Chapter 15
Issues of Mathematical Beauty, Revisited

With the application of the aesthetic as process theory in the previous chapters, we can now recapitulate and further discuss some of the issues and insights on mathematical beauty gained and pointed out throughout this book.

Although a non-literal interpretation of the term 'mathematical beauty' seems to be supported by attitudes like the two cultures divide, we found out that the arts/sciences divide is a cultural contingency. Moreover, we examined historical attempts to interpret mathematical beauty as a genuine aesthetic phenomenon. In addition, there are principled reasons against reinterpreting 'mathematical beauty'; metaphorical uses of terms such as 'beauty' or 'elegant' seem impossible. And, from a pragmatical perspective, literal approaches are in fact more fruitful. The most systematic and fruitful literal approach to beauty in science is McAllister's model of scientific development. However, we identified serious anomalies and theoretical drawbacks in McAllister's approach. We addressed those issues by introducing critical adequacy and robustness in a naturalistic revision of the aesthetic induction. This also led us to propose a matching naturalistic aesthetic theory, the aesthetic as process theory.

Now, the analysis of $y = e^x$ and the step-series interpretation of Cantor's diagonal proof illustrated that changes in aesthetic judgements not necessarily depend on a history of previous experiences, as assumed by McAllsiter's original approach. Our examples showed how judgements based on aesthetic inner experience can be the result of changes in the constitution of the experience itself.

Now, although this book does not endorses Gian-Carlo Rota's non-literal approach, the approach advanced here salutes and to some extent vindicates it, since it provided us with valuable insights: we learned that mathematical beauty is socio-historical, that properties like shortness play a role in mathematical beauty, and that familiarity with mathematics is necessary for appreciating mathematical beauty. These insights found a place in the aesthetic as process theory. For example, the fact that mathematical beauty depends on social and historical context is evident not only in the dynamic character of value, but also in the fact, illustrated in the three proceeding case analyses, that contextual knowledge plays a decisive role

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_15,

in allowing us to see the object of appreciation. Historical and social context are part of the background knowledge that is required in our interpretation of aesthetic mathematical intentional objects. However, I must emphasize that this dependence on context and history is not a feature that comes along with or results from the aesthetic character of mathematical beauty. Rather, this dependence on context and history is the result of the dependence on context and history of *understanding* in mathematics.

Now, aesthetic principles based on single properties such as order, uniformity or simplicity, also play a role in the aesthetic as process theory. We have seen that the role of properties like simplicity or step-parsimony, which are related to brevity or shortness, is to provide "extra" qualities on which our attention focuses so they are able to elicit affective responses. It must be noted that these properties are not part of the background understanding of mathematical items, but rather something extra; contingent virtues that we perceive in addition to the necessary characteristics of mathematical items. For example, the property we defined as derivative-symmetry is not a property of every function, or of the notion of derivative. The derivative of $y = e^x$ must have all properties of derivatives, but that $y = e^x$ is symmetric with respect to differentiation is an extra quality. Similarly, simplicity is not a necessary characteristic of proofs; that Cantor's diagonal proof is simple is an extra we appreciate, an extra to which we react affectively. The properties of uniformity or unity used by Shaftesbury or Hutchenson can be interpreted as some of these extra properties; as constitutive dimensions of aesthetic experience.

The need for familiarity with mathematics to appreciate mathematical beauty is addressed by the condition of a background-understanding dimension in phenomenological spaces. In this sense, familiarity with mathematics is necessary to locate, to "see", our objects of attention in a phenomenological space. This does not mean that familiarity is trivial for aesthetic response. Rather, the background understanding is what determines the particularity of mathematical aesthetic experience; it is what makes aesthetic experience a *mathematical* aesthetic experience. Just as seeing and hearing makes experiences of painting and music particular, background mathematical knowledge makes our aesthetic experience of mathematics particular. The background knowledge necessary for a mathematical aesthetic experience determines the modality of that experience.

We are now in position to address subtle issues such as the role of properties like shortness of steps in mathematical proofs. The second step of Cantor's diagonal proof (constructing the element complementary to the diagonal) is a good example of a short step in a proof. Shortness, in the context of the aesthetic experience of a proof, an experience that involves *active* content, facilitates the performance of activities and the checking that steps are related to each other. That is the case in Cantor's proof. In addition, properties like simplicity and parsimony allow us to focus our attention more effectively, and to have a more complete picture of an otherwise complex experience. These facts result in a more pleasurable performance of activities in our experience. The function of short steps in proofs can thus be accounted for by the roles played by such steps: they make a proof simpler, and they facilitate the active pleasure response.

Another subtle issue with Rota's view is the lack of a satisfactory explanation of the nature of mathematical ugliness. Rota explained that negative aesthetic judgements of mathematical items like proofs frequently result in further mathematical development: mathematicians keep working, looking for a more aesthetically acceptable proof. In our theory, the usage of aesthetic terms depends on the relations to their family of terms. The use of terms like mathematical beauty is linked to the usage of the entire family of interrelated terms (handsome, pretty, ugly, elegant, etc.), since their correct usage requires articulation. In addition, aesthetic judgements are locally terminal and they can participate in further aesthetic developments. The preferences held by a person or a community become public by uttering or publishing an aesthetic judgement. These aesthetic judgements can serve as a guide for further developments, telling us what is aesthetically meritorious and what not. If we try to develop more elegant proofs, for example, judgements of elegance of other proofs can show us which instances of proof are regarded as elegant. In a sense, mathematical aesthetic judgements can be compared to art criticism: they not only describe states of affairs regarding artworks, but also articulate processes in which personal preferences and social values interact with each other. Judgements of beauty or elegance, due to their term-family interdependence, set in place paradigms of beauty *and* elegance, and, by the same token, they also set corresponding negative paradigms. Thus, in encouraging further mathematical developments, paradigms of beauty (to be followed) are as valuable as paradigms of ugliness (to be avoided). Thus, ugliness as well as beauty and other interrelated aesthetic terms have a heuristic role; they set examples to be followed or avoided. This result is interesting also in the sense that it shows that the RSD model of aesthetic terms not only allows, but actually forces close relations among families of aesthetic terms and thus it entails closely related explanations of aesthetic phenomena. Hence the fact that ugliness as well as beauty have heuristic roles.

Furthermore, the use of different aesthetic terms ('elegant' instead of 'beautiful', for example) also play a role in refining our paradigms of aesthetic evaluation. The aesthetic as process theory permits complex processes of articulation. In the case of Cantor's proof, for instance, we have seen that whereas the use of the term beauty is possible, the term elegant is more accurate. Aesthetic judgements that include more accurate terms provide us with more sophisticated paradigms of articulation and thus with paradigms of more refined uses of terms. And, at another level, the usage of terms like 'elegant' provides us not only with paradigms of elegance, but also with paradigms of aesthetic articulation.