Chapter 7
Aesthetic Experience

In this chapter we address aesthetic experience. A person's inner experience of an observed object depends primarily, of course, on the features of the object of attention. But the experience is also deeply influenced by the subjects' particular skills, knowledge, attitudes, and so forth. As we have seen, Peter Kivy even links the content of our attention, intentional objects, to descriptions [44, pp. 81]. An aesthetic experience is a particular type of inner experience and it thus depends on the object's features and the subject's dispositions. The experience of music is very different from the experience of painting; the properties of a piece of music are very different from the properties of a painting. Moreover, what we know about music can affect how we perceive certain piece of music, but not necessarily how we perceive a painting, and vice versa. Hence, an analysis of the specifics of musical experience does not necessarily enlighten us about the nature of inner experiences in contemplating paintings. The same is true for the aesthetic experience of mathematics. Different aesthetic experiences should be addressed by concentrating on their own particularities, those particularities constitute the modality of the experience. In the following, I concentrate on the details relevant to aesthetic experience of mathematics. For convenience, I shall use again the term *phenomenological* to describe things related to the private first person perspective, the inner experience of a subject, which should not be confused with the technical Husserlian sense of describing "the structures of the experience as they present themselves to consciousness" [20, p. 2]. My approach is rather constrained in comparison: I merely advance a descriptive account of the intellectual, affective, and objective events and their relations relevant to the eliciting of aesthetic evaluations.

7.1 Characterizing Aesthetic Experience

I consider an aesthetic experience a collection of interrelated events that unfold over time, that is, a process. An aesthetic experience is not an independent process; it is a sub-process embedded in a larger aesthetic-process. Consider, for example,

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Euler's identity, $e^{i\pi} + 1 = 0$, named the most beautiful formula in mathematics by *The Mathematical Intelligencer* [94]. The aesthetic experience associated with Euler's identity depends not only on the person's inner events occurring during the act of contemplating the formula, but also on things like a person's knowing the mathematics which allows us to make sense of the sign $e^{i\pi} + 1 = 0$, the way a person's preferences were formed, other people's opinions, and so forth. The aesthetic experience of Euler's identity depends on events that are not necessarily occurring at the exact moment of the experience, but which have an influence on it; that is, the process of experiencing Euler's identity is embedded in a larger aesthetic-process. This is why I described that embedded sub-process as consisting of nodes 1 to 4 in the previous chapter. Node 4 includes an affective response, which I interpret as an evaluation. I distinguish between affective evaluation—an affective response—and aesthetic judgements—full-blown propositional aesthetic evaluations. Affective responses are characteristic of aesthetic episodes; this is one of the key features that distinguish them from other kinds of judgements.

The mathematical experience sub-process begins with a cognitive stimulus—mathematical experience is not perceptual, followed by a focusing of the attention on some relevant features of the stimulus, and by a further stage of active cognitive processing of the resultant object of attention. Consider Euler's identity, again, *The Mathematical Intelligencer* [93, 94] asked its readers to evaluate 24 theorems in terms of their beauty. Euler's identity ranked number one with an average score of 7.7, on a scale from 1 to 10. Now, the first event in our aesthetic experience of Euler's identity is an awareness of the mathematical formulation, by, for example encountering it in a textbook, or in the *Intelligencer*. However, more important than this initial awareness is the focusing of our attention on some relevant properties, such as the way the expression is composed: it comprises the constants e, i, π, 0, and 1, which are considered the most important constants in mathematics—I shall refer to the property of being composed by those constants as the *composition* of Euler's Identity. Another relevant feature of the expression is its simplicity. The occurrence of Euler's identity in a publication draws our attention, but it is the focusing of our attention on the composition and simplicity of Euler's identity that is important for eliciting an aesthetic response. Euler's identity is qualified as beautiful because its contemplation results in an affective response. This is why the editors of the *Intelligencer* deployed the predicate 'beautiful', rather than 'enlightening' to ask for the ranking. We experience some kind of affective response—we like it or dislike it—triggered by our contemplation of the formula. In the general case, the focusing of attention on some relevant aspects of the object of our attention results in responses of pleasure or displeasure. I shall use the term *enjoyment* to refer to the response of either pleasure or displeasure; that is, to the presence of any affective response.

We have seen that contemplation as well as active mental engagement can be pleasing or displeasing. We also have seen that preferences are formed and evolve influenced by diverse factors. Those elements must be taken into account to characterize aesthetic experience. I consider an aesthetic experience constituted by an intentional object, which I label the *content* of the experience, and an associated *enjoyment* elicited by that content. The enjoyment component—pleasure or displeasure—is necessary to distinguish it from mere inner mental representations of mathematical items, which are inner experiences, but not characteristically aesthetic. Taking into account the difference between contemplative and performative ways of eliciting enjoyment discussed in Chap. 3 and the dynamical character of preferences discussed in Chaps. 2, 4 and 5, at least three different types of aesthetic experience can be identified. Each of these types is characterized by a specific content-enjoyment relation. But before we can characterize the content-enjoyment relation, more details on intentional objects are needed.

The content of the experience as well as the way we actively deal with it are central in eliciting enjoyment. Now, in experiencing mathematics, perceptual stimuli are of little relevance. When mathematicians pass aesthetic judgements they are not referring to perceptual aspects of mathematical items. Mathematical beauty does not refer to things like the appearance of the sign $e^{i\pi} + 1 = 0$ printed on a page, or the diagrams illustrating a theory. The content that is important for aesthetic experience consists in the features found in the mental representations of mathematical items. For example, a non-perceptual feature in Euler's identity relevant for eliciting enjoyment is the feature that it comprises the most important constants in mathematics related in a simple manner. Our chief concern here must be this kind of features. An aesthetic intentional object is thus determined by a particular set of properties and relations *in a person's mental contents*. I have included a "formal build up", node 2, as part of a typical aesthetic-process. This takes into account the fact that the person's attention increasingly focuses on the relevant properties and relations—properties like simplicity or the composition of the object—of the cognitive stimulus, rather than on the whole collection of concrete features in the object. In the experience process our attention shifts from a concrete stimulus to a more specific set of features that constitute the intentional object relevant to aesthetic appreciation. For example, in Euler's identity, we first encounter and observe the expression $e^{i\pi} + 1 = 0$, but eventually our attention concentrates on properties like its composition and its simplicity, which are the significant properties for eliciting enjoyment.

7.2 Aesthetic Intentional Objects

Abstract objects are causally inefficacious; therefore, we concluded in Chap. 3, the content of our mind must be liable for our responses to mathematical objects. It is an intentional object which results in the affective response involved in an aesthetic-process. The intentional object consists of the relevant features that help to keep our attention focused and to elicit an affective response. Some features of this intentional object are the result of a natural process of abstraction. For example, in reading a story one extracts the propositional and then narrative information contained in the concrete characters printed on a page, or in the sounds uttered by a person. In listening to music one may extract information such as the pitch, and, if one is trained, even the name of a musical note, from the stream of sound reaching one's ears. In mathematics, when one contemplates an already abstract construct, such as Euler's identity, one contemplates it in an even more abstract manner: appreciating its remarkable composition and simplicity. A person can also discern further features in the object, resulting from the person's specific particularities; his skills, experience, knowledge, etc. For example, a person acquainted with a great deal of literature might realize that the novel he is reading sub-textually homages a famous Greek tragedy. Now, some of these individual peculiarities are influenced by changing external factors. Experiences provided by socialization and culture play a central role in forming our preferences. Those experiences modify the way we approach an object in the act of appreciation. They change our understanding of what are the relevant things to look for in an object, what things are acceptable and what are not. We unconsciously look for, and respond to those things. For example, in classical instrumental music we often look for patterns of temporal repetition; we learn that classic instrumental music is based on repeating patterns and, furthermore, within a single work is common to find that entire sections repeat themselves, in a sonata for instance. In mathematics, Gian-Carlo Rota points out that familiarity with different kinds of proofs helps us to recognize a good proof. In this respect, an interesting thing about mathematics is that this phenomenon is prevalent, not only in aesthetic appreciation, but in general, a large amount of knowledge is necessary to even see mathematics. We need to understand things like exponentiation, Euler's number, complex numbers, π, etc. in order to understand Euler's identity.

The way we perceive an object; that is, how we turn a concrete, or abstract, observed object into an intentional object depends primarily on the nature of the experience. Representational painting, for example, requires that the object of attention matches the object it depicts. But poetry or conceptual art, by contrast, require us to focus on the content of the text or the goals of the author. Mathematics usually requires a large amount of mathematical knowledge. Culture, via learning and training, plays a role in determining how we turn an observed object into and intentional object. This is why, as pointed out by Rota, familiarity with examples of mathematical beauty plays an important role in identifying other instances of mathematical beauty. How an intentional object is constituted is determined by the specifics of disciplines like narrative, music, painting and mathematics.

7.3 Mathematical Intentional Objects

It is now time to specify the features that characterize intentional objects in mathematical experience. I consider an intentional object the result of a shift of attention from a concrete initial stimulus to a specific set of properties associated with the object. An *aesthetic* intentional object is constituted by properties relevant for the eliciting of enjoyment—an affective response. To characterize aesthetic intentional objects in mathematics we need to avoid confusion between mathematical objects and objects of appreciation. Thus, a distinction must be drawn between mathematical objects, mathematical items and intentional objects: a *mathematical object* is an abstract object that appears as a referent in a mathematical theory—sets, functions, numbers, for example. I call a *mathematical item* any abstract[1] item that is characteristically part of mathematical *practice*.[2] A mathematical *intentional* object is the object in a person's inner experience resulting from focusing his attention on a mathematical item. If this attention results in a specific type of affective response (characterized below) the item is called an *aesthetic mathematical intentional object*. Those objects are the subject matter, the content, of an aesthetic experience.

7.4 A Notion of Aesthetic Mathematical Intentional Object

Aesthetic mathematical intentional objects are constituted by a set of properties, as perceived from a person's inner perspective, of course, and some structural relations among them. The set of properties comprises the properties that play a role in eliciting an affective response in the observer. For example, the simplicity and composition of Euler's identity, are relevant for our appreciation, but the property of, for instance, being a special case of Euler's formula is not. In eliciting enjoyment, not only the contemplation of properties plays a role, but also the mental activities in which a person engages. Therefore, the relations between properties that enable our attention to perform those activities are also relevant. In order to accommodate these features, I shall use, in a rather loose manner, the idea of space. I interpret intentional objects in aesthetic experience as objects existing in a *phenomenological space*—the space of a person's inner experience—with multiple dimensions. Dimension here is also interpreted rather loosely, as a parameter or piece of information necessary to specify the location of an object in the phenomenological space.[3] Intentional objects populate phenomenological spaces. The dimensions of phenomenological spaces correspond roughly to a relevant property of the intentional objects in our experience.

7.4.1 Dimensions and Properties

Consider a single mathematical result, Euler's identity, for instance. Its properties play the central role in eliciting enjoyment (in general that is the case in the

[1]In this way we exclude *concrete* indispensable items, like brains or mathematicians themselves.

[2]Rota and McAllister name several types of mathematical entities that are often qualified as beautiful, numbers, theorems, proofs, theories, and so forth. The above definition is adequate to cover those entities and some others not mentioned by them, such as derivations, or axiomatizations.

[3]Although the notions of space and dimension I utilize here resemble the ordinary concepts of physical space and dimension, they are rather closer to the formal notions of space and dimension. Unfortunately, a more formal treatment of these notions is beyond the scope of this book.

appreciation of single results like theorems, but the relations among them play a negligible role). By contrast, in the beauty of proofs and derivations, relations between single items play a major role since those relations are responsible for the emergence of structural properties of proofs or derivations like their simplicity, brevity or the shortness of their steps. For example, as we shall discuss below, the geometrical properties of complex numbers allow shorter and more elegant derivations of trigonometric identities.

In appreciating Euler's Identity, our attention must be focused in a specific way. A way in which we concentrate on some *extra* properties, properties that are not relevant to understand the formula, but that are necessary to aesthetically appreciate it. Euler's identity is very *simple*, but simplicity is not necessarily a property of all mathematical results; it is an extra quality that only some results possess. Properties like simplicity are the dimensions of the space in which our intentional object is located. An advantage of interpreting simplicity as a dimension is that it allows us to organize intentional objects according to its degree of simplicity. Now, dimensions not only organize intentional objects, but also enable us to see them, since they determine the different aspects of the objects that exist in the space. In order to allow dimensions to organize and determine objects, we need an idea of which features of the object the dimension indicates. For example, we have discussed that the property of simplicity may be ambiguous, as it can be interpreted in different ways and it can play different roles. A clear definition of simplicity is thus desirable if we are to conceptualize a space with that dimension. For this reason, it is best to interpret dimensions as explicit rules for interpreting relevant properties of objects. These rules can simply take the form of definitions that allow us to deal with the properties of objects in certain person's inner experience in a concrete situation. Thus, these rules can be simple declarations of the properties that constitute the intentional object. For example, if we need to introduce the dimension *simplicity*, in the phenomenological space in which an experience of Euler's identity is located, we need to define simplicity according to how the property appears in an intentional object. The definition can be as follows:

Simplicity = the feature of involving a minimum of operations and no non-relevant constants.

Euler's identity is simple in the sense defined above. It also has the very attractive extra quality of comprising the most important constants in mathematics. I have called this quality the composition of the formula. We can introduce another dimension into our space to account for how a person's attention focuses on this property by specifying the following interpretation:

Composition = the feature of being constituted by relevant items that are incorporated in a non-ad hoc manner.

In a phenomenological space with the dimensions of simplicity and composition, mathematical results are located in different spots, depending on how well they fit the definitions of the properties; that is, depending on how the dimensions order them. This models how our attention distinguishes and discriminates different mathematical results depending on how simple or well composed they are.

Among the things that substantially affect the forming of an intentional object is knowledge. As Kivy pointed out, a person acquainted with a famous actor sees that famous actor where some other person sees only a tall good looking man. A person who knows music theory hears a bold cadence where some other person may hear just some nice piece of music. Similarly, a mathematician sees the most beautiful theorem in mathematics where a lay person sees just an obscure formalism. In order to account for the role of knowledge in mathematical appreciation, we need to introduce a crucial dimension of mathematical phenomenological spaces: *background understanding*. Rota pointed out that to understand any piece of mathematics we need a great deal of mathematical knowledge. In order to appreciate a mathematical item, we first need to understand it. We can introduce a dimension that encapsulate the fact that we understand the mathematical item—and thus that such an item exists in our experience—simply by referring to the background knowledge necessary to understand it. For example, in order to understand Euler's identity we need to understand terms like p, e or i. More formally, we need to understand complex analysis. We can introduce a dimension CA corresponding to the property of being understandable only if complex analysis has been understood:

CA = the feature of being understandable only if complex analysis has been understood.

The dimensions that specify that mathematical understanding is required as background to appreciate a mathematical item shall be called *Background-Understanding* dimensions. At least one of these dimensions is necessary as part of any phenomenological space containing mathematical intentional objects. They are crucial to define the specificity of aesthetic experience in mathematics, and are analogous to the specific perceptual characteristic in other types of aesthetic experience. To appreciate painting we need sight; to appreciate music, hearing. And to appreciate mathematics, we need mathematical knowledge.

A background-understanding dimension is required for our experience to be about mathematics. But for our experience to be *aesthetic* we need extra properties that allow us to have an actual aesthetic response—simplicity or composition, for instance; properties that play a role in eliciting affective responses. To distinguish these properties I call them *aesthetically relevant* properties. In order to have an *aesthetic* object of attention, it is necessary that the phenomenological space in which it is located has at least one aesthetically relevant dimension. Thus, any mathematical phenomenological space must have at least two dimensions, and at least one must be aesthetically relevant. For example, the expression $x + x = 2x$, as an object of attention, requires background understanding (basic algebra) but it is aesthetically irrelevant, as it is not able to raise any kind of enthusiasm. Its properties are not able to elicit any kind of affective response. Thus, even if we can introduce different properties as dimensions of an "attention space", we cannot assign any aesthetic relevance to them because the affective character is absent, they do not constitute an *aesthetic* phenomenological space.

7.4.2 Activities and Relations

Consider the dimension *Complex Analysis*. This dimension allows us to understand, to see, so to speak, Euler's identity, but it also allows us to follow proofs or derivations involving complex functions. The proof of a theorem or the derivation of a result typically involves not only the passive contemplation of the result; rather, it consists in going through a series of steps and checking that the steps validly lead to the final result.

The expression $e^{i\pi} + 1 = 0$, for example, is a special case of:

$$e^{ix} = \cos x + i \sin x$$

Let $x = \pi$,

$$e^{i\pi} = \cos \pi + i \sin \pi$$

since $\cos \pi = -1$ and $sin\pi = 0$:

$$e^{i\pi} = -1$$

or

$$e^{i\pi} + 1 = 0$$

which is Euler's identity. In this very simple derivation, our attention is focused not on the properties of the resulting formula or the other individual expressions, but rather on how the successive steps lead us from the initial expression to the final one. This illustrates that the experience of mathematical items involves not only awareness of properties, but also the active engagement of our attention. The act of following this derivation is enabled by the properties and relations inherent in complex analysis, and thus by the background knowledge dimension of our phenomenological space.

The central element that determines an intentional object consists in the dimensions of the phenomenological space in which it is located. But from our discussion above is evident that there is a second important element: the set of relations that constrains the activities that can be performed by our attention in the phenomenological space (the dimensions of the space impose some constraints themselves, of course). In the general case, these properties and relations can be seen as rules of combination and transformation for the intentional objects existing in the phenomenological space—in order to keep my interpretation consistent, I defined dimensions also as *rules* of interpretation. This set of rules tells us how to obtain, or construct new intentional objects out of the original objects existing in the space. I call these rules *transforming operations*. Now, logic is the most fundamental set of rules of derivation in mathematics. All objects in a mathematical phenomenological space must have a background understanding dimension and are thus intrinsically regulated by logic. The second most important set of rules depends on the implicit relations of our background understanding dimension. For example, if our background understanding dimension is complex analysis, the identities and definitions involved in complex analysis are part of our transforming operations. Thus, we always have at least the rules of logic and of the particular background-understanding field of mathematics as transforming operations.

In mathematical appreciation we can have different operations working at different levels of appreciation and they are more relevant in performative (using the definitions introduced in Chap. 3 of *contemplative* and *performative* ways of eliciting affection) mathematical intentional objects such as derivations or proofs. For example, the introduction of the geometric interpretation of complex numbers by Caspar Wessel in 1799 allowed simpler derivations of already known results. Paul Nahin remarks:

> How beautifully simple is Wessel's idea. Multiplying by $\sqrt{-1}$ is, geometrically, simply a rotation by 90 degrees in the counter clockwise sense [...] Because of this property $\sqrt{-1}$ is often said to be the *rotation operator*, in addition to being an imaginary number. As one historian of mathematics has observed, the elegance and sheer wonderful simplicity of this interpretation suggests "that there is no occasion for anyone to muddle himself into a state of mystic wonderment over the grossly misnamed 'imaginaries.' " This is not to say, however, that this geometric interpretation wasn't a huge leap forward in human understanding. Indeed, it is only the start of a tidal wave of elegant calculations [68, pp. 54–55].

In the geometric interpretation "a complex number is either a point $a + ib$ in the so called *complex plane* or the directed radius vector from the origin to that point" [68, p. 48].

In addition to the representation $a + ib$, a complex number is sometimes represented by the associated length of its radius vector, called the *modulus* of the complex number, and the value of the polar angle $\arctan \frac{b}{a}$, called the *argument*. We can express this as follows [4]:

$$a + ib = \sqrt{a^2 + b^2}\angle \arctan \frac{b}{a}$$

[4]The angle notation, $\angle$, is very popular in fields like engineering. It is related to the polar form of complex numbers, the expression before the angle symbol represents its modulus and the expression after is the argument. This notation simplifies the visualization of operations: multiplication consists in multiplication of modulus and addition of arguments, exponentiation consists in exponentiation of modulus and multiplication of arguments.

Nahin's remarks on the geometric interpretation enthusiastically employ aesthetic adjectives. Nahin also stresses that the geometrical interpretation resulted in elegant calculations and even devotes a section of his book to presenting some of those calculations. De Moivre's theorem is instrumental in many of those calculations and it is an example of an elegant derivation itself [5]:

> With his wonderful deduction of the geometry of $\sqrt{-1}$ there was now no stopping Wessel with even more exotic calculations. For example, if you start with a unit radius vector of direction angle $\frac{\theta}{m}$, where m is an integer, then it follows immediately that
>
> $$\{1\angle\frac{\theta}{m}\}^m = \{\cos\frac{\theta}{m} + i\sin\frac{\theta}{m}\}^m = 1\angle\theta = \cos\theta + i\sin\theta$$
>
> Or turning this statement around by taking the mth root,
>
> $$\{\cos\theta + i\sin\theta\}^{\frac{1}{m}} = \cos\frac{\theta}{m} + i\sin\frac{\theta}{m}$$
>
> This result is not original with Wessel (although this elegantly simple derivation of it was), and it is commonly known as "DeMoivre's theorem" [68, p. 56].

The above derivation of DeMoivre's theorem is composed of several individual expressions, the steps of the derivation. In order to see the derivation as a single item we need to connect all those individual expressions. We do this by seeing the steps of the derivation as resulting from the application of logic or other inference rules implicit in complex analysis (or other relevant field). This illustrates that our object of attention is determined not only by its visible properties but also by how we *actively* deal with it. Furthermore, Wessel's geometric interpretation is a mathematical item that also has methodological repercussions: it results in elegant calculations. Historically, the fact that the geometric interpretation resulted in simpler derivations contributed to our appreciation of complex numbers. However, the fact that Wessel's geometric interpretation results in elegant calculations is not a property we can immediately see in the mere proposal of the interpretation. We can see that Wessel's proposal is simple, but to realize that it also results in elegant calculations we need to see the derivations themselves; that is, the property of *resulting in elegant calculations* is not immediately apparent by just directing our attention to the geometric interpretation. We need to perform further activities to realize the role it plays in, for example, the elegant derivation of DeMoivre's theorem. In other words, *resulting in elegant calculations* is not a property observable within a phenomenological space that includes Wessel's 1799 geometric interpretation of complex numbers. Now, this phenomenon occurs also in the arts. Features not observable in an artwork itself can help us to appreciate

[5] We shall see below that calculations, derivations or proofs belong to a different class of experience than formulas or theorems, since they are more "performative". Furthermore, this example involves not only active attention but also the fact that the person's history of experiences enables him to see some properties; it thus fits better in a third class comprising evaluations formed by a person's history of encounters with different mathematical items.

it better. Jeneffer Robinson illustrates this with the second movement of Carl Nielsen's Sixth Symphony (1925), which, according to some musical critics, is an expression of his bitterness and disappointment due to his failure to reach an international audience. However, when one listens to the music instead of bitter or disappointing, it sounds playful, humorous and even buffoonish. Robinson argues that this illustrates that a correct interpretation of some artworks must be grounded not only on observable features in the work, but also on extra knowledge, such as knowledge of the artist's life. We can hear Nielsen's music as an expression of bitterness only if we have extra musical knowledge about Nielsen himself. Robinson points out that what an artwork expresses may be *manifest* to us only if we have some information about the artist himself. Robinson concludes that "we cannot tell a work is an expression of bitterness, disappointment, and exasperation in its author just by paying close attention to 'the work itself' independently of its wider context" [76, p. 249]. The wider context provided by derivations such as DeMoivre's theorem's helps us to see new properties in Wessel's interpretation just as biographical context help us to see new properties in music.

The introduction of new transforming operations allows us to address this issue, since operations allow us to construct new objects like derivations or proofs, and thus to see new properties that are not originally visible on the object from within the phenomenological space. The introduction of further operations enables us to see further properties. I label this new type of operations *meta-intentional transforming operations*, whereas the rules implicit in our background understanding are labeled *implicit transforming operations*.

Meta-intentional operations allow us to introduce properties not visible within the phenomenological space. These operations must be consistent with our space, thus they must comply with two conditions: first they must be aesthetically-conservative; that is, they must preserve internal consistency and the aesthetic properties of the dimensions; they cannot change any of the properties responsible for eliciting enjoyment of the intentional objects in the space. For example, meta-intentional operations cannot introduce mathematical theorems that contradict the theorem on which our attention is focused, because that would amount to introducing an inconsistency, which is against logic. And they cannot introduce properties that contradict the properties already present in the object, either. For instance, an operation that turns Euler's Identity into a complicated theorem cannot be allowed. Second, meta-intentional operations must help, or be relevant to, obtaining aesthetically relevant properties, or procedures conducing to them. This is what enables these operations to facilitate seeing new properties. For example, in the case of Wessel's geometric interpretation we can introduce the operator *simplification* by, for instance, specifying how operations like multiplication, exponentiation and other calculations can be achieved by simpler means. And once we apply the *simplification* operation, we obtain a transformed intentional object. With these ideas, the power of Wessel's interpretation can be added to the properties that elicit enjoyment in aesthetically appreciating it: Nahin judges the derivation of DeMoivre's theorem elegant. This has to do with the fact that the theorem can be derived by very simple means. But realizing this kind of simplicity depends on different intellectual activities than realizing the simplicity of Euler's formula. In the case of the derivation we need to actively supervise that all steps are correct and that they lead to the theorem in a natural way. The fact that the derivation involves only a few steps and the steps themselves do not involve complicated manipulations contributes to see the derivation as simple. The fact that this simplicity is also connected with the simplicity of the geometric interpretation further enhances the aesthetic effect: the connection between a simple idea and its power is not only practically appealing, but also causes an affective response in us. We express these facts by using aesthetic terms, 'elegant', instead of just factually descriptive terms like 'brief' or 'fruitful'.

7.4.3 A Model of Aesthetic Mathematical Intentional Objects

The following model characterizes intentional objects in aesthetic experiences of mathematics, accounting for the issues discussed above:

(A) Aesthetic Mathematical Intentional Object (AMIO):
An intentional object is called *mathematical* when it is an intentional object associated to a mathematical item. A Mathematical Intentional Object (MIO) is called *aesthetic* when it is determined by a set of aesthetically relevant properties and structural relations; more specifically, when its associated phenomenological space (PS) and transforming operations (TO) comply with the following characterization:

(B) Phenomenological Space (PS):
A Phenomenological Space is a collection of at least two different properties, referred to as the dimensions of the space.

- (B.1) A dimension is a property introduced by an explicit interpretation or definition.
- (B.2) Every PS has at least one background-understanding dimension.
  - (B.2.1) A background-understanding dimension is a property that specifies the theoretical knowledge necessary to understand the mathematical item that constitutes the AMIO.
- (B.3) Every non-background understanding dimension is aesthetically relevant.

(C) Transforming Operation (TO):
A Transforming Operation is a set of rules that AMIOs follow in order to construct new AMIOs.

- (C.1) A TO is called *implicit* when it consists in the rules of logic and mathematical background knowledge.
- (C.2) A TO is called *meta-intentional* when it is not an implicit TO, and it is aesthetically conservative and intentionally relevant.

(C.2.1) A TO is *aesthetically conservative* in a phenomenological space PS if it is consistent with all the rules that define the dimensions of PS.

(C.2.2) A TO is *intentionally relevant* if it allows us to establish aesthetically relevant properties not present in a PS.

7.4.4 Aesthetic Form

Aesthetic mathematical intentional objects can be interpreted as a particular type of aesthetic forms. We must not confuse the notion of form in the arts and aesthetics with the technical notion of form in mathematics. Form in mathematics is usually interpreted as what remains invariant under the transformations of a given context. But in art disciplines, form usually refers to something different. In music, musical forms are the abstract structures that norm the organization of musical material, and even sound material; sonatas, rondos, cadences are examples of such structures. Poetic forms are also structures that norm the organization of words into lines and entire works; stanzas, sonnets, or haikus are instances of poetic forms. Forms in painting, sculpture or architecture are less abstract, as they are closely related to concrete spatial shapes in architecture and sculpture; or they are devised to mimic visual shapes in painting.

Now, aesthetic forms are closely related to our inner representations of the objects we are observing; that is, they are closely connected with intentional objects. Intentional objects are largely influenced by the modality, the type of experience—visual, auditory or intellectual. In general, all kinds of aesthetic forms are profoundly related to the modality of the experience involved in an aesthetic-process. The aesthetic form of a painting or a sculpture is closely related to its concrete visual or spatial structure; but the relation between the form of a poem or of a symphony is less closely related to the concrete visual properties of printed words or of heard stimuli. This is perhaps the most crucial feature that distinguishes one particular aesthetic experience from another. As far as I can tell, there is no single feature or set of features of intentional objects that can be used to characterize all possible aesthetic experiences. However, a significant insight can be gained by conducting local analyses of them if we complement it by locating it in the context of a wider theoretical framework like the aesthetic as process approach. I cannot offer a unified *notion* of aesthetic form that covers form in all kinds of artistic disciplines, however, I can offer a unifying *role* that aesthetic forms perform in aesthetic-processes. Even if aesthetic experience is different for different modalities of experience, aesthetic form performs the same role in all of them: it serves as the *focus and source* of the aesthetic experience. Aesthetic experience is constituted by its content and its associated pleasure response. Aesthetic form serves as the focus of attention; in this way it confers the aesthetic experience unity, even if the experience is performative, comprising dynamically changing mental activities. Aesthetic form is also the cause of the enjoyment (pleasure or displeasure) associated with the content of experience; in this way it lends the experience its aesthetic character.

7.5 Types of Experience

Aesthetic mathematical intentional objects are the characteristic content of mathematical aesthetic experiences. The peculiarities of these objects distinguish aesthetic experiences in mathematics from other kinds of aesthetic experiences. In addition to this, we can use the way the content of experience is involved in eliciting an affective response to further categorize mathematical aesthetic experiences. In this regard, I have identified three different types of experience based on the ways in which the content elicits affective responses.[6] The way in which content elicits responses induces a finer characterization of experiences. The content-response relations are differentiated by the particular way affective responses are elicited. I call those particular ways *appreciation responses*.

In the first type of appreciation response, enjoyment is elicited by passive contemplation, due to biologically conditioned affective responses to a stimulus. In the second type, the response is elicited by the performance of intellectual activities. In the third type, the response is elicited by *acquired* preferences, that is, by the preferences that have been modulated by the history of experiences of an individual. I label these aesthetic appreciation phenomena *basic*, *performative* and *adaptive*, respectively. Each appreciation response characterizes a different type of mathematical aesthetic experience, which we can label contemplative, performative, and adaptive.

7.5.1 Basic Appreciation Response

In basic aesthetic appreciation response (or basic response, for short) the affective response is the result of readily available affective responses to passive intentional objects, that is, objects not involving active mental contents.

Affective responses are involved in a wide class of behavioural and psychological phenomena, including emotions. I shall employ this fact to interpret the aesthetic response in mathematics. Emotions are systems of response to the environment that exhibit characteristic patterns of development consisting of an initial affective assessment of the situation followed by physiological changes and a further cognitive assessment of the situation [16, 21, 56, 76, 97]. As we have seen, the patterns of emotional response are not sequential in general; they unfold along relatively independent temporal paths. For our purposes here, the most relevant feature of emotional responses is the initial stage. This stage is an affective, *non-cognitive*, evaluation of a stimulus [28,53,73,96]. This crude initial appraisal classifies stimuli as belonging to one of two opposing categories: stimuli are classified as desirable

[6] It is not unlikely that further types of experience can be identified, but the three discussed here are very relevant for discussing the dynamics of aesthetic value, and the nature of aesthetic terms later on. Discussing further types of experience is a task better suited for future follow-up works.

or undesirable events; as praiseworthy or blameworthy agents; or as appealing or unappealing objects [73, 74]. Stimuli are thus classified in terms of *valences*. The desirable/undesirable and the appealing/unappealing valences are apt to deal with the kind of affective response we find in aesthetic experiences of mathematical items. It must be emphasized that this valences are not cognitive judgements. Although they can be expressed in verbal terms, that fact is merely a way of conceptualizing the automatic response that sets in motion an emotional episode *before* cognition sets in. For the purpose of characterizing aesthetic experience, we can make the simple assumption that the affective responses involved in our aesthetic experiences are similar to the non-cognitive appraisals associated to the automatic affective responses related to the desirable/undesirable or the appealing/unappealing valences. For the sake of brevity, I refer only to the appealing/unappealing valence hereafter.

My proposal is thus to view the pleasure response in basic responses as an affective response to a passive content of attention. Pleasure (or displeasure) is elicited as an automatic response, due to a cognitive input being classified on the appealing (or unappealing) side of the valence. The mere contemplation of the input stimulus results in a good feeling, a feeling of "I like it!". A similar mechanism is responsible for displeasure: the initial cognitive input is classified on the unappealing side of the valence; its mere presence results in a bad feeling, a feeling of "I don't like it!".

Basic aesthetic appreciation response thus involves the intentional objects able to elicit the affective responses associated with the valence polarity pair appealing/unappealing. We can characterize the first type of aesthetic experience as the experience-processes whose content involves basic appreciation responses.

Definition 1. An aesthetic experience is constituted by a *basic* aesthetic appreciation response if and only if the passive content of the experience can be classified by means of the appealing/unappealing valence.

A passive content of experience is a mental content that does not involve intellectual activities unfolding from one item to another. Theorems or formulas are examples of passive contents since they are items that can be contemplated without actively shifting attention to other items. Derivations and proofs are instances of items that require active attention, since in order to follow a derivation or a proof we need to shift our attention from one step of the derivation or proof to the next. For example, in addition to Euler's identity, Wells' list [94] includes theorems like 'π is transcendental', 'the number of primes is infinite' or the four-colour theorem. Interestingly enough, none of the entries in the list is a proof, the items are mainly contemplative.

Unlike proofs, theorems require merely contemplative attention. It is true that understanding any piece of mathematics requires a range of different passive and active kinds of attention. But in the case of single results, the appreciation of their extra properties does not involve further mental activities and in many cases the affective response is automatically elicited by the mere content of our attention, which makes them instances of basic appreciation phenomena.

Let us examine Euler's identity to illustrate basic responses. Our aesthetic experience of Euler's identity does not consist in the perception of concrete stimuli or the awareness of particular instances of the formula; rather, it consists in our awareness of its aesthetically relevant properties. In the aesthetic as process theory this fact corresponds to seeing the *intentional* Euler's identity as an intentional object existing in a phenomenological space whose dimensions are *CA* (complex analysis as background understanding), *simplicity*, and *composition*. Complex analysis allows us to "see" the object, whereas *simplicity* and *composition* play the role of eliciting an affective response, they are the aesthetically relevant dimensions. The contemplation of the intentional Euler's identity involves the awareness of its properties of simplicity and composition. Simplicity and composition are attractive, appealing, properties, we are prone to like them rather than to dislike them. That is, we are prone to affectively classify objects that possess these properties on the appealing side of the valence polarity; we experience an "I like it!" feeling when we are presented with such properties. Now, not all mathematical formulae are simple, nor do they involve the most important constants in mathematics; Euler's formula is and does. These facts are encapsulated in the properties of simplicity and composition and when we contemplate them, when we make Euler's identity our object of attention, we respond affectively to it. This contemplation does not involve further activities, since making it our object of attention (in the aesthetic sense) consists precisely in realizing its simplicity and composition.

In the case of Euler's identity, all we need for an affective response is passive contemplation (in the sense that our attention does not shift from item to item), and thus its experience involves a *basic* appreciation response.

The conception of basic response embraced here offers two advantages: first, it is clearly related to the affective response associated with having preferences for certain items. Affective responses play a central role in our aesthetic theory. Second, it differentiates aesthetic responses from emotions, but, at the same time, it allows us to establish a connection between them.

7.5.2 Non "Inductive" Response

Basic appreciation responses are characteristically mathematical because the intentional objects involved in them are characteristically mathematical. If we obviate this fact and think in terms of a broader class of intentional objects, music, narratives, poems, or other cognitive objects of attention, we can learn something about them.

The affective responses in basic aesthetic appreciation responses are connected with biologically conditioned responses. The existence of this kind of responses is evident in our preference for sweetness or aversion to bitterness. One of the features of basic experiences is that the *response* is non-cognitive; the response associated with the cognitive content—a theorem or a result—is an affective response. Another characteristic of these experiences is that, in the general case, even if the responses are elementary and non-cognitive; the inputs that trigger those responses are not necessarily elementary or non-cognitive. A green dot is simple in comparison with the sight of a natural landscape with green trees and grass; however, the landscape is more likely to trigger an affective response of "I like it!" than the mere green dot. The *basic* in basic appreciation response does not refer to the intentional object; it is the *mechanism* that triggers the experiences that is referred to as basic. In mathematics, the experience in our basic response must be triggered by a complex cognitive input; for example, an intentional object containing the mathematical item $e^{i\pi} + 1 = 0$.

By using a basic biologically conditioned mechanism to characterize basic aesthetic appreciation responses, my approach contrasts, again, with McAllister's. In McAllister's view, aesthetic preferences are formed through a history of experiences with certain properties via the aesthetic induction. Basic appreciation responses do not require a history of experiences, since the preferences involved are readily available, since they are biologically determined. Past experiences are not necessary to explain the preferences associated with basic appreciation phenomena. This further highlights that there are preferences that depend on a history of experiences and preferences that do not. My characterization of aesthetic experience takes this into account, as we shall see in the following sections.

7.5.3 Performative Appreciation Responses

In performative aesthetic appreciation responses the affective response is elicited by an *active* content. The main difference from basic responses is that in performative responses the affective response is elicited as the result of performing *intellectual activities* involving the content of attention. We have already discussed and explored mathematical proofs as instances of "active" mathematical items. That discussion can be generalized to mathematical items that involve shifts of attention and intellectual engagement similar to the ones found in mathematical proofs.

We have learnt that a significant role in eliciting enjoyment is played by a person's active mental engagement. Two factors contribute to elicit enjoyment: first, the mental activities performed can be pleasing (or displeasing) in themselves. Second, performing certain activities can modify an intentional object and the resulting object may possess new aesthetic properties; that is, our mental activities help us uncover (or, rather, construct) aesthetic properties not exhibited by the original intentional object.

Now, active engagement is also the source of enjoyment in literature or film. For example, people enjoy the act of anticipating the unfolding of events in the plot of a novel or film, and then witnessing their actual development. But in order to "see" the plot of a story one needs to know the entire set of individual events that constitute the plot. We need to "construct" the plot by mentally assembling it with the individual events presented to us. In music, performative mental engagement plays an even more central role. Peter Kivy argues that seeking hidden patterns and motifs is the main source of pleasure in listening to purely instrumental music [39, 41, 43, 44].

Mathematical proofs or derivations are strings of discrete statements which we have to follow to arrive to a conclusion, I exploited this fact in the narrative analogy in Chap. 3. In following a proof, we actively connect those discrete items. Appreciating a proof is not merely contemplating properties but it involves further mental activities. If the *active* part of the experience elicits an affective response we say we have a *performative* aesthetic appreciation response.

Let us examine an example. We have already presented the derivation of De Moivre's theorem. I now present another version of the theorem involving only integer exponents; I shall refer to it as *De Moivre's formula*:

$$(\cos x + \sin x)^n = \cos nx + i \sin nx$$

Its derivation from Euler's formula serves to illustrate the active content of experience:
Euler's formula:

$$e^{ix} = \cos x + i \sin x$$

The exponential law states:

$$(e^{ix})^n = (e^{inx})$$

Rewriting e^{inx}:

$$e^{inx} = e^{i(nx)}$$

Substituting in the first expression above yields:

$$e^{i(nx)} = \cos nx + i \sin nx$$

In this simple example our attention is focused not on a single item. Rather, it successively shifts from one item to another. We begin by focusing our attention on Euler's formula, then on the exponential law, then on associativity and finally on De Moivre's formula. Performing these activities is necessary to understand how De Moivre's formula is related to Euler's formula and thus to understand the derivation as a derivation. Theorems or formulae establish certain states of affairs concerning certain mathematical objects. Euler's identity establishes the identity between complex exponents and trigonometric functions. But a derivation, or a proof, for that matter, establishes the logical relations between different results. Now, the derivation not only establishes logical relations, it also helps us understand the results themselves: once we "see" the steps that takes us from Euler's identity to De Moivre's formula we see that, for example, the appearing of the term n in different places in the formula—once as an exponent and once as an argument of trigonometric functions—is not arbitrary: we understand why the term appears in both places, we understand their connection. Now, in order to gain this understanding, we need to actively follow the steps of the derivation, we must understand what each step does, we must check that each step makes sense, that there are no contradictions, that there are no errors, etc. The nature of derivations and proofs involves an active engagement of our attention.

Active engagement, of course, turns our experience of the mathematical item into an active experience. But aesthetic experience is characterized by an affective response. It is the affective response elicited by performing mental activities what makes performative experiences characteristically aesthetic. For example, the derivation of De Moivre's theorem from Wessel's geometric interpretation[7] of complex numbers is qualified as elegant by authors like Nahin. Let us remember that the derivation requires only the geometric interpretation of complex numbers; starting with a unit radius vector of direction angle $\frac{\theta}{m}$. It follows that

$$\{1\angle\frac{\theta}{m}\}^m = \{\cos\frac{\theta}{m} + i\,\sin\frac{\theta}{m}\}^m = 1\angle\theta = \cos\theta + i\,\sin\theta$$

Or turning this statement around by taking the mth root,

$$\{\cos\theta + i\,\sin\theta\}^{\frac{1}{m}} = \cos\frac{\theta}{m} + i\,\sin\frac{\theta}{m}$$

Something we immediately notice is that in this derivation our point of departure is more fundamental. While Euler's formula already establishes complicated and non-obvious relations, the geometric interpretation establishes a simple way to deal with complex numbers. Interestingly enough, this simple definition of complex numbers results in a shorter derivation of De Moivre's theorem. Now, noticing the simplicity of the geometric interpretation can be done only after we have gone through the entire set of steps of the derivation. The simplicity of the derivation is not a property of any of the individual steps, it is a property that emerges after seeing the derivation as a whole. And it is the simplicity of the derivation as a whole that results in an affective response. The simplicity in the derivation as a whole is of a different type than the simplicity in Euler's formula, furthermore, the affective response to the derivation-simplicity is not elicited by the merely passive contemplation of the definition or of any of the items involved in the derivation, but rather by facts related to our active engagement in following it: each of the steps makes simple assumptions or is simple in itself, and there are only few step in the derivation.

[7]Eulers' formula was proved in 1714 by Roger Cotes, and published in its current form by Euler in 1748. Wessel introduced his interpretation in 1799 in the *Royal Danish Academy of Sciences and Letters* but it remained obscure for some time.

This example thus exhibits two differences from the example of Euler's identity. First, the object of attention is of a different type: while in Euler's formula we have a single object of attention, in this derivation our attention focuses successively on the geometric interpretation, on the unit radius, on the first line of the derivation, and so forth. There is no single intentional object on which our attention focuses; the derivation is rather composed of multiple objects that are linked by logic, and, in our inner experience, by attention shifts. Second, we appreciate the brevity of the derivation, but this brevity does not appear as a property of any of the individual objects of our attention—the steps of the derivation, it is rather a property of the collection of those objects. Since the properties involved in eliciting an affective response are related to active engagements of our attention, the affective response cannot be elicited by mere passive contemplation.

Active attention is central to the appreciation of derivations, proofs and other mathematical items. We have seen that mathematical derivations or proofs resemble some aspects of narratives: they consist of a collection of individual events that develop in a coherent way to arrive to a closure. Now, just as not all stories are good, not all mathematical derivations or proofs are beautiful; this fact is many times related to the quality of their narratives. Only derivations or proofs whose narratives are able to elicit positive affective responses can be properly qualified as elegant or beautiful. The derivation of De Moivre's formula based on the geometric interpretation seems simpler than the one based on Euler's identity: it is easier to follow and it has a more fundamental and simpler premise than the derivation based on Euler's formula. Its narrative is thus more suitable to be qualified as pleasingly simple yet effective; that is, as *elegant*.

This elegance is related to the way the derivation "tells" its story; that is, to the way the derivation shifts our attention (the "plot" of the derivation) and how it reaches its conclusion (the "resolution" of the plot). It is not only the mere presence of an intentional object that is liable for the enjoyment of this experience, but rather the activities involved in following the "story".

Now, I have proposed two types of possible operations in phenomenological spaces: implicit and meta-intentional operations. These operations can model the intellectual activities that result in an affective response, or that allow us to "see" new properties that result in affective responses. The operations allow us to see properties in active experiences because they allow us to construct new intentional objects in a phenomenological space. For example, the derivation of DeMoivre's theorem from the geometric interpretation presented above is qualified as elegant due in part to its simplicity. But that simplicity is a property of the derivation as a whole. In order to take the effect of this simplicity into account, we need to interpret the derivation as a composite intentional object consisting of the collection of steps of the derivation. We can use, for example, the notion of logical consequence to link each new step to the previous one. This is permitted in our phenomenological space since the most basic type of implicit operation consists of the rules of logic. By including logical transformation operations, our experience consists not only of successive individual intentional objects, the steps of the derivation, but also of the object resulting from connecting all these steps by logical consequence.

We can attribute properties, like simplicity, to this newly constructed object that we cannot attribute to any of the individual steps. We can, for example, consider the influence of the number of steps involved in the derivation. We can introduce the phenomenological space dimension of *step-parsimony* defined as the property of consisting of very few steps. Of course, none of the individual steps can be qualified as being step-parsimonious, since they do not themselves have steps. But the composed intentional object resulting from the logical consequence operation can be qualified as step-parsimonious. *Step-parsimony* is one of the reasons why our derivation is qualified as elegant.

Our active experience of the derivation of the De Moivre's theorem occurs in a phenomenological space that allows the construction of a composed intentional object via the logical consequence implicit operation. This composed object, due to the property of *step-parsimony*, elicits an affective response. Our new phenomenological space thus includes the following dimensions—or properties visible in our experience: *algebra and analytic geometry*, as background understanding; *conceptual-simplicity*, defined, for example, as being understandable by a single mental act and without ad hoc concepts; and *step-parsimony*. It also includes the operation *logical-consequence*. Such a phenomenological space contains both the single intentional objects for each step and the resulting composed object for the whole derivation. In this case simplicity and step-parsimony together elicit an affective response. It must be noted that our appreciation of step-parsimony depends on our active construction of the composed intentional object; contemplating the derivation is not merely passive contemplation.

Just as there are affective responses resulting from the passive contemplation of intentional objects, there are affective responses associated with performing intellectual activities: just as we like or dislike certain stimuli, we like or dislike performing certain activities. Peter Kivy even argues that this is the source of our appreciation of purely instrumental music. We derive amusement and pleasure (or frustration and displeasure) from performing tasks such as seeking patterns and variation of patterns and motifs in a piece of music [39,41,43,44]. Thus, in addition to the passive enjoyment associated with the mere presence of an intentional object, there is an active enjoyment associated with the activities performed in an aesthetic experience and with new properties resulting from those activities.

We can now characterize the second way content and enjoyment relate to each other in aesthetic experience. In performative aesthetic appreciation responses the objects of attention involved in an active experience elicit an affective response, or the performance of the activities results in eliciting an affective response. In other words :

Definition 2. An aesthetic experience is constituted by a *performative* aesthetic appreciation response if and only if (1) The resulting content—an intentional object constructed in our inner experience—can be classified by means of the appealing/unapealling valence, or (2) The intellectual activities involved in the experience can be classified by means of the appealing/unapealling valence.

Table 7.1 Performative combinations

Combination no.	Passive content response	Active content response
1	Pleasure	Pleasure
2	Pleasure	Displeasure
3	Pleasure	None
4	Displeasure	Pleasure
5	Displeasure	Displeasure
6	Displeasure	None
7	None	Pleasure
8	None	Displeasure
9	(Non valid)	(Non valid)

Relation Content-Enjoyment of Performative Responses

There are two possible sources of enjoyment in active experiences: a passive, contemplative, source, resulting from the presence of an intentional object in our inner experience, and an active, performative, source, resulting from the constructed intentional objects or from performing the constructive activities. The relation between content and enjoyment is thus more complex in performative responses than in basic responses. In both cases the intentional object elicits an affective response, but in active experiences the content is active and the performed *activities* themselves can be a source of enjoyment. That is, in basic responses the only source of enjoyment is the object; in performative response both object and mental activities can be responsible for the enjoyment.

Objects and mental activities in our inner experience are irreducible to each other, just as physical objects and activities are irreducible to each other. The enjoyment derived from the objects and activities are conditioned by the features of objects and activities, and are thus also irreducible to each other. Even if constructed objects in performative response could be reduced to basic objects, there is a distinctive enjoyment associated with *performing* the activities that cannot be reduced to enjoyment of objects. The enjoyment derived from objects and activities are thus different. This yields a distinctive relation between the content and the enjoyment in performative appreciation responses.

As in basic appreciation responses, the active content of the experience is accompanied by a corresponding affective response of pleasure or displeasure. However, since I have used an inclusive-or in my definition, in performative responses we must consider the cases in which one of the elements of content does not elicit a response at all. This means that for the passive content the possible responses can be the eliciting of pleasure, displeasure or none. The same is true for active content. This results in the possible combinations shown in Table 7.1.

Response 9, no affective response whatsoever, is not actually possible, since that would amount to a non-affective, hence non-aesthetic, experience. The rest of the combinations are composed responses. Composed responses can be illustrated by the derivation of DeMoivre's theorem. In this derivation we experience a response of pleasure caused mainly by the parsimony of the derivation. This parsimony is not a property of any single step of the derivation, but of the derivation as a whole; it is a property of our constructed intentional object. Of course, there is an active element in the content, related to following the derivation, but these activities are not necessarily the source of our affective response, for the sake of argument we can assume that this active element results in no response. This situation corresponds to combination 3.

The best pleasure eliciting combination is, of course, combination 1; it is a "full pleasure" combination in which both the passive contemplation and the active element are pleasing. The relevant issue here is that the possible responses in performative responses constitute a more complex set of combinations than the mere set of pleasure and displeasure in basic responses.

7.5.4 Adaptive Appreciation Response

The third class of aesthetic experience in mathematics is characterized by adaptive aesthetic appreciation responses. In this type of responses we must take into account the fact that preferences change over time and that this change is influenced by a history of experiences. In adaptive aesthetic appreciation responses (or adaptive responses, for short) the passive or active character of the content is less relevant: the distinctive feature of adaptive aesthetic appreciation response is the *mechanism* liable for forming the preferences involved in eliciting the affective response. In adaptive responses the affective responses are the result of *acquired* preferences. As we know, the eliciting of our responses is affected by our histories of previous experiences.

Basic and performative responses are characterized by the fact that their content is able to elicit an affective response; the passive or active content of the experience triggers a response, a feeling of pleasure or displeasure. In this type of circumstances I shall say that the content of the experience *invokes* an affective response. In adaptive responses, the response is not elicited as a result of a readily available preference, but as a result of a preference we have *acquired*; that is, the intentional object possesses properties to which we have adapted to like or dislike. This is an acquired eliciting of enjoyment. In these circumstances, I shall say that the content of the experience *evokes* an affective response. It must be noted that an acquired responses may become strongly internalized. In fact, acquired responses (of fear, for instance) exhibit the same patterns of physiological arousal as biologically conditioned responses [28, 54, 76].

As McAllister pointed out, there is abundant evidence of evolving preferences in mathematics. Complex analysis offers interesting examples. Imaginary numbers were not fully understood until the sixteenth century. Its introduction was plagued with suspicion, caution and even aesthetic revulsion [68, pp. 16–17]. But imaginary and complex numbers eventually became part of the basics of mathematics and, as we have seen, a source of many "elegant calculations" [68, pp. 48–55]. Now, two factors played a role in changing our appreciation of complex numbers. First, complex numbers allow us to achieve shorter, more easily understandable mathematical derivations and proofs. That is precisely the case with the introduction of the geometric interpretation that led to a more elegant derivation of De Moivre's theorem discussed above. Second, mathematicians developed a familiarity with complex numbers and the mathematical items (proofs, derivations, theorems, etc.) that involve them. This familiarity, as is evident in the mere-exposure effect, eventually resulted in a change in our preferences. In this respect, an accurate model of preference evolution is relevant here. For example, as we know, the aesthetic induction or even the mere-exposure effect are insufficient to account for some patterns of evolution. Familiarity in mathematics exhibits one of those patterns. Familiarity with complex numbers may not result in an increase in preference. Even worse, the properties and interpretation of complex numbers can become so familiar that we may end up finding them unremarkable. For example, Le Lionnais bears witness to the fact that not all mathematicians find the "most beautiful formula of mathematics", Euler's Identity, so remarkable:

> Euler's formula [...] establishes what appeared in its time to be a fantastic connection between the most important numbers in mathematics, 1, $\sqrt{-1}$, π, and e. It was generally considered "the most beautiful formula of mathematics." The brilliance of this expression is due to the nearly perfect elimination of every element foreign to the three numbers just cited. Today the intrinsic reason for this compatibility has become so obvious that the same formula now seems, if not insipid, at least entirely natural [58, p. 128].

David Wells' readers, who ranked the formula as number one in the list of the most beautiful, obviously disagree with calling the formula insipid. Furthermore, Wells himself employs Le Lionnais' opinion to show that aesthetic preferences in mathematics change over time [94, pp. 38–39]. This disagreement also illustrates the variety of adaptive aesthetic responses. Le Lionnais' opinion is a case of negative acquired preference: according to him, the initial attractive composition of Euler's identity lost its appeal as we accumulated experiences with the items and principles involved in the formula. In these circumstances, our response is not elicited via a readily available response, but via a preference shaped by an evolution mechanism that involves past experiences and acquaintances with similar or related items.

To stress the contrast in the way the content of an experience elicits enjoyment, I introduced the terms 'invoking' for readily available eliciting, and 'evoking' for acquired eliciting. Evoked enjoyment is thus closely related to the mechanism that drives preference evolution. We have seen that some preferences vary depending on the recurrent presence of certain properties in empirically adequate theories. Some other preferences tend to remain stable; these preferences can be seen as closely related to invoked enjoyment.

If we concentrate on preferences which change driven by the "inductive" element in the evolution mechanism—the ones possessing a low degree of robustness, as discussed in Chap. 5—we can characterize the evolution of evoked enjoyment in a simple way: the exposure to certain stimuli or certain intellectual activities can induce a change in the elicitation of our feeling of pleasure (or displeasure). For example, in the case of Le Lionnais' response to Euler's formula, his response has been shaped by the familiarity he has with the formula and the results that underlie it, as he recognizes himself. But in the absence of familiarity, “in its time” as Le Lionnais puts it, the response was very enthusiastic; the formula was considered, as he states, the most beautiful formula of mathematics. This is a case in which familiarity has an adverse effect.

The effect of familiarity is well known in psychology. But using familiarity to explain preferences is not alien to aesthetics either. For example, Romantic approaches to music explain its emotional impact by arguing that the development of music resembles the development of emotions, or the development of life itself. More contemporarily, Jenefer Robinson’s theory of expression asserts that an object—a pictorial representation, for example—expresses a certain emotion if it holds appropriate similarities to the way the world appears to a person experiencing the emotion.

We can now characterize the third way in which content and enjoyment relate to each other in aesthetic experiences: Adaptive aesthetic appreciation responses occur when the passive or active content of the experience evokes an affective response.

Definition 3. An aesthetic experience is characterized by a *adaptive* aesthetic appreciation response if and only if (1) The content of the experience can evoke an affective response of pleasure or displeasure, or (2) The mental activities involved in the experience can evoke an affective response of pleasure or displeasure.

For example, Euler’s identity seems to elicit a positive affective response, except when one is too familiar with it. We have seen that the intentional Euler’s identity is located in a phenomenological space with *complex-analysis*, *simplicity* and *composition* as dimensions. All these dimensions should be part of Le Lionnais’ experience, since neither our background understanding nor the complexity (simplicity) or the components (composition) of the formula have changed. The properties of the intentional object are the same; the object is thus the same: Wells and Le Lionnais experience a similar intentional Euler’s identity in a similar phenomenological space. But due to Le Lionnais’ familiarity with the formula, it appears to him unremarkable and even insipid. The difference is not in the passive or active content of the experience, the difference is in how past experiences with the content have changed the effect of the properties of the object for Le Lionnais. Whereas for Wells and his readers the simplicity and composition of Euler’s formula are remarkable, for Le Lionnais they are too natural and too obvious. In Le Lionnais’s case, his acquired preferences play the central role in constituting his experience. This is a case of adaptive appreciation response, as the passive contemplation of the formula evokes, rather than invokes, the response of insipidness.

Adaptive Content-Response Relation

The relation between content and enjoyment in adaptive response is a little more complicated than the relation in performative responses. We can start by

Table 7.2 Adaptive combinations

Combination no.	Passive content evokes	Active content evokes	Passive content invokes	Active content invokes
1	Pleasure Pleasure	X	X	X
2	Pleasure	Displeasure	X	X
3	Pleasure	None	X	None
4	Pleasure	None	X	Pleasure
5	Pleasure	None	X	Displeasure
6	Displeasure	Pleasure	X	X
7	Displeasure	Displeasure	X	X
8	Displeasure	None	X	None
9	Displeasure	None	X	Pleasure
10	Displeasure	None	X	Displeasure
11	None	Pleasure	None	X
12	None	Pleasure	Pleasure	X
13	None	Pleasure	Displeasure	X
14	None	Displeasure	None	X
15	None	Displeasure	Pleasure	X
16	None	Displeasure	Displeasure	X
Non valid	(None)	(None)	X	X

establishing an analogy between the way the content elicits pleasure or displeasure in performative response and the way the content in adaptive response elicits a response just by replacing 'invoking' with 'evoking' an affective response. We can have purely passive content; the experience of Euler's formula, for example, does not involve active content. We can also have active content, in proofs or derivations, for instance. But from the inclusive-or in Definition 3, it follows there are cases in which the passive or active component that does not evoke a response still can invoke a response. In principle, we can have a case in which the passive content evokes a response and the active content does not evoke a response, but this active component still can invoke a response. Taking this into account, the possible combinations that constitute the relation between content and enjoyment in adaptive response can be summarized as shown in Table 7.2.

Note that the option of invoking a response for a given active or passive content is only available if there is no response evoked by that same content, otherwise the response should be considered as performative; I have employed the symbol 'X' to represent that the respective response in the table is not possible. For example, a combination of four pleasure responses is absent from Table 7.2, for the two invoked pleasure responses entail the two evoked responses. Although such a combination is possible in principle, it is better classified as a performative, rather than a adaptive combination.

Adaptive responses are more complex than performative responses. They not only have more possible combinations, they also include what I call *confusing* combinations. In the case of performative responses I pointed out that the full-pleasure combination (Pleasure, Pleasure) seems to render a clear pleasure response even for complex active experiences like proofs. However, in adaptive responses we have three different cases—1, 4 and 12—that render a similar full-pleasure response. The same phenomenon occurs with full-displeasure responses (Displeasure, Displeasure)—combinations 7, 10 and 16. This fact illustrates the complexity of aesthetic experience itself and it has consequences in other elements of aesthetic-processes particularly concerning aesthetic terms and aesthetic judgement, as we shall discuss later, in the respective chapters.

7.6 The Pleasure-Relation

Basic, performative and adaptive appreciation responses constitute the ways in which content and enjoyment relate in mathematical aesthetic experience. We have seen that in the case of basic responses the relation between the content of the experience and its affective response is very simple: an object of attention elicits either a pleasure or a displeasure response. In performative responses, due to its passive and active components, the possible response consists of eight possible response combinations. In adaptive responses we have sixteen possible response combinations. We can use these facts to model aesthetic pleasure.

It is easy to see that content and enjoyment in aesthetic experience have very different constraints. The content of experience is relatively independent of the experience itself, since an intentional object is determined by features in the original mathematical item and subjective dispositions already present in the observer. Pleasure, by contrast, is triggered by the passive presence of intentional objects or by performing of mental activities. In other words, the enjoyment depends on the content. We have also determined the possible ways in which pleasure is related to the content of experience. All this information can be summarized by saying that the content of an experience is an independent variable, and that pleasure is a variable that depends on the content. There is a dependence relation between pleasure and the content of experience.

The relation between the intentional object and the affective response resembles a function in the sense that it expresses the dependence between two entities. However, there is an important difference: a function associates a single output with an input, but in the case of the pleasure-relation, an intentional object, the input, can result in different affective responses, the outputs, depending on the context—the same object, a beautiful proof, for instance, can be involved in performative or adaptive responses. Several components of an aesthetic-process—the change of value over time, for instance—play a role in determining the affective response. Despite this large-scale dependence, the local dependence of affective response on intentional objects still captures some important features—the multiplicity and complexity of the possibilities of response, for example—that later on shall help us clarify some issues related to the production of aesthetic judgements. For this reason, I devote this section to modelling the local features of the pleasure response.

Pleasure—an affective response—is the result of the content of experience. The content can be seen as a variable with two components: the passive and active content. Pleasure, the affective response, can be seen as a variable with two components corresponding to the passive and active responses. We can define aesthetic pleasure as a relation that maps the content of experience into a set of ordered pairs that represent the possible combinations of responses. As before, we can call these responses or combinations of responses simply *enjoyment*.

We can thus define enjoyment simply as the set *ENJOYMENT* of all possible combinations of affective responses. This includes the 2 possibilities pleasure and displeasure for basic response; the 8 combinations for performative response; and the 16 for adaptive response. Aesthetic pleasure in mathematical aesthetic experiences can be informally characterized as follows[8]:

$$f: content \rightarrow ENJOYMENT$$

7.6.1 Formalization

Let us now formalize the idea of the pleasure-relation, starting by defining our vocabulary.

$$PC = \{Pas | Pas \text{ is an Aesthetic Mathematical Intentional Object}\}$$

$$AC = \{Act | Act \text{ is a mental activity related to an Aesthetic Mathematical Intentional Object}\}$$

$$PR = AR = \{P, D, Ep, Ed, N\}$$

where:

Pas: the variable for the passive content of the experience. The range of Pas consists of all possible Aesthetic Mathematical Intentional Objects; including no content at all, that is, no object of attention. In this context the symbol $\varnothing$ represents empty content, no object of attention.

Act: the variable for the active content of the experience. The range of Act consists of all possible intellectual activities performed by our attention, including no activity at all. In this context the symbol $\varnothing$ represents no activity.

Rp: the variable for the passive affective response.

Ra: the variable for the active affective response. The range of Rp and Ra is the set $PR = AR = \{P, D, Ep, Ed, N\}$

[8] Although the notation f is usually employed to refer to proper functions, I shall retain it instead of r, for example, in order to avoid confusion with other occurrences of 'r' in the discussion.

where:

P: an invoked pleasure response.
D: an invoked displeasure response.
Ep: an evoked pleasure response.
Ed: an evoked displeasure response.
N: no affective response.

We can now define all the ways of relating possible contents of experience to possible responses. This relation includes contents of attention that are completely empty $\langle \varnothing, \varnothing \rangle$ and responses that have no affective response $\langle N, N \rangle$. These cases cannot be categorized as *aesthetic*, since they amount to either an empty experience (in that case we cannot talk about aesthetic *experience*, since there is no content at all), or no affective response (if there is an actual content but no affective response, we have a kind of experience that is not *aesthetic*). There are many ways in which our attention can become engaged without arousing any affective response, but these are just episodes of *attention*, not episodes of aesthetic experience. Experiences with no content or with no affective response do not participate in aesthetic-processes. However, the cases of non-aesthetic experience can be considered limiting cases of "attention experience". Since the relation that admits aesthetic as well as non-aesthetic episodes of attention does not characterize aesthetic experiences but rather all kinds of episodes of attention, I call this relation *attention-relation*. The attention-relation contains all ordered pairs $\langle \mathit{content}, \mathit{response} \rangle$, including the limiting non-aesthetic and empty cases.
The attention-relation is defined as follows:

$$\mathit{Attention}\colon (PC \times AC) \times (PR \times AR)$$

This relation comprises all possibilities, including basic, performative and adaptive aesthetic appreciation responses. Appreciation responses are thus proper subsets of the attention-relation.

Now, we can define a general pleasure-relation as the subset of the attention-relation such that its elements consist of the ordered pairs in which the first coordinate is not empty and the second coordinate is an actual affective response. A non-empty first coordinate is any possible ordered pair $\langle \mathit{Pass}, \mathit{Act} \rangle$ in which at least one coordinate is a non-empty content; in other words any possible pair except $\langle \varnothing, \varnothing \rangle$. An actual affective response is any possible ordered pair $\langle Ra, Rp \rangle$ in which one of the coordinates is an actual response; in other words any possible pair except $\langle N, N \rangle$. In this way, we guarantee that our general pleasure-relation contains only non-empty experiences that do elicit affective responses. Experiences with solely an active content $\langle \varnothing, Act \rangle$ cannot be characterized as mathematical in our model and must also be excluded. Finally, we must exclude cases in which an affective response has no associated content. For example, if the content of attention is just an aesthetic mathematical intentional object and we contemplate it passively—that is, there is no active content—the associated response cannot have an active response.

Thus we must exclude combinations like $\langle\langle Pas, \varnothing\rangle, \langle P, P\rangle\rangle$. We can use a material implication ($\Rightarrow$) to express this condition as follows:

$$(Pas = \varnothing \Rightarrow Rp = \varnothing) \wedge (Act = \varnothing \Rightarrow Ra = \varnothing)$$

I label this condition the *Causal Condition*. In order to simplify the notation I write the symbol CC to stand for (or, being equivalent to, $\equiv$) the causal condition stated above:

$$CC \equiv \forall Pas, Act, Rp, Ra((Pas = \varnothing \Rightarrow Rp = \varnothing) \wedge (Act = \varnothing \Rightarrow Ra = \varnothing))$$

We can thus define the general pleasure-relation as follows:

$$\begin{aligned} Pleasure = \{\langle x, y\rangle | \langle x, y\rangle \in Attention \\ \wedge\, x \neq \langle\varnothing, \varnothing\rangle \wedge y \neq \langle N, N\rangle \ \wedge \forall Act(x \neq \langle\varnothing, Act\rangle) \wedge CC\} \end{aligned}$$

We can now define pleasure-relations for our three kinds of appreciation responses. I label them basic, performative and adaptive pleasure-relations.

The basic pleasure-relation consists of the ordered pairs whose first coordinate contains no active content, and the second coordinate contains no evoked response:

$$\begin{aligned} BasicPleasure = \{\langle x, y\rangle | \langle x, y\rangle \in Pleasure \wedge \exists Pas(x = \langle Pas, \varnothing\rangle) \\ \wedge \exists Rp(y = \langle Rp, \varnothing\rangle) \wedge Rp \notin \{Ep, Ed\}\} \end{aligned}$$

The performative pleasure-relation consists of the ordered pairs whose first coordinate contains an active content, and the second coordinate contains no evoked response:

$$\begin{aligned} PerformativePleasure = \{\langle x, y\rangle | \langle x, y\rangle \in Pleasure \wedge \forall Pas(x \neq \langle Pas, \varnothing\rangle) \\ \wedge \forall Pas(x \neq \langle\varnothing, Pas\rangle) \\ \wedge \exists Rp, Ra(y = \langle Rp, Ra\rangle \wedge Rp, Ra \notin \{Ep, Ed\})\} \end{aligned}$$

The adaptive pleasure-relation consists of the ordered pairs whose second coordinate contains an evoked response:

$$\begin{aligned} AdaptivePleasure = \{\langle x, y\rangle | \langle x, y\rangle \in Pleasure \\ \wedge \exists Rp, Ra(y = \langle Rp, Ra\rangle \\ \wedge (\exists Rp, Ra(Rp \in \{Ep, Ed, N\} \\ \vee Ra \in \{Ep, Ed, N\}))))\} \end{aligned}$$

It should be noted that the general pleasure-relation does not characterize a mathematical aesthetic experience, since there are many ways of characterizing the relation between content and responses. Only the pleasure-relations defined above characterize mathematical aesthetic experience.

We can characterize the modality of mathematical aesthetic experience as follows:
An aesthetic experience is a *mathematical aesthetic experience* if and only if $Pas \in \{O|O$ is an Aesthetic Mathematical Intentional Object$\}$ and there is a relation between the content of experience ce and its affective response ar such that:

$$\langle ce, ar\rangle \in BasicPleasure \cup PerformativePleasure \\ \cup AdaptivePleasure$$

We can now reformulate our pleasure-relation. Let us define:

$$Content = \{x|\exists x(\langle x, y\rangle \in BasicPleasure \cup PerformativePleasure \\ \cup AdaptivePleasure)\}$$

$$ENJOYMENT = \{y|\exists x(\langle x, y\rangle \in BasicPleasure \cup PerformativePleasure \\ \cup AdaptivePleasure)\}$$

A pleasure-relation P_r for mathematical aesthetic experience is defined by:

$$P_r \subseteq Content \times ENJOYMENT$$

Pleasure and Aesthetic Terms

The interpretation of aesthetic pleasure above yields an interesting insight. One of the arguments Rota presents for reinterpreting mathematical beauty is that ‘beauty’ is a concept that does not admit degrees [78]. This idea, however, is discredited by facts such as the existence of comparatives of the type “A is more beautiful than B”. Now, the idea of measuring the degree of beauty might sound strange. Fortunately, my interpretation of the pleasure-relation allows us to conceptualize the degree of beauty without the need for a measure. In the characterization, we can see that, in the general case, there are several possibilities for affective response. We certainly have cases where a pleasure-response does not seem to admit degrees—perhaps this is why Rota believes beauty does not admit degrees: in the case of experiences of basic response the pleasure-relations renders only pleasure $\langle P, N\rangle$ or displeasure $\langle D, N\rangle$. However, in the general case, the relation renders composed responses—$\langle D, P\rangle$, for instance. It also seems clear that the total enjoyment in full-pleasure combinations $\langle P, P\rangle$ is more enjoyable, so to speak, than combinations in which only one argument renders pleasure. Comparing different combinations can be consistently done by using comparatives, without the need for attributing specific degrees of beauty.

There is another consequence of this interpretation of aesthetic pleasure. If we try to link opposed predicates like ‘beautiful’ and ‘ugly’ to an appropriate pleasure-relation we get an interesting puzzle: As discussed in Chap. 3, the most obvious association we can establish, of course, is to link beauty to pleasure and ugliness to displeasure. This works well for basic responses, since we have only two possibilities. But for performative responses the situation is more complicated. We have eight possible combinations. A compromise here would be to associate beauty with full-pleasure $\langle P, P\rangle$ responses and ugliness with full-displeasure $\langle D, D\rangle$ responses. However, in the case of adaptive responses we have three full-pleasure combinations $\langle E_p, E_p\rangle$, $\langle E_p, P\rangle$, $\langle P, E_p\rangle$, and three full-displeasure combinations $\langle E_d, E_d\rangle$, $\langle E_d, D\rangle$, $\langle D, E_d\rangle$. Now, the problem is not how to assign a predicate to a response in the relation. There is nothing that prevents us from making the assignments to an arbitrary full-pleasure response. However, if we try to interpret what it means that the predicate ‘beautiful’ is the opposite of ‘ugly’ in terms of its associated pleasure-relation, some questions arise: what do the non-assigned possibilities in the relation mean? How should we interpret the negation of the predicate beauty; should we interpret it as one, several, or all the possible responses of the relation, or just as the complementary set of responses? Furthermore, do we need an inverse relation for ugliness? What is the relation between “mirror” responses (responses that have the opposite position for pleasure and displeasure) of the relation? Now, I believe this problem does not arise from the interpretation of the predicate in terms of the pleasure-relation, but rather from the assumption that the relation between beauty and ugliness is just that of their being opposites. I believe there is a significant relation between those predicates, but conceptualizing them merely as opposites is insufficient. In addition, there remains the fact that even if we assign terms like ‘beautiful’ or ‘ugly’ to particular responses, there is a whole range of unassigned combinations (the remaining possible responses) in the pleasure-relation. I think all this can be construed as showing that the conceptualization of aesthetic experiences in terms of predicates is open to many interpretations. I shall further explore these ideas later on, in the chapters on aesthetic terms and judgement. These issues also show that aesthetic issues are intimately connected with each other, but for the sake of organization, I shall stop the discussion of aesthetic experience here and further it in the coming chapters when necessary.