Chapter 1
On Non-literal Approaches

The issue of the meaning of the term 'mathematical beauty' shall serve as the *leitmotif* of the discussion throughout this book. We start by discussing the two possible ways of approaching this issue: we can take the term at face value; or interpret it figuratively, as meaning something else. That is, we can interpret the term in a literal or non-literal way. This discussion shall also serve to survey various views on beauty in mathematics that illuminate some aspects of it.

In this chapter, we examine three possible reasons for embracing a non-literal interpretation of mathematical beauty: first, the mutual exclusion of humanistic and scientific disciplines. Second, the epistemic character of mathematics. And, third, the rational character of mathematics. We shall see that none of these reasons is very compelling, for reinterpreting mathematical beauty does very little for the causes of mathematics' allegiance to science, its epistemic soundness or its rationality.

1.1 The Two Cultures, Shaftesbury and Hutchenson

As mentioned in the introduction, in his very influential Rede Lecture at Cambridge *The Two Cultures* [83], the physicist and novelist Charles Percy Snow denounced the fact that the western intellectual world is split into two cultures: the humanities and the sciences. Snow pointed out a fact that seems to be evident still today in many spheres of western culture. Regarding our discussion, the split manifests itself in the fact that the average person seems to see artistic and scientific disciplines as excluding each other; this phenomenon is the so-called *arts/sciences divide*. As we have discussed, mathematically lay people are quite justified to ask: isn't it truth, and not beauty, the goal of mathematics? And, what is the difference between beauty and ugly mathematics?

The questions are further justified by the fact that mathematics is a highly technical discipline with a very rich, and often confusing, jargon. For example, when mathematicians speak of natural, irrational or real numbers, the terms 'natural',

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‘irrational’ and ‘real’ mean something quite different from what those words mean to a lay person. Those terms have technical, specially defined meanings. The same occurs with terms such as ‘space’, ‘ring’, ‘group’, ‘category’, and many, many others.

Well, one might argue, since art and science inhabit separate realms and the mathematical jargon is confusing; it is reasonable to think that the term ‘beauty’ we find in mathematics is like the terms ‘real’ and ‘space’: a technical term with a special meaning. So, it might be the case, after all, that mathematicians do not use the term ‘beauty’ in its literal sense, but rather in some obscure technical sense. Perhaps Hardy’s statement—“beauty is the first test: there is no permanent place in this world for ugly mathematics” [33, p. 85]—conveys, say, a methodological precept, intended to be interpreted by the professionals familiar with the mathematical jargon. ‘Mathematical beauty’ might be a mere metaphor or a stand-in word which does not refer to a genuine aesthetic feature of mathematics after all.

Now, if we examine carefully the arts/sciences divide we shall find the foregoing conclusion unconvincing. We shall find out that there is no sound basis to believe that genuine aesthetic phenomena cannot occur in mathematics. The arts/sciences divide is more a cultural attitude than an intrinsic fact. In addition, the reason why non-mathematicians find the term ‘mathematical beauty’ inaccessible is independent of whether the term is interpreted literally or non-literally. Let us elaborate.

1.1.1 Unsound Divide

Although we all may be familiar with, or even take the arts/sciences divide as granted, once we take a more rigorous stance, our conviction quickly diffuses. In his “Two Cultures” lecture, Snow himself does not advocate the split between sciences and humanities. On the contrary, Snow sees the split as a hindrance to address humanity’s problems. Snow does not see the divide as something intrinsic to the culture, the sciences, or the humanities. He argues that the cultural split has its historical roots in the division of labour that began with the industrial revolution and that it was further crystallized in the nineteenth century by cultural movements such as Romanticism.

There is plenty of evidence that supports Snow’s argument. Earlier historical periods seemed to crossover the two cultures in a natural fashion. For example, early aesthetics—the discipline that studies topics such as the nature of beauty and ugliness, taste and art—did not exclude cognitive or rational phenomena from its field of study. Contrary to our contemporary attitude, during the early stages of aesthetics, intellectual phenomena, science and mathematics were regarded as genuine objects of aesthetic analysis.

Mathematical Beauty in the Eighteenth Century

Mathematical beauty is mostly disregarded by contemporary aesthetics. But it was addressed in a natural fashion by early eighteenth century aesthetics. In 1735 Alexander Gottlieb Baumgarten introduced for the first time the term *aesthetics* to denote the philosophical study of beauty and art. This event marks the birth of modern aesthetics [32, p. 15], and it had two significant consequences—perhaps adverse—that are relevant to mathematical beauty: first, aesthetics devoted itself to articulating a philosophy of art; it focused on the disciplines we now call the fine arts, which became its distinctive subjects of discussion. Second, it contributed to set in place sharper disciplinary boundaries. Moral issues, knowledge and beauty were conceived as independent from each other. As a matter of fact, the first characterizations of aesthetic phenomena were made by distinguishing them from cognition and volition [32, p. 16–17]. The hallmark of modern aesthetics, perhaps foretelling the Two Cultures divide, was the conception of the aesthetic response as independent from cognition and volition. Not surprisingly, the stance of modern aesthetics seems consistent with the arts/sciences divide. However, the precursors of modern aesthetics, Shaftesbury and Hutchenson, saw mathematics as a genuine bearer of beauty.

1.1.2 Shaftesbury

Anthony Ashley Cooper (1671–1713), the Third Earl of Shaftesbury, introduced for the first time the idea of disinterestedness as the chief characteristic of aesthetic responses. Shaftesbury characterizes aesthetic response as disinterested pleasure in the *order and proportion* manifested to our senses. Since order and proportion are features that are clearly represented in numbers and other mathematical entities, one can expect that, once disinterest is provided, they are capable of eliciting an aesthetic response. Shaftesbury himself points this out:

> Nothing surely is more strongly imprinted in our minds or more closely interwoven with our souls than the idea or sense of order and proportion. Hence all the force of numbers and those powerful arts founded on their management and use! What a difference there is between harmony and discord, cadence and convulsion! What a difference between composed and orderly motion and that which is ungoverned and accidental, between the regular and uniform pile of some noble architect and a hip of sand and stones, between an organized body and a mist or cloud driven by the wind! [15, p. 272]

The 'sense of order', according to Shaftesbury, is a feature that human beings characteristically possess. Shaftesbury further identifies order with *design* and he claims that what we love in order is the *designer*: the mind or intelligence responsible for that order; the source of order. For Shaftesbury the ultimate source of order is God. Our moral and aesthetic senses have thus the same source. They seem to be just different modalities of one and the same virtue. Numbers and their application are paradigmatic cases of order; hence the beauty of numbers. But the numbers' order is not completely independent: the true source of mathematical beauty is the designer behind its order. Beauty in numbers is just another manifestation of God.

Shaftesbury's account of beauty in numbers links moral, ontological and aesthetic matters. This stance contrast sharply with our own contemporary attitude that emphasizes disciplinary boundaries, and also shows that the arts/sciences divide is contingent upon historical circumstances.

1.1.3 Hutchenson

Francis Hutchenson (1694–1746), one of the founding fathers of the Scottish Enlightenment, also addresses the beauty of mathematics. He argues that the qualities of objects are distinct from and causes of *ideas*. Ideas are the sole materials of sensory awareness. Beauty is one of these ideas; it occurs in the mind caused by the property of *uniformity amidst variety* of external objects. Hutchenson represents a further modernization of aesthetics, since he endorses a more explicit conception of aesthetics as independent from volition and cognition. This is evident in Hutchenson's characterization of the response to beauty as:

> consisting in an immediate gratification in perceptual form that is free of the influence of all other forms of thought and value [36, p. 11].

For Hutchenson, the way we perceive beauty is different from our faculties of cognition and volition. He argues, for example, that knowledge does not affect our perception of beauty and concludes that our response to beauty can only be a sense:

> This Superior Power of Perception is justly called a Sense, because its affinity to the other Senses in this, that the Pleasure is different from any Knowledge of Principles, Proportions, Causes, or of the Usefulness of the Object, we are struck at the first with the beauty; nor does the most accurate Knowledge increase this Pleasure of Beauty [36, p. 11].

Hutchenson classifies the objects for this "sense of beauty" into three main types, which can be seen as referring to natural, conceptual and artistic beauty. Possessing the quality of unity amidst variety is the unifying principle behind all these types of objects and thus the characteristic feature of all beauty. Interestingly enough, mathematical theorems figure among Hutchenson's examples of conceptual beauty: uniformity amidst variety in perceptual forms is the source of "Original or Absolute Beauty" [36, p. 1]; uniformity amidst variety in conceptual contents is the source of the "Beauty of Theorems" [36, p. 30], and 'Relative or Comparative Beauty', which is "that which is apprehended in any Object, commonly considered as an Imitation of some Original and our pleasure in this beauty too is founded on a Conformity or a kind of Unity between the Original and the Copy" [36, p. 39]. Although Hutchenson's sense of beauty is not a cognitive faculty, there is room in it for mathematical theorems, for in theorems we find unity amidst variety. The pleasure elicited by the beauty of theorems does not have to do with the content of the theorems, but simply "with the most exact Agreement [of] an infinite Multitude of particular Truths" in a theorem [36, p. 30].

Now, the theories of Shaftesbury and Hutchenson address mathematical beauty and show that there is no inherent arts/sciences divide. Literal approaches to mathematical beauty are not only possible, but they can also be part of aesthetic theories. Now, Shaftesbury's and Hutchenson's ideas are certainly illuminating, but they fail to account for the use mathematicians make of the term 'mathematical beauty'. For example, in order to understand Hardy's statement that "beauty is the first test: there is no permanent place in the world for ugly mathematics" [33, p. 85] we must be able to contrast mathematical beauty and ugliness in mathematics. Hardy's statement illustrates that different aesthetic terms are used to evaluate mathematical entities. This evaluative aspect, however, is absent from Shaftesbury's and Hutchenson's approaches: all mathematics is characterized as beautiful in their definitions. A more comprehensive literal approach to mathematics, a proper aesthetics of mathematics, must incorporate the insights provided by authors such as Shaftesbury and Hutchenson, but it must also be able to account for the evaluative aspect of all sort of aesthetic terms in mathematics. At any rate, the ideas surveyed above greatly undermine the arts/sciences divide as an argument for reinterpreting mathematical beauty.

The Source of the Unintelligibility of Mathematical Beauty

The arts/sciences divide and the technical character of mathematical jargon motivated the idea that mathematical beauty should be interpreted in a non-literal way. We have shown the historical contingency of the arts/sciences divide. It is not difficult to show that the technical nature of mathematical jargon does not provide a sound reason for reinterpretation either: understanding mathematical jargon requires a high degree of technical proficiency *independently of how the term 'beauty' is interpreted*. Furthermore, it can also be shown that there are genuine aesthetic phenomena that require technical proficiency to be appreciated. Let us elaborate.

Gian-Carlo Rota, for example, points out that in order to appreciate mathematical beauty one must be able to understand mathematics: "[f]amiliarity with a huge amount of background material is the condition for understanding mathematics." [79, p. 179]. Even professional mathematicians specialized in a certain field might find results or proofs in other fields obscure. Moreover, the meaning of notations and symbols change from field to field. Wiles' long and complex proof of Fermat's Last Theorem is told to have been understood only by few specialist when it first appeared in 1994, partly due to its specialized notation and symbols.[1] Now, the need

[1] I must emphasize that the notation was not what made the proof hard to understand. The proof was hard to understand primarily because of its ideas were novel and it involved intricate and very abstract machinery that was alien to the field. That resulted in a notation that appeared obscure even to the specialist. But this simple need of having to learn a new notation makes technical

for technical proficiency is independent of how one interprets evaluative terms like ‘beautiful’ or ‘elegant’. Mathematics remains highly technical regardless of what meaning one gives to the term ‘mathematical beauty’. And the fact that mathematical jargon is highly technical does not imply that all terms mathematicians use are technical. Furthermore, evaluative terms, which are meant to express the worth of the item they evaluate, do not need to convey a specialized meaning, for their purpose is to clarify the stance of the speaker regarding the worth he attributes to the evaluated item.

On the other hand, some genuine aesthetic expressions can be fully appreciated only if the observer has certain proficiency in technical matters. For example, university courses on art appreciation are very common. It is a well known fact that knowledge about artistic styles, or even about a particular author’s biography and style, changes the way we perceive and appreciate painting, for instance. In music, knowledge about technical details such as the differences between things such as cadences, progressions or chords changes the way we appreciate music. The most elementary musical description is ridden with technical terms like ‘bar’, ‘interval’, ‘tonic’, etc. Moreover, some musical properties, such as the symmetry of a fugue or a sonata, are even simply “invisible” without certain degree of technical knowledge [39, 41, 42, 77]. This shows that the requirement of technical proficiency to even be able to observe certain events does not preclude the existence of genuine aesthetic phenomena associated to such events. Thus, the fact that technical proficiency is necessary to “see” mathematical entities does not preclude the possibility that such entities can be genuine aesthetic subjects.

The foregoing discussion shows that the Two-Cultures argument against a literal interpretation has a very weak basis. But perhaps issues deeper than the arts/sciences divide may provide a sounder basis. We must explore this avenue.

1.2 The Epistemic Character of Mathematics

The arts/sciences divide is not inherent to our culture or to mathematics. But there are things that are inherent to mathematics. Science and mathematics are conceived as having knowledge, truth and understanding as their goals, and as attempting to achieve those goals by rational means. If contingent cultural attitudes cannot provide a solid argument for reinterpreting mathematical beauty, perhaps its inherent characteristics may. Thus, a second motivation for a non-literal interpretation of mathematical beauty might be the epistemic character of mathematics.

The chief goals of mathematics are directly associated with knowledge. One might argue that the most valuable properties of mathematics are those conducing to justify, refine or achieve mathematical knowledge. Now, aesthetic qualities in

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subjects very opaque to the non-specialist. This is true not only for the lay person, but also for mathematicians themselves.

general are in principle ineffectual to achieve the epistemic goals of mathematics. Nonetheless, judgements of beauty are pervasive in mathematics. A possible explanation for this is that mathematical beauty possesses a hidden epistemic character after all. Therefore, we should not take the term ‘mathematical beauty’ at face value. Rather, we should search for an appropriate reinterpretation, one in accord with the epistemic goals of mathematics. This conclusion appears to be behind the approaches of authors like Gian-Carlo Rota.

1.2.1 Rota’s Interpretation of Mathematical Beauty

One of the most interesting attempts to tackle the issue of mathematical beauty is Gian-Carlo Rota’s 1997 article “The Phenomenology of Mathematical Beauty” [78]. In that article, Rota presents an articulated analysis of mathematical beauty—instead of the invariable anecdotal account that constitutes almost the entirety of publications dealing with beauty in mathematics. Rota attempts to reconcile the use of the term ‘mathematical beauty’ with the epistemic precepts of mathematical practice. Rota concludes that when mathematicians use the term ‘beauty’ they are actually referring to the enlightenment that a certain piece of mathematics provides. Enlightenment is a kind of understanding consisting in realizing the role of a certain piece of mathematics in a broader theoretical context. The concept of enlightenment, according to Rota, is fuzzy and mathematicians dislike fuzzy concepts; this is why they employ the term ‘beautiful’ instead of ‘enlightening’. Now, in his discussion Rota’s uncovers some significant characteristics of mathematical beauty that are worth taking notice here.

Rota notes that although mathematics’ chief concern is truth, there is an ambiguity in mathematical practice; for mathematicians often claim that “beauty is the *raison d’etre* of mathematics, or that mathematical beauty is the feature of the mathematical enterprise that gives mathematics a unique standing among the sciences” [78, p. 180]. An understanding of mathematical beauty is thus vital for a full understanding of mathematics. Rota’s thus sets himself “to uncover the sense of the term ‘beauty’ as it is currently used by mathematicians” [78, p. 171]. He begins by identifying five kinds of mathematical items often qualified as beautiful: “Theorems, proofs, entire mathematical theories, a short step in the proof of some theorem, and definitions are at various times thought to be beautiful or ugly by mathematicians” [78, p. 171]. Rota argues that *properties* like the shortness of a step in a proof are sometimes associated with mathematical beauty. Shortness is also associated with the beauty of proofs, theorems or definitions. He, however, is sceptical about properties such as the unexpectedness and inevitability of arguments.[2] Rota argues that the unexpectedness of an argument cannot be

[2]Rota does not mention it explicitly, but he is referring to G. H. Hardy’s view [33] that the unexpectedness and inevitability of a theorem or proof are the sources of mathematical beauty.

identified with its beauty since we can find examples of unexpected arguments that are not regarded as beautiful. Rota believes that the source of beauty is more complex. To support this, he points out that mathematical beauty depends on context:

> [. . .] the beauty of a piece of mathematics is strongly dependent upon schools and periods of history. A theorem that is in one context thought to be beautiful may in a different context appear trivial. [. . .] Undoubtedly, many occurrences of mathematical beauty eventually fade or fall into triviality as mathematics progresses [78, p. 175].

Despite this dependence on context, Rota thinks that beauty is an *objective* property, in the same fashion as mathematical truth or falsehood are objective properties [78, p. 175]. For Rota, mathematical beauty does not consist merely in the subjective feelings of a mathematician. The distinction between beauty and truth is not the distinction between subjective and objective properties: they are both objective, they are equally observable characteristics of mathematical items. The truth of a theorem does not possess a greater degree of objectivity than its beauty; rather, they are different in the sense that they are different "phenomena in an objective world" [78, p. 175]. Rota's emphasis on objectivity indicates that he is determined to defend the epistemic character of mathematics by eliminating subjectivity and reinterpreting beauty in epistemic terms.

Rota also addresses other aesthetic judgements in mathematics: judgements of ugliness and elegance. Mathematical ugliness, he stresses, plays an important role in encouraging mathematical research: an ugly proof often encourages the development of alternative, more aesthetically appealing proofs. Rota believes that lack of beauty is linked to lack of definitiveness, for a beautiful proof often becomes the definitive proof. A beautiful theorem, Rota argues, is not often improved on or generalized. Mathematical elegance, by contrast, plays a rather minor role: elegance has to do merely with the presentation of results and not often with content [78, pp. 178–179].

Returning to beauty, Rota points out that many instances of mathematical beauty depend on familiarity and comparison; they depend on background knowledge and an acquaintance with similar instances of mathematics. In general, familiarity with a large amount of background material is a precondition for understanding any piece of mathematics. And, in order to appreciate the beauty of a piece of mathematics, we need to contrast it with other pieces: "Very frequently, a proof is viewed as beautiful only after one is made aware of previous clumsy and longer proofs" [78, p. 170]. Rota connects this fact with what he calls the "Light bulb mistake" [78, p. 179–180], which is instrumental in his account of mathematical beauty. He argues that we manage to understand a piece of mathematics only after having gone through the great pains needed to acquire the necessary background knowledge and understanding of mathematics. But in our recollection, we remember the instances of mathematical beauty as "if they had been perceived by an instantaneous realization, in a moment of truth, like a light-bulb suddenly being lit" [78, p. 180]. Once we have understood a theorem, for example, we forget about all the effort we invested in the background understanding required to understand its proof. The difficulties we encountered seem to disappear. Our recollection retains only an “image of an instant flash of insight, of a sudden light in the darkness” [78, p. 180].

It is against this background that Rota advances his interpretation of mathematical beauty. He argues that mathematicians often show their disapproval of a certain piece of mathematics by asking the question “What is this good for?” The question shows that they do not see the point of, for example, re-stating something that has already been logically verified to be true. Rota points out that logical verification alone does not enable us to see “the role that a statement plays within the theory. It does not explain how such a statement relates to other results, nor make us aware of the relevance of the statement in various contexts” [78, 181]. Logical truth does not *enlighten* us about the deep significance of a mathematical statement. Rota argues that when mathematicians ask the question “What is this good for?” they are not looking for truth, but for enlightenment. Under this interpretation, enlightenment is what drives the mathematical enterprise, and what distinguish mathematics from other scientific disciplines.

Enlightenment, however, is not explicitly acknowledged by mathematicians. Rota gives two reasons for this: First, enlightenment is not easily formalized. Second, enlightenment, unlike truth, does admit degrees; some mathematical results or proofs are more enlightening than others. The concept of beauty, according to Rota, is appealing because it does not admit degrees. Rota argues that mathematicians “universally dislike any concepts admitting degrees, and will go to any length to deny the logical standing of any such concepts” [78, p. 181]. ‘Mathematical beauty’ is the term that mathematicians “have resorted to in order to obliquely admit the phenomenon of enlightenment, while avoiding to acknowledge the fuzziness of this phenomenon” [78, p. 181]. In Rota’s view, when mathematicians call a theorem beautiful, they really mean that the theorem is enlightening. Similarly, they call a proof beautiful when it “gives away the secret of the theorem, when it leads us to perceive the actual, not the logical inevitability of the statement that is being proved” [78, 182].

1.2.2 Rota’s Problems

Rota’s work offers us very valuable insights: Mathematical beauty is not merely subjective feeling; it is rather an objective property. It depends on historic-social context. It also depends highly on background knowledge and experiences. Aesthetic considerations play a role in encouraging mathematical development.

Despite these insights, we can identify serious problems in Rota’s account. The most obvious being that Rota’s own argument against unexpectedness can be used against enlightenment: there are instances of mathematical beauty that are not enlightening. For example, beautiful methods of proof such as *reductio* *ad absurdum*[3] are among the most beloved methods of proof ever since Euclid. G. H. Hardy eloquently testifies this:

> [...] *reductio ad absurdum*, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game [33, p. 94].

Reductio ad absurdum has been regarded as one of the most beautiful methods of proof ever since Euclid, but a proof by *reductio ad absurdum* is not necessary enlightening, as it relies in showing that assuming a false premise leads to contradiction. Intuitionism, one of the three classic schools of philosophy of mathematics in the twentieth century, rejects some proofs by *reductio ad absurdum* precisely because the method is very opaque in respect of the mathematical constructs involved in the theorem, since it does not construct or show the existence of mathematical entities. In this sense, *reductio ad absurdum* is quite the opposite to being enlightening.[4]

The problem is evident even in Rota's own example of beauty in "a short step in the proof of some theorem" [78, p. 171]. The property of being enlightening is the property of being able to bring understanding of the role of a certain piece of mathematics in a more general context (the role of a theorem in a theory, for example) or understanding of the relation of a certain statement to other similar statements. In the case of steps in a proof it is not easy to see how a short step can be enlightening in the above sense. A single step in a proof, especially if it is short, is too small a piece of information to be credited with being enlightening by itself. Furthermore, steps in proofs often depend on information or results from previous steps in the same proof. A proof is a way of going from certain premises to a certain conclusion; a way of connecting premises and conclusion. A step in a proof is just an element bridging premises and conclusion. These steps do not always need to have meaning by themselves (the substitution of a logical formula for an equivalent one, for example) and as such they are very unlikely to be able to bring understanding, in the sense of showing the role of the step in a more general context. One could reply that steps, being the links that bridge premises and conclusion, actually help to connect the premise with the conclusion; and as such, the steps in a proof help us understand the relation between premises and conclusion. But if we grant this interpretation we must accept that *all* steps in a proof help us understand the larger premises-conclusion relation. In that case, every step in a proof should be qualified as enlightening and thus beautiful in Rota's sense.

Thus, the property of being enlightening is insufficient to account for all cases of mathematical beauty. In this sense, enlightenment is similar to unexpectedness and

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[3]In a proof by contradiction, or *reductio ad absurdum*, one assumes the negation of the statement to be proven and shows that it leads to a contradiction.

[4]It must be noted that intuitionism does not reject all proofs by *reductio ad absurdum*, but only those that intend to establish an existential claim, or those that intend to move from not-not-A to A. At any rate, the kind of proofs rejected by intuitionism can be considered characteristically non-enlightening.

inevitability. Since Rota rejects properties such as unexpectedness and embraces enlightenment, it seems that Rota has an epistemological agenda driving him to be selective about which properties should used to account for 'mathematical beauty'. Rota is aware of the enormous stock of anecdotal accounts of mathematical beauty, but he is very sceptical about them:

> Following this mistaken conviction, several attempts have been made to string together beautiful mathematical results, and to present them in the form of books bearing such attractive titles as "The one hundred most beautiful theorems of mathematics". Such anthologies are seldom to be found on any mathematician's bookshelf [78, p. 180].

He is also aware of the fact that a common strategy in anecdotal accounts is to try to reduce mathematical beauty to some unproblematical property. This is why he addresses, although without mentioning names or sources, G. H. Hardy's view that the properties of unexpectedness and inevitability are the sources of beauty. Rota seems less concerned with the fact that unexpectedness and inevitability are not sufficient conditions for beauty than with the fact that those properties have no relevant epistemic role. In this sense, Shaftesbury's order-based and Hutchenson' unity-based views probably would be equally unacceptable for Rota.

A second problem with Rota's approach is that it is quite implausible that mathematicians universally fail to apply the term 'beautiful' in an appropriate manner: if average people routinely use the term 'beautiful' correctly, and mathematicians are unable to do this, this is an extraordinary fact crying out for explanation. Rota's sole explanation is that mathematicians are not willing to deal with the fuzziness of enlightenment. Even if we lend credibility to this explanation we must face the problem that beauty is also fuzzy. Although Rota argues that beauty does not admit degrees, it is easy to show the contrary. Rota himself argues that an ugly proof often encourages the search for alternative proofs. Of course the result of this search is not always a beautiful proof, but it is not hard to envisage that a mathematician would be content, at least for a while, with a more aesthetically meritorious proof: a more beautiful proof or at least a less ugly proof. But the very fact that we can qualify some proofs as *more* beautiful than others shows that beauty admits degrees. In general, the predicate 'beautiful' can be employed very naturally in comparative judgements; there is nothing wrong with sentences like 'A is more (or less) beautiful than B'. Furthermore, in actuality some mathematical results are regarded as more beautiful than others: David Wells [94] even offers a ranking of some of the most beautiful theorems, based on a poll of mathematicians. In summary, beauty is a concept that *does* admit degrees; it is a fuzzy concept. If Rota were correct in arguing that mathematicians dislike fuzzy concepts, they would try to avoid the predicate 'beautiful' as much as the predicate 'enlightening'.

A third problem is that Rota mistakes beauty for the properties that evoke beauty. We have seen that Hutchenson already distinguishes the idea of beauty, which occurs in the mind, from the properties that cause that idea, which appear in observed objects. This distinction is in accord with our everyday experience: we all know that the contemplation of properties such as symmetry or simplicity many times elicits aesthetic pleasure. Rota fails to distinguish between the properties responsible for eliciting an experience of beauty and the property of beauty itself.

For example, when we pass an aesthetic judgement that a very symmetric object is beautiful, we are not erroneously using the term "beautiful" to refer to the property of symmetry; we are expressing the pleasure elicited by the contemplation of that symmetric object. No one would say that when we say "beautiful" we really mean "symmetric". Rota, however, claims that such is the case with enlightenment. Although enlightenment can elicit responses of aesthetic pleasure, it does not follow that when mathematicians call a mathematical item beautiful, they do so because they do not know how to use the term 'beautiful' or because they do not want to deal with statements like "this is enlightening". It is more reasonable to assume that they are trying to express an aesthetic experience of beauty. It is quite possible that such a perception of beauty is often derived from epistemic qualities like being enlightening. But that does not mean that mathematical beauty should be reduced to those properties. In addition, in general there is no way to reduce aesthetic properties to their associated non-aesthetic properties [82].

A fourth problem for Rota's approach is that it does not provide a unified criterion to interpret things like ugliness or elegance in mathematics. Statements like Hardy's "beauty is the first test: there is no permanent place in the world for ugly mathematics" [33, p. 85], show that there is a connection between judgements of beauty and ugliness. In general, the motivations we have to judge some things as beautiful are related to the motivations we have to judge some other things as ugly. However, although Rota sees beauty as related to enlightenment, he relates mathematical ugliness to an entirely different phenomenon, to non-definitiveness:

> The lack of beauty in a piece of mathematics is of frequent occurrence, and it is a strong motivation for further mathematical research. Lack of beauty is associated with lack of definitiveness. A beautiful proof is more often than not the definitive proof (though a definitive proof need not be beautiful); a beautiful theorem is not likely to be improved upon or generalized [78, p. 178].

Moreover, according to Rota, mathematical elegance in proofs "has to do with the presentation of mathematics, and only tangentially does it relate to content" [78, p. 178]. Rota's accounts of beauty, ugliness, and elegance seem disconnected from each other: for him, beauty has an epistemic role, ugliness has a heuristic role, and elegance has only a pedagogical role.

Finally, if our initial motivation for reinterpreting mathematical beauty is to defend the epistemic character of mathematics, it is not clear whether Rota's reinterpretation can accomplish that. The problems discussed above show that Rota's approach does not satisfactorily reinterpret mathematical beauty. Instances of mathematical beauty such as *reductio ad absurdum* or short steps in proofs remain unaccounted for. Furthermore, mathematical elegance and ugliness are not even accounted for in epistemological terms. Even if we grant that enlightenment explains many cases of mathematical beauty, a defence of the epistemic character of mathematics is a principled matter; it does not matter if we account for only few or many cases, we should be able to account for *all* cases. Furthermore, we should be able to do so in a unified manner, not by resorting to epistemological, heuristic or pedagogical arguments whenever things get complicated. In addition, there is no principled reason to think that epistemic and aesthetic concerns cannot coexist.

Reinterpreting mathematical beauty in epistemic terms does not really seem to do much for the epistemic goals of mathematics, and it rather introduces unnecessary complications.

1.3 The Rational Character of Mathematics

Mathematics' epistemological ends are achieved by rational means. We have seen that reinterpreting mathematical beauty, if certainly illuminating, does not really contribute much to defending the epistemic character of mathematics. Now, there is no principled reason why mathematics' epistemic and aesthetic considerations cannot coexist. But if we consider the other inherent characteristic of mathematics, that its results are achieved by rational means, the issue becomes more problematical.[5]

The rationalist image of science maintains that there is a set of precepts for conducting science whose justification is to a great extent principled and extra-historical. Those precepts are the norms of rationality [50,52,62,72,75]. Science and mathematics have a deep allegiance to rationality. Judgements of beauty, however, are subjective. Subjective judgements are incompatible with the norms of rationality. Mathematicians placidly invoking subjective judgements of beauty to reformulate, for example, a proof or an axiomatization is thus very troubling. One way to address this problem is, again, to argue that judgements of beauty should not be taken at face value, but rather they should be reinterpreted in a fashion consistent with the norms of rationality. After all, this reinterpreting strategy has been successful in dealing with another problem for scientific rationality: scientific revolutions.

One of the major sources of discussion in philosophy of science during the second half of the twentieth century has been the rational character of science. Kuhn's view on scientific revolutions, or at least the radical reading of it, played a central role in igniting the discussion, to the extent that the nature of scientific revolutions has become the core argument of schools of thought that deny that science is rational, such as relativism or post-modernism.

According to the radical reading of Kuhn's *The Structure of Scientific Revolutions*, a scientific revolution is a paradigm shift. Scientific paradigms are constituted, among other things, by sets of criteria for evaluating theories. The paradigm adopted after a revolution is incommensurable with the one relinquished by the revolution. Therefore, there is no common set of criteria that allows scientists to assess the worth of the old pre-revolution theories compared to the new post-revolution theories. With no common criteria to evaluate pre- and post-revolution theories there is no rational ground to choose between those theories. Thus, undergoing a scientific revolution, that is, choosing the new theories and relinquishing the old ones, has no rational ground. Under this interpretation, scientific revolutions are non-rational phenomena that challenge the depiction of science as a rational discipline.

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[5] Discussing the rationality of science or mathematics is beyond the scope of this book; here I address solely the issues relevant to the interpretation of beauty.

Arguments like the foregoing reveal that the rationality of science is debatable. If a rational depiction of science is to be embraced, those arguments need to be contested. A way to do that is to dispute Kuhn's radical interpretation of scientific revolutions by arguing that, in a revolution, only some of the criteria for theory evaluation change [52, 62, 72]. Some crucial criteria, such as the criteria for evaluating empirical adequacy or logical soundness, are preserved. In other words, the core of the norms of rationality persists across scientific revolutions. Thus, a rationalist image of science must advocate a much less radical reading of Kuhn; in other words, it must *reinterpret* the notion of scientific revolution.

Now, if the rationality of science can be safeguarded by reinterpreting scientific revolutions, it makes sense to attempt to address the threat aesthetic evaluations pose to rationality by reinterpreting aesthetic evaluations. One might thus argue that, in the case that concerns us here, mathematical beauty is not really subjective because the term 'mathematical beauty' does not really refer to the property of being beautiful, but to something consistent with the rationality of mathematics.

Now, although I agree that the presence of subjective judgements certainly constitutes a problem, and that we should find a way to deal with it, embracing a non-literal interpretation of mathematical beauty not necessarily solves the problem. Reinterpreting aesthetic evaluations does not guarantee that their associated subjectivity is eliminated; rather, as we shall see, it merely displaces subjectivity momentarily out of sight. To see this clearly, let us examine some actual reinterpretation strategies.

1.3.1 Rational Reinterpretations

Arguments about the incompatibility of rationality and aesthetic judgements in science take different forms. For example, logical positivism, a rationalistic variety of empiricism that emerged in the first part of the twentieth century, maintained that the norms of rationality allows only logical and empirical criteria as acceptable criteria for evaluating scientific theories. Criteria of beauty or ugliness are independent from and even inconsistent with logical and empirical criteria. Logical positivism held a stance regarding aesthetic evaluations, clearly expressed by Herbert Feigl:

> A few words on some misinterpretations stemming from predominant concern with the history and especially the psychology of scientific knowledge. In the commendable (but possibly Utopian) endeavor to bring the "two cultures" closer together (or to bridge the "cleavage in our culture") the more tender-minded thinkers have stressed how much the sciences and the arts have in common. The "bridges" [...] are passable only in regard to the *psychological* aspects of scientific [...] creation [...]. Certainly, there are aesthetic aspects of science [...]. But [...] what is primary in the appraisal of scientific knowledge claims is (at best) secondary in the evaluation of works of art –and vice versa [22].

As we can see, logical positivism admits that there is a sense in which aesthetic considerations can coexist with epistemic ones, but the proper evaluation of science cannot admit judgements of beauty; it thus denies legitimacy to such judgements.

As James McAllister comments: according to logical positivism "there exist no such phenomenon as scientists' aesthetic evaluation of their theories and therefore no such phenomenon that need trouble philosophers of science" [62, p. 13].

The strategy of dismissing aesthetic judgements in science can be successful only if it is in accord with the actual practice of science. But McAllister has compellingly documented that aesthetic evaluations in science are not mere anecdotal episodes or expressions of the scientists' private idiosyncrasies. Rather, they have played and still play a role in the choices of theories scientists make, and thus in the actual development of science [62–64]. Now, despite the fact that the actual practice of science does not support logical positivism's attitude and that logical positivism itself has been superseded within the philosophy of science, its distrust of judgements of beauty seems to be still influential. Helge Kragh illustrates this, by remarking that subjectivity is indeed troubling and difficult to address:

> The principle of mathematical beauty, like related aesthetic principles, is problematical. The main problem is that beauty is essentially subjective and hence cannot serve as a commonly defined tool for guiding or evaluating science. It is, to say the least, difficult to justify aesthetic judgment by rational arguments. Within literary and art criticism there is, indeed, a long tradition of analyzing the idea of beauty, including many attempts to give the concept an objective meaning. Objectivist and subjectivist theories of aesthetic judgment have been discussed for centuries without much progress, and today the problem seems as muddled as ever. Apart from the confused state of art in aesthetic theory, it is uncertain to what degree this discussion is relevant to the problem of scientific beauty. I, at any rate, can see no escape from the conclusion that aesthetic judgment in science is rooted in subjective and social factors. The sense of aesthetic standards is pan of the socialization that scientists acquire; but scientists, as well as scientific communities, may have widely different ideas of how to judge the aesthetic merit of a particular theory. No wonder that eminent physicists do not agree on which theories are beautiful and which are ugly.

If aesthetics itself cannot find an objective way to deal with beauty, the use of aesthetic evaluations by scientists is certainly a troubling challenge to the rationality of science. The influence of these ideas manifest itself not only among philosophers of science but also among scientists, as illustrated by the following warning against taking beauty in science at face value by Nobel Prize winning physicist Steven Weinberg:

> A physicist who says that a theory is beautiful does not mean quite the same thing that would be meant in saying that a particular painting or a piece of music or poetry is beautiful. It is not merely a personal expression of aesthetic pleasure; it is much closer to what a horse-trainer means when he looks at a racehorse and says that it is a beautiful horse. The horse-trainer is of course expressing a personal opinion, but it is an opinion about an objective fact: that, on the basis of judgements that the trainer could not easily put into words, this is the kind of horse that wins races [...] The physicist's sense of beauty is also supposed to serve a purpose –it is supposed to help the physicist select ideas that help us to explain nature [92].

Like Rota, Weinberg believes that aesthetic evaluations should not be conceived as referring to the standard subjective experience of beauty, but rather as referring to an objective property.

Now, Weinberg's reinterpretation exhibits some interesting characteristics: Weinberg does not explicitly formulate a new meaning for the term 'beauty'. Rather, he establishes an analogy to clarify the meaning: in qualifying a race horse as beautiful, any average person would refer to the fact that the horse's observable shape is pleasing; but an specialist, a professional horse-trainer, might refer to the fact that the horse is good at performing the task the trainer expects it to perform. A similar difference in meaning may be inferred, if my reading is correct, between the literal application of the term beauty by an average person and the figurative application by the scientific specialist: beautiful theories are merely theories which are good at doing what they are expected to do. The most significant feature of Weinberg's reinterpretation is that it explicitly intends to displace the subjectivity of the judgement of beauty: instead of relying on the usage of the term 'beautiful' by the average person, the analogy points out that we should rely on the usage of the term as intended by the specialist.

Now, the problem with this strategy is that subjectivity does not disappear. Weinberg recognizes that the specialist finds it difficult to put his judgement in objective terms (although one wonders why Weinberg does it so easily). So, the new meaning of beauty is not free from subjectivity itself. Furthermore, the analogy itself is rather idiosyncratic, for the analogy is valid for many cases of beautiful physical theories, but it is also valid for any accepted physical theory. Any accepted physical theory explains nature to some extent, but it is well documented that physicist prefer simpler theories instead of complicated ones. In this sense Weinberg's notion of the physicists' sense of beauty cannot explain why if complicated theories are good at explaining nature, physicist still prefer simpler theories. It cannot explain either why physicists disagree about which theories are beautiful and which are ugly, as pointed out by Kragh. If Weinberg analogy is valid it is *trivially* valid, and thus if we choose to accept it as an explanation of beauty in physics, it requires us to be selective in the cases in which it will be applied. And since we have no objective guidelines, this selectiveness is again *subjective*.

Thus, Weinberg's view does not really eliminates the subjectivity of the judgements of beauty, but it only distributes, so to speak, it among the new non-literal meaning and the analogy itself. If we are interested in safeguarding rationality by eliminating subjectivity, the above strategy cannot guarantee that. Subjectivity was merely displaced from a very visible place to a place out of immediate sight.

Now, unlike Rota's, Weinberg's reinterpretation is by no means articulated. Furthermore, that reinterpretation is useless for mathematics, since mathematics is an abstract discipline that does not depend on empirical facts. For these reasons, and for the sake of brevity, it may be better to recourse once again to Rota. If we assume that acceptable epistemic properties must be in accord with the norms of rationality, we can see Rota's approach as a way of reinterpreting mathematical beauty to defend rationality. Let us disregard our previous objections and see if Rota's interpretation can safeguard the rationality of mathematics. It is true that enlightenment seems appropriate to account for certain uses of the term mathematical beauty. For example, according to David Wells [94], Euler's identity is regarded beautiful precisely because it connects in a single equation the most important constants in mathematics. This is in agreement with Rota's enlightenment interpretation. However, other usages of the word beauty do not lend themselves to be interpreted in Rota's way. Hardy's famous quotation "beauty is the first test, there is no place in the world for ugly mathematics", for example. Rota's reinterpretation is only partially correct. Now, our problem here is not the principled issue of reducing beauty to enlightenment, but rather that in order to achieve a sensible interpretation one must make idiosyncratic choices depending on one's goals. Choosing which aspects of the use of the term are relevant and which are not is also a decision made by the person making the reinterpretation, in this sense the reinterpretation is also idiosyncratic.

Now, the partial accuracy of Rota's reinterpretation is less a flaw of Rota's particular tactic for understanding beauty than a problem induced by the very nature of the term beauty. Most philosophers acknowledge that there are no norms governing the application of terms such as beautiful, ugly or elegant [11, 13, 82]. This is in accord with Kragh's comment that judgements of beauty in science are subjective and idiosyncratic. Rota's approach simply illustrates that any reinterpretation has to deal with the fact that the use of the word beauty is subjective and idiosyncratic. Rota's selective interpretation is a way of dealing with this issue, another way is Weinberg's trivially correct interpretation.

We can see now that reinterpreting 'mathematical beauty' does not really eliminate subjectivity. We have two tactics to deal with the subjectivity of beauty: a partial reinterpretation, like Rota's, or an interpretation that covers all cases at the expense of covering them trivially, like Weinberg's. As we have seen, both tactics involve idiosyncratic choices, which is precisely what we are trying to avoid. The reinterpretations of beauty that are able to accurately encapsulate beauty's idiosyncrasy merely displace the subjectivity to the reinterpretation itself. The goal of reinterpreting beauty is to eliminate subjectivity, but, as we have seen, it only manages to displace it somewhere else.

To conclude this chapter let us quickly recapitulate. We have discussed three reasons why one should embrace a non-literal interpretation of mathematical beauty. I have argued that cultural attitudes like the Two Cultures split provide no sound basis for thinking that aesthetic phenomena are alien to mathematics; as a matter of fact, early proto-aesthetics even offered accounts of mathematical beauty. I also argued that two of the most central characteristic of mathematics do not provide a good argument for reinterpretation either. It is certainly troubling that aesthetic properties are epistemologically inert, and inconsistent with scientific rationality. However, I argued, reinterpreting beauty does not help to solve these problems, for that strategy cannot account for all instances of mathematical beauty and it cannot satisfactorily eliminate the subjective and idiosyncratic character of aesthetic judgements. In the next chapter we shall survey a possible explanation of these facts: aesthetic terms withstand non-literal usage. We shall also see that embracing a literal interpretation of beauty can actually help us to advance a defence of rationality.