} \mathrm{d}w = \frac{x}{b_0 + b_1 x}.$$

The pdf of the random variable $\Delta = W_i - Y_i \sigma_u^2 \beta_1$ is denoted as p_Δ. So, we have

$$\int_{-\infty}^{\infty} p_\Delta(w)\,\mathrm{d}w = 1,$$
$$\int_{-\infty}^{\infty} \frac{\beta_0 + \beta_1 X}{\sqrt{2\pi}\sigma_u^2 |\beta_1|} \operatorname{mills}\left(\frac{\beta_0 + \beta_1 w}{|\beta_1|\sigma_u}\right) \mathrm{e}^{-\frac{(w-X)^2}{2\sigma_u^2}} \mathrm{d}w = 1,$$
$$\frac{1}{|\beta_1|\sigma_u} \int_{-\infty}^{\infty} \operatorname{mills}\left(\frac{\beta_0 + \beta_1 w}{|\beta_1|\sigma_u}\right) \frac{1}{\sqrt{2\pi}\sigma_u} \mathrm{e}^{-\frac{(w-X)^2}{2\sigma_u^2}} \mathrm{d}w = \frac{1}{\beta_0 + \beta_1 X}.$$

Hence, we can set

$$f_{c1}(w, \mathbf{b}) = \frac{1}{|b_1|\,\sigma_u} \operatorname{mills}\left(\frac{b_0 + b_1 w}{|b_1|\,\sigma_u}\right),$$

since this function satisfies (8) for all x such that $b_0 + b_1 x > 0$.

As

$$\mathsf{E}[b_0 f_{c1}(W, \mathbf{b}) + b_1 f_{c2}(W, \mathbf{b})] = 1, \qquad W \sim N(x, \sigma_u^2),$$

we can find $f_{c2}(w, \mathbf{b})$ from the equation

$$b_0 f_{c1}(w, \mathbf{b}) + b_1 f_{c2}(w, \mathbf{b}) = 1. \tag{9}$$

Hence,

$$f_{c2}(w, \mathbf{b}) = \frac{1 - b_0 f_{c1}(w, \mathbf{b})}{b_1} = \frac{1}{b_1} - \frac{b_0}{|b_1|\, b_1 \sigma_u} \operatorname{mills}\left(\frac{b_0 + b_1 w}{|b_1|\sigma_u}\right).$$

If $\beta_1 \sigma_u = 0$, then

$$f_{c1}(w, \mathbf{b}) = \frac{1}{b_0 + b_1 w}, \qquad f_{c2}(w, \mathbf{b}) = \frac{w}{b_0 + b_1 w}.$$

Remark 1. The functions $f_{c1}(w, \mathbf{b})$ and $f_{c2}(w, \mathbf{b})$ are shown to satisfy Eq. (7). According to Theorem 1 (see Appendix), if $\sigma_u^2 > 0$ and $\mathbf{b} \in \mathbb{R}^2 \setminus \{(b_0, b_1) \mid b_0 \leq 0, b_1 = 0\}$, all solutions to (7) coincide almost everywhere.

The system of estimating equations for Corrected Score estimator is

$$\begin{cases} \sum_{i=1}^{n} f_{c1}(W_i, \mathbf{b}) = \sum_{i=1}^{n} Y_i, \\ \sum_{i=1}^{n} f_{c2}(W_i, \mathbf{b}) = \sum_{i=1}^{n} Y_i W_i. \end{cases} \tag{10}$$

We add the first equation multiplied by b_0 to the second one multiplied by b_1. Taking (9) into account, we obtain

$$n = \sum_{i=1}^{n} Y_i (b_0 + b_1 W_i). \tag{11}$$

This equation is equivalent to Eq. (5). Both the Sufficiency estimator and the Corrected Score estimator satisfy Eq. (11).

System of Eq. (10) may have many solutions. We chose one in the same way as for Sufficiency estimator.

Remark 2. In error-free model, the Sufficiency estimator and the Corrected Score estimator, as well as some Conditional Score estimators (Quasi-Score, Conditional-Linear, and Optimal estimators), coincide with Maximum Likelihood estimator.

8 Simulation

We choose the sample sizes $n = 10{,}000$ and $n = 1{,}000{,}000$. Simulate a random sample of X_i, $i = 1, \ldots, n$ from lognormal law, $\ln X_i \sim N(0, 1)$. This sample will be the same for all simulations.

In each simulation, generate $W_i \sim N(X_i, \sigma_u^2)$ and $Y_i \sim \exp(\beta_0 + \beta_1 X_i)$ with true parameters $\sigma_u^2 = 0.6$, $\beta_0 = 2$, and $\beta_1 = 5$. We evaluate Sufficiency estimator and Corrected Score estimator. Also we evaluate Maximum Likelihood estimator in error-free model in order to show the impact of errors on quality of estimators. The estimates are averaged over 1,000 realizations. The simulation results are presented in Table 1.

In our simulation measurement error essentially degrades the efficiency of the estimators. The Corrected Score estimator has recognizable bias and larger standard deviation than the Sufficiency estimator. We mention that for sample size $n = 10{,}000$ for 20 realizations out of 1,000, the estimating equation for Corrected Score estimator has no solution.

Table 1 Simulation results

True value		$\beta_0 = 2$		$\beta_1 = 5$	
Sample size	Error variance	Means and standard deviations of estimates over 1,000 simulations of β_0		of β_1	
n	σ_u^2	mean	std	mean	std
Maximum Likelihood estimator in error-free model					
10,000	0	1.9997	0.0748	5.0004	0.0963
1,000,000	0	2.0001	0.0074	4.9998	0.0100
Sufficiency estimator					
10,000	0.6	1.9626	0.2440	5.0459	0.3168
1,000,000	0.6	2.0005	0.0228	4.9999	0.0293
Corrected Score estimator					
10,000	0.6	1.3623	0.8823	5.7490	1.0419
1,000,000	0.6	1.8540	0.2097	5.1707	0.2452

9 Conclusion

The estimating functions for the Sufficiency estimator, for the Corrected Score estimator, and for some estimators from Conditional Score class are written explicitly in exponential regression model with 'reciprocal' link function $\mathsf{E}Y = \frac{1}{\beta_0+\beta_1 X}$ and Gaussian measurement error in regressor.

The estimating functions for these estimators are unbiased. In structural model, they are unbiased under some conditions (not shown here) on the distribution of the true regressor X. However, the proof of consistency of the estimators is a matter of further investigation.

The Sufficiency estimator and the Corrected Score estimator are compared numerically. The Sufficiency estimator seems to be better than the Corrected Score estimator.

Appendix

In this section we show the uniqueness of the solution to deconvolution problem.

Definition 1. Let $(\Omega, \mathcal{F}, \mathbb{P}_\theta, \theta\in\Theta)$ be a statistical space. The statistics $T : \Omega \to A$ is called *complete* if for any Borel measurable function $f : A \to \mathbb{R}$ the equality $\forall\theta \in \Theta : \mathsf{E}_\theta f(T) = 0$ implies $\forall\theta \in \Theta : \mathsf{P}_\theta(f(T){=}0) = 1$.

Completeness of a statistics characterizes the family of induced distributions $\{\mathsf{P}_\theta T^{-1},\ \theta \in \Theta\}$.

Example 1. Let $(\Omega, \mathcal{F}, \mathbb{P}_\eta, \eta\in\Xi)$ be a statistical space, and let the parameter set $\Xi \subset \mathbb{R}$ contain internal points. Let Y be a random variable whose density with respect to some σ-finite measure on $\mathbb{R}$ is defined by formula (1) with known dispersion parameter ϕ, i.e., the distributions of Y form a one-parameter exponential family with canonical parameter η. Then Y is a complete statistics for η. This is a particular case of Theorem 6.22 from Lehmann and Casella [4] or Theorem 2.1 from Shao and Zhang [6].

A family of normal distributions with common variance is a one-parameter exponential family.

Example 2. Let $\sigma^2 > 0$ be a known number and $\Theta = (a, b)$ be an interval on $\mathbb{R}$. The random variable $\zeta \sim N(\mu, \sigma^2)$ is a complete statistics for the parameter $\mu \in \Theta$.

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue measurable function, and let $\sigma^2 > 0$ and $a < b$. If*

$$\forall x \in (a, b) : \int_{-\infty}^{\infty} f(t) \exp\left\{-\frac{(t-x)^2}{2\sigma^2}\right\} \mathrm{d}x = 0, \tag{12}$$

where the latter integral is Lebesgue integral (and converges absolutely), then $f(t) = 0$ a.e. on $\mathbb{R}$.

Proof. The function $f(t)$ is equal to some Borel measurable function a.e. Hence, we can assume that $f(t)$ is a Borel measurable function. Equality (12) implies that for all $\mu \in (a, b)$

$$\mathsf{E} f(\xi) = 0, \qquad \xi \sim N(\mu, \sigma^2).$$

As ξ is a complete statistics for μ, $f(\xi) = 0$ a.s. and $f(t) = 0$ a.e. □

Note that both Example 2 and Theorem 1 above hold true for infinite interval (a, b). Thus, a and b can be arbitrary such that $-\infty \le a < b \le +\infty$.

Theorem 1 is used in the proof of uniqueness of solutions $f_{c1}(w, \mathbf{b})$ and $f_{c2}(w, \mathbf{b})$ to the deconvolution problem (7).

Example 3. Consider a single observation (see Sect. 4). Assume that parameters $\boldsymbol{\beta}$ and σ^2 are fixed and known, and X is an unknown parameter satisfying $\beta_0 + \beta_1 X > 0$. Random variables $W \in \mathbb{R}$ and $Y > 0$ have joint density (2). Then the random variable $\Delta = W - Y\beta_1\sigma_u^2$ is a complete statistics for X. Indeed, it is clear from (3) that the distribution of Δ belongs to an exponential family with canonical parameter X.

References

1. Carroll, R.J., Ruppert, D., Stefanski, L.A., Crainiceanu, C.M.: Measurement Error in Nonlinear Models—A Modern Perspective. Chapman & Hall, Boca Raton (2006)
2. Greene, W.H.: Econometric Analysis. Prentice-Hall, New Jersey (2011)

3. Kukush, A., Schneeweiss, H.: Comparing different estimators in a nonlinear measurement error model. Part I. Math. Methods Stat. **14**, 53–79 (2005)
4. Lehmann, E.L., Casella, G.: Theory of Point Estimation. Springer, New York (1998)
5. Nakamura, T.: Corrected-score functions for error-in-variables models. Biometrika **77**, 127–132 (1990)
6. Shao, J., Zhang, H.: Lecture Notes for Statistics 709. University of Wisconsin. http://www.stat.wisc.edu/~doksum (2008). Accessed 13 July 2013
7. Stefanski, L.A.: Unbiased estimation of nonlinear function of a normal mean with application to measurement error model. Commun. Stat. Theory Methods **18**, 4335–4358 (1989)
8. Stefanski, L.A., Carroll, R.J.: Conditional scores and optimal scores for generalized linear measurement-error models. Biometrika **74**, 703–716 (1987)