Synthese Library 370

Ulianov Montano

Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics

Springer

Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics

SYNTHESE LIBRARY

STUDIES IN EPISTEMOLOGY,
LOGIC, METHODOLOGY, AND PHILOSOPHY OF SCIENCE

VOLUME 370

For further volumes:
http://www.springer.com/series/6607

Ulianov Montano

Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics

Springer

Ulianov Montano
Mexico City, Mexico

ISBN 978-3-319-03451-5 ISBN 978-3-319-03452-2 (eBook)
DOI 10.1007/978-3-319-03452-2
Springer Cham Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013956239

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Contents

Introduction

Mathematicians often and naturally evaluate certain pieces of mathematics using words like beautiful, elegant, or even ugly.[1] Such evaluations are prevalent; they appear in discussions, in teaching contexts and in the literature. However, rigorous investigation of them, of mathematical beauty in general, is much less common. Among the few authors that address this issue is James McAllister, who has extensively documented [62, 64] that when mathematicians talk about mathematical beauty they are being "serious" mathematicians: McAllister shows that beauty in science does not confine itself to anecdotes or personal idiosyncrasies, but rather it had played a role in shaping the development and progress of science [62, 63]. To interpret aesthetic judgements in mathematics while taking their seriously aesthetic character is no simple matter, however. Addressing aesthetic phenomena is no simple endeavour in itself, and if we take into account that aesthetic phenomena are traditionally associated with things such as art and expression and considered alien to science, the difficulty of the task is only enhanced. The belief that aesthetic matters and mathematics are alien to each other seems to be confirmed to the mathematically lay person every time mathematicians qualify an obscure formalism as "beautiful". That formalism is to the lay person as impenetrable as any other "non-beautiful" piece of mathematics. Lay people can surely find such evaluations strange. The aim of this book is to develop a rigorous and comprehensive theory of beauty in mathematics capable of explaining this and other problems while taking into account the role of beauty in the development of mathematics and explaining how aesthetic judgements in mathematics are genuinely aesthetic. Perhaps a caveat

[1]Many other words can be found in aesthetic evaluations in mathematics. For example: neat, handsome, harmonious, charming, pristine, tidy, simple, clumsy, horrible, nasty, crude, rough, dirty, etc. However, some of those terms, clumsy, for instance, can be argued to be more metaphorical than aesthetic. Others, simple, for instance, can be argued to be literally descriptive but not necessarily aesthetic. Terms such as beautiful, elegant or ugly are much more obviously aesthetic. The issue of metaphorical and controversially aesthetic terms shall be addressed at length later on, but, for the sake of clarity, in most of this book I shall refer to the more obvious, and in fact most frequently used, aesthetic terms.

is in order here; the enterprise in this book is of a theoretical nature, the reader interested in detailed historical accounts, examples of beauty in mathematics or more light-hearted discussions is advised to look elsewhere. Fortunately, that reader can rest assured that there is abundant literature covering such issues. The present book intends to contribute to a rigorous *science* of aesthetic phenomena in mathematics.

Art, Science and *The Two Cultures*

Mathematician's aesthetic judgements may puzzle people unacquainted with mathematics for good reasons. After all, in the western world the arts have been traditionally perceived as separated from the sciences. This tradition has been pointed out by some authors at least since the nineteenth century. But the most influential comment on the arts/sciences divide is C. P. Snow's 1959 Rede lecture "The Two Cultures", which denounced and lamented the divide into the "literary" and the "scientific" cultures [83]. Snow placed on the foreground a phenomenon that has been perceived and discussed long before in various spheres of culture and academia. And even after more than half a century of heated discussion ignited by Snow's denouncement, the two cultures divide seems to be still in place. The two cultures divide manifests itself in the fact that the average person usually sees artistic and scientific disciplines as alien to each other. The average person finds this perception further confirmed by the fact that mathematical beauty is usually inaccessible to the non-mathematician. Mathematically lay people find it difficult to even see what mathematicians mean when they refer to pieces of mathematics in aesthetic terms. For example, G. H. Hardy famously declared: "beauty is the first test: there is no permanent place in this world for ugly mathematics" [33, p. 85]. A person professionally acquainted with mathematics usually understands that Hardy is referring to certain preferences mathematicians have. But a layman might ask, justifiably: isn't it truth, and not beauty, the goal of mathematics? And, moreover, what is the difference between beautiful and ugly mathematics? To the lay person, beauty in mathematics greatly differs from beauty in everyday life. He is thus perfectly justified to ask the question addressed in this book: what do mathematicians mean when they talk about beauty?

Now, it is not only the issue of the two cultures divide that poses puzzles for understanding beauty in mathematics. Perhaps the most notable of those issues is that there are metaphorical usages of aesthetic terms and that they might very easily lead us to believe that all aesthetic terms in mathematics are metaphorical. Sometimes mathematicians, or scientists in general, intend to emphasize the heuristic or practical importance of certain results or theories by qualifying them as beautiful. In those cases, beauty is a sort of figure of speech that refers to a general kind of virtue, perhaps a practical or methodological virtue, and not necessarily to an aesthetic quality. Richard Feynman, for instance, has famously referred to Euler's identity as "the most remarkable formula of mathematics—our jewel" [24,

p. 23]. Although an aesthetic interpretation of this statement is possible, I believe that a different, non-aesthetic interpretation is equally plausible. Euler's identity has enormous repercussions in fields like physics or engineering. The reference to "our jewel" might be interpreted as simply implying the great practical value of the identity, not necessarily its aesthetic value. We can contrast this with a more obvious and explicitly literal use of aesthetic terms by Bertrand Russell: "Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture" [81].

Figurative usages of aesthetic terms may seem convincing that there is no proper beauty in mathematical beauty but merely a metaphor when seen from our twenty-first century point of view. However, earlier historical periods did not distinguish so sharply between what we nowadays know as the arts and the sciences. It is well known that for Pythagoreans the harmonic nature of the world was clearly manifested in numbers as well as in music. In the middle ages, scientific education involved *music*, along with arithmetic, geometry and astronomy; these disciplines formed the quadrivium: the group of scientific subjects taught in universities, after the literary education of the trivium, which consisted of grammar, dialectic and rhetoric. The inclusion of music among the scientific disciplines was not arbitrary; it was grounded on a specific conception of knowledge. The Greek philosopher Proclus Lycaeus (412–485) dealt with the subjects of the quadrivium by explicating them based on the notion of quantity. For Proclus, the quadrivium's subjects were mathematical subjects [31]. By dissociating discrete from continuous quantity, Proclus believed that an arithmetical fact had its analogue in geometry and *vice versa,* and that a musical fact had its analogue in astronomy and *vice versa*. The scientific subjects were thus characterized as follows: Arithmetic is discrete quantity at rest. Geometry is continuous quantity at rest. Music is continuous quantity in motion. Astronomy is discrete quantity in motion [31, pp. 71–72].

The sixth century philosopher Anicius Manlius Severinus Boethius (ca. 480–524 or 525) was who introduced the word "quadrivium"—which means four-fold path [61, p. 14]. He translated Euclid's and Ptolemy's texts used in the teaching of the quadrivium. Boethius even wrote his own treatise on music, the *Principles of Music*. He distinguished three types of music: instrumental, human and cosmic, all of which involved the study of harmonic ratios. Boethius was not concerned with the practice of music but with its principles—something roughly similar to what we call music theory nowadays. Boethius' principles of music embodied "ideal structures of the world" [61, p. 15]. Boethius believed that human beings find a natural joy in music, and he connected this fact with Plato's view that the world is structured according to musical intervals. He also endorsed Plato's view of the power that music has to change people's moods and behaviour. As to Proclus, music to Boethius is a mathematical subject, and yet music was characterized by the same features we attribute to it today: the power to deliver enjoyment and to affect our emotions. To Boethius there seemed to be no conflict, but rather a natural connection between beauty and mathematics.

The gap between music and mathematics deepened as the disciplines of the arts and the sciences matured. The Renaissance and Modernity granted less importance to the aesthetic aspects of mathematics. At the same time, aesthetic problems began to be seen as independent from cognitive phenomena. The birth of modern aesthetics is characterized by the introduction of the view that the phenomena related to our perceptions of beauty are independent of any practical or cognitive concerns [32]. As the empirical component of knowledge gained importance in science, and its power of description and prediction was notably enhanced by mathematics, mathematics became more closely associated with the other sciences. Mathematics' old relationship with beauty and music lost relevance. By the end of the nineteenth century, the epistemic problems were at the heart of the philosophical debate on mathematics [14].

The relation between mathematics and beauty would not be investigated any more. The peculiarities of mathematics, a formal discipline isolated from empirical events and governed solely by logic, posed the most serious difficulties for philosophers at the end of the nineteenth century. Gottlob Frege, for instance, considered that arithmetic and games of chess were very alike:

> An arithmetic with no thought as its content will also be without possibility of application. Why can no application be made of a configuration of chess pieces? Obviously because it expresses no thought. Why can arithmetic equations be applied? Only because they express thoughts. How could we possibly apply an equation which expressed nothing and was nothing more than a group of figures, to be transformed into another group of figures in accordance with certain rules? Now, it is the applicability alone which elevates arithmetic from a game to the rank of science [27].

A purely formalist approach to mathematics does not allow us to justify the place of mathematics among the sciences. This is a worrying picture for anyone who regards mathematics as a serious discipline. Frege concluded that it is applicability which elevates arithmetic from a game to a science. Now, mathematical beauty seems a frivolous concern when such urgent problems are at hand. If mathematical beauty was to be addressed, a serious mathematician should address it in such a way that it does not compromise the scientific character of mathematics. This concern, I believe, may explain why contemporary mathematicians might be interested in interpreting mathematical beauty in terms of scientific precepts. Gian-Carlo Rota [79]—whose ideas shall be discussed in detail later on—offers a notorious example of a scientific-character-preserving approach to mathematical beauty. Rota believes that when mathematicians employ the term "mathematical beauty", they are actually referring to the objective property of being enlightening.

Now, a different trend gained appeal during the second half of the twentieth century. The arts/sciences divide began to be seriously questioned. We witnessed attempts to make reason and beauty meet again. Nelson Goodman [30], for example, proposed that cognitive processes play an important role in aesthetic appreciation. Susan Langer [51] addressed music by analyzing topics like language, abstraction, and knowledge. Authors in the philosophy of science have also shown an interest in the topic of beauty in science. James McAllister [62] developed a rationalist picture of scientific change based on a mechanism he called the aesthetic induction. Theo Kuipers [49], when further developing the idea of the aesthetic induction, explored the idea that beauty can even play a role in the scientific search for truth.

A rapprochement between aesthetics and mathematics can also be found among twentieth century mathematicians. Francois Le Lionnais [57], for instance, resorts to the history of art to illustrate different kinds of mathematical beauty. He argues that beauty appears in every branch of knowledge, but nowhere with more force than in mathematics: “Has not the Western world confirmed the opinion of ancient Greece which up to the time of Euclid considered mathematics more an art than a science?” [58, p. 121]. He presents an interesting list of authors who enthusiastically expressed their aesthetic views on mathematics. But, even more importantly, Le Lionnais recognizes the need to deal with the subject in a more rigorous manner, not just by pointing out or demonstrating the existence of mathematical beauty:

> If some great mathematicians have known how to give lyrical expression to their enthusiasm for the beauty of their science, nobody has suggested examining it as if it were the object of an art—mathematical art—and consequently the subject of a theory of aesthetics, the aesthetics of mathematics [58, p. 122].

Le Lionnais emphasizes that there is much work to do to accomplish an aesthetics of mathematics. His own work “has no intention of establishing [an aesthetics of mathematics]; it aspires only to prepare the way for it” [58, p. 122]. Le Lionnais thus calls for developing a more rigorous approach to mathematical beauty, and not being content with pre-theoretical displays of enthusiasm. If we take Le Lionnais seriously, then an aesthetics of mathematics must not only address mathematical beauty in a literal way—not merely figuratively—but also in a fully theoretical way.

Now, pre-theoretical approaches to mathematical beauty can illuminate some aspects of it. For example, Russell’s analogy with sculpture implies the rough idea that mathematical beauty is contemplative; related, perhaps, to the static character of abstract mathematical objects. But we must keep in mind that many pre-theoretical approaches are just brief remarks accompanying non-technical works. Paul J. Nahin [69], for instance, interprets mathematical beauty as related to the fact that mathematics is *disciplined* reasoning:

> The reason I think that Einstein’s theory is still beautiful (despite currently being replaced by quantum-mechanics-compatible equations) is that it is the result of disciplined reasoning. Einstein created new physics [. . .], but [his] work was done while satisfying certain severe restrictions [. . .] a theory that satisfies such a broad constraint must, I think, be beautiful [69, p. xix].

In this respect, Le Lionnais’s ideas are more theoretically developed. He distinguishes between “classical” and “romantic” beauty in mathematical propositions and methods. According to Le Lionnais, a piece of mathematics possesses classical beauty “when we are impressed by its austerity or its mastery over diversity, and even more so when it combines these two characteristics in a harmoniously arranged structure” [58, p. 124]. Regularity is the property more clearly associated with Le Lionnais’ classical beauty. He thinks that the geometry of the triangle, cycloids and the logarithmic spiral exemplify classical beauty. In contrast, a piece of mathematics possesses romantic beauty when its beauty consists in the “glorification of violent emotion, non-conformism and eccentricity” [58, p. 130]. The notion of asymptote, complex numbers, and Cantor’s notion of infinity are some examples of romantic beauty. Le Lionnais's approach in terms of the art-historical distinction between classical and romantic[2] beauty is certainly interesting and more developed, but I believe it has serious limitations: it deals with mathematical beauty not by offering an explanation of aesthetic phenomena in mathematics, but by dealing with mathematics as an art. It resorts to the history of art rather than to an aesthetic theory. The distinction between classical and romantic art refers to a difference in style; it does not refer to the nature of the aesthetic phenomenon, but to some differences among classes of objects that embody that phenomenon.

The question of how to interpret what mathematicians mean when they use aesthetic expressions is best addressed in the context of an aesthetic theory that deals with more fundamental and general issues. Approaching mathematics as if it were an art seems to me like rushing matters. It is better to start by addressing basic issues. This is precisely my tactic in this book. Thus, it is a good idea to clarify the conceptual apparatus that will be employed to tackle our problems.

Let us first identify our task: I shall address the specific problem of giving an interpretation of aesthetic judgements in mathematics. Now, the mathematical beauty that is our subject matter is not the beauty that the discipline as a whole possesses, as a result of, for example, mathematics being a rigorous or disciplined endeavour. Rather, I shall deal with the phenomena involved in mathematician's judgements, like "Cantor's notion of infinity is beautiful" or "proofs by cases are cumbersome". I accept mathematicians' judgements and assume that beauty, ugliness, elegance, etc. are qualities of some pieces of mathematics. Some pieces are beautiful, some others are ugly, elegant, aesthetically indifferent, and so forth. Hence, in addition to beauty, notions like ugliness or elegance are also part of the subject matter of this work.

Let us now clarify the conceptual apparatus that shall be employed. Two elements are relevant here: the theoretical tenets that shall be endorsed and the methodologies that shall be employed. I consider that a healthy trend in contemporary philosophy and aesthetics is that they no longer distance themselves from natural science. My approach shall thus cohere as much as possible with empirical findings. I aim to be consistent with science, and, when no scientific results are available, to employ a scientifically informed common sense. Philosophically, this work sympathizes with analytic philosophy's commitment to precision, thoroughness and conceptual rigour, and thus with methods like formalization and conceptual analysis.

Within this framework I shall tackle the task of providing a literal interpretation of aesthetic judgements in mathematics by proposing an aesthetic theory that allows us to interpret mathematical aesthetic judgement as *bona fide* aesthetic judgements. Again, it must be emphasized that this book does not attempt to show the reader

[2]Western art movements are customarily classified by periods. Three of the most conspicuous are the Baroque, the Classical and the Romantic. Le Lionnais adopts the standard distinction between classical and romantic movements in art history, but the baroque is missing from his discussion. He does not give a reason for that, thus, Le Lionnais's approach seems to leave some room for us to ask about the existence of "baroque" mathematical beauty and how it should be characterized.

instances of mathematical beauty or to highlight the significance of it; there exists a large literature devoted to that.

Now, to substantiate the thesis that mathematical aesthetic judgements are genuine aesthetic judgements, the strategy is to advance a theoretical framework that accounts for mathematical aesthetics in the same fashion as it accounts for regular aesthetics. The leitmotif of this book is thus to provide an interpretation of aesthetic terms (words like beautiful, elegant, ugly, etc.) and judgements that accounts for their usage in mathematics in the same manner as it accounts for their everyday usage. Key to this interpretation shall be to approach the subject of aesthetics from a systemic perspective: I shall propose to see things like aesthetic pleasure, aesthetic experience or aesthetic judgements, as elements that interact with each other, forming a larger system. This system shall be labelled an *aesthetic-process*. The aesthetic theory I propose in this book interprets aesthetic events as elements of aesthetic-processes. The working of such processes shall be explicated by presenting accounts of three central topics: aesthetic experience, aesthetic value and aesthetic judgement. To give a clearer picture of this book, perhaps a summary of chapters is in order.

Summary of Chapters

The book is divided into three parts. Part I presents the antecedents, Part II is the core of this work; it develops the aesthetic theory that allows us to explain mathematical aesthetic judgements. Part III applies the theory to concrete cases of aesthetic evaluations in mathematics and further elucidates aspects of it. The chapter plan is as follows:

Part I

Chapter 1 discusses in more detail the idea that mathematical beauty should be reinterpreted to preserve mathematics' stand among the sciences. Three reasons to reinterpret mathematical beauty are examined: the two cultures divide, the epistemic character of mathematics and, finally, its rational character. We shall see that the reasons for endorsing a non-literal interpretation of mathematical beauty are rather feeble. The discussion also serves to introduce and examine the historical conceptions of mathematical beauty by Shaftesbury, Hutchenson and, much more recently, Rota. Lord Shaftesbury, an eighteenth century philosopher, addresses beauty in numbers using the idea that order is the principle of beauty. Francis Hutchenson, another important eighteenth century philosopher of value, argues that beauty is an idea aroused in our minds by the property of uniformity amidst variety. The common problem with these accounts is that they lead one to conclude that every mathematical item is beautiful. However, mathematicians would agree that there is more than beautiful mathematics; there are elegant as well as clumsy proofs, for example. We also discuss the contemporary mathematician Gian-Carlo Rota's ideas. Rota claims that "mathematical beauty" is a term mathematicians employ to refer to the enlightenment provided by some pieces of mathematics. I discuss some of the shortcomings of Rota's non-literal approach. For example, this approach cannot coherently account for mathematical ugliness or elegance; or for the fact that mathematicians (experts in employing and introducing exotic terms and meanings) choose to employ the term "beauty" rather than a less confusing one.

Chapter 2 discusses literal interpretations of beauty in mathematics and science. We shall see that according to some authors, aesthetic terms cannot be used metaphorically, thus providing a principled reason for taking aesthetic judgements in mathematics at face value. In addition, we shall see that a literal interpretation can yield interesting results; for example, McAllister's model of scientific progress. McAllister interprets scientific change in terms of aesthetic canons rather than in terms of Kuhnian paradigms. McAllister's most attractive insight is the idea of the aesthetic induction, which accounts for historical changes in aesthetic preferences: preferences for certain properties of scientific theories increase as a scientific community witnesses recurrent appearances of those properties in empirically adequate theories. Although McAllister insights are very significant, the idea of the aesthetic induction has some limitations.

Chapter 3 discusses some of the limitations of McAllister's ideas by analysing the ugliness of computer assisted proofs, an example proposed by McAllister himself. We shall see that an account of beauty based merely on the passive contemplation of properties of objects is insufficient to account for mathematical items that involve the active use of our attention, such as proofs or derivations. Chapter 3 stresses the centrality of mental contents and mental activities in mathematical beauty; thus, the notion of *intentional object* is introduced.

Chapter 4 digs deeper into the mechanism of the aesthetic induction. We shall see that the aesthetic induction has conceptual problems as well as significant explanatory anomalies. We shall see that historical evidence supports the existence of historical constants—properties whose preferences remain relatively unchanged throughout history, and that this contradicts the aesthetic induction.

In Chap. 5, I propose to naturalize the aesthetic induction in order to solve its problems. We discuss how this course of action is suggested by Theo Kuipers' approach to the aesthetic induction. Kuipers attempts to naturalistically substantiate the aesthetic induction by interpreting it in terms of the *mere exposure effect*; which is the unconscious development of preferences for familiar stimuli rather than for unfamiliar ones. The idea of using empirical findings to address beauty in science is quite appealing, therefore I utilize it, along with the evidence discussed in Chap. 4 and a rudimentary naturalistic aesthetic theory, to develop a more accurate model of the aesthetic induction, which I label *constrained* aesthetic induction.

Part II

Although appealing, the aesthetic induction is insufficient to accurately account for mathematical beauty by itself, as demonstrated in Chap. 3. Chapter 6 is therefore devoted to set the theoretical foundations of a proper account of aesthetic phenomena. A naturalistic aesthetic theory based on the notion of aesthetic-process is advanced. An aesthetic theory is needed not only to address the shortcomings of previous attempts to explain mathematical beauty, but also to address the issue that mathematics is not a traditional subject of aesthetics. The best way of making up for this absence of a tradition is to offer a theoretical justification for the aesthetic character of mathematical aesthetic judgements. Moreover, we shall further substantiate the theory with an a posteriori justification by applying it in Part III to concrete cases of mathematical aesthetic evaluations.

The aesthetic theory proposes to see the different kinds of aesthetic things—such as aesthetic experience, aesthetic value, aesthetic descriptions, etc.—as interconnected by the fact that they all are elements of a process in which objective properties, subjective reactions and social influences and contexts interact with each other. I call this process an *aesthetic-process*. This idea allows us to interpret the mark of the aesthetic—that is, the feature shared by things like aesthetic judgements, aesthetic value, aesthetic experience and so forth—as the feature of being applied meaningfully to the kinds of things that characteristically participate in aesthetic-processes. Aesthetic events should not be understood in isolation but as part of a process, of a system that unfolds by following different pathways over different times.

Among the events involved in an aesthetic-process, I discuss three central ones: aesthetic experience, aesthetic value and aesthetic judgements. I devote Chaps. 7 through 10 to discuss them.

Chapter 7 discusses aesthetic experience. Aesthetic experience is interpreted as an embedded sub-process that involves changes in the focus and content of our attention and the eliciting of affective responses.

Aesthetic value, discussed in Chap. 8, is interpreted as a relation between sets of properties and mental activities, and subjective reactions. Its evolution is governed by a variety of constrained aesthetic induction.

Chapters 9 and 10 address aesthetic terms and judgements. Chapter 9 discusses the nature of aesthetic terms and judgements; Chap. 10 discusses the functions aesthetic terms and judgements play in aesthetic-processes.

Aesthetic judgements are interpreted as expressions of subjective states, characterized by the application of aesthetic terms. Their functions are to articulate aesthetic experience and to share the result of such articulation. In Chap. 9, aesthetic terms are characterized in terms of what I label the RSD model of aesthetic terms. The RSD model maintains that the correct application of aesthetic terms requires a simultaneous relation among a space of possible affective responses, a family of terms, and a referential domain.

In Chap. 9, the functions of aesthetic terms are examined. The use of aesthetic terms not only expresses a subjective state, but it helps to constitute the experience itself, I label this function *articulation*. The second function is to broadcast information which enables other people to undergo their own processes of articulation. These functions, along with the characterization of aesthetic terms, allow us to characterize not only aesthetic judgements as used to evaluate artworks, but also aesthetic terms as used by mathematicians.

Chapter 11 discusses how the aesthetic as process theory accounts for mathematical aesthetic judgements and also presents a review of the most relevant insights afforded by the theory. The question "what do mathematicians mean when they talk about beauty?" is answered in a simple way: mathematicians employ aesthetic terms in a literal sense, but their use is closer to the way aesthetic terms are used by specialists, critics or artists, rather than by an average person. Appreciation depends profoundly on a great deal of background mathematical knowledge. Our goal of literally interpreting aesthetic judgements in mathematics is achieved with the following interpretation: *mathematical aesthetic judgements are articulated expressions of subjective states (aesthetic experiences) which result from an affective engagement of our attention to a mathematical item. The affective reaction reflects our preferences (our aesthetic values), which in turn are modulated by our natural tendencies and cultural influences*.

Part III

After introducing the brand new aesthetic theory, I showcase it in action. Chapters 12 to 14 apply it by examining examples of aesthetic judgements in mathematics. Chapter 12 addresses mathematical beauty by means of a very basic example, the function $y = e^x$.

Chapter 13 addresses a more refined judgement. I analyze an *elegant* proof: Cantor's diagonal argument. This example allows us to see how the notions of aesthetic experience and subjective articulation can account for the nuances involved in the application of the closely related notions of beauty and elegance.

Chapter 14 revisits mathematical ugliness, discussed in Chap. 3, in a more theoretically developed way. By doing so, we shall have covered the most relevant aesthetic terms employed in mathematics. Analysing computer-assisted proofs, again, shall serve to display the advantages of my systemic approach, as it shall be clear that aesthetic judgements in mathematics depend not only on inductive changes in value but also on changes in the constitution of our experience. It shall also show that the aesthetic as process theory is able to make predictions: the theory coherently predicts that computer-assisted proofs have little chance of being regarded as beautiful in the future, contrary to what James McAllister conjectures.

Finally, Chap. 15 is devoted to review the insights gained by the theory, and how they relate to the issues discussed throughout the book.

Part I
Antecedents

Chapter 1
On Non-literal Approaches

The issue of the meaning of the term 'mathematical beauty' shall serve as the *leitmotif* of the discussion throughout this book. We start by discussing the two possible ways of approaching this issue: we can take the term at face value; or interpret it figuratively, as meaning something else. That is, we can interpret the term in a literal or non-literal way. This discussion shall also serve to survey various views on beauty in mathematics that illuminate some aspects of it.

In this chapter, we examine three possible reasons for embracing a non-literal interpretation of mathematical beauty: first, the mutual exclusion of humanistic and scientific disciplines. Second, the epistemic character of mathematics. And, third, the rational character of mathematics. We shall see that none of these reasons is very compelling, for reinterpreting mathematical beauty does very little for the causes of mathematics' allegiance to science, its epistemic soundness or its rationality.

1.1 The Two Cultures, Shaftesbury and Hutchenson

As mentioned in the introduction, in his very influential Rede Lecture at Cambridge *The Two Cultures* [83], the physicist and novelist Charles Percy Snow denounced the fact that the western intellectual world is split into two cultures: the humanities and the sciences. Snow pointed out a fact that seems to be evident still today in many spheres of western culture. Regarding our discussion, the split manifests itself in the fact that the average person seems to see artistic and scientific disciplines as excluding each other; this phenomenon is the so-called *arts/sciences divide*. As we have discussed, mathematically lay people are quite justified to ask: isn't it truth, and not beauty, the goal of mathematics? And, what is the difference between beauty and ugly mathematics?

The questions are further justified by the fact that mathematics is a highly technical discipline with a very rich, and often confusing, jargon. For example, when mathematicians speak of natural, irrational or real numbers, the terms 'natural',

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_1,

'irrational' and 'real' mean something quite different from what those words mean to a lay person. Those terms have technical, specially defined meanings. The same occurs with terms such as 'space', 'ring', 'group', 'category', and many, many others.

Well, one might argue, since art and science inhabit separate realms and the mathematical jargon is confusing; it is reasonable to think that the term 'beauty' we find in mathematics is like the terms 'real' and 'space': a technical term with a special meaning. So, it might be the case, after all, that mathematicians do not use the term 'beauty' in its literal sense, but rather in some obscure technical sense. Perhaps Hardy's statement—"beauty is the first test: there is no permanent place in this world for ugly mathematics" [33, p. 85]—conveys, say, a methodological precept, intended to be interpreted by the professionals familiar with the mathematical jargon. 'Mathematical beauty' might be a mere metaphor or a stand-in word which does not refer to a genuine aesthetic feature of mathematics after all.

Now, if we examine carefully the arts/sciences divide we shall find the foregoing conclusion unconvincing. We shall find out that there is no sound basis to believe that genuine aesthetic phenomena cannot occur in mathematics. The arts/sciences divide is more a cultural attitude than an intrinsic fact. In addition, the reason why non-mathematicians find the term 'mathematical beauty' inaccessible is independent of whether the term is interpreted literally or non-literally. Let us elaborate.

1.1.1 Unsound Divide

Although we all may be familiar with, or even take the arts/sciences divide as granted, once we take a more rigorous stance, our conviction quickly diffuses. In his "Two Cultures" lecture, Snow himself does not advocate the split between sciences and humanities. On the contrary, Snow sees the split as a hindrance to address humanity's problems. Snow does not see the divide as something intrinsic to the culture, the sciences, or the humanities. He argues that the cultural split has its historical roots in the division of labour that began with the industrial revolution and that it was further crystallized in the nineteenth century by cultural movements such as Romanticism.

There is plenty of evidence that supports Snow's argument. Earlier historical periods seemed to crossover the two cultures in a natural fashion. For example, early aesthetics—the discipline that studies topics such as the nature of beauty and ugliness, taste and art—did not exclude cognitive or rational phenomena from its field of study. Contrary to our contemporary attitude, during the early stages of aesthetics, intellectual phenomena, science and mathematics were regarded as genuine objects of aesthetic analysis.

Mathematical Beauty in the Eighteenth Century

Mathematical beauty is mostly disregarded by contemporary aesthetics. But it was addressed in a natural fashion by early eighteenth century aesthetics. In 1735 Alexander Gottlieb Baumgarten introduced for the first time the term *aesthetics* to denote the philosophical study of beauty and art. This event marks the birth of modern aesthetics [32, p. 15], and it had two significant consequences—perhaps adverse—that are relevant to mathematical beauty: first, aesthetics devoted itself to articulating a philosophy of art; it focused on the disciplines we now call the fine arts, which became its distinctive subjects of discussion. Second, it contributed to set in place sharper disciplinary boundaries. Moral issues, knowledge and beauty were conceived as independent from each other. As a matter of fact, the first characterizations of aesthetic phenomena were made by distinguishing them from cognition and volition [32, p. 16–17]. The hallmark of modern aesthetics, perhaps foretelling the Two Cultures divide, was the conception of the aesthetic response as independent from cognition and volition. Not surprisingly, the stance of modern aesthetics seems consistent with the arts/sciences divide. However, the precursors of modern aesthetics, Shaftesbury and Hutchenson, saw mathematics as a genuine bearer of beauty.

1.1.2 Shaftesbury

Anthony Ashley Cooper (1671–1713), the Third Earl of Shaftesbury, introduced for the first time the idea of disinterestedness as the chief characteristic of aesthetic responses. Shaftesbury characterizes aesthetic response as disinterested pleasure in the *order and proportion* manifested to our senses. Since order and proportion are features that are clearly represented in numbers and other mathematical entities, one can expect that, once disinterest is provided, they are capable of eliciting an aesthetic response. Shaftesbury himself points this out:

> Nothing surely is more strongly imprinted in our minds or more closely interwoven with our souls than the idea or sense of order and proportion. Hence all the force of numbers and those powerful arts founded on their management and use! What a difference there is between harmony and discord, cadence and convulsion! What a difference between composed and orderly motion and that which is ungoverned and accidental, between the regular and uniform pile of some noble architect and a hip of sand and stones, between an organized body and a mist or cloud driven by the wind! [15, p. 272]

The 'sense of order', according to Shaftesbury, is a feature that human beings characteristically possess. Shaftesbury further identifies order with *design* and he claims that what we love in order is the *designer*: the mind or intelligence responsible for that order; the source of order. For Shaftesbury the ultimate source of order is God. Our moral and aesthetic senses have thus the same source. They seem to be just different modalities of one and the same virtue. Numbers and their application are paradigmatic cases of order; hence the beauty of numbers. But the numbers' order is not completely independent: the true source of mathematical beauty is the designer behind its order. Beauty in numbers is just another manifestation of God.

Shaftesbury's account of beauty in numbers links moral, ontological and aesthetic matters. This stance contrast sharply with our own contemporary attitude that emphasizes disciplinary boundaries, and also shows that the arts/sciences divide is contingent upon historical circumstances.

1.1.3 Hutchenson

Francis Hutchenson (1694–1746), one of the founding fathers of the Scottish Enlightenment, also addresses the beauty of mathematics. He argues that the qualities of objects are distinct from and causes of *ideas*. Ideas are the sole materials of sensory awareness. Beauty is one of these ideas; it occurs in the mind caused by the property of *uniformity amidst variety* of external objects. Hutchenson represents a further modernization of aesthetics, since he endorses a more explicit conception of aesthetics as independent from volition and cognition. This is evident in Hutchenson's characterization of the response to beauty as:

> consisting in an immediate gratification in perceptual form that is free of the influence of all other forms of thought and value [36, p. 11].

For Hutchenson, the way we perceive beauty is different from our faculties of cognition and volition. He argues, for example, that knowledge does not affect our perception of beauty and concludes that our response to beauty can only be a sense:

> This Superior Power of Perception is justly called a Sense, because its affinity to the other Senses in this, that the Pleasure is different from any Knowledge of Principles, Proportions, Causes, or of the Usefulness of the Object, we are struck at the first with the beauty; nor does the most accurate Knowledge increase this Pleasure of Beauty [36, p. 11].

Hutchenson classifies the objects for this "sense of beauty" into three main types, which can be seen as referring to natural, conceptual and artistic beauty. Possessing the quality of unity amidst variety is the unifying principle behind all these types of objects and thus the characteristic feature of all beauty. Interestingly enough, mathematical theorems figure among Hutchenson's examples of conceptual beauty: uniformity amidst variety in perceptual forms is the source of "Original or Absolute Beauty" [36, p. 1]; uniformity amidst variety in conceptual contents is the source of the "Beauty of Theorems" [36, p. 30], and 'Relative or Comparative Beauty', which is "that which is apprehended in any Object, commonly considered as an Imitation of some Original and our pleasure in this beauty too is founded on a Conformity or a kind of Unity between the Original and the Copy" [36, p. 39]. Although Hutchenson's sense of beauty is not a cognitive faculty, there is room in it for mathematical theorems, for in theorems we find unity amidst variety. The pleasure elicited by the beauty of theorems does not have to do with the content of the theorems, but simply "with the most exact Agreement [of] an infinite Multitude of particular Truths" in a theorem [36, p. 30].

Now, the theories of Shaftesbury and Hutchenson address mathematical beauty and show that there is no inherent arts/sciences divide. Literal approaches to mathematical beauty are not only possible, but they can also be part of aesthetic theories. Now, Shaftesbury's and Hutchenson's ideas are certainly illuminating, but they fail to account for the use mathematicians make of the term 'mathematical beauty'. For example, in order to understand Hardy's statement that "beauty is the first test: there is no permanent place in the world for ugly mathematics" [33, p. 85] we must be able to contrast mathematical beauty and ugliness in mathematics. Hardy's statement illustrates that different aesthetic terms are used to evaluate mathematical entities. This evaluative aspect, however, is absent from Shaftesbury's and Hutchenson's approaches: all mathematics is characterized as beautiful in their definitions. A more comprehensive literal approach to mathematics, a proper aesthetics of mathematics, must incorporate the insights provided by authors such as Shaftesbury and Hutchenson, but it must also be able to account for the evaluative aspect of all sort of aesthetic terms in mathematics. At any rate, the ideas surveyed above greatly undermine the arts/sciences divide as an argument for reinterpreting mathematical beauty.

The Source of the Unintelligibility of Mathematical Beauty

The arts/sciences divide and the technical character of mathematical jargon motivated the idea that mathematical beauty should be interpreted in a non-literal way. We have shown the historical contingency of the arts/sciences divide. It is not difficult to show that the technical nature of mathematical jargon does not provide a sound reason for reinterpretation either: understanding mathematical jargon requires a high degree of technical proficiency *independently of how the term 'beauty' is interpreted*. Furthermore, it can also be shown that there are genuine aesthetic phenomena that require technical proficiency to be appreciated. Let us elaborate.

Gian-Carlo Rota, for example, points out that in order to appreciate mathematical beauty one must be able to understand mathematics: "[f]amiliarity with a huge amount of background material is the condition for understanding mathematics." [79, p. 179]. Even professional mathematicians specialized in a certain field might find results or proofs in other fields obscure. Moreover, the meaning of notations and symbols change from field to field. Wiles' long and complex proof of Fermat's Last Theorem is told to have been understood only by few specialist when it first appeared in 1994, partly due to its specialized notation and symbols.[1] Now, the need

[1] I must emphasize that the notation was not what made the proof hard to understand. The proof was hard to understand primarily because of its ideas were novel and it involved intricate and very abstract machinery that was alien to the field. That resulted in a notation that appeared obscure even to the specialist. But this simple need of having to learn a new notation makes technical

for technical proficiency is independent of how one interprets evaluative terms like 'beautiful' or 'elegant'. Mathematics remains highly technical regardless of what meaning one gives to the term 'mathematical beauty'. And the fact that mathematical jargon is highly technical does not imply that all terms mathematicians use are technical. Furthermore, evaluative terms, which are meant to express the worth of the item they evaluate, do not need to convey a specialized meaning, for their purpose is to clarify the stance of the speaker regarding the worth he attributes to the evaluated item.

On the other hand, some genuine aesthetic expressions can be fully appreciated only if the observer has certain proficiency in technical matters. For example, university courses on art appreciation are very common. It is a well known fact that knowledge about artistic styles, or even about a particular author's biography and style, changes the way we perceive and appreciate painting, for instance. In music, knowledge about technical details such as the differences between things such as cadences, progressions or chords changes the way we appreciate music. The most elementary musical description is ridden with technical terms like 'bar', 'interval', 'tonic', etc. Moreover, some musical properties, such as the symmetry of a fugue or a sonata, are even simply "invisible" without certain degree of technical knowledge [39, 41, 42, 77]. This shows that the requirement of technical proficiency to even be able to observe certain events does not preclude the existence of genuine aesthetic phenomena associated to such events. Thus, the fact that technical proficiency is necessary to "see" mathematical entities does not preclude the possibility that such entities can be genuine aesthetic subjects.

The foregoing discussion shows that the Two-Cultures argument against a literal interpretation has a very weak basis. But perhaps issues deeper than the arts/sciences divide may provide a sounder basis. We must explore this avenue.

1.2 The Epistemic Character of Mathematics

The arts/sciences divide is not inherent to our culture or to mathematics. But there are things that are inherent to mathematics. Science and mathematics are conceived as having knowledge, truth and understanding as their goals, and as attempting to achieve those goals by rational means. If contingent cultural attitudes cannot provide a solid argument for reinterpreting mathematical beauty, perhaps its inherent characteristics may. Thus, a second motivation for a non-literal interpretation of mathematical beauty might be the epistemic character of mathematics.

The chief goals of mathematics are directly associated with knowledge. One might argue that the most valuable properties of mathematics are those conducing to justify, refine or achieve mathematical knowledge. Now, aesthetic qualities in

---

subjects very opaque to the non-specialist. This is true not only for the lay person, but also for mathematicians themselves.

general are in principle ineffectual to achieve the epistemic goals of mathematics. Nonetheless, judgements of beauty are pervasive in mathematics. A possible explanation for this is that mathematical beauty possesses a hidden epistemic character after all. Therefore, we should not take the term ‘mathematical beauty’ at face value. Rather, we should search for an appropriate reinterpretation, one in accord with the epistemic goals of mathematics. This conclusion appears to be behind the approaches of authors like Gian-Carlo Rota.

1.2.1 Rota’s Interpretation of Mathematical Beauty

One of the most interesting attempts to tackle the issue of mathematical beauty is Gian-Carlo Rota’s 1997 article “The Phenomenology of Mathematical Beauty” [78]. In that article, Rota presents an articulated analysis of mathematical beauty—instead of the invariable anecdotal account that constitutes almost the entirety of publications dealing with beauty in mathematics. Rota attempts to reconcile the use of the term ‘mathematical beauty’ with the epistemic precepts of mathematical practice. Rota concludes that when mathematicians use the term ‘beauty’ they are actually referring to the enlightenment that a certain piece of mathematics provides. Enlightenment is a kind of understanding consisting in realizing the role of a certain piece of mathematics in a broader theoretical context. The concept of enlightenment, according to Rota, is fuzzy and mathematicians dislike fuzzy concepts; this is why they employ the term ‘beautiful’ instead of ‘enlightening’. Now, in his discussion Rota’s uncovers some significant characteristics of mathematical beauty that are worth taking notice here.

Rota notes that although mathematics’ chief concern is truth, there is an ambiguity in mathematical practice; for mathematicians often claim that “beauty is the *raison d’etre* of mathematics, or that mathematical beauty is the feature of the mathematical enterprise that gives mathematics a unique standing among the sciences” [78, p. 180]. An understanding of mathematical beauty is thus vital for a full understanding of mathematics. Rota’s thus sets himself “to uncover the sense of the term ‘beauty’ as it is currently used by mathematicians” [78, p. 171]. He begins by identifying five kinds of mathematical items often qualified as beautiful: “Theorems, proofs, entire mathematical theories, a short step in the proof of some theorem, and definitions are at various times thought to be beautiful or ugly by mathematicians” [78, p. 171]. Rota argues that *properties* like the shortness of a step in a proof are sometimes associated with mathematical beauty. Shortness is also associated with the beauty of proofs, theorems or definitions. He, however, is sceptical about properties such as the unexpectedness and inevitability of arguments.[2] Rota argues that the unexpectedness of an argument cannot be

[2] Rota does not mention it explicitly, but he is referring to G. H. Hardy’s view [33] that the unexpectedness and inevitability of a theorem or proof are the sources of mathematical beauty.

identified with its beauty since we can find examples of unexpected arguments that are not regarded as beautiful. Rota believes that the source of beauty is more complex. To support this, he points out that mathematical beauty depends on context:

> [. . .] the beauty of a piece of mathematics is strongly dependent upon schools and periods of history. A theorem that is in one context thought to be beautiful may in a different context appear trivial. [. . .] Undoubtedly, many occurrences of mathematical beauty eventually fade or fall into triviality as mathematics progresses [78, p. 175].

Despite this dependence on context, Rota thinks that beauty is an *objective* property, in the same fashion as mathematical truth or falsehood are objective properties [78, p. 175]. For Rota, mathematical beauty does not consist merely in the subjective feelings of a mathematician. The distinction between beauty and truth is not the distinction between subjective and objective properties: they are both objective, they are equally observable characteristics of mathematical items. The truth of a theorem does not possess a greater degree of objectivity than its beauty; rather, they are different in the sense that they are different "phenomena in an objective world" [78, p. 175]. Rota's emphasis on objectivity indicates that he is determined to defend the epistemic character of mathematics by eliminating subjectivity and reinterpreting beauty in epistemic terms.

Rota also addresses other aesthetic judgements in mathematics: judgements of ugliness and elegance. Mathematical ugliness, he stresses, plays an important role in encouraging mathematical research: an ugly proof often encourages the development of alternative, more aesthetically appealing proofs. Rota believes that lack of beauty is linked to lack of definitiveness, for a beautiful proof often becomes the definitive proof. A beautiful theorem, Rota argues, is not often improved on or generalized. Mathematical elegance, by contrast, plays a rather minor role: elegance has to do merely with the presentation of results and not often with content [78, pp. 178–179].

Returning to beauty, Rota points out that many instances of mathematical beauty depend on familiarity and comparison; they depend on background knowledge and an acquaintance with similar instances of mathematics. In general, familiarity with a large amount of background material is a precondition for understanding any piece of mathematics. And, in order to appreciate the beauty of a piece of mathematics, we need to contrast it with other pieces: "Very frequently, a proof is viewed as beautiful only after one is made aware of previous clumsy and longer proofs" [78, p. 170]. Rota connects this fact with what he calls the "Light bulb mistake" [78, p. 179–180], which is instrumental in his account of mathematical beauty. He argues that we manage to understand a piece of mathematics only after having gone through the great pains needed to acquire the necessary background knowledge and understanding of mathematics. But in our recollection, we remember the instances of mathematical beauty as "if they had been perceived by an instantaneous realization, in a moment of truth, like a light-bulb suddenly being lit" [78, p. 180]. Once we have understood a theorem, for example, we forget about all the effort we invested in the background understanding required to understand its proof. The difficulties we encountered seem to disappear. Our recollection retains only an "image of an instant flash of insight, of a sudden light in the darkness" [78, p. 180].

It is against this background that Rota advances his interpretation of mathematical beauty. He argues that mathematicians often show their disapproval of a certain piece of mathematics by asking the question "What is this good for?" The question shows that they do not see the point of, for example, re-stating something that has already been logically verified to be true. Rota points out that logical verification alone does not enable us to see "the role that a statement plays within the theory. It does not explain how such a statement relates to other results, nor make us aware of the relevance of the statement in various contexts" [78, 181]. Logical truth does not *enlighten* us about the deep significance of a mathematical statement. Rota argues that when mathematicians ask the question "What is this good for?" they are not looking for truth, but for enlightenment. Under this interpretation, enlightenment is what drives the mathematical enterprise, and what distinguish mathematics from other scientific disciplines.

Enlightenment, however, is not explicitly acknowledged by mathematicians. Rota gives two reasons for this: First, enlightenment is not easily formalized. Second, enlightenment, unlike truth, does admit degrees; some mathematical results or proofs are more enlightening than others. The concept of beauty, according to Rota, is appealing because it does not admit degrees. Rota argues that mathematicians "universally dislike any concepts admitting degrees, and will go to any length to deny the logical standing of any such concepts" [78, p. 181]. 'Mathematical beauty' is the term that mathematicians "have resorted to in order to obliquely admit the phenomenon of enlightenment, while avoiding to acknowledge the fuzziness of this phenomenon" [78, p. 181]. In Rota's view, when mathematicians call a theorem beautiful, they really mean that the theorem is enlightening. Similarly, they call a proof beautiful when it "gives away the secret of the theorem, when it leads us to perceive the actual, not the logical inevitability of the statement that is being proved" [78, 182].

1.2.2 Rota's Problems

Rota's work offers us very valuable insights: Mathematical beauty is not merely subjective feeling; it is rather an objective property. It depends on historic-social context. It also depends highly on background knowledge and experiences. Aesthetic considerations play a role in encouraging mathematical development.

Despite these insights, we can identify serious problems in Rota's account. The most obvious being that Rota's own argument against unexpectedness can be used against enlightenment: there are instances of mathematical beauty that are not enlightening. For example, beautiful methods of proof such as *reductio* *ad absurdum*[3] are among the most beloved methods of proof ever since Euclid. G. H. Hardy eloquently testifies this:

> [...] *reductio ad absurdum*, which Euclid loved so much, is one of a mathematician's finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game [33, p. 94].

Reductio ad absurdum has been regarded as one of the most beautiful methods of proof ever since Euclid, but a proof by *reductio ad absurdum* is not necessary enlightening, as it relies in showing that assuming a false premise leads to contradiction. Intuitionism, one of the three classic schools of philosophy of mathematics in the twentieth century, rejects some proofs by *reductio ad absurdum* precisely because the method is very opaque in respect of the mathematical constructs involved in the theorem, since it does not construct or show the existence of mathematical entities. In this sense, *reductio ad absurdum* is quite the opposite to being enlightening.[4]

The problem is evident even in Rota's own example of beauty in "a short step in the proof of some theorem" [78, p. 171]. The property of being enlightening is the property of being able to bring understanding of the role of a certain piece of mathematics in a more general context (the role of a theorem in a theory, for example) or understanding of the relation of a certain statement to other similar statements. In the case of steps in a proof it is not easy to see how a short step can be enlightening in the above sense. A single step in a proof, especially if it is short, is too small a piece of information to be credited with being enlightening by itself. Furthermore, steps in proofs often depend on information or results from previous steps in the same proof. A proof is a way of going from certain premises to a certain conclusion; a way of connecting premises and conclusion. A step in a proof is just an element bridging premises and conclusion. These steps do not always need to have meaning by themselves (the substitution of a logical formula for an equivalent one, for example) and as such they are very unlikely to be able to bring understanding, in the sense of showing the role of the step in a more general context. One could reply that steps, being the links that bridge premises and conclusion, actually help to connect the premise with the conclusion; and as such, the steps in a proof help us understand the relation between premises and conclusion. But if we grant this interpretation we must accept that *all* steps in a proof help us understand the larger premises-conclusion relation. In that case, every step in a proof should be qualified as enlightening and thus beautiful in Rota's sense.

Thus, the property of being enlightening is insufficient to account for all cases of mathematical beauty. In this sense, enlightenment is similar to unexpectedness and

[3] In a proof by contradiction, or *reductio ad absurdum*, one assumes the negation of the statement to be proven and shows that it leads to a contradiction.

[4] It must be noted that intuitionism does not reject all proofs by *reductio ad absurdum*, but only those that intend to establish an existential claim, or those that intend to move from not-not-A to A. At any rate, the kind of proofs rejected by intuitionism can be considered characteristically non-enlightening.

inevitability. Since Rota rejects properties such as unexpectedness and embraces enlightenment, it seems that Rota has an epistemological agenda driving him to be selective about which properties should used to account for 'mathematical beauty'. Rota is aware of the enormous stock of anecdotal accounts of mathematical beauty, but he is very sceptical about them:

> Following this mistaken conviction, several attempts have been made to string together beautiful mathematical results, and to present them in the form of books bearing such attractive titles as "The one hundred most beautiful theorems of mathematics". Such anthologies are seldom to be found on any mathematician's bookshelf [78, p. 180].

He is also aware of the fact that a common strategy in anecdotal accounts is to try to reduce mathematical beauty to some unproblematical property. This is why he addresses, although without mentioning names or sources, G. H. Hardy's view that the properties of unexpectedness and inevitability are the sources of beauty. Rota seems less concerned with the fact that unexpectedness and inevitability are not sufficient conditions for beauty than with the fact that those properties have no relevant epistemic role. In this sense, Shaftesbury's order-based and Hutchenson' unity-based views probably would be equally unacceptable for Rota.

A second problem with Rota's approach is that it is quite implausible that mathematicians universally fail to apply the term 'beautiful' in an appropriate manner: if average people routinely use the term 'beautiful' correctly, and mathematicians are unable to do this, this is an extraordinary fact crying out for explanation. Rota's sole explanation is that mathematicians are not willing to deal with the fuzziness of enlightenment. Even if we lend credibility to this explanation we must face the problem that beauty is also fuzzy. Although Rota argues that beauty does not admit degrees, it is easy to show the contrary. Rota himself argues that an ugly proof often encourages the search for alternative proofs. Of course the result of this search is not always a beautiful proof, but it is not hard to envisage that a mathematician would be content, at least for a while, with a more aesthetically meritorious proof: a more beautiful proof or at least a less ugly proof. But the very fact that we can qualify some proofs as *more* beautiful than others shows that beauty admits degrees. In general, the predicate 'beautiful' can be employed very naturally in comparative judgements; there is nothing wrong with sentences like 'A is more (or less) beautiful than B'. Furthermore, in actuality some mathematical results are regarded as more beautiful than others: David Wells [94] even offers a ranking of some of the most beautiful theorems, based on a poll of mathematicians. In summary, beauty is a concept that *does* admit degrees; it is a fuzzy concept. If Rota were correct in arguing that mathematicians dislike fuzzy concepts, they would try to avoid the predicate 'beautiful' as much as the predicate 'enlightening'.

A third problem is that Rota mistakes beauty for the properties that evoke beauty. We have seen that Hutchenson already distinguishes the idea of beauty, which occurs in the mind, from the properties that cause that idea, which appear in observed objects. This distinction is in accord with our everyday experience: we all know that the contemplation of properties such as symmetry or simplicity many times elicits aesthetic pleasure. Rota fails to distinguish between the properties responsible for eliciting an experience of beauty and the property of beauty itself.

For example, when we pass an aesthetic judgement that a very symmetric object is beautiful, we are not erroneously using the term “beautiful” to refer to the property of symmetry; we are expressing the pleasure elicited by the contemplation of that symmetric object. No one would say that when we say “beautiful” we really mean “symmetric”. Rota, however, claims that such is the case with enlightenment. Although enlightenment can elicit responses of aesthetic pleasure, it does not follow that when mathematicians call a mathematical item beautiful, they do so because they do not know how to use the term ‘beautiful’ or because they do not want to deal with statements like “this is enlightening”. It is more reasonable to assume that they are trying to express an aesthetic experience of beauty. It is quite possible that such a perception of beauty is often derived from epistemic qualities like being enlightening. But that does not mean that mathematical beauty should be reduced to those properties. In addition, in general there is no way to reduce aesthetic properties to their associated non-aesthetic properties [82].

A fourth problem for Rota’s approach is that it does not provide a unified criterion to interpret things like ugliness or elegance in mathematics. Statements like Hardy’s “beauty is the first test: there is no permanent place in the world for ugly mathematics” [33, p. 85], show that there is a connection between judgements of beauty and ugliness. In general, the motivations we have to judge some things as beautiful are related to the motivations we have to judge some other things as ugly. However, although Rota sees beauty as related to enlightenment, he relates mathematical ugliness to an entirely different phenomenon, to non-definitiveness:

> The lack of beauty in a piece of mathematics is of frequent occurrence, and it is a strong motivation for further mathematical research. Lack of beauty is associated with lack of definitiveness. A beautiful proof is more often than not the definitive proof (though a definitive proof need not be beautiful); a beautiful theorem is not likely to be improved upon or generalized [78, p. 178].

Moreover, according to Rota, mathematical elegance in proofs “has to do with the presentation of mathematics, and only tangentially does it relate to content” [78, p. 178]. Rota’s accounts of beauty, ugliness, and elegance seem disconnected from each other: for him, beauty has an epistemic role, ugliness has a heuristic role, and elegance has only a pedagogical role.

Finally, if our initial motivation for reinterpreting mathematical beauty is to defend the epistemic character of mathematics, it is not clear whether Rota’s reinterpretation can accomplish that. The problems discussed above show that Rota’s approach does not satisfactorily reinterpret mathematical beauty. Instances of mathematical beauty such as *reductio ad absurdum* or short steps in proofs remain unaccounted for. Furthermore, mathematical elegance and ugliness are not even accounted for in epistemological terms. Even if we grant that enlightenment explains many cases of mathematical beauty, a defence of the epistemic character of mathematics is a principled matter; it does not matter if we account for only few or many cases, we should be able to account for *all* cases. Furthermore, we should be able to do so in a unified manner, not by resorting to epistemological, heuristic or pedagogical arguments whenever things get complicated. In addition, there is no principled reason to think that epistemic and aesthetic concerns cannot coexist.

Reinterpreting mathematical beauty in epistemic terms does not really seem to do much for the epistemic goals of mathematics, and it rather introduces unnecessary complications.

1.3 The Rational Character of Mathematics

Mathematics' epistemological ends are achieved by rational means. We have seen that reinterpreting mathematical beauty, if certainly illuminating, does not really contribute much to defending the epistemic character of mathematics. Now, there is no principled reason why mathematics' epistemic and aesthetic considerations cannot coexist. But if we consider the other inherent characteristic of mathematics, that its results are achieved by rational means, the issue becomes more problematical.[5]

The rationalist image of science maintains that there is a set of precepts for conducting science whose justification is to a great extent principled and extra-historical. Those precepts are the norms of rationality [50,52,62,72,75]. Science and mathematics have a deep allegiance to rationality. Judgements of beauty, however, are subjective. Subjective judgements are incompatible with the norms of rationality. Mathematicians placidly invoking subjective judgements of beauty to reformulate, for example, a proof or an axiomatization is thus very troubling. One way to address this problem is, again, to argue that judgements of beauty should not be taken at face value, but rather they should be reinterpreted in a fashion consistent with the norms of rationality. After all, this reinterpreting strategy has been successful in dealing with another problem for scientific rationality: scientific revolutions.

One of the major sources of discussion in philosophy of science during the second half of the twentieth century has been the rational character of science. Kuhn's view on scientific revolutions, or at least the radical reading of it, played a central role in igniting the discussion, to the extent that the nature of scientific revolutions has become the core argument of schools of thought that deny that science is rational, such as relativism or post-modernism.

According to the radical reading of Kuhn's *The Structure of Scientific Revolutions*, a scientific revolution is a paradigm shift. Scientific paradigms are constituted, among other things, by sets of criteria for evaluating theories. The paradigm adopted after a revolution is incommensurable with the one relinquished by the revolution. Therefore, there is no common set of criteria that allows scientists to assess the worth of the old pre-revolution theories compared to the new post-revolution theories. With no common criteria to evaluate pre- and post-revolution theories there is no rational ground to choose between those theories. Thus, undergoing a scientific revolution, that is, choosing the new theories and relinquishing the old ones, has no rational ground. Under this interpretation, scientific revolutions are non-rational phenomena that challenge the depiction of science as a rational discipline.

[5] Discussing the rationality of science or mathematics is beyond the scope of this book; here I address solely the issues relevant to the interpretation of beauty.

Arguments like the foregoing reveal that the rationality of science is debatable. If a rational depiction of science is to be embraced, those arguments need to be contested. A way to do that is to dispute Kuhn's radical interpretation of scientific revolutions by arguing that, in a revolution, only some of the criteria for theory evaluation change [52, 62, 72]. Some crucial criteria, such as the criteria for evaluating empirical adequacy or logical soundness, are preserved. In other words, the core of the norms of rationality persists across scientific revolutions. Thus, a rationalist image of science must advocate a much less radical reading of Kuhn; in other words, it must *reinterpret* the notion of scientific revolution.

Now, if the rationality of science can be safeguarded by reinterpreting scientific revolutions, it makes sense to attempt to address the threat aesthetic evaluations pose to rationality by reinterpreting aesthetic evaluations. One might thus argue that, in the case that concerns us here, mathematical beauty is not really subjective because the term 'mathematical beauty' does not really refer to the property of being beautiful, but to something consistent with the rationality of mathematics.

Now, although I agree that the presence of subjective judgements certainly constitutes a problem, and that we should find a way to deal with it, embracing a non-literal interpretation of mathematical beauty not necessarily solves the problem. Reinterpreting aesthetic evaluations does not guarantee that their associated subjectivity is eliminated; rather, as we shall see, it merely displaces subjectivity momentarily out of sight. To see this clearly, let us examine some actual reinterpretation strategies.

1.3.1 Rational Reinterpretations

Arguments about the incompatibility of rationality and aesthetic judgements in science take different forms. For example, logical positivism, a rationalistic variety of empiricism that emerged in the first part of the twentieth century, maintained that the norms of rationality allows only logical and empirical criteria as acceptable criteria for evaluating scientific theories. Criteria of beauty or ugliness are independent from and even inconsistent with logical and empirical criteria. Logical positivism held a stance regarding aesthetic evaluations, clearly expressed by Herbert Feigl:

> A few words on some misinterpretations stemming from predominant concern with the history and especially the psychology of scientific knowledge. In the commendable (but possibly Utopian) endeavor to bring the "two cultures" closer together (or to bridge the "cleavage in our culture") the more tender-minded thinkers have stressed how much the sciences and the arts have in common. The "bridges" [...] are passable only in regard to the *psychological* aspects of scientific [...] creation [...]. Certainly, there are aesthetic aspects of science [...]. But [...] what is primary in the appraisal of scientific knowledge claims is (at best) secondary in the evaluation of works of art –and vice versa [22].

As we can see, logical positivism admits that there is a sense in which aesthetic considerations can coexist with epistemic ones, but the proper evaluation of science cannot admit judgements of beauty; it thus denies legitimacy to such judgements.

As James McAllister comments: according to logical positivism "there exist no such phenomenon as scientists' aesthetic evaluation of their theories and therefore no such phenomenon that need trouble philosophers of science" [62, p. 13].

The strategy of dismissing aesthetic judgements in science can be successful only if it is in accord with the actual practice of science. But McAllister has compellingly documented that aesthetic evaluations in science are not mere anecdotal episodes or expressions of the scientists' private idiosyncrasies. Rather, they have played and still play a role in the choices of theories scientists make, and thus in the actual development of science [62–64]. Now, despite the fact that the actual practice of science does not support logical positivism's attitude and that logical positivism itself has been superseded within the philosophy of science, its distrust of judgements of beauty seems to be still influential. Helge Kragh illustrates this, by remarking that subjectivity is indeed troubling and difficult to address:

> The principle of mathematical beauty, like related aesthetic principles, is problematical. The main problem is that beauty is essentially subjective and hence cannot serve as a commonly defined tool for guiding or evaluating science. It is, to say the least, difficult to justify aesthetic judgment by rational arguments. Within literary and art criticism there is, indeed, a long tradition of analyzing the idea of beauty, including many attempts to give the concept an objective meaning. Objectivist and subjectivist theories of aesthetic judgment have been discussed for centuries without much progress, and today the problem seems as muddled as ever. Apart from the confused state of art in aesthetic theory, it is uncertain to what degree this discussion is relevant to the problem of scientific beauty. I, at any rate, can see no escape from the conclusion that aesthetic judgment in science is rooted in subjective and social factors. The sense of aesthetic standards is pan of the socialization that scientists acquire; but scientists, as well as scientific communities, may have widely different ideas of how to judge the aesthetic merit of a particular theory. No wonder that eminent physicists do not agree on which theories are beautiful and which are ugly.

If aesthetics itself cannot find an objective way to deal with beauty, the use of aesthetic evaluations by scientists is certainly a troubling challenge to the rationality of science. The influence of these ideas manifest itself not only among philosophers of science but also among scientists, as illustrated by the following warning against taking beauty in science at face value by Nobel Prize winning physicist Steven Weinberg:

> A physicist who says that a theory is beautiful does not mean quite the same thing that would be meant in saying that a particular painting or a piece of music or poetry is beautiful. It is not merely a personal expression of aesthetic pleasure; it is much closer to what a horse-trainer means when he looks at a racehorse and says that it is a beautiful horse. The horse-trainer is of course expressing a personal opinion, but it is an opinion about an objective fact: that, on the basis of judgements that the trainer could not easily put into words, this is the kind of horse that wins races [...] The physicist's sense of beauty is also supposed to serve a purpose –it is supposed to help the physicist select ideas that help us to explain nature [92].

Like Rota, Weinberg believes that aesthetic evaluations should not be conceived as referring to the standard subjective experience of beauty, but rather as referring to an objective property.

Now, Weinberg's reinterpretation exhibits some interesting characteristics: Weinberg does not explicitly formulate a new meaning for the term 'beauty'. Rather, he establishes an analogy to clarify the meaning: in qualifying a race horse as beautiful, any average person would refer to the fact that the horse's observable shape is pleasing; but an specialist, a professional horse-trainer, might refer to the fact that the horse is good at performing the task the trainer expects it to perform. A similar difference in meaning may be inferred, if my reading is correct, between the literal application of the term beauty by an average person and the figurative application by the scientific specialist: beautiful theories are merely theories which are good at doing what they are expected to do. The most significant feature of Weinberg's reinterpretation is that it explicitly intends to displace the subjectivity of the judgement of beauty: instead of relying on the usage of the term 'beautiful' by the average person, the analogy points out that we should rely on the usage of the term as intended by the specialist.

Now, the problem with this strategy is that subjectivity does not disappear. Weinberg recognizes that the specialist finds it difficult to put his judgement in objective terms (although one wonders why Weinberg does it so easily). So, the new meaning of beauty is not free from subjectivity itself. Furthermore, the analogy itself is rather idiosyncratic, for the analogy is valid for many cases of beautiful physical theories, but it is also valid for any accepted physical theory. Any accepted physical theory explains nature to some extent, but it is well documented that physicist prefer simpler theories instead of complicated ones. In this sense Weinberg's notion of the physicists' sense of beauty cannot explain why if complicated theories are good at explaining nature, physicist still prefer simpler theories. It cannot explain either why physicists disagree about which theories are beautiful and which are ugly, as pointed out by Kragh. If Weinberg analogy is valid it is *trivially* valid, and thus if we choose to accept it as an explanation of beauty in physics, it requires us to be selective in the cases in which it will be applied. And since we have no objective guidelines, this selectiveness is again *subjective*.

Thus, Weinberg's view does not really eliminates the subjectivity of the judgements of beauty, but it only distributes, so to speak, it among the new non-literal meaning and the analogy itself. If we are interested in safeguarding rationality by eliminating subjectivity, the above strategy cannot guarantee that. Subjectivity was merely displaced from a very visible place to a place out of immediate sight.

Now, unlike Rota's, Weinberg's reinterpretation is by no means articulated. Furthermore, that reinterpretation is useless for mathematics, since mathematics is an abstract discipline that does not depend on empirical facts. For these reasons, and for the sake of brevity, it may be better to recourse once again to Rota. If we assume that acceptable epistemic properties must be in accord with the norms of rationality, we can see Rota's approach as a way of reinterpreting mathematical beauty to defend rationality. Let us disregard our previous objections and see if Rota's interpretation can safeguard the rationality of mathematics. It is true that enlightenment seems appropriate to account for certain uses of the term mathematical beauty. For example, according to David Wells [94], Euler's identity is regarded beautiful precisely because it connects in a single equation the most important constants in mathematics. This is in agreement with Rota's enlightenment interpretation. However, other usages of the word beauty do not lend themselves to be interpreted in Rota's way. Hardy's famous quotation "beauty is the first test, there is no place in the world for ugly mathematics", for example. Rota's reinterpretation is only partially correct. Now, our problem here is not the principled issue of reducing beauty to enlightenment, but rather that in order to achieve a sensible interpretation one must make idiosyncratic choices depending on one's goals. Choosing which aspects of the use of the term are relevant and which are not is also a decision made by the person making the reinterpretation, in this sense the reinterpretation is also idiosyncratic.

Now, the partial accuracy of Rota's reinterpretation is less a flaw of Rota's particular tactic for understanding beauty than a problem induced by the very nature of the term beauty. Most philosophers acknowledge that there are no norms governing the application of terms such as beautiful, ugly or elegant [11, 13, 82]. This is in accord with Kragh's comment that judgements of beauty in science are subjective and idiosyncratic. Rota's approach simply illustrates that any reinterpretation has to deal with the fact that the use of the word beauty is subjective and idiosyncratic. Rota's selective interpretation is a way of dealing with this issue, another way is Weinberg's trivially correct interpretation.

We can see now that reinterpreting 'mathematical beauty' does not really eliminate subjectivity. We have two tactics to deal with the subjectivity of beauty: a partial reinterpretation, like Rota's, or an interpretation that covers all cases at the expense of covering them trivially, like Weinberg's. As we have seen, both tactics involve idiosyncratic choices, which is precisely what we are trying to avoid. The reinterpretations of beauty that are able to accurately encapsulate beauty's idiosyncrasy merely displace the subjectivity to the reinterpretation itself. The goal of reinterpreting beauty is to eliminate subjectivity, but, as we have seen, it only manages to displace it somewhere else.

To conclude this chapter let us quickly recapitulate. We have discussed three reasons why one should embrace a non-literal interpretation of mathematical beauty. I have argued that cultural attitudes like the Two Cultures split provide no sound basis for thinking that aesthetic phenomena are alien to mathematics; as a matter of fact, early proto-aesthetics even offered accounts of mathematical beauty. I also argued that two of the most central characteristic of mathematics do not provide a good argument for reinterpretation either. It is certainly troubling that aesthetic properties are epistemologically inert, and inconsistent with scientific rationality. However, I argued, reinterpreting beauty does not help to solve these problems, for that strategy cannot account for all instances of mathematical beauty and it cannot satisfactorily eliminate the subjective and idiosyncratic character of aesthetic judgements. In the next chapter we shall survey a possible explanation of these facts: aesthetic terms withstand non-literal usage. We shall also see that embracing a literal interpretation of beauty can actually help us to advance a defence of rationality.

Chapter 2
Beautiful, Literally

In Chap. 1 I contested a set of reasons to reinterpret mathematical beauty. In this chapter, I examine reasons to embrace a literal interpretation. First, we shall examine a principled reason to reject non-literal interpretations. Next, I shall argue that literal interpretations of beauty are appealing as they can be instrumental in understanding the progress of science: we shall see that in James McAllister's approach a literal interpretation of beauty is the cornerstone in his defense of a rationalist model of scientific development.

2.1 Metaphor and Aesthetic Terms

As we have seen, reinterpreting the term 'beauty' in the context of evaluating scientific and mathematical theories does not help us to safeguard their epistemic or rational character. The reason for this is that any reinterpretation must deal with the subjectivity and idiosyncrasy of the use of terms like 'beautiful', 'ugly', or 'elegant'. Those characteristics seem to be resilient to reinterpretation, as they manifest themselves again at some point in the reinterpretation. Now, that judgements of taste are characteristically subjective has been a tenet of modern aesthetic ever since Kant. Contemporary authors like Rafael De Clerq have explored some semantic aspects of aesthetic terms closely related to their subjectivity. Some of those ideas might help us to explain why attempts to reinterpret mathematical beauty are doomed to failure and, thus, they may spare us the trouble of contesting any more non-literal interpretations.

According to De Clercq, aesthetic terms, terms such as 'beautiful', 'elegant' or 'ugly', possess a salient feature in common:

> their *resistance to metaphorical usage*. In other words, aesthetic terms cannot be turned into metaphors. For instance, it makes no sense to say that something is beautiful "metaphorically speaking." Likewise, it does not make sense to say that something is metaphorically elegant, metaphorically harmonious, or metaphorically sublime [17, p. 27].

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_2,

De Clercq suggests an explanation for this feature:

> Aesthetic terms do not have a particular area of application associated with them. There is not a particular *kind* of object to which they are to be applied. As a result, it is not possible to commit something like a "category mistake" with respect to such terms. By contrast, terms for animal species such as 'elephant' and 'crocodile' can be applied only within the animal kingdom: to apply them outside this area is to commit a "category mistake" (which may of course result in a metaphor) [17, p. 27].

If aesthetic terms cannot be used metaphorically, that explain why attempts to reinterpret mathematical beauty fail. Thus, if we intend to safeguard the epistemic or rational character of mathematics, or to understand the meaning and nature of mathematical beauty, we should find a strategy different from reinterpreting mathematical beauty.

Now, De Clercq makes refinements to his view that are further illuminating: some aesthetic terms, 'balanced', for instance, are already metaphors, and some others, such as 'garish', are not universally applicable. De Clercq argues that in the case of already metaphorical terms his characterization in terms of metaphoric resistance is still valid, for these terms cannot be turned into metaphors. There is no such thing, he argues, as a second order metaphor. As for not universally applicable aesthetic terms, De Clercq suggests that we should regard them as "semi-aesthetic" terms [17, pp. 28–29].

Now, even if one finds De Clercq's characterization in terms of metaphorical resilience too strong, for our purposes his weaker assertion that aesthetic terms such as 'beautiful' or 'elegant' can be properly applied to any domain of objects without incurring in a category mistake is still appealing, since it suffices to substantiate a principled scepticism against reinterpreting the term 'beauty'. For example, one might contest De Clercq's view by arguing that instances such as using the expression "a beautiful horse" to mean "a race winning horse" show that aesthetic terms can be turned into metaphors, or at least that they can be used in a non-literal manner.[1] I agree that the usage of expressions like "this horse is beautiful" to refer to the fact that such a horse wins races is non-literal. However, as De Clercq points out, it always makes sense to use terms like 'beautiful' to qualify any kind of entities—including mathematical constructs and entities—in a literal sense. For example, in qualifying racing horses or scientific theories as beautiful, one may encounter some cases in which such expressions are not genuine aesthetic evaluations; nonetheless, such expression can *always* be interpreted in a literal way. The expression "a beautiful something" may mean sometimes *a race winning horse* or *an understanding promoting theory* given the appropriate context. However, without any further information, it is always possible to interpret the expression in a literal way, meaning that the horse or theory elicit a positive aesthetic experience. This is evident in the fact that in order to understand what a "beautiful horse" means in a non-aesthetic sense we need an explanation that tells us what the specialist mean

[1] In fact, some dictionaries includes non-literal definitions like "being advantageous" or "being apt" in their entries for beauty.

by 'beautiful'. But to understand the expression as an aesthetic evaluation, we need no explanation, even if one is not familiar with horses, since we all are familiar with the use of 'beautiful' as a genuine aesthetic term. In order to understand the expression literally there is nothing needed other than acquaintance with language. So, even if we admit that some uses of the term 'beautiful' for evaluating mathematical entities might be intended to convey a non-aesthetic meaning, genuine aesthetic evaluations are always possible in principle. Thus, even if we have a non-literal interpretation of aesthetic evaluations in mathematics acceptable in certain contexts, we still need to deal with the genuine instances of aesthetic evaluations that are possible in principle. Furthermore, the most interesting uses of aesthetic terms in evaluative contexts—such as Hardy's or Russell's, as discussed in the introduction—seem to be genuine aesthetic evaluations. At any rate, non-literal interpretations cannot rule out the possibility of genuine aesthetic evaluations, and thus a thorough analysis of mathematical beauty must address genuine aesthetic evaluations. Any non-literal interpretation is bounded to be insufficient, and it must be supplemented with a way of addressing literal aesthetic evaluations. Now, addressing genuine aesthetic evaluations is not only inevitable, but, as we shall now see, it is also advantageous, since it can provide us with significant insights and tools. Those insights and tools are put to good use by James McAllister.

2.2 McAllister's Approach

In *Beauty and Revolution in Science* [62], by embracing a literal interpretation of aesthetic evaluations in science, James McAllister formulates a rationalist model of scientific change and scientific revolutions. I shall not address the details of that model here; rather I shall concentrate on the issues relevant for our purposes.

McAllister takes an explicit theoretical stance. The results accomplished by that approach illustrates not only that a literal interpretation of beauty can be more coherent and fruitful than a non-literal interpretation, but also that to gain insight into the role of beauty in science a minimal theoretical basis is necessary. McAllister claims that the evolution of scientists' aesthetic preferences for some theories is closely related to the evolution of science itself; that in addition to logical and empirical criteria, aesthetic criteria play a role in the evaluation of scientific theories. So, let us take a closer look.

McAllister abundantly documents the fact that aesthetic evaluations are pervasive in science, which, as we have seen, constitutes a perplexing intrusion of the irrational into science. He also documents the even more perplexing fact that prominent scientists, Dirac, for example [19, p. 10], have endorsed the idea that the beauty of theories plays a significant role in science. In spite of this, McAllister does not opt for reinterpreting beauty; rather he maintains that a rationalist view of science that includes both aesthetic evaluations and scientific revolutions can be defended. Central to McAllister's account is the notion of the aesthetic induction, a mechanism that connects empirical and aesthetic evaluations of theories through an inductive relation.

McAllister shows that throughout the history of science aesthetic evaluations of scientific theories appear not as mere biographical anecdotes or as the result of private idiosyncrasies, but rather as an influential element in the scientists' professional work. In his book [62], McAllister develops a model of scientific change in which the two salient intrusions of irrationality into science, aesthetic evaluations and scientific revolutions, fit coherently into a rational picture of science. To achieve this, McAllister elaborates three closely related theses: first, that the scientists' aesthetic preferences play an actual role in the development of science. Second, that such preferences evolve driven by the aesthetic induction. And third, that scientific revolutions are aesthetic ruptures; that is, episodes in which the set of aesthetic criteria held to by a scientific community is replaced by a new one. We shall concentrate on the second thesis, since that is the most relevant for our purposes.

For McAllister's defence of a rational depiction of science it is essential to establish that the aesthetic evaluations scientists use to choose theories are not irrational. To accomplish this, McAllister argues that there is a non-reductive connection between the scientists' aesthetic evaluations and their empirical evaluations. He argues that scientists come to increase their appreciation for the aesthetic properties of theories that have shown to be empirical adequate in the past because they inductively project that when a new theory exhibits those properties, the theory will be empirically adequate [62, pp. 77–79]. This connection, which he conceptualizes as the aesthetic induction, is the key to give aesthetic evaluations and scientific revolutions a rational basis. Since McAllister's takes the applications of terms such as 'elegant' or 'beautiful' to scientific theories at face value, and that act constitutes the basis for his entire project, a minimal theory of the nature of beauty seems necessary. That theory is our concern here.

2.2.1 McAllister's Aesthetic Theory

McAllister interprets beauty literally; he does not try to find the "true" meaning of beauty in science. Rather, he attempts to gain understanding of, and to some extent systematize, the aesthetic phenomena that affect the way scientific theories are evaluated and chosen by scientists. McAllister's employs the basics of an aesthetic theory to make sense of aesthetic phenomena. However, the degree of rigour with which he discusses the theory is not homogeneous. McAllister utilizes Hutchenson's aesthetic theory to formulate his general approach to the notion of beauty, but he also resorts to more pragmatical and semi-theoretical tactics to address the notion of aesthetic property and to describe the mechanism of evolution of aesthetic preferences.

McAllister endorses some of Francis Hutchenson's ideas to allow him to accommodate scientists' aesthetic judgements in the broader context of aesthetic phenomena. The issue of where value is located and, more specifically, the debate between objectivism, the view that value is available in the world, and projectivism, the view that value is projected into the world by observers, occupies the central place in McAllister's discussion. Following Hutchenson, McAllister endorses a projectivist approach: beauty is projected into objects by the beholder [62, pp. 31–32]. Moreover, McAllister believes that Hutchenson accounts for the most relevant issues of beauty: Hutchenson tells us what beauty is; an idea caused by the feature of uniformity amidst variety. Hutchenson's ideas also allows us to distinguish between the beauty of a theory and the beauty of other phenomena; for he points out the relevant feature (uniformity amidst variety) that might lead scientists to regard a theory as beautiful. However, McAllister finds Hutchenson's account not completely satisfactory, since McAllister does not believe there is any property that all scientists throughout history recognize as guaranteeing beauty in theories [62, pp. 22–23]. For McAllister, beauty is a dynamic rather than a static concept. Evidence from the history of science supports this idea. Thus, the issue of accounting for the dynamics of the concept of beauty in science becomes central in McAllister's approach. To formulate a model of the evolution of beauty in science McAllister utilizes the evidence provided by the usage of aesthetic evaluations throughout the history of science. His formulation is thus based on the evolution of the scientists' aesthetic preferences. To articulate such formulation, McAllister uses the notions of aesthetic property, aesthetic criteria and aesthetic canon.

2.2.2 Aesthetic Properties, Aesthetic Criteria and Aesthetic Canon

McAllister defines an aesthetic property as "one that evokes aesthetic responses in observers" [62, p. 35] and proposes two criteria to identify which properties of scientific theories are aesthetic.
His first criterion:

> First, I shall judge a property of a theory to be an aesthetic property if scientists in the relevant disciplines react to it publicly as aesthetic, for example by declaring that they attach aesthetic value to it, by citing it in an act of theory evaluation that they describe as aesthetic, or by applying to it standard terms of aesthetic appreciation, such as "beautiful," "elegant," "pleasing," or "ugly." I regard these acts as amounting to aesthetic responses to the property in question, so any property of theories that prompts these acts in scientists satisfies in a straightforward way my definition of aesthetic property [62, p. 36].

His second criterion:

> a property is aesthetic if, in virtue of possessing that property, a scientific theory is liable to strike beholders as having a high degree of aptness. The justification of this criterion is that, in many philosophies of art, the beauty of an object is explicated as its aptness or the aptness of its elements [62, p. 37].

Interestingly enough, McAllister's second criterion covers Weinberg's non-literal interpretation of beauty as referring to the fact that something performs well the task it is expected to perform. McAllister, however, acknowledges that some usages of beauty in that sense are genuinely pseudo-aesthetic. At any rate, as we have discussed above, the most problematic cases are the genuine usages of aesthetic terms, and we shall concentrate on them. Now, the issue of aesthetic properties is a very contentious one. McAllister's definition is rather pragmatic and, of course, debatable, but I shall assume here that it is sufficient for his purposes; in the next chapters I shall discuss whether or not that is the case.

In McAllister's view, aesthetic properties are essential to explain aesthetic evaluations. McAllister explains aesthetic evaluations in a projectivist way, In such explanation, the links between aesthetic properties and aesthetic evaluations are the aesthetic criteria. In McAllister's projectivist approach, making aesthetic evaluations depends on two factors: the presence of an object—a theory, for instance—bearing aesthetic properties, and the presence of values in the person—a scientist, for instance—observing the object. Scientists are moved to project beauty into a theory as a consequence of their "holding to one or more aesthetic criteria that attach aesthetic value to properties of the theory" [62, pp. 34–35]. Aesthetic criteria are the criteria that attach aesthetic value to specific properties, thus an aesthetic criterion embodies a person's preference, or degree of preference, for certain aesthetic property. For example, let us assume that visual symmetry is a desirable property in things like buildings. McAllister's theory tells us that in that case there is an aesthetic criterion associated to the property of symmetry which is responsible for attributing beauty to symmetrical buildings. Different people hold to different aesthetic criteria, and they do so in different degrees—or with different intensities. This explains the diversity of aesthetic responses in different individuals evoked by the same object. Scientists are not different in this respect: they ascribe beauty to particular scientific theories because they hold to aesthetic criteria that attribute value to the properties of those theories. Now, the aesthetic preferences of an individual or a community change over time. The patters of change of those preferences determine the dynamics of beauty—which is Mcallister's chief interest—and the aesthetic criteria provide a convenient way of modelling those patterns.

2.2.3 The Aesthetic Canon

The aesthetic criteria held by a scientist or a scientific community determine what the scientist or community consider as aesthetically valuable. The exhaustive collection of aesthetic criteria of a scientist or community is called their *aesthetic canon*. In order to model the evolution of aesthetic preferences McAllister expresses the aesthetic canon as an exhaustive, perhaps infinite, list of properties and the numbers representing the intensity of the preference for such properties.

More specifically, McAllister proposes that for every possible property exhibited by a theory, we can conceive a corresponding aesthetic criterion. For example, a property P might have an associated criterion of the type:

If a theory has P, attach more aesthetic value to it than, if other circumstances are equal, it did not.

These aesthetic criteria ground evaluations of theories, since they are guidelines for theory assessment. According to McAllister, such criteria are actually used to choose among theories. Ideally, we can assume that a scientist holds to as many aesthetic criteria as properties to which he responds aesthetically. The scientist's aesthetic canon comprises all such criteria. The different aesthetic criteria possess associated weightings which assess the relative worth the scientist attributes to the property involved in the criterion, and the influence of the criterion in theory choice. Some properties are better regarded than some others; their associated weightings should reflect this fact. In other words, one criterion might weigh or be worth more than another criterion within the canon. To capture these features, McAllister represents aesthetic criteria by means of pairs of items of information: the property P, and its associated weighting W_P. McAllister thus formulate a fully expressed canon as a list of such pairs, as follows:

$$\begin{array}{l} P, W_P \\ Q, W_Q \\ R, W_R \\ \vdots \end{array}$$

The aesthetic canon may comprise an infinite number of entries, one for each *possible* property of scientific theories. Most of the criteria will carry a weighting of zero, as scientists typically value only a few properties and are indifferent to the rest. The advantage of this conception of the canon is that any change in aesthetic preferences can be represented simply as a change in the weightings of the criteria [62, pp. 34–35].

All these ideas serve to introduce in an articulated manner McAllister's central notion: the aesthetic induction.

2.2.4 The Aesthetic Induction

Aesthetic preferences are subject to change. That fact is central in McAllister's approach. His key claim is that this change is connected with *empirical* evaluations of theories. In McAllister's view the standard idea that aesthetic phenomena are independent from pragmatic issues is challenged by evidence in the history of art, and, of course, the history of science. McAllister points out that aesthetic preferences do change, evolve, even in the arts. He draws our attention to the case of cast-iron, steel and concrete structures in architecture. These materials were introduced for practical reasons; they were increasingly used in buildings due to their structural advantages. But eventually they gained the appreciation of architects and the public [62, Chap. 9]. So, properties that appeared in buildings due to their practical success, gained in aesthetic preference on its own merit; this influence of pragmatic factors on aesthetic preference is a variety of what McAllister calls the aesthetic induction.

Now, McAllister's goal of defending a rationalistic picture of science depends on showing that the aesthetic evaluations that scientists use to choose theories are not irrational. To accomplish this, McAllister's strategy is to connect the scientists' aesthetic evaluations of some theories with the empirical evaluations they use to choose those theories. Regarding that connection McAllister discusses "two erroneous views of scientists' aesthetic judgments," which he labels *autonomism* and *reductionism* [62, Chap. 4]. Autonomism "regards scientists' aesthetic and empirical evaluations as wholly distinct from and irreducible to one another, whereas reductionism views them as nothing but aspects of one another" [62, p. 61]. McAllister rejects both views, and offers an alternative: scientists come to increase their appreciation for the aesthetic properties that recurrently appear in theories that have shown to be empirical adequate in the past. The reason for this is that they inductively project that when a new theory exhibits those properties, the theory will be empirically adequate [62, p. 77–79]. The change in the scientists' aesthetic preferences is thus the result of the recurrent appearance of certain properties associated with empirically successful theories, this phenomenon is the *aesthetic induction*, and is the link between empirical and aesthetic evaluations which warrants rationality. McAllister describes the aesthetic induction as follows:

> A community compiles its aesthetic canon at a certain date by attaching to each property a weighting proportional to the degree of empirical adequacy then attributed to the set of current and recent theories that have exhibited that property. The degree of empirical adequacy of a theory is, of course, judged by applying the community's empirical criteria for theory evaluation. I name this procedure the aesthetic induction [62, p. 78].

Since there exists a link between empirical and aesthetic evaluations, the possibility that aesthetic properties may be indicators of empirical adequacy cannot be ruled out; and, thus, it is rational to choose theories based on empirical criteria when those theories are empirically equivalent. In this way, aesthetic evaluations are not completely idiosyncratic or subjective. More specifically, in situations where scientists have to choose between empirically equivalent theories, they prefer theories that bear properties with the highest weighting within the aesthetic canon [62, pp. 78–81].

McAllister expands this model of theory choice into a model of scientific change and scientific revolutions by describing different scenarios: the periods in which the aesthetic criteria evolve gradually over time within an aesthetic canon are analogous to what Kuhn calls periods of normal science. The episodes in which an aesthetic canon is replaced by a different one are episodes of aesthetic rupture. McAllister characterize scientific revolutions as episodes of aesthetic rupture. The existence of these scenarios is substantiated and documented by McAllister with a range of historical cases that show the effect of the evolution of aesthetic preferences and episodes of aesthetic rupture in the development of science [62, Chaps. 10 and 11], unfortunately, surveying them is beyond the scope of this book.

To point out an important feature of his model of scientific change, McAllister gives the following illustration of the aesthetic induction at work:

> A scientific community looks back over the recent history of a particular branch of science. It perceives that some theories, which are to a notable degree visualizing (rather

> than abstract) theories, have been empirically very successful, whereas others, which lend themselves to mechanistic analogies, have won little empirical success. Both visualization and tractability by mechanistic analogies are aesthetic properties of theories. In consequence of the empirical success of the visualizing theories, the property of visualization will obtain an increased weighting in the aesthetic canon for theory evaluation that the community will hereafter apply. By contrast, the property of being tractable by mechanistic analogies will receive a lowered weighting in the canon, in virtue of the scarce empirical success of recent theories that displayed this property [62, pp. 78–79].

According to McAllister, "[t]he aesthetic induction is an instance of inductive projection, since it amounts to consulting the properties of past good theories to determine which future theories should be expected to be good" [62, p. 79].

The aesthetic induction induces a bias toward the properties of successful theories:

> By imagining the aesthetic induction in operation, we can infer how a community's set of aesthetic preferences among theories will evolve in particular circumstances. A theory that achieves significant empirical success will cause its community's aesthetic canon to be remodeled to a certain extent, in such a way, that the canon comes to attribute a greater weighting to that theory's aesthetic properties. The canon will therefore acquire a bias in favor of any future theories that exhibit the aesthetic properties of current successful theories. In other words, by their empirical success, theories can predispose the community to choosing future theories with properties similar to their own. A future theory will then win the endorsement from the aesthetic canon in the measure to which it shares the aesthetic properties of current theories that have been attributed high degrees of empirical adequacy. If, on the other hand, a new theory shows properties different from those currently entrenched in the canon. It will be denied endorsement by the aesthetic canon [62, p. 79].

In summary, successful theories greatly contribute to determining which theories will later be welcomed by the scientific community. This type of inductive projection is particular in that the properties involved in past and future theories are not empirical properties, but *aesthetic* properties of theories.

2.2.5 Aesthetic Induction in Mathematics

Although McAllister characterizes the aesthetic induction in terms of empirical adequacy, he argues that a variant of the aesthetic induction influences the development of mathematics [64]. The aesthetic induction operates in mathematics in a fashion similar to how it operates in the empirical sciences:

> [. . .] evidence that conceptions of mathematical beauty evolve under the influence of the aesthetic induction is provided by the gradual acceptance of new classes of numbers in mathematics, such as negative, irrational, and imaginary numbers. Each of these classes of numbers had to undergo a gradual process of acceptance: whereas initially each new class of numbers was regarded with aesthetic revulsion, in due course –as it demonstrated its empirical applicability in mathematical theorizing— it came to be attributed growing aesthetic merit [64, p. 29].

McAllister documents various types of mathematical entities that mathematicians do regard, or have regarded, as beautiful, numbers, theorems, theories; turning, in the end, to focus on mathematical proofs, as they appear to be specially illuminating. McAllister argues that in Antiquity a prototypical proof was defined as a short, simple series of logical inferences from a set of axioms to the theorem. The series of inferences was required to be sufficiently short and simple that a mathematician could grasp it in a single act of mental apprehension [64, p. 19]. The graspability of a proof is closely related to how well the proof lends itself to being understood:

> Mathematicians' views about beauty in proofs have been influenced by their familiarity with classical proofs. Mathematicians have customarily regarded a proof as beautiful if it conformed to the classical ideals of brevity and simplicity. The most important determinant of a proof's perceived beauty is thus the degree to which it lends itself to being grasped in a single act of mental apprehension [64, p. 22].

McAllister points out that, in recent decades, two new types of proofs have appeared: long proofs, such as Wiles' 108-page-long proof of Fermat's last theorem; and computer-assisted proofs, such as Appel and Haken's proof of the four-colour theorem. These types of proof challenge the classical conception of proof, since they are not graspable in the classical sense, and may even exhibit a logical structure different from the classical proof. McAllister speculates that they might even alter our conception of beautiful proof. In this respect, he focuses on the aesthetic merit of computer-assisted proofs: even if computer-assisted proofs have settled important questions, mathematicians have received them with aesthetic revulsion; but that, McAllister speculates, might change as they become more acceptable.

Since mathematics is not an empirical science, McAllister proposes the acceptability of proofs as the factor involved in the aesthetic induction in mathematics: in the same fashion as the evolution of the beauty of empirical theories depends on their empirical adequacy, the evolution of the beauty of proofs depends on their acceptability. Now, if mathematical beauty indeed evolves driven by the aesthetic induction, the preference for computer-assisted proofs must be driven by it as well:

> [. . .] the criteria that determine whether a theory is deemed to provide an understanding of phenomena may evolve in response to the empirical success of theories, in accord with the aesthetic induction. If this is true, a deep link exists between the concept of scientific understanding and conceptions of the beauty of scientific theories. On the basis of the reception of computer-assisted proofs, I conjecture that the evolution of aesthetic criteria applied to mathematical proofs is also governed by the aesthetic induction [64, pp. 28–29].

Currently, computer-assisted proofs are regarded as ugly, but in McAllister's view that is merely a contingency. The aesthetic induction allows the possibility that as computer-assisted proofs become more accepted and recurrently succeed in providing results, mathematicians' preferences for them shall evolve, perhaps even to the point of finding them beautiful.

2.2.6 McAllister in Summary

As we have seen, McAllister not only endorses a literal interpretation of beauty in science, but it does so by formulating a sophisticated theory of the nature and role of beauty in science—although its application to mathematics needs a small variation. That theory can be summarized as follows: (1) Projectivism: McAllister rejects objectivism, which is the view that beauty is an objective property of objects. Beauty is not interpreted as an objective property, but as a value that observers project into objects. A value is something that is considered good, important or desirable. (2) Aesthetic Properties Evoke Aesthetic Responses: objects, including scientific theories, may possess intrinsic properties that evoke aesthetic responses in the observer and lead to project aesthetic value into those objects. These properties are the aesthetic properties. (3) Aesthetic criteria: A person is moved to project beauty into an object when he holds to aesthetic criteria that attribute value to the properties of that object. (3.1) Beauty in science: A scientist, or a mathematician, is moved to project beauty into a theory, or other mathematical entity, when he holds to aesthetic criteria that attribute value to the properties of that object. (4) Aesthetic induction: scientists' aesthetic preferences evolve modulated by an inductive mechanism, the aesthetic induction, in which properties recurrently appearing in empirically adequate, or mathematically acceptable, theories gain in preference. (4.1) Aesthetic induction in Mathematics: mathematicians' aesthetic preferences evolve modulated by the aesthetic induction induced by the acceptability of mathematical entities. (5) Beauty and Scientific Change: scientist often choose between equally empirically adequate theories by opting for theories bearing the properties with the greater degree of aesthetic preference. (6) Beauty and Revolution: Scientific revolutions are episodes of aesthetic rupture [62, pp. 30–34].

Let us conclude this chapter here. In the previous chapter we saw that attempts to eliminate the subjectivity of aesthetic evaluations in science by reinterpreting beauty in science and mathematics have had little success. In this chapter we discussed an argument that should round up and give closure to our discussion of the reasons for embracing a non-literal approach. We encountered reasons to be sceptic about any temptations of reinterpretation. If we take into account ideas like De Clercq's, non-literal interpretations of beauty seem to be hopeless, or at least insufficient in principle. Moreover, McAllister's work gives us further reasons to abandon the search for a non-literal interpretation of beauty in mathematics, since, contrary to any nonliteralist concerns we may have, McAllister's approach shows that a literal interpretation of beauty in science can be used in an articulated and fruitful manner to achieve ambitious goals like defending the rationality of science. McAllister's approach is not only pragmatically appealing, but also very illuminating about the way in which a systematic approach to beauty should be conducted: McAllister's analysis is supported by historical evidence and a proper aesthetic theory. McAllister's work is also illuminating about the possibilities of a coherent approach: it aims to achieve very significant goals, and it is even capable of grounding plausible conjectures. Our goals here are different from McAllister's, but his way of dealing with the subject is inspiring. For that reason, a careful discussion of not only its insights but also its problems is in order. I devote the next two chapters to discuss some problems with McAllister's ideas. Addressing those problems shall prove to be very important for developing an adequate aesthetics of mathematics.

Chapter 3
Ugly, Literally

There are principled and pragmatic reasons to interpret beauty in mathematics literally. But that is not necessarily good news. On the contrary, formulating a plausible theory of mathematical beauty is a much more daunting endeavour than searching for a putative true meaning of mathematical beauty. To illustrate this, in this chapter we examine a literal interpretation of the beauty of mathematical proofs that yields results which contradict McAllister's approach.

We have surveyed, for different reasons, approaches to mathematical beauty by Shaftesbury, Hutchenson, Rota and McAllister. A feature common to all these approaches is that the *properties* of the object being evaluated play the central role in accounting for the aesthetic merit of the object. A problem with the first three authors is that their approaches cannot account for the evaluative character of aesthetic descriptions. Under Shaftesbury's and Hutchenson's interpretations, any piece of mathematics should be regarded as beautiful. But, as we know, aesthetic evaluations go both ways; some pieces of mathematics are judged as ugly. In the following, I show that the evaluative character of aesthetic evaluations can be better understood if we take into account not only the properties of objects, but also the way our attention engages in experiencing those objects.

3.1 Ugliness in Mathematics

As a way to emphasize the relevance of the way we experience mathematical items, I address mathematical ugliness, since an articulated account of this issue is missing in all accounts reviewed so far: Shaftesbury and Hutchenson do not account for it, Rota does not deal with mathematical ugliness in the same terms as he deals with beauty, and McAllister gives ugliness only the peripheral role of a possible initial condition in a process of aesthetic induction. In [67] an approach able to address both mathematical beauty and ugliness on the same grounds is presented. This is achieved by extending the literal approach to mathematical beauty by treating

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_3,

mathematical proofs as if they were one of the usual objects of aesthetic judgement, such as narrative or music, and drawing conclusions from it. The idea of investigating the similarities between mathematical proofs and things like narratives is by no means far-fetched. Authors such as Robert Thomas or Alan Cain have explored the connections between narrative and mathematical proofs. Thomas [88] explores the parallels between proof and narrative arguing that "[l]ogical consequence is the gripping analogue in mathematics of narrative consequence in fiction; all physical causes, personal intentions, and logical consequences in stories are mapped to implication in mathematics" [88, p. 45]. Thomas is concerned with the ontological thesis of fictionalism, the view that mathematical objects are merely fictions, and his mapping of most features of fictional stories into logical consequence serves him well. Alan Cain utilizes ideas similar to Thomas', but he is more interested in aesthetic matters. Cain's goal is to explicate Hardy's interpretation of mathematical beauty—as unexpectedness and inevitability—in terms of narrative devices; more specifically in terms of avoidance of *deus ex machina*.[1] I [67] explore an approach different from Thomas's and Cain's, although consistent with their findings. I do not map features of narratives into features of mathematical proofs, since, as Thomas correctly concludes, most narrative features map into logical consequence. Rather, I propose a simpler *analogy* between narratives and proofs which yields richer results. Mathematical proofs logically proceed from one statement to another with the goal of logically arriving to the statement to be proved; this fact enables this analogy: a mathematical proof can be seen as a string of "events" intended to reach an "ending". The "events" are the steps of the proof; the individual statements that take us from the initial assumption to the conclusion, the "ending". In this sense, proofs are very similar to narratives, to stories. A story is a string of events that has a beginning, develops following certain logic, and reaches an ending. Thomas and Cain's mapping-based approaches can only address the local features, the "events", of narratives that map into logical consequence, global events—such as the quality of a proof's "plot"—cannot be addressed by such approaches. The analogy formulated above allows us to address local as well as global elements in narratives and proofs, and, thus, to explore their roles in eliciting aesthetic responses.

In a story, there are several kinds of elements from which we derive pleasure. Two of the most obvious are the plot and the individual events depicted by the narrative. A plot is a sequence of events that outlines a story; it is the rough content of the story without specific details. The events that constitute the plot are, of course, part of the whole narrative, but the whole narrative usually includes events that are not needed for the plot; for instance, details that describe the geographical, historical,

[1] *Deus ex machina* is the plot device in which the resolution of an apparently unsolvable situation is achieved by introducing an *ad hoc* event or character. *Deus ex machina* means literally "God from the machine"; it is derived from ancient Greek drama, where such a plot device was implemented by the intervention of a god, with the actor playing the god being lowered onto the stage by a crane, the *machine*. Since Aristotle, authors have complained of the poor aesthetic merit of such a plot device.

or emotional set-up of the story. These individual events are often found enjoyable in themselves and may have been created just for their independent attractiveness.

A mathematical proof exhibits features analogous to both the plot and the individual events of a narrative. For example, we can summarize a film or a novel by telling only their plot and omitting minor details. In a similar way, we can formulate a heuristic argument that summarizes a mathematical proof by pointing out its crucial ideas and omitting the details. This summarizing argument is a sort of plot of the proof. On the other hand, the individual events of a narrative also have an obvious analogue in a proof: the proof's individual steps. Now, regarding the aesthetic merit of these features, a summary of the argument of a proof is usually not acceptable as an instance of mathematical beauty, for it is only the complete proof which is considered a proper mathematical entity [9, 10, 25]. A summarizing argument, the plot of a proof, may have uses in pedagogical or other contexts, but a proof is, by its very nature, a rigorous sequence of uncontroversial or logically justified statements. A summarizing argument is just not a proper mathematical entity and thus is not eligible to have mathematical beauty ascribed to it. As for the individual steps of a proof, as we saw in Chap. 1, Rota [78, p. 185] points out that sometimes they are considered beautiful if they are short, and this accords with the classical notion that briefer proofs are more meritorious and more likely to be beautiful. Now, so far, our analogy with narrative plots and individual events have been inconclusive or led back to issues already addressed. The analogy seems to have offered little additional insight into mathematical beauty. Fortunately, a third aspect of narrative, its global structural properties, is more helpful in this respect.

The structural properties of things like novels, short stories, or films are important sources of pleasure. We find enjoyable the way a story is structured: how the events are linked to each other, and how they anticipate and resolve into each other. An unexpected but consistent ending, for example, is often pleasing.[2] The ending of a story and the way the narrative progresses to reach it are considered of special value; so much so that when someone tells us in advance the ending of a novel or a film, we may consider it has been spoiled. The quality of the *storytelling*—that is, how events develop and resolve, how a situation leads to another, how fitting is the ending, etc.—is an important element we enjoy in a narrative. Mathematical proofs exhibit qualities that are analogous to a narrative's storytelling, since they too develop from one step to the next following a rigorous logic to reach a conclusion. But there are many ways in which a proof can progress: we find frequently, or perhaps usually, that a mathematical statement can be proved in more than one way. The different ways of proving a statement differ, obviously, in the steps that constitute the proof, but they also differ in how one step leads to another. For example, one of the proofs that are most often cited as an instance of mathematical beauty is Cantor's diagonal proof that the real numbers are uncountable [26].

[2] Hardy [33, pp. 29–30] points out something similar when he asserts that an unexpected theorem and its proof are sometimes considered beautiful, although, as we have seen, Rota contests this idea.

Cantor, however, devised two different proofs [26] before he introduced the famous diagonal argument. The diagonal argument proves a profound result in a very simple way, it is remarkably powerful, clever, and simple, and it has been used to prove several other theorems. Cantor's diagonal argument is simple not only in the sense that it is very parsimonious, involving only a few steps, but also in the sense that understanding it requires very little mathematical background. Now, the conclusion that the real numbers are uncountable is significant by itself. But what makes the diagonal proof remarkable, in contrast to Cantor's earlier proofs, is *the way we reach that conclusion*; in other words the proof's *storytelling*. A narrative plot can be told in different ways; similarly, a mathematical statement can be proved in different ways; and in both cases the quality of those ways, the quality of their storytelling, is something that we may find pleasing and enjoyable.[3] Furthermore, since the storytelling of Cantor's diagonal proof is *pleasingly simple yet effective*, it is natural to describe it in the same way we describe other simple yet effective pleasing things: as *elegant*; which is in fact the way the diagonal proof is often described [26].

3.2 Computer-Assisted Proofs

The analogy to narrative introduced above allows us to account not only for beauty, but also for ugliness. To show this, I now address Appel and Haken's computer-assisted proof of the four-color theorem [4, 5, 7]. Let us remember that proof is the subject of a very interesting conjecture resulting from McAllister's aesthetic induction; he asserts that as computer-assisted proofs accumulate a track record of success, mathematicians might end up judging them beautiful.

Appel and Haken's proof, or at least the first part of it, is an instance of the method of proof by cases. Proofs by cases are not the favourites of mathematicians: ever since Euclid, a tenet implicit in mathematical practice is a sort of Occam's Razor principle regarding ontological, methodological, and epistemological commitments. A simpler proof is preferred to a complicated one, and the same is true about definitions or axiomatizations. This tenet may be the reason why mathematicians show aversion to proofs by cases. The simplicity of a proof is thus a desirable quality, and its complexity an undesirable one.[4] Appel and Haken's proof involves checking almost 2,000 independent cases: not a parsimonious proof by any standards. However, the early negative aesthetic judgements of the proof seem to be grounded not on its being a very large proof by cases, but rather on the intervention

[3] This feature of mathematical proofs, that their structural properties are more critical to aesthetic merit than the plot or the individual events, puts them in closer analogy to music than to narratives. Unfortunately, to elaborate on the analogy with music is beyond the scope of this book. I have, however, addressed the parallels between music and mathematics in [66].

[4] Perhaps this is also related, in part, to the fact that simplicity facilitates understanding the proof, whereas complexity hinders it.

of a computer in proving the cases [86,89]. When the proof was first communicated, mathematicians found the computer-assisted steps in the proof objectionable for methodological reasons: computer programs may contain errors and computers may malfunction [89]. And since we cannot manually check the computer's results, we cannot be certain of the correctness of the results [86]. There were also epistemic concerns; since the results of a computer cannot be justified a priori—that is, by means of a purely logical argument, they are considered as empirical results [18]. Mathematical knowledge is independent of experience, and thus computer results are not in accord with the a priori nature of mathematics. Moreover, a proof that admits results from a computer has a different logical structure from a traditional proof; for it includes steps that are justified empirically. As we discussed in Chap. 2, McAllister has shown that in the history of mathematics such concerns have accompanied the introduction of new entities and methods, but have dissipated as the new entities are proven useful. McAllister explains this phenomenon as the result of the aesthetic induction, which links beauty to acceptability. Thus, we might be tempted to explain the ugliness of Appel and Haken's proof as the result of mathematicians' initial concerns about the proof's soundness. But this explanation is unsatisfactory for the following reason: when the critics qualified the proof as ugly, they were conceding the proof was a *genuine* proof, for the concept of mathematical proof, like mathematical truth, does not admit degrees. The acknowledgement that the computer-assisted proof is a genuine proof is an implicit admission that the epistemic concerns can be satisfactorily met [89]. The issue here is not whether the proof is genuine or not, but rather that it is ugly. The ugliness of the proof is thus not explained by mathematicians' epistemic or other technical concerns.

Now, the analogy with narrative can offer a simpler explanation of the negative aesthetic judgments passed about computer-assisted proofs: *the assistance of the computer impairs, or rather wrecks, the proof's storytelling*. To see this, imagine a novel or a film with a complicated plot, and suppose that just before the ending we are told that the remaining events are so complicated that they cannot be entirely narrated, but that we have good reasons to trust that they lead to the expected happy ending. That way of telling a story is very disappointing, even frustrating; for it initially engages us in following a series of events and then thwarts us in midstream, preventing us from following the crucial events to reach the ending. It is natural to feel cheated. Something analogous occurs with a computer-assisted proof: the computer, in certain crucial steps, provides results that would be impossible to obtain by traditional means. In Appel and Haken's proof, the assistance of the computer consists in checking the almost 2,000 cases [4–7], which exhausts all the possibilities anticipated by the proof's opening argument. In following the proof we get engaged, at first, in an experience similar to following a traditional proof, but at a crucial moment, we are asked to stop following the proof and trust, with very good reasons, the results provided by a computer program. As in the case of narratives, this interruption is likely to elicit a feeling of disappointment, or even frustration, which naturally translates into an expression of aesthetic aversion. In summary, when we are presented with a result provided by a computer right in the middle of a proof, we find that fact displeasing, for the proof does not offer us a complete "story" to follow. This is completely independent from our unfamiliarity with proofs of this type, and from our concerns about their soundness. The intervention of a computer in a proof has denied us some of the links, or "events", necessary for a complete "story". The results provided by a computer are thus "narrative gaps" in the proof. Narrative gaps, in our analogy, impair all three elements we enjoy in a narrative: plot, individual events, and global structural qualities. This impairment is independent of the acceptability we lend to the results from the computer, for the origin of the impairment is not that we do not trust the results, but rather that they *alienate us from the act of following the proof*. However willing we are to accept results from a computer, we cannot follow them in the same way as we follow the steps of a traditional proof. Even if a computer-assisted proof is perfectly sound in methodological and epistemic terms, it is still objectionable in aesthetic terms, for no degree of soundness can fill the narrative gaps in the proof's storytelling. Under this interpretation, judgments of ugliness about Appel and Haken's proof are not concealed doubts about the proof's soundness, but genuine expressions of the fact that the proof is disappointing *as a story*.

Now, this conclusion contradicts McAllister's conjecture that as computer-assisted proofs become increasingly accepted and accumulate a track record of epistemic success, they might become instances of mathematical beauty through a process of aesthetic induction. Under the interpretation presented here, no degree of acceptability or epistemic soundness can alleviate the factors, the narrative gaps, responsible for the ugliness of those proofs. Contrary to McAllister's, the view formulated here does not allow the possibility that the aesthetic evaluation of computer-assisted proofs improves much in the future. Now, this result is certainly unfortunate for our project, since two different literal approaches to mathematical beauty yield contradictory results. So far, in reviewing approaches to mathematical beauty, things seem to have progressed relatively smoothly, as the insufficiency of those approaches seemed to have a reasonable explanation, and the approaches, although diverse, did not seem to clash with each other. McAllister even endorsed and amended Hutchenson's view. But now we are faced with a head-on clashing of views, as our conclusion here contradicts McAllister's. We are not yet in position to propose a way to reconcile these conflicting views, but in order to achieve that reconciliation we need to examine carefully what are the factors responsible for the conflict.

3.3 Phenomenological Factors

The most salient factor that differentiates the narrative analogy from McAllister's approach has to do with inner experience. By treating mathematics as a subject of aesthetic judgement, the narrative analogy endorsed a literal interpretation of mathematical beauty. Now, affective responses, positive or negative, are typically involved in aesthetic phenomena [77], authors like Malcolm Budd [12] even argue that there is a distinctive feeling associated to aesthetic experience, the "aesthetic feeling". We do not need to endorse strong claims like the existence of an aesthetic feeling in mathematics, but it is reasonable to assume that affective responses are some of the phenomena involved when mathematics becomes a subject of aesthetic judgement. Terms like 'beautiful' or 'ugly' are deployed to evaluate objects based on the way we perceive their effects from our first-person subjective perspective [35,82,99]. Calling something beautiful or ugly is a report of an affective response, hence of a subjective phenomenon belonging to the realm of private inner experience. As a rough way of characterizing beauty and ugliness, we can utilize their respective associate positive and negative affective responses as signposts. This characterization differs from the ones reviewed so far in that it relies on private inner experience. Things related to the content of the consciousness as experienced in the first person point of view are called *phenomenological* [20]. I will use this term[5] to distinguish *inner experience* from things external to the consciousness such as publicly available properties, social influences, experiences in the empirical sense, and even abstract objects. This clarifies the first factor that differentiates the narrative analogy from McAllister's approach. McAllister, as most authors reviewed so far, is concerned with factors external to the consciousness. In contrast, the analogy with narrative exploits the fact that our inner experience in following a proof is a relevant factor in passing aesthetic judgements about mathematical proofs. The main aspect in which the narrative analogy differs from approaches like McAllister's is that it is a phenomenological approach. Let us examine this in further detail.

3.3.1 Intentional Objects

A phenomenological understanding of the experience of beauty, in general, casts the object of the experience as an *intentional object*. An intentional object is, roughly speaking, the mental content that fills our attention in an episode of appreciation. For example, in listening to a piece of music, our mind is filled with musical events. Music becomes the content of our attention. That content is a mental phenomenon, different from the physical phenomena that stimulate our ears. Similarly, the content of our mind when we follow a mathematical proof, rather than the proof as an abstract object independent of us, is what is relevant to its aesthetic appeal. Peter Kivy characterizes an intentional object as an "object perceived under a specific description" [44, pp. 81]. He illustrates this with the following example:

> You and I might both be looking at a man. I believe the man to be a well-known actor. You don't know him at all: he is just a tall, good-looking man to you. The 'intentional object'

[5] I will use the term only in this narrow sense, and only for the sake of brevity. It should not be confused with the wider sense of *phenomenology* that refers to a particular school of philosophical though and methodology.

> of my gaze is a 'tall, good-looking man who is a well-known actor, famous for his Hamlet.' Your 'intentional object' is merely 'a tall, good-looking man.' We both see the 'same man'; but, depending upon what we know, or believe about the man, we see 'different men'; we see different 'intentional objects.' [44, pp. 81]

This illuminating characterization differentiates between the physical object that a person observes and our phenomenological experience of it; that is, the content of the observer's inner experience. Furthermore, Kivy's characterization brings out a significant feature of an intentional object: since the object is perceived under a *certain description*, the intentional object does not necessarily possess the same properties the actual object possesses; it may possess only the properties relevant for the description. Moreover, the intentional object may also possess properties that depend on the individual characteristics of the observer—his skills, experiences, knowledge, etc.—which the actual observed object not necessarily possess. An observed object and its corresponding representation in our inner experience, the intentional object, may differ in many aspects, depending on our individual peculiarities. Even if Kivy's emphasis on description may sound too strong, it serves to clarify that to understand aesthetic phenomena it is important to distinguish between physical objects and intentional objects. Intentional objects are constituted mostly by properties that are relevant for the experiences of which they are the content. The properties that are relevant to constitute aesthetic experiences play a role, eliciting affective responses, in giving an aesthetic character to the experience.

3.3.2 Intentional Proofs

Returning to our analysis of proofs, we can now see that it is important to distinguish between the *proof as a mathematical object* and the proof as the mental content we bear in our inner experience; that is, the *proof as an intentional object*—or *intentional proof*, for short. This is particularly important due to the abstract nature of mathematics. Under the standard interpretation [10], a mathematical proof is an abstract object: an object with no spatio-temporal location and no causal interaction with the physical universe. Abstract objects cannot interact causally with us. But only a causally efficacious entity—a concrete object or, in the case of mathematics, the content of our mind when we think of some mathematical entity—can elicit a response in us. Thus, the intentional proof must be what is involved in our appreciating of mathematical beauty, or ugliness, for that matter.

Now, in an intentional proof, as in a narrative, our mind undergoes a process that unfolds over time. This makes proofs interesting intentional objects: in following a proof our mind focuses successively on its different steps, and performs the appropriate mental operations that allow us to see the connection between the successive steps. Those operations typically include depicting a hypothetical situation, manipulating symbols, or performing a logical inference [90, 91]. When we bear a proof in mind, we do not just observe the intentional proof and discern properties like its simplicity. Rather, we *construct* the intentional object that constitutes the intentional proof by an active and dynamic process of considering the successive steps and linking them logically. In a sense, our mind is the observer as well as the constructor of the intentional proof. This fact allows us to identify two ways in which an intentional proof can elicit an affective response of pleasure (or displeasure, for that matter). I label these ways the *contemplative* and the *performative* ways of appreciating mathematical beauty. The first way, the contemplative way, is by contemplating the proof as a whole and being affected by its aesthetic properties. Pleasure can be derived from contemplating, for example, the simplicity of the proof. We can say that the *content* of our mind is responsible for the aesthetic quality of our experience. The second way, the performative way, in which an intentional proof can elicit an affective response is by taking pleasure (or displeasure) in the act of constructing it; in other words, in the very act of following the proof. The mental operations involved in following the proof can be pleasing (or displeasing) to us, making the process of constructing the intentional proof pleasing (or displeasing). We can say that the *activity* of our mind is responsible for the aesthetic quality of our experience. Interestingly enough, this phenomenon also occurs in the appreciation of narrative (which should not be a surprise) and music [44], since we enjoy being actively engaged in understanding a story or a piece of music, and, furthermore, some of the structural properties of narratives and music can only be appreciated when our mind is actively engaged in discerning how the events in a story are related to each other [3, 44]. We shall further explore these issues in ulterior chapters.

McAllister, as all authors discussed so far, focuses on the *contemplative* way of eliciting responses, that is, on the appreciation of the properties of mathematical wholes. The properties of a proof, such as its symmetry or simplicity, are certainly capable of eliciting aesthetic responses, and certainly some instances of mathematical beauty can be accounted for in terms of contemplative properties. The narrative analogy shifts attention to the *performative* aspect, which is almost completely neglected in the literature.[6] In the performative way of appreciating mathematical beauty, the source of pleasure is the activity of the mind itself. We take pleasure in the experience of re-constructing a proof, not merely in the mental photograph, so to speak, of what has been laid out for us. This further clarifies my conclusion on the ugliness of computer-assisted proofs: they deprive us of the pleasure we would take in the activity of working through a proof, for at a crucial point our mind is not allowed to do anything but accept results from an external source. This is certainly disappointing from the point of view of our inner experience. And since this disappointment is a negative affective response, our experience is expressed as a negative judgement. It should be even clearer now that such judgement is independent of the soundness of the proof, as that soundness is external to our experience. It is also clear now that judgements of ugliness can have sources (in this case, our inner experience) beyond the *properties* involved in judgements of beauty, although closely related to them. This is consistent with the use of aesthetic terms

[6] Although Hardy [33] seems to have something similar in mind when he speaks of unexpectedness, he does not elaborate it.

by mathematicians and with the intuitive idea that the reason we have to qualify something as beautiful are connected with the reason we have to qualify something as ugly.

3.3.3 Why the Conflict?

We are now in position to point out the source of conflict between McAllister's approach and the one formulated in this chapter: McAllister's conjecture that computer-assisted proofs might come to be regarded as beautiful, by the mechanism of the aesthetic induction, deals only with the contemplative way of appreciating mathematical beauty. The conclusion I reached by using the narrative analogy, by contrast, has to do with the performative aspect of the experience; that is, with the pleasure elicited by our mental activity. The joy of thinking drives much of the work of mathematicians (this fact is neglected when we focus only on properties of mathematical entities observed outside ourselves); they have every right to be frustrated at proofs that deprive them of that joy. The phenomena utilized by McAllister and by my narrative analogy, although related, have different characteristics, summarized in the contemplative/performative distinction, and the emphasis on those differences results in contradicting conjectures about the future of the aesthetic merit of computer-assisted proofs.

To summarize our discussion, in this chapter I concluded that mathematicians dislike computer-assisted proofs not because they believe they are methodologically or epistemologically unsound, but because those proofs are crippled intentional objects; their storytelling is deformed by their narrative gaps. The assistance of the computer deprives mathematicians of the *experience* of conducting their job in following a proof, which is frustrating. Frustration is a negative affective response that can naturally lead to an aesthetic judgement: "the proof is ugly!".

The approach presented in this chapter also brings out the fact that approaches as different as Rota's and McAllister's share the feature of grounding their interpretations of mathematical beauty in the *properties* of abstract mathematical entities. In the approach formulated above, our *active engagement* in following a proof plays the central role. Mathematical ugliness, which has not been satisfactorily accounted for by Shatesbury's, Hutchenson's or Rota's approaches and which played a peripheral role in McAllister's account, can be addressed in a natural manner if we take into account phenomenological factors. The analogy presented above accounts for mathematical ugliness in the same way as it accounts for beauty, for the activity of following a mathematical proof can be pleasing, but also displeasing. The bad news is that by taking phenomenological factors into account, things become more complicated, as illustrated by the fact that the approach above seriously undermines the plausibility of McAllister's conjecture on the future of our appreciation of computer-assisted proofs.

To conclude, in this chapter, I have tried to highlight the role of the inner experience of mathematicians in their evaluations of mathematical entities. This factor, as is evident in Chaps. 1 and 2, is largely neglected in the literature, and can even be perceived as a threat to rationality. However, in practice, much of what motivates mathematicians to tackle a problem, or to find alternative proofs for theorems already proven, or even to approach mathematics in the first place, has to do with their inner experience: with things like the feeling of curiosity, the pleasure of solving a problem, or the joy of gaining understanding. The very existence of mathematics is closely connected with our subjective life. Although the introduction of our subjective life into our discussion is illuminating, we are still far from having a satisfactory and coherent account of mathematical beauty. I am convinced that the insights provided by the many authors discussed so far can be reconciled. But before attempting that reconciliation, we must return to McAllister, since a critical discussion of his position is still pending and, more importantly, that discussion shall reveal hints on how to coherently reconcile the diverse approaches.

Chapter 4
Problems of the Aesthetic Induction

In Chap. 2 we surveyed McAllister's study of beauty in science. That study is the most sophisticated available on the topic, and its claim that evaluations play a role in the development of science is supported by compelling evidence. McAllister's work is insightful not only in the sense that it accounts for phenomena like scientific revolutions and the role of aesthetic evaluations in science, but also in that it shows the advantages of having an explicit aesthetic theory to make sense of aesthetic phenomena in an methodologically sound way. In this respect, McAllister's idea of the aesthetic induction is particularly interesting, as it makes possible an account of aesthetic evaluations compatible with the rationalist image of science. However, I believe that there is room for improving McAllister's ideas and that further insight can be gained by addressing the weakest points in those ideas. In this chapter, I shall identify some problems with McAllister's approach. In the next chapter, I shall address those problems by introducing a more accurate model of aesthetic evaluations in science.

4.1 Two Kinds of Problems

Since McAllister's work is the first attempt to formulate an articulated model of the role of beauty in science and it has very ambitious goals, it is no surprise that some problems can be identified in it. Here, we shall concentrate on issues that directly concern our goal—the formulation of a consistent theory of mathematical beauty. In this respect, I have identified two types of problems with McAllister's approach: explanatory anomalies and theoretical tensions.

4.1.1 Anomalies

McAllister's central idea, the aesthetic induction, has a significant relevance to our discussion not only because it allows us to capture the dynamic character of beauty,

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_4,

but also because it allows us to explain episodes of theory choice and even make predictions such as McAllister conjecture on computer-assisted proofs. As we saw in the previous chapter, predictions like the fate of the aesthetic merit of computer-assisted proofs can be contested if we extend our approach to consider the role of our inner experience. But this is not the problem I would like to address here—after all, only time can tell whether or not a conjecture is correct; the problem I have in mind has to do with the explanations of historical episodes offered by the aesthetic induction. The problem is that there exist a whole class of patterns of evolution of aesthetic preferences that cannot be explained by the aesthetic induction; that is, the aesthetic induction has significant explanatory anomalies.

The aesthetic induction cannot account for the patterns of evolution of what I call *historical constants* (especially, of *negative* historical constants). I elaborate: in the aesthetic induction, the track record of experiences with certain property determines the intensity of the preference for that property, at least in principle. The aesthetic induction, as presented by McAllister, is equally valid for all properties. It does not differentiate between, for example, the property of simplicity and its opposite, complexity. But in actuality, as we shall see below, the aesthetic induction seems to affect different properties in different ways. Properties of theories such as being an abstract theory (in the sense of relying on abstract mathematical models), being a visualizing theory (in the sense of not relying on abstract mathematical models, but rather on offering a visualization of phenomena), or being tractable by mechanistic analogy, according to McAllister's own illustrations, seem to evolve in great accord with the aesthetic induction: they have exhibited varying degrees of preference in different historical periods, in close association with their empirical success. How these properties fare historically in terms of preference is a contingent matter. I label this type of properties *historical contingencies*.[1] In contrast, properties such as harmony, symmetry or simplicity seem to consistently exhibit high degrees of preference throughout history. McAllister himself recognizes that this feature may even mislead us into thinking that the beauty associated with those properties is an objective property—let us remember that McAllister endorses projectivism—or even that it might have some metaphysical basis [62, Chaps. 3 and 7].[2] The nature of such properties is a fascinating topic in itself, but one beyond the scope of this book. Here, our concern is that these properties exist, and that their degrees of preference remain constant over time. I label these properties *historical constants*.

[1] As a matter of fact, from today's perspective, it is difficult to see how properties such as tractability by mechanistic analogy, abstractness or being visualizing can be regarded as *aesthetic* qualities of theories. This is precisely because the appreciation of such properties is determined by contingent historical circumstances. Different historical contexts influence what properties are seen as aesthetically appealing by a scientist living in such contexts. Our contemporary context is one in which tractability by mechanistic analogy or being visualizing seem simply deprived of any aesthetic appeal. This fact supports my labelling them *contingencies*.

[2] McAllister claims that if there is any relation between beauty and truth such a relation must be established empirically.

McAllister's work shows that the aesthetic induction can account for the evolution of historical contingencies such as the property of being visualizing: this property increased its degree of preference as theories that relied on visualizing phenomena accumulated a track record of empirical success. However, it is more difficult to account for the pattern of evolution of historical constants. Consider, for example, the properties of simplicity and complexity. Already in Ancient Greece, simple theories were preferred over complicated ones. A similar situation can be found throughout history and among contemporary scientists, and this is perhaps even more evident in mathematics: *The Elements* of Euclid, which set the standards of rigour and logical structure that characterize mathematics, shows great commitment to proving theorems in the simplest possible way. This commitment to simplicity is appreciated even today, as it is testified by the fact that Euclid's proof of the infinity of primes is the very first item in *Proofs From The Book* [2, p. 3], a contemporary compilation of the most beautiful mathematical proofs.[3] It is not only in particular proofs, but also in Euclid's general style where we find a preference for simplicity:

> To prove a good theorem with the weakest possible tools is rather like landing a large trout on an old and beloved silk line. It does not make for speed or brevity, but it has an undeniable charm. Euclid is not always given to swiftness, but he is rather devoted to the task of getting as much as he can with as little as he can get away with [1, p. 57].

In the second century, astronomers also showed an explicit preference for simplicity:

> The classical and Alexandrian astronomers not only constructed theories but fully realized that these theories were not the true design but just descriptions that fit the observations. Ptolemy says in the Almagest that in astronomy one ought to seek as simple as possible a mathematical model [45, p. 159].

Newton saw simplicity as a prominent precept in the investigation of nature: it appears as the first of the "rules of reasoning in philosophy" in his *Principia* [71, p. 3]. Contemporary scientists still value simplicity. Stephen Weinberg makes this evaluation: "Einstein's general theory of relativity [...and...] Newton's theory of gravity [...] are equally beautiful" [23, p. 107]. He argues that the simplicity of Einstein's general relativity makes it as beautiful as Newton's gravity, even if they exhibit different kinds of simplicity [23, pp. 107–108]. Philosophers of science are also aware of the importance of simplicity, as Donald Hillman remarks: "Principles of simplicity have been abundant, from Occam's Razor in fourteenth century philosophy all the way down to various twentieth-century attempts to interpret simplicity in its scientific connection" [34, p. 226].

Simplicity has enjoyed a high degree of preference throughout history. McAllister does not see this as problem with the aesthetic induction. After all, the evolution of the preference for simplicity does not directly contradict the

[3]The compilation was inspired by the mathematician Paul Erdos, who used to say that God had a book that contained all the most beautiful mathematical proofs. Erdos used to exclaim "This is a proof from the Book" whenever he found a proof he considered extraordinarily beautiful.

mechanism of the aesthetic induction, since simple theories do have a track record of empirical success to explain their high degree of preference. What is peculiar about simplicity is that, although preferences change constantly over time, the preference for simplicity seems to remain unchanged, even across scientific revolutions. The aesthetic induction may be consistent with the evolution of the preference for simplicity, but it cannot account for the fact that such preference was already present at the very beginnings of the study of nature, nor that the preference never seems to lose momentum, despite the ever changing historical contexts. McAllister himself seems to recognize that there is something anomalous in properties like simplicity, since he devotes an entire chapter [62, Chap. 7] to discuss simplicity: he concludes that simplicity plays a complex role that involves empirical and non-empirical criteria for theory choice. We shall not address the nature of simplicity here, but it is worth mentioning that simplicity certainly plays a diversity of roles in scientific practice. The simplicity of a theory, of an explanation or of a mathematical formalism has epistemological, pragmatic and methodological advantages. For example, Karl Popper relates the degree of simplicity of a theory to its degree of falsifiability, arguing that simple statements are highly prized "because they tell us more; because their empirical content is greater; and because they are better testable" [75, p. 128]. Pragmatically and methodologically, a simple mathematical formulation, for instance, enables quicker and more accurate calculations, as well as further formal development. In addition, some authors interpret simplicity as an indicator of empirical adequacy and, thus, as a valid empirical criterion for theory choice [34, pp. 225–226]. The intricate nature of simplicity might somehow explain why the aesthetic induction seems to play a marginal role in its evolution. But if we focus on the instances of simplicity that have an aesthetic character, its evolution still poses questions for the aesthetic induction.

Now, the pattern of evolution of the preference for simplicity may pose questions for the aesthetic induction, but at least it is consistent with it. Much more problematic are properties that exhibit patterns of evolution inconsistent with the aesthetic induction, as we shall see now. First of all, recall that in McAllister's model, the aesthetic canon includes all possible aesthetic properties [62, pp. 78–79]. This means that the aesthetic canon involves aesthetic properties that evoke positive, negative or neutral (indifferent) responses. Thus, we can classify our aesthetic properties not only as historically contingent or constant, but also, as positive, negative or neutral. Since the aesthetic induction does not differentiate properties, let us consider simplicity's opposite: *complexity*. The history of the preference for complexity is, of course, the mirror image of the history of simplicity: the unappealing character of complexity remains unchanged throughout history; even across scientific revolutions. Now, a significant fact about the preferences for simplicity and complexity is that they are often overlooked to achieve empirical and epistemic success. This means that in the history of science there are not only simple theories with a track record of success, but also complicated theories with a track record of success. This would not be a problem if complexity were a historical contingency, but it is a *constant*. Complexity in mathematics provides us with clear examples of this. We know that Greek mathematicians had a strong predilection for simplicity. However, in order to further advance the discipline, they had to sacrifice their aesthetic prejudices. Morris Kline comments:

> By insisting on a unity, a completeness, and a simplicity for their geometry, and by separating speculative thought from utility, classical Greek geometry became a limited accomplishment. It narrowed people's vision and closed their minds to new thoughts and methods. It carried within itself the seeds of its own death. The narrowness of its field of action, the exclusiveness of its point of view, and the aesthetic demands on it might have arrested its development, had not the influences of the Alexandrian civilization broadened the outlook of Greek mathematicians [45, p. 175].

In general, mathematicians are well aware that complicated theories or methods are necessary to achieve epistemic success. However, the aesthetic merit of complicated mathematics has not increased over time. The different methods of proof illustrate this. There are several methods of proof that mathematicians utilize on a regular basis. There are simple, beautiful methods such as *reductio ad absurdum*, which we discussed earlier. But there are also complicated, brute force methods of proof. Complicated methods, despite their undeniable epistemic soundness, were not found appealing by Greek mathematicians, and that is still the case today. An example of this are proofs by cases.[4] G.H. Hardy comments on the complexity of proofs by cases:

> We do not want many 'variations' in the proof of a mathematical theorem: 'enumeration of cases', indeed, is one of the duller forms of mathematical argument. A mathematical proof should resemble a simple and clear-cut constellation, not a scattered cluster in the Milky Way [33, p. 113].

This type of proof is even the target of harsh words. A page later, Hardy comments on the soundness of proofs by cases:

> [...] All this is quite genuine mathematics, and has its merits; but it is just that 'proof by enumeration of cases' [...] which a real mathematician tends to despise [33, p. 114].

It must be emphasized that the different methods of proof are equally sound: a result proved by *reductio ad absurdum* is just as true as one proved by cases. The point is that mathematicians abhor proofs by cases despite the fact that *they are a sound method of proof*. Furthermore, throughout history, prominent mathematicians achieved many relevant results by proofs by cases or by methods involving proving special cases: proofs by cases appear already in *The Elements* of Euclid (about 300 B.C.). Cardano (1501–1576), who presented for the first time in history a method for solving cubic equations, justified his method by treating separately the many cases of cubic and quadratic auxiliary equations involved in the solution [45, pp. 265–266]. Leibniz (1646–1716) conceived the solution to the problem of orthogonal trajectories by tackling special cases of it. Jacob (1654–1705) and Johann Bernoulli (1667–1748) devised solutions to special cases of isoperimetric problems, which

[4] In a proof by cases one divides the statement to be proven into a finite number of mutually exclusive cases, and then shows and documents independently that in each case the statement holds.

eventually led to a general solution [47, pp. 575–577]. For many years Fermat's last theorem, $x^n + y^n = z^n$, was approached by attempting to prove it for special cases of n; by Euler (1707–1783), Lagrange (1736–1813), Legendre (1752–1833), Gauss (1777–1855), and Dirichlet (1805–1859) among others [46, 47]. More recently proofs by cases have become conspicuous by achieving spectacular results and arising heated controversies. As we have seen, Appel and Haken's 1976 computer-assisted proof of the four-color theorem is a proof by cases, with almost 2000 cases, which posed all kinds of questions about its validity. Unflattering adjectives are still regularly applied to this proof:

> [...] this particular "proof" is almost always what mathematicians think of when asked "What is an example of ugly mathematics?" [70]

Proofs by cases have a very long history of success. But mathematicians' preference for them does not seem to increase. This contradicts the aesthetic induction. Furthermore, Gian-Carlo Rota, as we have seen, suggests that ugly proofs play a significant role in the development of mathematics as incentives to look for alternative proofs [80, pp. 9–10].

Complicated yet successful theories can be found not only in mathematics. In physics, the Standard Model, one of the greatest achievements of science, is not necessarily regarded as a paradigm of beauty despite its great success:

> At present, there has been no experimental deviation from the Standard Model. Thus, it is perhaps the most successful theory ever proposed in the history of science. However, most physicists find the Standard Model unappealing because it is exceptionally ugly and asymmetrical. [...] The reason why the Standard Model is so ugly is that it is obtained by gluing, by brute force, the current theories of the electromagnetic force, the weak force, and the strong force into one theory [37, p. 75].

An aversion to complicated theories and methods not only seems to be a constant fact throughout history, but it also seems that scientists are aware of this fact: the complexity of the Standard Model is not the source of an increased appreciation, but rather, in a fashion analogous to Rota's suggestion, an incentive to search for simpler alternatives, as it is evident in the struggle for achieving simplicity in Great Unification Theories. Mathematicians seem even more aware that aversion to complicated mathematics and predilection for simple theories and methods are timeless classics, so to speak. This is why mathematical ugliness motivates further research.

There is enough evidence that complexity is a historical constant, a negative one. We have seen that the pattern of evolution of positive historical constants poses questions. But that pattern is at least somehow consistent with the aesthetic induction. The evolution of negative historical constants, by contrast, contradicts it directly: complicated mathematical methods have a long track record of success, but their degree of preference has not increased. A similar argument can be substantiated for properties that are the opposites of positive historical constants like symmetry, harmony or unity. The aesthetic induction has no means to deal with negative historical constants satisfactorily, as it treats all properties equally. Furthermore, another consequence of that egalitarian stance is that the aesthetic induction allows predictions such as McAllister's conjecture on computer-assisted proofs, which seems rather implausible, once again, given the historical evidence compiled above.

Computer-assisted proofs are regarded as ugly proofs. As we saw in Chap. 2, in McAllister's view that is merely a historical contingency: the aesthetic merit of computer-assisted proofs might improve as they become acceptable. But our discussion in Chap. 3 and the foregoing one do not support that conjecture. Complicated methods of proof have been accepted by mathematicians ever since Antiquity. This acceptability, however, did not result in an increase in the preference for those methods. Such is the case of proofs by cases. Computer-assisted proofs are instances of proof by cases. Thus, in addition to our analysis in Chap. 3, we have now further reasons to doubt McAllister's conjecture: the history of the method of proof by cases seems to show that the aesthetic induction will do little to improve the aesthetic merit of their contemporary incarnations in computer-assisted proofs.

Now, we have seen that different properties of theories exhibit different patterns of evolution. We focused on historical constants; properties whose degrees of preference seem to remain constant throughout history. Those properties, especially negative historical constants, are problematic for the aesthetic induction. The aesthetic induction predicts an increase in preference for any property associated with theories or methods that enjoy a track record of success. Now, if a property is a historical constant, it very probably enjoys some degree of success; otherwise it would not remain constant in the ever changing landscape of science. The aesthetic induction should increase their aesthetic merit. However, the key characteristic of negative historical constants is that they lack aesthetic merit and they remain that way. The pattern of evolution of properties such as complexity contradicts the aesthetic induction. In general, negative historical constants constitute the clearest type of anomalies in the aesthetic induction, since they exhibit the following characteristics: (1) A long history of presence in science. (2) Their historical track record, due to the ever changing nature of science, includes necessarily some degree of success. (3) Contrary to what the aesthetic induction predicts, their degrees of preference remain small; otherwise they would not be *constants*.

In the next chapter, I will propose a way to dispose of the anomalies in the aesthetic induction. But right now, let us discuss the second type of problems with McAlister's approach: its theoretical tensions.

4.2 Theoretical Tensions

In addition to the explanatory anomalies, I have identified issues of a more theoretical nature in McAllister's approach. Most of them have to do with McAllister's theoretical assumptions. In this section, I shall discuss four issues with the aesthetic induction: first, the aesthetic induction is not a genuine case of induction. Second, it does not distinguishes between the problem of beauty and the problem of the aesthetic. Third, it incurs in an inconsistency regarding objectivism and projectivism. And fourth, McAllister's aesthetic theory plays no role in accounting for the evolution of aesthetic preferences.

4.2.1 Induction

McAllister sees the aesthetic induction as a special case of inductive projection. However, a simple analysis reveals that is not the case. Induction is a type of inference in which the features of an unobserved instance are predicted based on the features of a finite set of observed instances. More formally, induction is an inference in which we conclude a general or universal proposition from a set of finite instances of it. The general form of induction is:

Given that
$a_1, a_2, a_3, \ldots, a_n$ are all P that are also Q,
we conclude that
All P are Q.

There is a variety of induction in which from a finite number of instances we predict the next instance. This variety is called *inductive projection*. Its form is as follows:

1. $a_1, a_2, a_3, \ldots, a_n$ are all P that are also Q,
2. a_{n+1} is P,
   We conclude that
3. a_{n+1} is also Q.

Now, if the aesthetic induction were a special case of projective induction, it would have the following form, which for convenience I label *Idealized Aesthetic Induction* (IAI):

IAI:

1. $a_1, a_2, a_3, \ldots, a_n$ are all A that are also E,
2. a_{n+1} is A,
3. a_{n+1} is also E.

where A is an aesthetic property of scientific theories and E is the property of being empirically adequate.

Now, IAI is adequate to model the *reason* why a scientist chooses a theory based on its aesthetic properties. However, this inference does not model McAllister's conception of the aesthetic induction: "a community attaches to each property of theories a degree of aesthetic value proportional to the degree of empirical success of the theories that have exhibited that property" [62, p. 4, see also p. 78]. In other words, the aesthetic induction is the mechanism that determines the degree of preference W_A associated with the property A. This is very different from what is expressed by IAI. McAllister seems to use the term 'aesthetic induction' to refer to both the *mechanism* that determines the weightings W_A, and the *inference* scientists use to justify their theory choices. Consider, for example, the case in which a scientist chooses a theory S over a competing theory T based not on empirical grounds but on the fact that S is symmetric. Symmetry is preferred over other properties because it possesses a higher degree of preference. In McAllister's model, this degree of preference is the result of the fact that symmetric theories had been empirically adequate in the past. This process somehow resembles inductive projection. However, the act of choosing theory S is not an inductive procedure. Rather, it is merely the result of using the scientist's aesthetic criteria, which is a simple deductive process of comparing degrees of preference W_P and selecting the highest weighting.

IAI expresses something completely different from the foregoing. It expresses that since symmetric theories have been empirically adequate in the past, we can project that a new symmetric theory might also be empirically adequate. The function of IAI is to *justify* that scientists act rationally when they base their theory choices on aesthetic criteria. IAI makes no reference to degrees of preference W_P or to how to determine such degrees. What McAllister calls the aesthetic induction corresponds to a stage prior to the justification of the theory choice. In such prior stage, the degree of preference for certain property is determined by the track record of empirical success of the theories that exhibited such property.

It is clear now that the aesthetic induction as formulated by McAllister is not a special case of induction, but rather a mechanism with at least three discernible stages: a first stage that determines the degrees of preference; a second stage in which those degrees are employed to choose a theory; and a final stage in which inductive projection is used to rationally justify that choice. To clearly see the differences between IAI and McAllister's actual ideas, I present now a more accurate rendering of McAllister's model. For convenience, I label this rendering *Actual Aesthetic Induction* (AAI):

AAI:

(AAI.1) An aesthetic canon is compiled by following this procedure: for every property P there is an associated weighting W_P such that:

$$W_P = CD$$

where:

W_P: is the weighting associated to property P.
D: is the degree of empirical adequacy as estimated by the history of success of P-bearing theories.
C: is a constant of proportionality between the degrees of empirical adequacy and the weightings W_P.

(AAI.2) Given two equally empirically adequate competing theories T and S which exhibit the aesthetic properties A and B respectively, a scientist will choose T over S only if $W_A > W_B$.

(AAI.3) The scientist makes that choice because he believes that IAI is correct; that is, he believes that: (AAI.3.1) $a_1, a_2, a_3, \ldots, a_n$, are all A that are also E; (AAI.3.2) $a_{n+1}$is A, and (AAI.3.3) a_{n+1} is also E.

Where: a_i is a theory, A is an aesthetic property of scientific theories and E is the property of being empirically adequate.

Given the problems with the idea of induction, in the next chapter I shall abandon the idea that the aesthetic induction is a special case of induction, and propose an alternative view.

4.2.2 Beauty and Aesthetic

A central assumption in the idea of the aesthetic induction is that aesthetic evaluations in science are genuinely aesthetic. In McAllister's view, this assumption enables us to distinguish between empirical and aesthetic evaluations and to establish a non-reductive relation between them. As we have seen, McAllister attempts to substantiate the empirical/aesthetic distinction by providing an aesthetic theory that characterizes aesthetic properties. McAllister, however, fails to distinguish between the problem of characterizing *the aesthetic* and the problem of elucidating the notion of beauty. This is evident when he addresses the issue of aesthetic properties. He addresses aesthetic properties in two separate occasions: the first time, following Hutchenson, he defines aesthetic properties as properties that move the observer to project beauty into an object [62, pp. 32–33]. Thus, aesthetic properties are defined in terms of beauty. McAllister addresses aesthetic properties in a second occasion, while discussing the aesthetic canon. This time, he suggests two criteria for identifying aesthetic properties: the first criterion is that a property is aesthetic if it appears in a public aesthetic expression uttered by a scientist. The second criterion is that "if in virtue of possessing that property, a scientific theory is liable to strike beholders as having a high degree of aptness" [62, p. 37]. In this occasion, McAllister seems to be concerned with the relation between aesthetic properties and aesthetic responses, and with what makes an aesthetic property aesthetic. Although these ways of addressing aesthetic properties do not contradict each other, the way McAllister uses them shows that he does not distinguish between the problem of beauty and the problem of identifying the mark of the aesthetic, or of identifying what makes an aesthetic property *aesthetic*. These problems are different. Understanding the nature of beauty is one of the central problems of aesthetics, but the problem of finding the mark of *the aesthetic* is much broader and relatively independent from the problem of beauty. The problem of the aesthetic has to do with discerning what things such as aesthetic judgements, aesthetic concepts, aesthetic value, and so forth, have in common; what is that make them all *aesthetic*. The problem of the nature of beauty can be addressed by offering suitable definitions such as Shaftesbury's, Hutchenson's or even Rota's definitions in terms of properties like order, unity, or enlightenment. However, that tactic is useless to explain, for example, what makes predicates such as ‘beautiful’ or ‘elegant’ *aesthetic* predicates. Addressing the problem of the aesthetic needs a completely different strategy. For example, Nick Zangwill [100] starts by defining the notion of aesthetic judgement and then defines the remaining notions in terms of it: aesthetic properties are properties attributed by aesthetic judgements; aesthetic concepts are concepts used in aesthetic judgements; an aesthetic experience is what motivates the passing of an aesthetic judgement; and so forth. McAllister seems to use a mixture of strategies. He defines aesthetic properties in terms of beauty. But he also defines them in terms of public aesthetic expressions. Now, the issue of characterizing aesthetic properties is a contentious question in aesthetics; we should not expect a definitive answer in this context. But, for that very reason, a more consistent treatment of the problem is desirable.

4.2.3 Objectivism/Projectivism Inconsistency

McAllister’s aesthetic theory exhibits some inconsistencies. Recall that the theory rejects objectivism and endorses projectivism. However, in his criterion for identifying aesthetic properties McAllister resorts to an objectivist tactic, since the criterion relies on the property of aptness. This tactic is similar to Shaftesbury’s, Hutchenson’s and Rota’s tactics of explaining beauty in terms of a non-aesthetic property. McAllister endorses projectivism as a way to avoid the metaphysical complications of objectivism. However, he seems tempted to offer objectivist explanations when it seems suitable, as in the case of aptness. Now, projectivism is not the only available way to circumvent metaphysical complications. In the next chapter we shall see that Theo Kuipers, for example, opts for a naturalistic interpretation of McAllister’s ideas; an approach that later on I explore and further develop.

4.2.4 Theory and Modelling

The aesthetic induction intends to connect aesthetic and empirical evaluations. But that connection is a little weak as formulated by McAllister. The details of how the aesthetic induction operates are obtained solely by using historical evidence. The aesthetic principles endorsed by McAllister play no role in shaping the aesthetic induction as a model of the mechanism that drives the evolution of aesthetic preferences. This is evident if we examine McAllister’s tenet that the aesthetic terms used by scientists to evaluate theories refer to genuine aesthetic properties. The purpose of this tenet is to help us to distinguish aesthetic from empirical evaluations. Now, the aesthetic induction models the mechanism of evolution of aesthetic preferences. However, modelling such evolution does not require a literal interpretation of aesthetic terms. The model of the aesthetic induction itself, as formulated by McAllister, does not involve things like affective responses, aesthetic pleasure or any of the characteristics usually attributed to aesthetic phenomena. The model depends only on the historical evidence documented by McAllister. In this sense, a perfectly good model of the evolution of preferences can be obtained by attending to McAllister's evidence without resorting to his empirical/aesthetic distinction, since the only thing we need is a much weaker empirical/non-empirical distinction. Thus, McAllister's aesthetic theory is not really necessary for his aesthetic induction. To emphasize his commitment to a literal interpretation of beauty in science, McAllister even attempts to show the existence of the aesthetic induction in the arts, when he draws our attention to the case of cast-iron, steel and concrete structures in architecture. But even if the existence of aesthetic induction in the arts supports McAllister's ideas, this does not give his aesthetic theory a role in the aesthetic induction. A closer relationship between aesthetic theory and preference evolution modelling is desirable if a non-reductivist and *genuinely aesthetic* account of beauty in science and mathematics is to be achieved.

Addressing this and the other issues discussed above shall illuminate some relevant aspects to formulate an articulated aesthetics of mathematics. We shall begin this task in the next chapter.

Chapter 5
Naturalizing the Aesthetic Induction

In this chapter I propose solutions to the anomalies and the theoretical tensions of McAllister's approach discussed in the previous chapter. We have seen that McAllister's approach also conflicts with the analysis presented in Chap. 3 and [67]. The reason why I intend to salvage McAllister's approach despite its problems is that it provides very valuable insights. In particular, the idea that beauty, or rather our set of aesthetic preferences, is not a static notion, but one that changes and evolves over time is very compelling. The historical evidence and our own experience support that idea. In addition, authors like Theo Kuipers [49] argue that empirical evidence support the idea of the aesthetic induction. The idea of using empirical insights, as suggested by Kuipers, is the first hint I shall use to propose solutions to McAllister's problems.

5.1 The Mere-Exposure Effect and the Nature of the Aesthetic Induction

Impressed by McAllister's findings, Theo Kuipers endorses McAllister's ideas [49] and conjectures an empirical explanation of the aesthetic induction. In the aesthetic induction, recurrent occurrences of a property in empirically adequate theories tend to increase the scientists' preference for that property. Kuipers [49] points out that this feature is similar to what in experimental psychology is known as the *mere-exposure effect*. Kuipers uses this finding to substantiate and further utilize the idea of the aesthetic induction. He proposes that a hypothetical variant of the *mere-exposure effect* (MEE hereafter) can account for the aesthetic induction. According to Kuipers, the MEE is "the fact that an increasing number of presentations of the same item tends to increase the aesthetic or, at least, affective appreciation of that item" [49, p. 297]. The MEE has "first a phase of monotone increasing aesthetic appreciation with the number of confrontations", followed by a second phase of decreasing appreciation [49, p. 297]. This is the so-called inverted

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_5,

U-shape feature of the MEE. The switching point of this inverted U-shape can be prompted or retarded, depending on experimental conditions. Kuipers points out two experimental conditions that have not yet been studied: successive variation of the same stimulus; and introducing some kind of reinforcement. He conjectures that they might retard the inverted U-shape switching point, he labels the MEE under these conditions *qualified mere-exposure effect*, and claims that it can explain the existence of the aesthetic induction:

> McAllister's notion of 'aesthetic induction' can be seen as a reinforcement variant of the mere-exposure effect. More specifically, McAllister claims that aesthetic induction is triggered by empirical success, i.e., in psychological terms, empirical success functions as a kind of reinforcement. If the number of empirically successful theories with a certain nonempirical feature increases the aesthetic appreciation of that feature increases. Similarly, if increasingly many empirically successful revisions of a theory have a constant nonempirical feature, that feature becomes aesthetically more and more appreciated. This phenomenon naturally leads to McAllister's idea of an 'aesthetic canon' of received aesthetic features in a certain phase of a discipline that may be replaced by a different one after a scientific revolution [49, p. 299].

Kuipers explicate the role of beauty as a signpost of truth by utilizing his theory of truth approximation [48] and an extensive formal analysis of the scenarios studied by McAllister. In his formal analysis, Kuipers interprets the aesthetic induction as consisting in the co-occurrence of two inductive mechanisms he labels *proper affective induction* and *cognitive meta-induction*. Affective induction is an inference-like process driven by his qualified mere-exposure effect; that is, the psychological phenomenon of enhancing affective responses under the circumstances of a recurrent appearance of properties in empirically successful theories. Cognitive meta-induction is an inference similar to inductive projection—although with an extra psychological component—and it is closer to a traditional cognitive mechanism of induction [49, pp. 300–302].

Now, although Kuipers' work makes the aesthetic induction an even more appealing idea to understand the role of beauty in science, it inherits McAllister's problems. Here we shall focus on his most relevant insight to our purposes: the possibility of a naturalistic explanation of the aesthetic induction. Like McAllister, Kuipers believes that the aesthetic induction is a special case of induction. Paul Thagard, however, has pointed out that Kuipers' idea of aesthetic induction, like McAllister's, does not really have the logical form of induction [87]. In addition, as I have shown in the last chapter, there are properties whose patterns of evolution cannot be explained by an inductive mechanism. Since we have no reason to insist in seeing the aesthetic induction as a special case of inductive projection, my first step in addressing its problems is to abandon the very idea of induction.

5.1.1 Naturalizing Preference Evolution

I propose a wider view of the mechanism that governs the evolution of aesthetic preferences: I interpret the aesthetic induction as a *natural phenomenon* that should be modelled by taking into account their historical, empirical and formal features. To naturalize our model, an adequate theoretical foundation is necessary. Thus, we must address the faint connection between McAllister's aesthetic theory and the model of the aesthetic induction. In this respect, it should be pointed out that McAllister resorts to traditional aesthetics, as it is evident in his endorsement of the objectivist/projectivist dilemma, which brings along its large stock of problems. To circumvent those problems, and to better support the idea that aesthetic phenomena in science are genuinely aesthetic, I shall employ a naturalistic aesthetic theory.

Endorsing naturalism in interpreting the aesthetic induction not only shall serve to address the problems of the aesthetic induction, but also sets in place the position endorsed in this book; recall that one of the goals of this book is to provide the first proper theory of aesthetic phenomena in mathematics. It must be emphasized that the evolution of aesthetic preferences is only one of several elements involved in the appreciation of mathematical beauty. Thus, a complete aesthetic theory can be proposed only after we have discussed those factors. Nonetheless, we can formulate a rough picture of how such a theory shall look like in order to embed our model of evolution of aesthetic preferences in a consistent framework. This rough picture of a naturalistic aesthetic theory comprises the following assumptions:

1. Aesthetic phenomena are elements in a particular type of process of interaction between an individual and his natural and social environment.
2. This interaction is grounded on characteristic *affective* episodes, which constitute the core of what is commonly known as aesthetic experience.
3. The predicate *aesthetic* that appears in notions like aesthetic judgement, aesthetic concept, aesthetic value, and so on, must be interpreted as indicating that the things that it qualifies—judgements, concepts, values, etc.—play an indispensable role in the development of the process of aesthetic interaction.
4. Terms such as 'beautiful', 'elegant', 'ugly', and so forth, which appear in aesthetic evaluations, are *aesthetic terms*. Aesthetic terms play the role of elucidating, articulating and expressing the affective state of an individual engaged in appreciating an object.
5. Aesthetic terms that qualify scientific theories or entities must be taken at face value; scientists utilize aesthetic terms in the same manner as any other person.
6. Finally, since aesthetic episodes are natural phenomena, the evolution of aesthetic preferences is influenced by three factors: first, by the history of the development of such phenomena. Second, by the interaction between the subject and his community. And third, by the inherent natural factors involved in the affective phenomena that ground aesthetic episodes. The description of the evolution of aesthetic preferences is not a theoretical matter, but an empirical one. Thus, in such description, historical and scientific evidence must be taken into account.

Refining and developing this rough theory shall be carried out in the following chapters, but the details presented above should suffice to address the problems of McAllister's approach. The theory avoids the theoretical tensions discussed in the last chapter: the characteristic mark of the aesthetic—being an element in a specific type of transaction between individuals and their environment—is clearly distinguished from the use of the term 'beautiful'. The objectivism/projectivism dilemma is not relevant to the theory, since a process of aesthetic interaction involves objective as well as subjective phenomena (affective responses, for example). Finally, this naturalistic theory not only provides room for empirical input, but it relies on it. A model of evolution of aesthetic preferences based on the available evidence fits thus naturally into it.

5.1.2 Naturalizing the Aesthetic Induction

In the context of our naturalistic aesthetic theory, the aesthetic induction must be interpreted as a form of interaction between the subject and his environment. More specifically, as a *long-term* interaction between the individual and his *social* environment. The interaction is thus highly history-dependent. Now, the aesthetic induction drives the evolution of aesthetic preferences. Since we interpret the aesthetic induction as a natural phenomenon, the patterns of evolution it induces are influenced by social as well as natural factors. McAllister's aesthetic induction models a great deal of the social factors with the idea of aesthetic canon, but the natural factors do not figure in the model. Kuipers' approach integrates some of the natural factors, however, he insists on seeing the aesthetic induction as a special case of induction. Abandoning that idea, as I have done, allows us to integrate more of the natural factors. A wider range of empirical findings are now available to us.

5.1.3 Preference, Affection and Emotion

To model the evolution of aesthetic preferences in science we already have the historical evidence provided by McAllister's work. But our naturalistic approach must also consider the natural factors that influence those preferences. In the last decades, much progress has been done in the empirical study of preferences. Preferences as basic as our predilection for sweetness, or our aversion to bitterness have been studied in detail. Several factors influence the formation of preferences, including inherent biological factors, which render a depiction of aesthetic preferences very different from McAllister's. In this respect, Robert Zajonc comments:

> Preferences are formed by diverse processes. Some objects, by their inherent properties, induce automatic attraction or aversion. Sucrose is attractive virtually at birth, whereas bitter substances –quinine, for example— are universally aversive. Preferences may also be established by classical or operant conditioning. If a child is rewarded when she sits in a particular corner of the crib, that corner will become a preferred location for her. An office worker whose colleagues notice his new tie will develop a preference for similar ties. Preferences can also be acquired by virtue of imitation, a social process that emerges in fashions. Preferences also arise from conformity pressures. In economics, preference is

> regarded as the product of rational choice –a deliberate computation that weighs the pros and cons of alternatives [98, p. 224].

The particular kind of preferences in which we are interested here are those that are accompanied by an affective response, as those are the preferences involved in aesthetic evaluations. Thus, an issue we must address is the influence of affection and emotion over the evolution of preferences. In this respect, one of the most useful findings, maintained by authors like Joseph LeDoux [55], is that emotions are innate systems of response whose function is to promptly prepare an organism for coping with its environment. The experimental basis of this idea comes from sources like the study of fear in rats. In responses of fear, an organism faces a potentially harmful stimulus. The stimulus triggers a process which includes the activation of two redundant neural pathways in the brain. The first pathway runs through a region in the brain known as the *amygdala*. The *amygdala* makes a rapid, 12 ms in a rat, but crude assessment of the situation. This assessment triggers a further series of physical, psychological and physiological responses which prepare the organism for dealing with the imminent danger. Simultaneously, a second neural pathway is activated. This second pathway runs through the cerebral cortex where a slower—twice as long as the *amygdala* response—but more refined assessment of the situation is conducted. This refined assessment enhances or inhibits the responses triggered by the *amygdala*'s rapid and crude assessment [55, p. 163–165].

There are three elements of LeDoux's findings that are particularly relevant to our purposes: first, emotions have a biological basis. Second, emotions have associated physiological responses. Third, emotions, in addition to a rapid non-cognitive component, have a cognitive one. Higher cognitive processes occur in the cortex. The cortical pathway in fear responses is thus a sort of cognitive control of the initial response [55, pp. 264–290]. The cognitive component of emotions is deeply associated with what I call the *plasticity of affection and emotion*. An instance of affective plasticity is the mere-exposure effect: affective and emotional responses are inherently determined by biology and do not depend on a history of prior exposure of triggering stimuli. Exposure of stimuli, however, does modify affective responses. This occurs even in the "absurdly simple" [98, p. 224] circumstance of mere repeated exposure of stimuli. Now, instances of emotional plasticity are even more interesting. Emotions such as fear exhibit a wide range of adaptability: it is natural to be afraid of dangerous animals, snakes, for example. It is also natural to feel fear when we see a snake-like shadow in the dark. These fear responses are biologically conditioned. But there are fear responses to much more abstract situations: fear of losing one's job, or fear of losing one's investments in the stock market, for example. Such responses illustrate the adaptability, the plasticity, of fear [77, pp. 72–74]. Those "cognitive" fears are the result of a process of gradual adaptation that involves the cortical circuits associated with the biologically determined response of fear [77, p. 73].

Now, according to McAllister and, especially, Kuipers, the history of experiences with certain property highly influences the evaluation of theories that exhibit that property. There are obvious similarities between this phenomenon and the plasticity of affective responses. Thus, it is reasonable to argue that affective plasticity is one of the factors behind the patterns of evolution of preferences modelled by the aesthetic induction.[1]

5.1.4 Constraints of Affective Plasticity

Although the aesthetic induction can model the influence that affective plasticity has over the evolution of preferences, the anomalies remain unaccounted for. Since affective plasticity is one among the many factors that influence the formation of preferences we have further features of affection to help us to refine our model. We must explore this avenue.

Our innate preference for sweetness and aversion to bitterness show that some preferences are not formed by a history of experiences with stimuli, they are biologically conditioned. Emotion and affection are also biologically conditioned; as their survival value depends on the fact that they are responses readily available to the organism, independently of its particular history of experiences. In this sense, some stimuli can be interpreted as possessing properties that elicit responses inherently, without the need of previous experiences. A way of incorporating this feature of affection into our model of evolution of preferences is to consider that historical constants might be associated with biologically determined affective responses. This interpretation allows us to account for the evolution of such properties in a more realistic manner, and without resorting to hypothesis such as the intricate nature of simplicity.

Now, affection and emotion characteristically involve physical and physiological changes in the organism. This introduces constraints that must be taken into account. If the inherent biological readiness of emotions is liable for the fact that they are independent of a history of experiences with stimuli, and the cognitive control mentioned above is liable for their plasticity, then the physical and physiological changes are liable for restricting such plasticity. In an emotional episode, the organism experiences changes in heart rate, in skin conductivity, in muscular blood flow; it also releases neurotransmitters, secrete hormones, and so forth [28, p. 155]. All these events occur in particular sequences that determine the development of

[1]In a sense, this approach is similar to Kuipers'. However, my approach intends to exploit more characteristics of affective phenomena. Much of the biologically determined characteristics of affective phenomena are absent from McAllister's and Kuipers' models. Although Kuipers includes an affective induction in his model, such induction is based on a hypothetical variant of the mere-exposure effect and, more importantly, it assumes that exposure of properties determines the evolution of preferences. The mechanism of affective induction is independent of the fact that some preferences are biologically conditioned. The term 'induction' in affective induction highlights the fact that experiences with instances of properties determine the outcome of the process. My aim is precisely to incorporate the non-inductive biologically determined characteristics of preferences, which are probably liable for the anomalies in McAllister's and Kipers' models.

the emotional episode. The secretion of hormones, for example, results in chemical changes in the organism. Those hormones and their associated chemical changes remain in the organism for periods of time that depend on physical and chemical factors. This means that their effects can be felt long after the emotional episode has ceased. This is why in episodes of intense fear or anger the organism is unable to return to a relaxed state even if the stimulus that triggered the response has vanished [28, pp. 133–135]. Physical and chemical changes can occur only within certain parameters. Thus, physiological changes in emotional episodes are constrained by factors such as the characteristics of chemical reactions or the way in which molecules are transported within the organism. This constrains the ways in which emotions develop. By the same token, the plasticity of emotions is constrained to remain within certain limits determined by the physical and physiological characteristics of emotional responses. When emotions undergo adaptations, those adaptations occur in a manner determined by the physiological parameters associated with the emotion. Furthermore, the different types of emotional response have different profiles of physiological arousal that accompany the emotion. These profiles manifest themselves even when the emotion is triggered not by a perceptual stimulus but by a cognitive input. This means that even if emotions such as fear are very plastic, their adaptation tends to remain within the range permitted by the profile of the emotion [8, 28, 55, 59]. Thus, the plasticity of emotions is inherently limited in two respects: the rate and range of adaptation. Since different emotions have different physical and physiological profiles and each organism possesses a particular biological constitution, the limits of plasticity vary depending on the type of emotion and the individual.

The constraints on plasticity can help us model the anomalies of McAllister's model of preference evolution. Recall, for example, that mathematicians' aversion to complicated methods of proof does not change throughout history despite the fact that the methods are sound, accepted, and have a long history of success. This pattern of evolution contradicts the aesthetic induction, but it is consistent with the constraints on plasticity: affective responses can adapt only within certain range and at a certain rates, depending on the physiological profile of the response. The fact that certain aversion remains unchanged should be seen as evidence that the type of affective response associated to that aversion has a limited degree of plasticity or that it has reached the limit of its plasticity. A model of the evolution of preferences that incorporates all these inherent characteristics of affection should be able to account for the anomalies in the original model. I now introduce that model.

5.2 Conceptualizing Preference Evolution

In the previous chapter, a simple analysis revealed that the aesthetic induction is not a genuine case of induction. In addition, the analysis made evident that a more formal approach is less prone to confusion. Thus, to formulate the new model of preference evolution I shall use a similar formal[2] approach and, later on, shall even utilize my rendering of McAllister's model as a template for the new model of preference evolution. Now, in order to incorporate the features of affection and emotion discussed above, we need to introduce some concepts to model those features.

5.2.1 The Aesthetic Canon as a System

McAllister's interpretation of an aesthetic canon as an exhaustive list of aesthetic criteria—pairs of properties P and weightings W_P—is very convenient, since any evolution in the canon can be represented simply as changes in the weightings. But listing aesthetic criteria is rather informal and limited. A more convenient way of dealing with a collection of criteria is by using set theoretical notions, as we can resort to all kinds of useful tools. For example, the notion of system is particularly useful when we have sets that change over time. A system is a set of elements—often called 'components'—and relations that allow us to model complex behaviours. In order to take advantage of the concepts available in systems theory, I interpret an aesthetic canon as a system of aesthetic criteria. Thus, I define an aesthetic canon, *Acanon*, as follows:

$$Acanon = \{(P, W_P(t))\}$$

where: P is an aesthetic property of theories, and $W_P(t)$ the degree of preference for P at a certain time t.

This interpretation allows us to describe how an aesthetic canon evolves simply by describing how the weightings change. Borrowing further ideas from systems theory, we can see the process of determining the weightings as a description of the *dynamics* of the system. In this respect, the notion of *evolution rule* is pertinent. The evolution rule of a system is the rule that describes what future states of the system result from its current state. For example, the first phase of my rendering of McAllister's model in the previous chapter, which sets the weightings W_P, can be seen as a description of the dynamics of McAllister's aesthetic canon. And its evolution rule, as rendered in AAI.1 in the previous chapter, is:

$$W_P = CD$$

Unlike McAllister's, our naturalistic approach must incorporate the effects of affection's inherent biological readiness and affective plasticity. Thus, we must consider appropriate parameters for those factors. I introduce the notions of *critical adequacy* and of *robustness of critical adequacy* to accomplish that.

[2]It must be noted that Kuipers [49] offers a formal analysis of the aesthetic induction as well. But since Kuipers' model suffers from the same problems as McAllister's, I pursue a different direction here.

5.2.2 Critical Adequacy

I define critical adequacy as follows:

Critical Adequacy: An object O is critically adequate if and only if there is a property P of O that warrants that an average person with the appropriate experience will pass a positive aesthetic judgement about O.

Critical adequacy embodies the fact that the presence of pleasing (or displeasing) properties motivates the eliciting of judgements. The person's experience involved in the definition may include, in the scientific context, considerations such as empirical adequacy. The inclusion of a person's experience warrants that critical adequacy can play a role analogous to the role played by empirical adequacy in McAllister's model.

Now, the influence of a property upon the aesthetic canon is represented by its weighting. In order to incorporate the influence of critical adequacy into our evolution rule we must represent it as a *parameter* in an evolution rule. A notion of critical adequacy that admits degrees is more suitable for this purpose. Thus, consider the following definition:

Degree of Critical Adequacy: An object O has a high degree of critical adequacy if and only if there is a property P of O whose presence makes very probable that an average person with the appropriate experience will pass a positive aesthetic judgement about O.

The degree of critical adequacy embodies the intensity with which an object with certain properties fits the taste of a person or community. Affective plasticity allows an aesthetic canon to evolve, and it explains that the aesthetic canon's dynamics is linked to the history of experiences with certain properties. The biological constitution of affective phenomena accounts for the fact that there are objects and properties capable of invoking affective responses regardless of any previous experience with such objects or properties. The notion of critical adequacy models this characteristic. The degree of critical adequacy can be used to model the dynamics of an aesthetic canon in a manner analogous to McAllister's evolution rule.

Now, the constraints on affective plasticity are still absent from our model. To address this issue, I introduce the notion of robustness of critical adequacy.

5.2.3 Robustness of Critical Adequacy

We have seen that the degrees of preference for properties like simplicity or complexity tend to remain unchanged over extended periods of time. The fact that preferences are greatly influenced by our biological constitution and by the constraints on affective plasticity can account for this pattern of evolution; or, at least, for the fact that preferences resist arbitrary or unconstrained changes. This characteristic resembles what in systems theory is known as the *robustness of a system*, which is the system's ability to remain unchanged or to sustain little change despite the perturbations induced by the environment. I borrow the idea of robustness to refine the description of the dynamics of an aesthetic canon. It must be pointed out that although I have offered a possible empirical explanation of robustness, the notion formulated below can be seen as having mostly a descriptive character. It allows us to model the pattern of evolution of historical constants without committing to a specific explanation of it. Consider thus the following definition:

Robustness of Critical Adequacy: The critical adequacy of a property P of an object O is robust if and only if P is able to motivate the same affective response despite changes in the history of experiences with P.

As before, a definition that admit degrees is better suited to be incorporated as a parameter into an evolution rule.

Degree of Robustness of Critical Adequacy: The critical adequacy of a property P of an object O is robust in a high degree if and only if in most cases P is able to motivate the same affective response despite changes in the history of experiences with P.

What this definition tells us is that properties with robust critical adequacy tend to maintain their degree of critical adequacy despite the fact that a history of experiences with such properties builds up over time. This is precisely the pattern of evolution of historical constants, which indicates that those properties possess a high degree of robustness. The degree of robustness introduces differences among properties regarding their patterns of evolution. Properties with a low degree of robustness change depending on the contingencies that affect the aesthetic canon, whereas properties with high degree of robustness remain relatively unchanged.

5.2.4 Dynamics of the New Model

With the concepts introduced above, we can now model the dynamics of an aesthetic canon as follows:

Naturalistic Dynamics of an Aesthetic Canon: The evolution of an aesthetic canon is governed by the following mechanism: A community compiles its aesthetic canon at a certain time t by attaching to each aesthetic property an associated weighting according to the following function, which I call *Naturalistic Evolution Rule* (NER):

Naturalistic Evolution Rule (NER):

$$W_P(t) = (1 - R_P)CA_P + R_P W_P(t-1)$$

where:

$W_P(t)$: is the weighting of P at a certain time t, resulting from the evolution of the aesthetic canon.

$W_P(t-1)$: is the original weighting of property P at a prior time $t-1$, before the evolution of the aesthetic canon.

A_P[3]: is the degree of critical adequacy of P, whose range is the closed unit interval [0,1].

R_P: is the degree of robustness of P, whose range is [0,1].

C: is a constant that gauges the ratio between the weightings and the degrees of critical adequacy.

This function has the desirable characteristic that if the robustness R_P is very low, the function is similar to McAllister's evolution rule. But if the robustness is high, the function mimics the tendency of certain preferences to remain constant over time. Consider the case in which robustness is ideally low, with $R_P = 0$. The function reduces to $W_P(t) = CA_P$. That is, the weighting associated to P is proportional to P's critical adequacy, which is a generalization of McAllister's evolution rule $W_P = CA$. Consider now the case in which robustness is ideally high, with $R_P = 1$. The function reduces to $W_P(t) = W_P(t-1)$. That is, the weighting remains unchanged, which is the pattern of evolution of an ideal historical constant. Of course, for non-ideal cases, the function yields values that reflect the effect of the various factors involved: the proportionality to critical adequacy and the effect of the robustness of each property. Now, NER is merely an illustration of a suitable evolution rule and it does not intend to be a factual model. It only intends to show that the "inductive" aspects, modelled by the proportionality $W_P(t) = CA_P$, and the "affective" aspects, modelled by the robustness $W_P(t) = W_P(t-1)$, can be integrated in a consistent manner.

Now, NER can account for the same cases as McAllister's rule, since NER has $W_P(t) = CA_P$ as a special case. The function can also account for McAllister's anomalies in a natural fashion. For example, the pattern of evolution of a negative historical constant can be interpreted as evidence that such property possesses a low degree of critical adequacy and a high degree of robustness. The naturalistic evolution rule yields more accurate descriptions than the original aesthetic induction. Thus, the model of evolution proposed here is more general than McAllister's not only in formal terms, but also in the sense that it covers its original as well as its anomalous cases.

---

[3] Strictly speaking, one should write $A_P(t)$, for an aesthetic canon changes over time and the degree of critical adequacy changes with it. However, an aesthetic canon changes at a much slower rate than the individual degrees of preference; for the sake of simplicity, the slow change in critical adequacy is neglected and the parameter treated as a constant.

5.3 The New Model: Constrained Aesthetic Induction

I shall use my rendering of McAllister's model of the role of aesthetic evaluations in science in Chap. 4 as a template for my new model. The model thus consists of three stages. The first stage describes the dynamics of the aesthetic canon. The second describes how a theory choice is conducted. The third justifies that choice. I label the new model *Constrained Aesthetic Induction* (CAI) and express it as follows:

5.3.1 Constrained Aesthetic Induction (CAI)

CAI is a natural process that occurs in the context of aesthetic episodes. Aesthetic episodes are natural processes of interaction between subjects and their natural and social environment. CAI unfolds through stages CAI.1–CAI.3.

(CAI.1) A community compiles its aesthetic canon, $Acanon = \{(P, W_P(t))\}$, at certain time t by associating to every aesthetic property P a weighting determined by the Naturalistic Evolution Rule (NER).

(CAI.2) Given two equally empirically adequate competing theories T and S, which exhibit the aesthetic properties E and F respectively, a scientist will choose T over S only if $W_E(t) > W_F(t)$.

(CAI.3) The scientist makes that choice because he holds (perhaps unconsciously) that: (CAI3.1) $a_1, a_2, a_3, \ldots, a_n$ are all P that are also Q; (CAI.3.2) a_{n+1} is P; and (CAI.3.3) a_{n+1} is also Q. Where a_i is a theory, P is an aesthetic property of scientific theories, and Q is the property of being empirically adequate.

The label aesthetic *induction* is not completely accurate to name the new model, since neither the evolution rule nor the model as a whole are proper instances of induction. Inductive projection is involved in the justification phase CAI.3, but its role is purely *epistemic*, not aesthetic. Although the term "induction" in the new model of "aesthetic induction" may be a little inaccurate, the model itself is a more accurate depiction of aesthetic evaluations in science, and it allows us to dispose of the problems of the original model.

5.3.2 Problems Addressed

Addressing the problems with the original model is now simple. First, the anomalies: Since the original aesthetic induction is a special case of the constrained aesthetic induction (CAI), the results that can be obtained by using the original aesthetic induction can also be obtained by using CAI. We only need to assume a low degree of robustness in the aesthetic canon. This last assumption, however, is not an accurate depiction of what occurs in actuality, as the existence of historical constants shows. For example, the pattern of evolution of simplicity is evidence that simplicity possesses a high degree of critical adequacy and a high degree of robustness. Similarly, the pattern of evolution of complexity is evidence that complexity possesses a low degree of critical adequacy and a high degree of robustness. In general, historical constants can be modelled as properties with high degrees of robustness, and historical contingencies as properties with low degrees of robustness. Historical constants are not anomalous in the new model.

The theoretical problems of the original model can also be addressed. The problems were: first, the aesthetic induction is not induction. Second, a confusion between beauty and the aesthetic. Third, an inconsistent projectivist position. And fourth, a faint link between theory and modelling. The first problem is not an issue since the new model is not based on inductive projection. The second problem is not an issue either: the new model is based on a naturalistic theory which differentiates beauty from the aesthetic: the aesthetic is interpreted in functional terms, whereas beauty is interpreted as an aesthetic term that articulates aesthetic experiences. As for the third problem: since aesthetic episodes are complex processes of interaction between the subject and the environment, the objectivism/projectivism dilemma does not play a significant role. Finally, the naturalistic aesthetic theory tightens the relation between theoretical and descriptive issues; as it offers plenty of room for empirical description in a consistent way: the findings of empirical science are incorporated into the new model as parameters in the evolution rule.

In this way, we now have the basics of a naturalistic aesthetic theory able to account for the evolution of aesthetic preferences in science. We also have a more accurate model of that evolution. In the following chapter we shall extend the discussion and use the model introduced here to develop an even more sophisticated aesthetics of mathematics.

Part II
An Aesthetics of Mathematics

Chapter 6
Introduction to a Naturalistic Aesthetic Theory

In this and the following chapters I address the chief task of this book: to provide a consistent literal interpretation of aesthetic judgements in mathematics. So far, we have surveyed diverse approaches to mathematical beauty and it has become clear that there are many ways of approaching the subject. For this reason, I begin by stating the stance taken here to approach mathematical beauty.

I endorse a literal interpretation of the term 'mathematical beauty'. It is true that there are idioms like "a beautiful steak" with non-aesthetic connotations, meaning simply that, for instance, 'the steak is good'—Weinberg's interpretation goes along this line. But McAllister's systematic study shows that the most common and interesting usages of the term 'beauty' in science are genuinely aesthetic. In addition, although finding non-literal interpretations of the term beauty does not seem to be very difficult, that tactic is neither free of problems nor fruitful. The really hard problem seems to be to address genuine aesthetic judgements in a coherent and fruitful manner, and this "hard problem" is the issue we shall address. Unless there is an indication that aesthetic evaluations in mathematics mean something like *a very apt or adequate entity* or *something good at doing its job*, I take aesthetic judgements in mathematics at face value, as expressing a genuine appreciation of aesthetic merit.

This book not only tackles the hard problem of mathematical beauty, but it also does so in the "hard way": I do not intend to interpret mathematical aesthetic judgements in pre-theoretical terms, but rather to advance a rigorous theory—or at least the first draft of it—of aesthetic phenomena in mathematics. This is not an arbitrary choice but a necessity: mathematics is not a traditional topic of aesthetics, therefore a justification of mathematics as a subject matter of aesthetics must be done as rigorously and systematically as possible. Thus, the insights gained so far shall be utilized, further developed and systematized.

So, let us briefly recapitulate the approaches to mathematical beauty discussed so far. We started by surveying reasons for embracing non-literal and literal approaches. Mathematical beauty was explained on the basis of single non-aesthetic properties by Shaftesbury as the result of order, and by Hutchenson as caused

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_6,

by uniformity amidst variety. These approaches, however, cannot account for mathematical elegance or ugliness. For Rota, mathematical beauty's true meaning is enlightenment; the use of the term 'mathematical beauty' by mathematicians is their way of avoiding the fuzziness of the concept of enlightenment. Rota's non-literal approach, however, has some important shortcomings: enlightenment cannot account for salient instances of mathematical beauty such as proofs by *reductio ad absurdum*. In general, Rota's view cannot account for mathematical beauty, elegance and ugliness in a consistent way. Neither can it explain why mathematicians—experts in the use of novel and exotic terms—choose the term 'beauty', rather than a less confusing one, to deal with enlightenment. McAllister formulated a very sophisticated literal approach to beauty in science; based on a dynamic view of aesthetic preferences. His central ideas are the aesthetic canon—a sort of extensional representation of preferences—and the aesthetic induction—the mechanism that governs the evolution of the aesthetic canon by consulting old preferences and the empirical track record of their associated theories. We discussed some general issues with McAllister's and the other approaches. I pointed out the relevance of phenomenological factors, an issue generally neglected. Taking into account inner experience yields a depiction of mathematical beauty significantly different from McAllister's. In that regard, issues like affective response or intentional objects took the floor. We saw that to explain aesthetic appreciation of causally inefficacious objects like mathematical objects we need intentional objects—mental objects or objects perceived under certain interpretation, according to Peter Kivy. Intentional objects are shaped by the subject's knowledge, skills, experiences, etc. The activity of the mind in following proofs was also addressed; we identified two ways in which affective responses can be elicited: by contemplating or by constructing intentional objects. As for issues specific to McAllister's approach we identified two kinds of problems: explanatory anomalies and theoretical tensions. Those problems were addressed by proposing a rudimentary naturalistic aesthetic theory and a more accurate model of preference evolution; the constrained aesthetic induction. In this respect, affection and emotion, which have been closely connected with aesthetic phenomena since Antiquity, took a central role. Some features of affection can explain the dynamic nature of aesthetic preferences. Affection, however, is not completely independent from cognition. The modern conception of emotion interprets it as a dynamic process involving psychological, physiological and cognitive phenomena. Affection, at least in the case of emotions, is not alien to cognition. Not only insights and findings from the study of emotions, but also concepts from systems theory were utilized to naturalize the idea of the aesthetic induction. Aesthetic criteria were modelled as abstract ordered pairs and the aesthetic canon as a set of pairs modulated by an evolution function describing the dynamics of the system. The model takes into account the influence of the plasticity of affection—via critical adequacy—and its constraints—via robustness of critical adequacy—enabling it to account for the anomalies in McAllister's approach, like negative historical constants.

In short, explanations of mathematical beauty are mostly grounded on observable properties and our responses to them—even Rota's and Weinberg's. McAllister further shows that the source of our responses, our preferences, evolves over time. I made things more complicated by introducing phenomenological factors into the picture; and revised the model of evolution of preferences by introducing empirical insights on affection and emotion. Let us now further develop all this insights. To accomplish that, I interpret beauty not in terms of the properties that grounds it, but rather as a phenomenon embedded in a larger system of characteristic events. The aesthetic theory proposed in this book is a description of that system, which, for convenience, shall be referred to as the *aesthetic-process* hereafter.

When an individual undergoes an aesthetic episode, he becomes engaged in a series of events that includes the perception of objective properties, the changes in the mental states of the individual, the effect of the individual's previous experiences, knowledge and skills, and the social factors that contribute to shaping our perceptions and reactions. An aesthetic episode is not the mere result of our perception of an objective property, nor just a private subjective occurrence. Rather, it consists in engaging oneself in the unfolding of interrelated events, whose global roles contribute to a phenomenon that is characteristically aesthetic. This series of events constitutes an aesthetic-process. The private aesthetic episode as experienced by the individual is thus a way in which the individual actively relates to his environment, culture, community and his own history of experiences. The idea of the aesthetic-process shall be central in elaborating our naturalistic aesthetics. For example, we shall see that although single-property based accounts, like Shaftesbury's beauty-is-order or Hutchenson's beauty-is-uniformity-amidst-variety, are not sufficient to explain aesthetic episodes, they play a very significant role in aesthetic-processes, as they constitute the basis of an aesthetic experience. Thus, my approach is not in opposition to property-based accounts; it just proposes that a broader framework is necessary to give us a better understanding of their role in aesthetic episodes.

I shall employ the idea of aesthetic-process to address three central issues: aesthetic experience, aesthetic value and aesthetic judgement in mathematics. These subjects have figured prominently in our discussion: the meaning of terms such as beautiful, elegant, or ugly in mathematics has been the chief issue driving the discussion so far.[1] And closely connected with the issue of aesthetic terms is the issue of aesthetic evaluation or, more technically, aesthetic judgement, which is the issue that sparkled all our discussion in the first place. The issue of aesthetic value features centrally in McAllister's discussion, as the objectivism/projectivism dilemma is one of the first issues McAllister addresses. That these issues feature prominently in our discussion is not fortuitous, aesthetic judgement and aesthetic value are crucial subjects in aesthetics. And to these issues I have added that of private inner experience. In trying to make sense of our affective response towards mathematical objects, we have learnt that it must be our mind's content

[1]McAllister discusses aesthetic terms in close connection with aesthetic properties. Perhaps recognizing that, in general, the semantic notion of 'predicate' is conceptualized as closely connected with the metaphysical notion of 'property'.

that elicits a response, since abstract objects are causally inefficacious. Even if we disregard this peculiarity of mathematics, inner experience is a central issue in aesthetics. Ever since Kant [32], aesthetic phenomena have been regarded as characteristically *subjective*, that is, dependent on the subject's perspective, feelings, beliefs, and desires [84, p. 900]. The particular way a person perceives, understands and responds to an object is central to understanding his aesthetic episode; and this centrality is only bolstered by the abstract nature of mathematics. An account of aesthetic experience in mathematics is crucial to account for mathematical beauty. Now, to discuss aesthetic experience, aesthetic value and aesthetic terms and judgments is necessary not only because it helps to shape a broad and consistent view of mathematical beauty, but also because the issues themselves are intimately connected, as it shall be evident later in our discussion.

6.1 The Mark of the Aesthetic

One of the motivations for proposing the integration of diverse aesthetic events into an aesthetic-process comes from an issue we encountered while discussing the theoretical problems with McAllister's approach: that the use of the predicate 'aesthetic' seems to imply the existence of a characteristic aesthetic feature, a "mark of the aesthetic", so to speak. The predicate 'aesthetic' is used to qualify different types of things: judgements, experiences, pleasure, concepts, properties, terms, and so forth. An important question about this predicate is whether there is a notion of 'the aesthetic' supporting this usage. I have proposed a preliminary answer to this question in the previous chapter, I expand it here: we can make sense of the notion of *the aesthetic* if we interpret the use of the predicate 'aesthetic' to qualify a certain kind of things as indicating that kind of things plays a significant role in the development of an aesthetic-process. In this way, the notion of *the aesthetic* can be interpreted in terms of aesthetic-processes. Since in an aesthetic-process the events involved are interrelated, there are systemic relations among the different phenomena that are qualified as aesthetic. These relations explain the affinity of concepts like aesthetic judgement and aesthetic value. An aesthetic concept (pleasure, judgement, value, for example), interpreted in this way, is aesthetic because it plays a significant role in one or several stages of an aesthetic-process.

Now, the idea of the aesthetic-process itself has been inspired by the study of emotions and other mental phenomena. Emotions, according to the modern empirical view, are complex response systems for coping with the environment that unfold continuously over time comprising multiple elements. Those elements include psychological, physiological, physical and even cognitive responses. What is particularly significant for our purposes is that this conception of emotions is the result of the synthesis of diverse and sometimes contradictory aspects of emotions. The complex nature of emotion has made it a very contentious subject matter. For many years, the most accepted theory of emotions, known as the judgement theory of emotions, maintained that cognitive evaluations, or propositional judgements, formed the core of emotions [16, 21, 76]. According to this view, emotions are a sort of judgement, that is, they are some sort of propositional content or, at least, they have that content as their source. Emotions are thus cognitive. This view was opposed by authors who believed that emotions were characteristically non-cognitive. William James, for example, pointed out that emotions are characterized by physiological changes, emotions are thus profiles of physiological arousal. Modern neuro-scientists conceive emotions in ways that open the possibility of reconciling cognitive and non-cognitive views by mapping them into the diverse neurological, physiological and physical components observed in an emotional episode [21, 76].[2] The idea of utilizing the complex neural, physiological and physical processes to explain the complexities of emotions has also philosophical appeal, as it is evident in approaches of philosophers of art like Jenefer Robinson. Robinson utilizes LeDoux's findings to devise a theory of emotions which allows her to explain artistic expression in music and literature [76]. My strategy here is to further extend the reach of this idea by utilizing not only empirical findings in the neurosciences, but also further insights from aesthetics.

Now, precursors of the idea of integrating different aspects of the aesthetic can be found not only in the empirical study of mental phenomena, but also in philosophical approaches to art and beauty. I have identified at least two approaches to integrating various kinds of aesthetic things that can be seen as precursors of the "aesthetic as process" theory proposed here. The first approach interprets all kinds of aesthetic things in terms of a single central concept. The second approach interprets complex concepts in terms of simpler concepts and their relations. I call the first approach the *centralist approach*; and the second, closer to my systemic approach, the *relationist approach*. We have encountered a centralist approach to the aesthetic in the previous chapter, deployed by Nick Zangwill:

> The predicate "aesthetic" can qualify many different kinds of things: judgments, experiences, concepts, properties, or words. It is probably best to take aesthetic judgments as central. We can understand other aesthetic kinds of things in terms of aesthetic judgments: aesthetic properties are those that are ascribed in aesthetic judgments; aesthetic experiences are those that ground aesthetic judgments; aesthetic concepts are those that are deployed in aesthetic judgments; and aesthetic words are those that are typically used in the linguistic expression of aesthetic judgments [100].

The centralist approach endorses the idea that the meaning of the word 'aesthetic' is somehow shared by the different kinds of things qualified as aesthetic. Even if we are not offered an explicit definition of the notion of the aesthetic, a sort of partial reducibility among the different kinds of things seems to be assumed. The centralist approach takes one kind of aesthetic thing as central—aesthetic judgements, for

[2]Many cognitive, sensorial and even motor phenomena in mammals and other species involve multiple neural pathways that interact with each other dynamically over time. Moreover, in recent years the idea that the cognitive, sensory and motor systems are deeply intermingled has become one of the most influential findings in shaping the scientific conception of mental phenomena. Views with telling names such as "situated cognition", "embedded cognition", or "embodied cognition" have ever increasingly gained a central stage in the debate on the nature of mental phenomena.

example—and interprets the rest of kinds—aesthetic properties, for example—in terms of the central notion. Peripheral notions are thus seen as reducible to, or definable in terms of, the central notion. The centralist approach has two problems. First, in practice, notions like aesthetic preference cannot be satisfactorily accounted for in terms of a central notion. The dynamic aspect of aesthetic preferences, for example, is disregarded by a centralist picture. Second, the choice of an item as central is rather arbitrary, since no argument is required to choose one over other concept, other than the a posteriori fact that the definition can be made.

Alan Goldman endorses a relationist approach in his 1995 book *Aesthetic Value* [29]. He interprets aesthetic values as relations between the properties of the object being evaluated and the reactions in the evaluator:

> evaluative aesthetic properties are constituted ultimately by relations between non evaluative properties of artworks, which we call base properties, and positive or negative reactions of certain observers [29, p. 45].

Goldman rejects the traditional objectivism/projectivism dilemma. He is dissatisfied with both objectivist and subjectivist accounts of aesthetic value and intends to present an alternative to the traditional conceptions of aesthetic value as being either grounded on the objective properties of artworks or purely subjectively projected by the observer. He opts for a compromise in the form of a relation between subjective reactions and objective properties. More importantly for us is that Goldman does not focus on the meaning of the concept of aesthetic value and its inter-reducibility; rather he introduces the elements he thinks are involved in it—objective properties and reactions in observers—and argues that the relation between them is what constitutes value. Goldman analyzes value in its elementary components and he then uses those components to explicate the more complex notion of aesthetic value.

As illustrated by Goldman, the relationist approach does not rely so much on the idea that aesthetic things and concepts are inter-definable. It addresses a concept by interpreting it as a relation between some more elementary relevant things. It approaches things like aesthetic value not by trying to exploit the fact that the meanings of different aesthetic kinds of things depend on each other, but rather by trying to find the objects, phenomena or ideas involved in the concept and elucidating the relations that help constitute the concept. A relationist approach depends on both conceptual analysis and synthesis. Something particularly appealing is that conceptual synthesis offers room for incorporating things like the dynamics of beauty. Now, although relationist approaches do not have the methodological limitations of centralist approaches, they lack the ability to integrate a wide range of aspects of the aesthetic, since a careful analysis of individual issues is more demanding than carrying out a reduction. The idea of aesthetic-processes aims precisely to expand the focus of the analysis in order to achieve a wider integration. Just as Goldman suggests that aesthetic value cannot be understood solely based on either objective properties or subjective responses, I suggest that the different aspects of the aesthetic cannot be understood in isolation. Rather, they should be understood as elements of a system of complex relations. The affinity of these elements, the fact that they all are aspects of “the aesthetic”, results from their relation to a single unified system: an aesthetic-process.

6.1.1 Phases of an Aesthetic-Process

The events comprised by an aesthetic-process include things like contemplating objects, experiencing pleasure elicited by this contemplation, attributing value, acquiring new values, passing judgements, etc. Although some events are closely related—like hearing music and experiencing the feeling it causes in us, in general there is no fixed sequence in which these events appear. For example, events related to the formation and acquisition of values—such as the preferences modelled in Chaps. 2, 4 and 5—occur during long periods of time—perhaps centuries—and depend on complex social relations. The eliciting of pleasure, in contrast, occurs in a very short period of time, usually following immediately the contemplation, for example, of certain properties of objects. Another feature that adds to the complexity of the structure of aesthetic-processes is that they are mixtures of natural and social phenomena. Their development involves natural aspects, like affective responses, and social aspects, like learning from a given culture. Aesthetic-processes are the result of complex interrelations that have evolved from our basic senses, feelings and social interactions. Hence, we cannot expect to find "aesthetic things", like aesthetic judgements or values at certain fixed points of the process. For example, we might have the experience of a musical piece at a certain time and, based on values acquired long ago, immediately articulate the experience in a judgement. But we can also have the same experience, but articulate it in a judgement only days or weeks later, once we have acquired some new elements—learning music theory, for instance—relevant for evaluation. Although very closely related, the events of experience, evaluation and judgement do not necessarily occur immediately, or one after the other, or in a fixed sequence.

Although the development of aesthetic-processes is non-fixed and non-sequential, there are subsets of events that appear and develop in a more orderly fashion. For example, affective responses are closely linked to the contemplation of an object, and the acquisition and development of aesthetic values, although it might occurs over extended periods of time, usually involve specific types of events like biologically constituted tendencies, the history of experiences of an organism, its social context, etc. These relatively fixed subsets of events constitute sub-processes within aesthetic-processes. We can use the organization of events in sub-processes to illustrate the idealized unfolding of a typical aesthetic-process. The unfolding involves the following phases—which I label *nodes* to stress that they merely indicate significant spots in sequences of events:

Short-term nodes:

1. Cognitive or Sensorial Input (Trigger)
2. Intentional Object Build-Up
3. Active Transformations
4. Affective Evaluation
5. Articulation
6. Judgement

Long-term nodes:

7. Aesthetic Criteria Repository
8. Value Dynamics

In this depiction of the process, the first six nodes are labelled "short-term nodes" because they represent sequences of very closely interrelated events that tend to appear in quick succession. Nodes 1 to 4 comprise the collection of events that tend to occur in that order in individual aesthetic experiences. Nodes 4 to 6 are closely related to the passing of aesthetic judgements. It must be noted, however, that passing aesthetic judgements does not necessarily follow aesthetic experiences immediately. We can have aesthetic experiences of pure contemplation without necessarily passing a judgement, and, conversely, we can pass aesthetic judgements without currently experiencing an aesthetic episode, based solely on our recollection of past experiences, for example. Long-term nodes represent events that usually occur over longer periods of time. Nodes 7 and 8 represent events like the acquisition and change of preferences or values; we have seen that periods as long as the entire history of science are important in determining if, for instance, certain property is a historical constant.

These nodes ideally develop as follows (in the general case, later on we shall concentrate on the case of mathematics):

Usually, the process starts by being triggered in a quasi-automatic way, without much intervention of the conscious will, in a fashion similar to how an affective episode is triggered by stimuli from the environment.[3] This occurs in node 1. The very first event of an aesthetic episode in a person consists in focusing his attention on an object, a stimulus, or a mental content. For example, when attending a concert we listen to the music as soon as it begins. But just before the beginning of the concert, the people in the concert hall do all sorts of things; they talk or walk around, for instance. It is only when the music starts that everyone remains silent and in their seats and they focus their attention on the sounds coming from the orchestra; music becomes the focus of attention. The process can also be triggered by a cognitive input: aesthetic experiences of narrative or poetry, for example, are associated solely with strings of *words*, that is, with purely cognitive inputs. Now, this triggering sub-process often includes the engagement of an automatic response that further focuses the attention of the individual on the event in question. This is because in many cases affective phenomena occur by the mere exposure of some stimuli to which we have a biologically determined or an acquired preference—for example, it is well know that music engage us in an affective and immediate way, even before we can cognitively process it.

Once initiated the process, the individual focuses his attention on certain features of the observed object depending on the individual's skills, attitudes, knowledge, and previous experiences; by doing this the individual starts to perceive the object

[3] In general an aesthetic response is triggered by environmental stimuli, but "cognitive" stimuli are, of course, possible.

in his own particular way, he shapes a personal inner representation of the object or, more technically, an *intentional object*.[4] This occurs in node 2. For example, returning to our illustration above, once we are listening to music in the concert hall, we begin to focus on things like the repetition or variation of motifs, the arrangement of musical sections, and so forth [39, 40, 42, 44]. During this stage the initial perceptual or cognitive trigger becomes an intentional object. The features of this object depend not only on the individual's particular skills, attitudes, knowledge, preferences, etc., but also on the *modality* of the experience (we shall elaborate on this later on).

Node 3 involves the mental *activities* undergone in the process. Up to node 2, the attention of the individual consists mostly in what in Chap. 3 I labelled the contemplative way of appreciating an object. The performative way of engaging in an aesthetic episode, that is, the *active* intellectual engagement, is important to understand aesthetic experiences of, for instance, narrative, poetry and music [42, 44]. In narrative and music, we not only become aware of the events depicted in the story or the musical events we listen to, but we also get engaged in having expectations and making predictions about how those events might develop. In reading a story, the features of a plot can only be appreciated after we have "assembled" the plot by reading all the relevant individual events the narrative comprises. *Active* attention must be considered part of aesthetic experience in the general case. Node 3 comprises the activities that our attention performs in dealing with the intentional object. Once we have "constructed" an intentional object we engage in an active "processing" it by further discerning its structural properties. For example, we cannot tell whether a story's plot is good or bad until we know all the events involved in the plot, which often means knowing the entire story. And whether or not a person likes the plot depends on the specific way in which the he deals with the plot. I shall discuss all this in greater detail in the chapter on aesthetic experience.

Following these initial experience nodes, there is an evaluation stage, node 4. Passive and active attention result in affective responses; these responses can be construed as affective evaluations of the object of attention. For example, one can enjoy a story or a piece of music, or one can dislike them. One may feel positively stimulated, or disappointed, or even bored. Now, let us remember that in an emotive response to a stimulus the initial response consists in an affective response which is, according to LeDoux, a rapid and crude evaluation of the stimulus [54, pp. 163–165]. We can see the initial response in aesthetic experience in an analogous manner: the positive (or negative) response can be thought of as an indicator of one's preferences and thus be conceptualized as an evaluation, an affective evaluation. In a sense, in this stage a sort of affective "aesthetic canon" governs our evaluations of the contents of experience. As we shall see in the chapters on aesthetic judgement, evaluation in an aesthetic-process differs from

[4]Recall Kivy's definition of intentional object as an "object perceived under a specific description" [44, p. 81].

non-aesthetic evaluation; aesthetic evaluations consist of affective responses that must be clarified and articulated before they can be expressed as propositional judgements. It is only when we have articulated the experience that the affective evaluations can be turned into something meaningful in the propositional sense. Thus, the evaluation comes only after a phase in which we make sense of the process: we articulate the process as a whole and get our experience ready to be expressed in propositions like 'This concert is beautiful'. There are several ways of articulating an aesthetic-process, but a very direct way is simply by summarizing the affective evaluation of our experience in an aesthetic description: an aesthetic judgement. This judgement is different from epistemic or moral judgements in that its primary function does not consist in conveying information about states of affairs, but rather in making us aware of the nature of the process we are undergoing: it consists in determining our internal state. But aesthetic judgements also have the function of conveying information. The actual public uttering of a judgement takes the form of a description 'A is B'. For example 'Last night's concert was beautiful'. These descriptions have a definite propositional content that carries information. This information carrying function shall be discussed in detail in Chap. 10 and shall be labelled the *broadcasting* function of aesthetic judgements. It occurs in node 6.

Nodes 5 and 6 can be seen, as comprising private and public dimensions of aesthetic judgements, respectively. Aesthetic judgements as represented by the events in node 6 carry information outside the individual that can be used for non-private purposes. In this public pathway of information, aesthetic judgements are significant for the social interactions that shape aesthetic preferences and values. People learn and modify their preferences and values under the influence of public aesthetic judgements. In social interaction, aesthetic judgements are "recycled" through a long-term process that modifies preferences and values. In a sense, individual public judgements, under the appropriate circumstances, might eventually be incorporated and assimilated into preferences and values.

Nodes 7 and 8 represent the series of events involved in the evolution of preferences and values. We have learnt that aesthetic preferences change over time and that some patterns of evolution can take very long periods of time. Aesthetic preferences outside science are not different. For example, it is very well known that musical dissonance is much more accepted nowadays than 200 years ago. Furthermore, perhaps under the influence of a variation of the mere-exposure effect, our preferences for certain kinds of music change depending on our experience. Atonal music, for instance, is something for which we have to *acquire* a taste. Despite their long-term character, events in nodes 7 and 8 determine events in short-term nodes. Passing aesthetic judgements and making affective evaluations can be seen as re-entry points of long-term feedback pathways between long- and short-term nodes. Since affective evaluations actualize preferences, node 4 can be seen as making a feedback pathway with node 7. And since aesthetic judgements serve as evidence of the presence of aesthetic values,[5] node 6 can be seen as forming a

[5]This is the basis of my interpretation of aesthetic value, as we shall discuss later on.

feedback pathway with node 8. Node 8 can be seen as a value “repository” that governs evaluations. The feedback pathway goes from our own and other persons’ past judgements to the long-term repository and then back to the evaluation and judgement nodes in an individual’s aesthetic-process.

Aesthetic experience is characterized mainly by nodes 1 to 4, aesthetic judgement by nodes 5 and 6 (although we could also include node 4). Aesthetic preferences and value are modelled by nodes 7 and 8. We have now a general idea of how the events unfold in an aesthetic-process, but we must account for the specifics of aesthetic experience, value and judgement, in order to ground our aesthetic theory, the following chapters are devoted to that.

Chapter 7
Aesthetic Experience

In this chapter we address aesthetic experience. A person's inner experience of an observed object depends primarily, of course, on the features of the object of attention. But the experience is also deeply influenced by the subjects' particular skills, knowledge, attitudes, and so forth. As we have seen, Peter Kivy even links the content of our attention, intentional objects, to descriptions [44, pp. 81]. An aesthetic experience is a particular type of inner experience and it thus depends on the object's features and the subject's dispositions. The experience of music is very different from the experience of painting; the properties of a piece of music are very different from the properties of a painting. Moreover, what we know about music can affect how we perceive certain piece of music, but not necessarily how we perceive a painting, and vice versa. Hence, an analysis of the specifics of musical experience does not necessarily enlighten us about the nature of inner experiences in contemplating paintings. The same is true for the aesthetic experience of mathematics. Different aesthetic experiences should be addressed by concentrating on their own particularities, those particularities constitute the modality of the experience. In the following, I concentrate on the details relevant to aesthetic experience of mathematics. For convenience, I shall use again the term *phenomenological* to describe things related to the private first person perspective, the inner experience of a subject, which should not be confused with the technical Husserlian sense of describing "the structures of the experience as they present themselves to consciousness" [20, p. 2]. My approach is rather constrained in comparison: I merely advance a descriptive account of the intellectual, affective, and objective events and their relations relevant to the eliciting of aesthetic evaluations.

7.1 Characterizing Aesthetic Experience

I consider an aesthetic experience a collection of interrelated events that unfold over time, that is, a process. An aesthetic experience is not an independent process; it is a sub-process embedded in a larger aesthetic-process. Consider, for example,

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_7,

Euler's identity, $e^{i\pi} + 1 = 0$, named the most beautiful formula in mathematics by *The Mathematical Intelligencer* [94]. The aesthetic experience associated with Euler's identity depends not only on the person's inner events occurring during the act of contemplating the formula, but also on things like a person's knowing the mathematics which allows us to make sense of the sign $e^{i\pi} + 1 = 0$, the way a person's preferences were formed, other people's opinions, and so forth. The aesthetic experience of Euler's identity depends on events that are not necessarily occurring at the exact moment of the experience, but which have an influence on it; that is, the process of experiencing Euler's identity is embedded in a larger aesthetic-process. This is why I described that embedded sub-process as consisting of nodes 1 to 4 in the previous chapter. Node 4 includes an affective response, which I interpret as an evaluation. I distinguish between affective evaluation—an affective response—and aesthetic judgements—full-blown propositional aesthetic evaluations. Affective responses are characteristic of aesthetic episodes; this is one of the key features that distinguish them from other kinds of judgements.

The mathematical experience sub-process begins with a cognitive stimulus—mathematical experience is not perceptual, followed by a focusing of the attention on some relevant features of the stimulus, and by a further stage of active cognitive processing of the resultant object of attention. Consider Euler's identity, again, *The Mathematical Intelligencer* [93, 94] asked its readers to evaluate 24 theorems in terms of their beauty. Euler's identity ranked number one with an average score of 7.7, on a scale from 1 to 10. Now, the first event in our aesthetic experience of Euler's identity is an awareness of the mathematical formulation, by, for example encountering it in a textbook, or in the *Intelligencer*. However, more important than this initial awareness is the focusing of our attention on some relevant properties, such as the way the expression is composed: it comprises the constants e, i, π, 0, and 1, which are considered the most important constants in mathematics—I shall refer to the property of being composed by those constants as the *composition* of Euler's Identity. Another relevant feature of the expression is its simplicity. The occurrence of Euler's identity in a publication draws our attention, but it is the focusing of our attention on the composition and simplicity of Euler's identity that is important for eliciting an aesthetic response. Euler's identity is qualified as beautiful because its contemplation results in an affective response. This is why the editors of the *Intelligencer* deployed the predicate 'beautiful', rather than 'enlightening' to ask for the ranking. We experience some kind of affective response—we like it or dislike it—triggered by our contemplation of the formula. In the general case, the focusing of attention on some relevant aspects of the object of our attention results in responses of pleasure or displeasure. I shall use the term *enjoyment* to refer to the response of either pleasure or displeasure; that is, to the presence of any affective response.

We have seen that contemplation as well as active mental engagement can be pleasing or displeasing. We also have seen that preferences are formed and evolve influenced by diverse factors. Those elements must be taken into account to characterize aesthetic experience. I consider an aesthetic experience constituted by an intentional object, which I label the *content* of the experience, and an associated *enjoyment* elicited by that content. The enjoyment component—pleasure or displeasure—is necessary to distinguish it from mere inner mental representations of mathematical items, which are inner experiences, but not characteristically aesthetic. Taking into account the difference between contemplative and performative ways of eliciting enjoyment discussed in Chap. 3 and the dynamical character of preferences discussed in Chaps. 2, 4 and 5, at least three different types of aesthetic experience can be identified. Each of these types is characterized by a specific content-enjoyment relation. But before we can characterize the content-enjoyment relation, more details on intentional objects are needed.

The content of the experience as well as the way we actively deal with it are central in eliciting enjoyment. Now, in experiencing mathematics, perceptual stimuli are of little relevance. When mathematicians pass aesthetic judgements they are not referring to perceptual aspects of mathematical items. Mathematical beauty does not refer to things like the appearance of the sign $e^{i\pi} + 1 = 0$ printed on a page, or the diagrams illustrating a theory. The content that is important for aesthetic experience consists in the features found in the mental representations of mathematical items. For example, a non-perceptual feature in Euler's identity relevant for eliciting enjoyment is the feature that it comprises the most important constants in mathematics related in a simple manner. Our chief concern here must be this kind of features. An aesthetic intentional object is thus determined by a particular set of properties and relations *in a person's mental contents*. I have included a "formal build up", node 2, as part of a typical aesthetic-process. This takes into account the fact that the person's attention increasingly focuses on the relevant properties and relations—properties like simplicity or the composition of the object—of the cognitive stimulus, rather than on the whole collection of concrete features in the object. In the experience process our attention shifts from a concrete stimulus to a more specific set of features that constitute the intentional object relevant to aesthetic appreciation. For example, in Euler's identity, we first encounter and observe the expression $e^{i\pi} + 1 = 0$, but eventually our attention concentrates on properties like its composition and its simplicity, which are the significant properties for eliciting enjoyment.

7.2 Aesthetic Intentional Objects

Abstract objects are causally inefficacious; therefore, we concluded in Chap. 3, the content of our mind must be liable for our responses to mathematical objects. It is an intentional object which results in the affective response involved in an aesthetic-process. The intentional object consists of the relevant features that help to keep our attention focused and to elicit an affective response. Some features of this intentional object are the result of a natural process of abstraction. For example, in reading a story one extracts the propositional and then narrative information contained in the concrete characters printed on a page, or in the sounds uttered by a person. In listening to music one may extract information such as the pitch, and, if one is trained, even the name of a musical note, from the stream of sound reaching one's ears. In mathematics, when one contemplates an already abstract construct, such as Euler's identity, one contemplates it in an even more abstract manner: appreciating its remarkable composition and simplicity. A person can also discern further features in the object, resulting from the person's specific particularities; his skills, experience, knowledge, etc. For example, a person acquainted with a great deal of literature might realize that the novel he is reading sub-textually homages a famous Greek tragedy. Now, some of these individual peculiarities are influenced by changing external factors. Experiences provided by socialization and culture play a central role in forming our preferences. Those experiences modify the way we approach an object in the act of appreciation. They change our understanding of what are the relevant things to look for in an object, what things are acceptable and what are not. We unconsciously look for, and respond to those things. For example, in classical instrumental music we often look for patterns of temporal repetition; we learn that classic instrumental music is based on repeating patterns and, furthermore, within a single work is common to find that entire sections repeat themselves, in a sonata for instance. In mathematics, Gian-Carlo Rota points out that familiarity with different kinds of proofs helps us to recognize a good proof. In this respect, an interesting thing about mathematics is that this phenomenon is prevalent, not only in aesthetic appreciation, but in general, a large amount of knowledge is necessary to even see mathematics. We need to understand things like exponentiation, Euler's number, complex numbers, π, etc. in order to understand Euler's identity.

The way we perceive an object; that is, how we turn a concrete, or abstract, observed object into an intentional object depends primarily on the nature of the experience. Representational painting, for example, requires that the object of attention matches the object it depicts. But poetry or conceptual art, by contrast, require us to focus on the content of the text or the goals of the author. Mathematics usually requires a large amount of mathematical knowledge. Culture, via learning and training, plays a role in determining how we turn an observed object into and intentional object. This is why, as pointed out by Rota, familiarity with examples of mathematical beauty plays an important role in identifying other instances of mathematical beauty. How an intentional object is constituted is determined by the specifics of disciplines like narrative, music, painting and mathematics.

7.3 Mathematical Intentional Objects

It is now time to specify the features that characterize intentional objects in mathematical experience. I consider an intentional object the result of a shift of attention from a concrete initial stimulus to a specific set of properties associated with the object. An *aesthetic* intentional object is constituted by properties relevant for the eliciting of enjoyment—an affective response. To characterize aesthetic intentional objects in mathematics we need to avoid confusion between mathematical objects and objects of appreciation. Thus, a distinction must be drawn between mathematical objects, mathematical items and intentional objects: a *mathematical object* is an abstract object that appears as a referent in a mathematical theory—sets, functions, numbers, for example. I call a *mathematical item* any abstract[1] item that is characteristically part of mathematical *practice*.[2] A mathematical *intentional* object is the object in a person's inner experience resulting from focusing his attention on a mathematical item. If this attention results in a specific type of affective response (characterized below) the item is called an *aesthetic mathematical intentional object*. Those objects are the subject matter, the content, of an aesthetic experience.

7.4 A Notion of Aesthetic Mathematical Intentional Object

Aesthetic mathematical intentional objects are constituted by a set of properties, as perceived from a person's inner perspective, of course, and some structural relations among them. The set of properties comprises the properties that play a role in eliciting an affective response in the observer. For example, the simplicity and composition of Euler's identity, are relevant for our appreciation, but the property of, for instance, being a special case of Euler's formula is not. In eliciting enjoyment, not only the contemplation of properties plays a role, but also the mental activities in which a person engages. Therefore, the relations between properties that enable our attention to perform those activities are also relevant. In order to accommodate these features, I shall use, in a rather loose manner, the idea of space. I interpret intentional objects in aesthetic experience as objects existing in a *phenomenological space*—the space of a person's inner experience—with multiple dimensions. Dimension here is also interpreted rather loosely, as a parameter or piece of information necessary to specify the location of an object in the phenomenological space.[3] Intentional objects populate phenomenological spaces. The dimensions of phenomenological spaces correspond roughly to a relevant property of the intentional objects in our experience.

7.4.1 Dimensions and Properties

Consider a single mathematical result, Euler's identity, for instance. Its properties play the central role in eliciting enjoyment (in general that is the case in the

[1] In this way we exclude *concrete* indispensable items, like brains or mathematicians themselves.

[2] Rota and McAllister name several types of mathematical entities that are often qualified as beautiful, numbers, theorems, proofs, theories, and so forth. The above definition is adequate to cover those entities and some others not mentioned by them, such as derivations, or axiomatizations.

[3] Although the notions of space and dimension I utilize here resemble the ordinary concepts of physical space and dimension, they are rather closer to the formal notions of space and dimension. Unfortunately, a more formal treatment of these notions is beyond the scope of this book.

appreciation of single results like theorems, but the relations among them play a negligible role). By contrast, in the beauty of proofs and derivations, relations between single items play a major role since those relations are responsible for the emergence of structural properties of proofs or derivations like their simplicity, brevity or the shortness of their steps. For example, as we shall discuss below, the geometrical properties of complex numbers allow shorter and more elegant derivations of trigonometric identities.

In appreciating Euler's Identity, our attention must be focused in a specific way. A way in which we concentrate on some *extra* properties, properties that are not relevant to understand the formula, but that are necessary to aesthetically appreciate it. Euler's identity is very *simple*, but simplicity is not necessarily a property of all mathematical results; it is an extra quality that only some results possess. Properties like simplicity are the dimensions of the space in which our intentional object is located. An advantage of interpreting simplicity as a dimension is that it allows us to organize intentional objects according to its degree of simplicity. Now, dimensions not only organize intentional objects, but also enable us to see them, since they determine the different aspects of the objects that exist in the space. In order to allow dimensions to organize and determine objects, we need an idea of which features of the object the dimension indicates. For example, we have discussed that the property of simplicity may be ambiguous, as it can be interpreted in different ways and it can play different roles. A clear definition of simplicity is thus desirable if we are to conceptualize a space with that dimension. For this reason, it is best to interpret dimensions as explicit rules for interpreting relevant properties of objects. These rules can simply take the form of definitions that allow us to deal with the properties of objects in certain person's inner experience in a concrete situation. Thus, these rules can be simple declarations of the properties that constitute the intentional object. For example, if we need to introduce the dimension *simplicity*, in the phenomenological space in which an experience of Euler's identity is located, we need to define simplicity according to how the property appears in an intentional object. The definition can be as follows:

Simplicity = the feature of involving a minimum of operations and no non-relevant constants.

Euler's identity is simple in the sense defined above. It also has the very attractive extra quality of comprising the most important constants in mathematics. I have called this quality the composition of the formula. We can introduce another dimension into our space to account for how a person's attention focuses on this property by specifying the following interpretation:

Composition = the feature of being constituted by relevant items that are incorporated in a non-ad hoc manner.

In a phenomenological space with the dimensions of simplicity and composition, mathematical results are located in different spots, depending on how well they fit the definitions of the properties; that is, depending on how the dimensions order them. This models how our attention distinguishes and discriminates different mathematical results depending on how simple or well composed they are.

Among the things that substantially affect the forming of an intentional object is knowledge. As Kivy pointed out, a person acquainted with a famous actor sees that famous actor where some other person sees only a tall good looking man. A person who knows music theory hears a bold cadence where some other person may hear just some nice piece of music. Similarly, a mathematician sees the most beautiful theorem in mathematics where a lay person sees just an obscure formalism. In order to account for the role of knowledge in mathematical appreciation, we need to introduce a crucial dimension of mathematical phenomenological spaces: *background understanding*. Rota pointed out that to understand any piece of mathematics we need a great deal of mathematical knowledge. In order to appreciate a mathematical item, we first need to understand it. We can introduce a dimension that encapsulate the fact that we understand the mathematical item—and thus that such an item exists in our experience—simply by referring to the background knowledge necessary to understand it. For example, in order to understand Euler's identity we need to understand terms like p, e or i. More formally, we need to understand complex analysis. We can introduce a dimension CA corresponding to the property of being understandable only if complex analysis has been understood:

CA = the feature of being understandable only if complex analysis has been understood.

The dimensions that specify that mathematical understanding is required as background to appreciate a mathematical item shall be called *Background-Understanding* dimensions. At least one of these dimensions is necessary as part of any phenomenological space containing mathematical intentional objects. They are crucial to define the specificity of aesthetic experience in mathematics, and are analogous to the specific perceptual characteristic in other types of aesthetic experience. To appreciate painting we need sight; to appreciate music, hearing. And to appreciate mathematics, we need mathematical knowledge.

A background-understanding dimension is required for our experience to be about mathematics. But for our experience to be *aesthetic* we need extra properties that allow us to have an actual aesthetic response—simplicity or composition, for instance; properties that play a role in eliciting affective responses. To distinguish these properties I call them *aesthetically relevant* properties. In order to have an *aesthetic* object of attention, it is necessary that the phenomenological space in which it is located has at least one aesthetically relevant dimension. Thus, any mathematical phenomenological space must have at least two dimensions, and at least one must be aesthetically relevant. For example, the expression $x + x = 2x$, as an object of attention, requires background understanding (basic algebra) but it is aesthetically irrelevant, as it is not able to raise any kind of enthusiasm. Its properties are not able to elicit any kind of affective response. Thus, even if we can introduce different properties as dimensions of an "attention space", we cannot assign any aesthetic relevance to them because the affective character is absent, they do not constitute an *aesthetic* phenomenological space.

7.4.2 Activities and Relations

Consider the dimension *Complex Analysis*. This dimension allows us to understand, to see, so to speak, Euler's identity, but it also allows us to follow proofs or derivations involving complex functions. The proof of a theorem or the derivation of a result typically involves not only the passive contemplation of the result; rather, it consists in going through a series of steps and checking that the steps validly lead to the final result.

The expression $e^{i\pi} + 1 = 0$, for example, is a special case of:

$$e^{ix} = \cos x + i \sin x$$

Let $x = \pi$,

$$e^{i\pi} = \cos \pi + i \sin \pi$$

since $\cos \pi = -1$ and $sin\pi = 0$:

$$e^{i\pi} = -1$$

or

$$e^{i\pi} + 1 = 0$$

which is Euler's identity. In this very simple derivation, our attention is focused not on the properties of the resulting formula or the other individual expressions, but rather on how the successive steps lead us from the initial expression to the final one. This illustrates that the experience of mathematical items involves not only awareness of properties, but also the active engagement of our attention. The act of following this derivation is enabled by the properties and relations inherent in complex analysis, and thus by the background knowledge dimension of our phenomenological space.

The central element that determines an intentional object consists in the dimensions of the phenomenological space in which it is located. But from our discussion above is evident that there is a second important element: the set of relations that constrains the activities that can be performed by our attention in the phenomenological space (the dimensions of the space impose some constraints themselves, of course). In the general case, these properties and relations can be seen as rules of combination and transformation for the intentional objects existing in the phenomenological space—in order to keep my interpretation consistent, I defined dimensions also as *rules* of interpretation. This set of rules tells us how to obtain, or construct new intentional objects out of the original objects existing in the space. I call these rules *transforming operations*. Now, logic is the most fundamental set of rules of derivation in mathematics. All objects in a mathematical phenomenological space must have a background understanding dimension and are thus intrinsically regulated by logic. The second most important set of rules depends on the implicit relations of our background understanding dimension. For example, if our background understanding dimension is complex analysis, the identities and definitions involved in complex analysis are part of our transforming operations. Thus, we always have at least the rules of logic and of the particular background-understanding field of mathematics as transforming operations.

In mathematical appreciation we can have different operations working at different levels of appreciation and they are more relevant in performative (using the definitions introduced in Chap. 3 of *contemplative* and *performative* ways of eliciting affection) mathematical intentional objects such as derivations or proofs. For example, the introduction of the geometric interpretation of complex numbers by Caspar Wessel in 1799 allowed simpler derivations of already known results. Paul Nahin remarks:

> How beautifully simple is Wessel's idea. Multiplying by $\sqrt{-1}$ is, geometrically, simply a rotation by 90 degrees in the counter clockwise sense [...] Because of this property $\sqrt{-1}$ is often said to be the *rotation operator*, in addition to being an imaginary number. As one historian of mathematics has observed, the elegance and sheer wonderful simplicity of this interpretation suggests "that there is no occasion for anyone to muddle himself into a state of mystic wonderment over the grossly misnamed 'imaginaries.' " This is not to say, however, that this geometric interpretation wasn't a huge leap forward in human understanding. Indeed, it is only the start of a tidal wave of elegant calculations [68, pp. 54–55].

In the geometric interpretation "a complex number is either a point $a + ib$ in the so called *complex plane* or the directed radius vector from the origin to that point" [68, p. 48].

In addition to the representation $a + ib$, a complex number is sometimes represented by the associated length of its radius vector, called the *modulus* of the complex number, and the value of the polar angle $\arctan \frac{b}{a}$, called the *argument*. We can express this as follows [4]:

$$a + ib = \sqrt{a^2 + b^2}\angle \arctan \frac{b}{a}$$

[4]The angle notation, $\angle$, is very popular in fields like engineering. It is related to the polar form of complex numbers, the expression before the angle symbol represents its modulus and the expression after is the argument. This notation simplifies the visualization of operations: multiplication consists in multiplication of modulus and addition of arguments, exponentiation consists in exponentiation of modulus and multiplication of arguments.

Nahin's remarks on the geometric interpretation enthusiastically employ aesthetic adjectives. Nahin also stresses that the geometrical interpretation resulted in elegant calculations and even devotes a section of his book to presenting some of those calculations. De Moivre's theorem is instrumental in many of those calculations and it is an example of an elegant derivation itself [5]:

> With his wonderful deduction of the geometry of $\sqrt{-1}$ there was now no stopping Wessel with even more exotic calculations. For example, if you start with a unit radius vector of direction angle $\frac{\theta}{m}$, where m is an integer, then it follows immediately that
>
> $$\{1\angle\frac{\theta}{m}\}^m = \{\cos\frac{\theta}{m} + i\sin\frac{\theta}{m}\}^m = 1\angle\theta = \cos\theta + i\sin\theta$$
>
> Or turning this statement around by taking the mth root,
>
> $$\{\cos\theta + i\sin\theta\}^{\frac{1}{m}} = \cos\frac{\theta}{m} + i\sin\frac{\theta}{m}$$
>
> This result is not original with Wessel (although this elegantly simple derivation of it was), and it is commonly known as "DeMoivre's theorem" [68, p. 56].

The above derivation of DeMoivre's theorem is composed of several individual expressions, the steps of the derivation. In order to see the derivation as a single item we need to connect all those individual expressions. We do this by seeing the steps of the derivation as resulting from the application of logic or other inference rules implicit in complex analysis (or other relevant field). This illustrates that our object of attention is determined not only by its visible properties but also by how we *actively* deal with it. Furthermore, Wessel's geometric interpretation is a mathematical item that also has methodological repercussions: it results in elegant calculations. Historically, the fact that the geometric interpretation resulted in simpler derivations contributed to our appreciation of complex numbers. However, the fact that Wessel's geometric interpretation results in elegant calculations is not a property we can immediately see in the mere proposal of the interpretation. We can see that Wessel's proposal is simple, but to realize that it also results in elegant calculations we need to see the derivations themselves; that is, the property of *resulting in elegant calculations* is not immediately apparent by just directing our attention to the geometric interpretation. We need to perform further activities to realize the role it plays in, for example, the elegant derivation of DeMoivre's theorem. In other words, *resulting in elegant calculations* is not a property observable within a phenomenological space that includes Wessel's 1799 geometric interpretation of complex numbers. Now, this phenomenon occurs also in the arts. Features not observable in an artwork itself can help us to appreciate

[5] We shall see below that calculations, derivations or proofs belong to a different class of experience than formulas or theorems, since they are more "performative". Furthermore, this example involves not only active attention but also the fact that the person's history of experiences enables him to see some properties; it thus fits better in a third class comprising evaluations formed by a person's history of encounters with different mathematical items.

it better. Jeneffer Robinson illustrates this with the second movement of Carl Nielsen's Sixth Symphony (1925), which, according to some musical critics, is an expression of his bitterness and disappointment due to his failure to reach an international audience. However, when one listens to the music instead of bitter or disappointing, it sounds playful, humorous and even buffoonish. Robinson argues that this illustrates that a correct interpretation of some artworks must be grounded not only on observable features in the work, but also on extra knowledge, such as knowledge of the artist's life. We can hear Nielsen's music as an expression of bitterness only if we have extra musical knowledge about Nielsen himself. Robinson points out that what an artwork expresses may be *manifest* to us only if we have some information about the artist himself. Robinson concludes that "we cannot tell a work is an expression of bitterness, disappointment, and exasperation in its author just by paying close attention to 'the work itself' independently of its wider context" [76, p. 249]. The wider context provided by derivations such as DeMoivre's theorem's helps us to see new properties in Wessel's interpretation just as biographical context help us to see new properties in music.

The introduction of new transforming operations allows us to address this issue, since operations allow us to construct new objects like derivations or proofs, and thus to see new properties that are not originally visible on the object from within the phenomenological space. The introduction of further operations enables us to see further properties. I label this new type of operations *meta-intentional transforming operations*, whereas the rules implicit in our background understanding are labeled *implicit transforming operations*.

Meta-intentional operations allow us to introduce properties not visible within the phenomenological space. These operations must be consistent with our space, thus they must comply with two conditions: first they must be aesthetically-conservative; that is, they must preserve internal consistency and the aesthetic properties of the dimensions; they cannot change any of the properties responsible for eliciting enjoyment of the intentional objects in the space. For example, meta-intentional operations cannot introduce mathematical theorems that contradict the theorem on which our attention is focused, because that would amount to introducing an inconsistency, which is against logic. And they cannot introduce properties that contradict the properties already present in the object, either. For instance, an operation that turns Euler's Identity into a complicated theorem cannot be allowed. Second, meta-intentional operations must help, or be relevant to, obtaining aesthetically relevant properties, or procedures conducing to them. This is what enables these operations to facilitate seeing new properties. For example, in the case of Wessel's geometric interpretation we can introduce the operator *simplification* by, for instance, specifying how operations like multiplication, exponentiation and other calculations can be achieved by simpler means. And once we apply the *simplification* operation, we obtain a transformed intentional object. With these ideas, the power of Wessel's interpretation can be added to the properties that elicit enjoyment in aesthetically appreciating it: Nahin judges the derivation of DeMoivre's theorem elegant. This has to do with the fact that the theorem can be derived by very simple means. But realizing this kind of simplicity depends on different intellectual activities than realizing the simplicity of Euler's formula. In the case of the derivation we need to actively supervise that all steps are correct and that they lead to the theorem in a natural way. The fact that the derivation involves only a few steps and the steps themselves do not involve complicated manipulations contributes to see the derivation as simple. The fact that this simplicity is also connected with the simplicity of the geometric interpretation further enhances the aesthetic effect: the connection between a simple idea and its power is not only practically appealing, but also causes an affective response in us. We express these facts by using aesthetic terms, 'elegant', instead of just factually descriptive terms like 'brief' or 'fruitful'.

7.4.3 A Model of Aesthetic Mathematical Intentional Objects

The following model characterizes intentional objects in aesthetic experiences of mathematics, accounting for the issues discussed above:

(A) Aesthetic Mathematical Intentional Object (AMIO):
An intentional object is called *mathematical* when it is an intentional object associated to a mathematical item. A Mathematical Intentional Object (MIO) is called *aesthetic* when it is determined by a set of aesthetically relevant properties and structural relations; more specifically, when its associated phenomenological space (PS) and transforming operations (TO) comply with the following characterization:

(B) Phenomenological Space (PS):
A Phenomenological Space is a collection of at least two different properties, referred to as the dimensions of the space.

- (B.1) A dimension is a property introduced by an explicit interpretation or definition.
- (B.2) Every PS has at least one background-understanding dimension.
  - (B.2.1) A background-understanding dimension is a property that specifies the theoretical knowledge necessary to understand the mathematical item that constitutes the AMIO.
- (B.3) Every non-background understanding dimension is aesthetically relevant.

(C) Transforming Operation (TO):
A Transforming Operation is a set of rules that AMIOs follow in order to construct new AMIOs.

- (C.1) A TO is called *implicit* when it consists in the rules of logic and mathematical background knowledge.
- (C.2) A TO is called *meta-intentional* when it is not an implicit TO, and it is aesthetically conservative and intentionally relevant.

(C.2.1) A TO is *aesthetically conservative* in a phenomenological space PS if it is consistent with all the rules that define the dimensions of PS.

(C.2.2) A TO is *intentionally relevant* if it allows us to establish aesthetically relevant properties not present in a PS.

7.4.4 Aesthetic Form

Aesthetic mathematical intentional objects can be interpreted as a particular type of aesthetic forms. We must not confuse the notion of form in the arts and aesthetics with the technical notion of form in mathematics. Form in mathematics is usually interpreted as what remains invariant under the transformations of a given context. But in art disciplines, form usually refers to something different. In music, musical forms are the abstract structures that norm the organization of musical material, and even sound material; sonatas, rondos, cadences are examples of such structures. Poetic forms are also structures that norm the organization of words into lines and entire works; stanzas, sonnets, or haikus are instances of poetic forms. Forms in painting, sculpture or architecture are less abstract, as they are closely related to concrete spatial shapes in architecture and sculpture; or they are devised to mimic visual shapes in painting.

Now, aesthetic forms are closely related to our inner representations of the objects we are observing; that is, they are closely connected with intentional objects. Intentional objects are largely influenced by the modality, the type of experience—visual, auditory or intellectual. In general, all kinds of aesthetic forms are profoundly related to the modality of the experience involved in an aesthetic-process. The aesthetic form of a painting or a sculpture is closely related to its concrete visual or spatial structure; but the relation between the form of a poem or of a symphony is less closely related to the concrete visual properties of printed words or of heard stimuli. This is perhaps the most crucial feature that distinguishes one particular aesthetic experience from another. As far as I can tell, there is no single feature or set of features of intentional objects that can be used to characterize all possible aesthetic experiences. However, a significant insight can be gained by conducting local analyses of them if we complement it by locating it in the context of a wider theoretical framework like the aesthetic as process approach. I cannot offer a unified *notion* of aesthetic form that covers form in all kinds of artistic disciplines, however, I can offer a unifying *role* that aesthetic forms perform in aesthetic-processes. Even if aesthetic experience is different for different modalities of experience, aesthetic form performs the same role in all of them: it serves as the *focus and source* of the aesthetic experience. Aesthetic experience is constituted by its content and its associated pleasure response. Aesthetic form serves as the focus of attention; in this way it confers the aesthetic experience unity, even if the experience is performative, comprising dynamically changing mental activities. Aesthetic form is also the cause of the enjoyment (pleasure or displeasure) associated with the content of experience; in this way it lends the experience its aesthetic character.

7.5 Types of Experience

Aesthetic mathematical intentional objects are the characteristic content of mathematical aesthetic experiences. The peculiarities of these objects distinguish aesthetic experiences in mathematics from other kinds of aesthetic experiences. In addition to this, we can use the way the content of experience is involved in eliciting an affective response to further categorize mathematical aesthetic experiences. In this regard, I have identified three different types of experience based on the ways in which the content elicits affective responses.[6] The way in which content elicits responses induces a finer characterization of experiences. The content-response relations are differentiated by the particular way affective responses are elicited. I call those particular ways *appreciation responses*.

In the first type of appreciation response, enjoyment is elicited by passive contemplation, due to biologically conditioned affective responses to a stimulus. In the second type, the response is elicited by the performance of intellectual activities. In the third type, the response is elicited by *acquired* preferences, that is, by the preferences that have been modulated by the history of experiences of an individual. I label these aesthetic appreciation phenomena *basic*, *performative* and *adaptive*, respectively. Each appreciation response characterizes a different type of mathematical aesthetic experience, which we can label contemplative, performative, and adaptive.

7.5.1 Basic Appreciation Response

In basic aesthetic appreciation response (or basic response, for short) the affective response is the result of readily available affective responses to passive intentional objects, that is, objects not involving active mental contents.

Affective responses are involved in a wide class of behavioural and psychological phenomena, including emotions. I shall employ this fact to interpret the aesthetic response in mathematics. Emotions are systems of response to the environment that exhibit characteristic patterns of development consisting of an initial affective assessment of the situation followed by physiological changes and a further cognitive assessment of the situation [16, 21, 56, 76, 97]. As we have seen, the patterns of emotional response are not sequential in general; they unfold along relatively independent temporal paths. For our purposes here, the most relevant feature of emotional responses is the initial stage. This stage is an affective, *non-cognitive*, evaluation of a stimulus [28,53,73,96]. This crude initial appraisal classifies stimuli as belonging to one of two opposing categories: stimuli are classified as desirable

[6] It is not unlikely that further types of experience can be identified, but the three discussed here are very relevant for discussing the dynamics of aesthetic value, and the nature of aesthetic terms later on. Discussing further types of experience is a task better suited for future follow-up works.

or undesirable events; as praiseworthy or blameworthy agents; or as appealing or unappealing objects [73, 74]. Stimuli are thus classified in terms of *valences*. The desirable/undesirable and the appealing/unappealing valences are apt to deal with the kind of affective response we find in aesthetic experiences of mathematical items. It must be emphasized that this valences are not cognitive judgements. Although they can be expressed in verbal terms, that fact is merely a way of conceptualizing the automatic response that sets in motion an emotional episode *before* cognition sets in. For the purpose of characterizing aesthetic experience, we can make the simple assumption that the affective responses involved in our aesthetic experiences are similar to the non-cognitive appraisals associated to the automatic affective responses related to the desirable/undesirable or the appealing/unappealing valences. For the sake of brevity, I refer only to the appealing/unappealing valence hereafter.

My proposal is thus to view the pleasure response in basic responses as an affective response to a passive content of attention. Pleasure (or displeasure) is elicited as an automatic response, due to a cognitive input being classified on the appealing (or unappealing) side of the valence. The mere contemplation of the input stimulus results in a good feeling, a feeling of "I like it!". A similar mechanism is responsible for displeasure: the initial cognitive input is classified on the unappealing side of the valence; its mere presence results in a bad feeling, a feeling of "I don't like it!".

Basic aesthetic appreciation response thus involves the intentional objects able to elicit the affective responses associated with the valence polarity pair appealing/unappealing. We can characterize the first type of aesthetic experience as the experience-processes whose content involves basic appreciation responses.

Definition 1. An aesthetic experience is constituted by a *basic* aesthetic appreciation response if and only if the passive content of the experience can be classified by means of the appealing/unappealing valence.

A passive content of experience is a mental content that does not involve intellectual activities unfolding from one item to another. Theorems or formulas are examples of passive contents since they are items that can be contemplated without actively shifting attention to other items. Derivations and proofs are instances of items that require active attention, since in order to follow a derivation or a proof we need to shift our attention from one step of the derivation or proof to the next. For example, in addition to Euler's identity, Wells' list [94] includes theorems like 'π is transcendental', 'the number of primes is infinite' or the four-colour theorem. Interestingly enough, none of the entries in the list is a proof, the items are mainly contemplative.

Unlike proofs, theorems require merely contemplative attention. It is true that understanding any piece of mathematics requires a range of different passive and active kinds of attention. But in the case of single results, the appreciation of their extra properties does not involve further mental activities and in many cases the affective response is automatically elicited by the mere content of our attention, which makes them instances of basic appreciation phenomena.

Let us examine Euler's identity to illustrate basic responses. Our aesthetic experience of Euler's identity does not consist in the perception of concrete stimuli or the awareness of particular instances of the formula; rather, it consists in our awareness of its aesthetically relevant properties. In the aesthetic as process theory this fact corresponds to seeing the *intentional* Euler's identity as an intentional object existing in a phenomenological space whose dimensions are *CA* (complex analysis as background understanding), *simplicity*, and *composition*. Complex analysis allows us to "see" the object, whereas *simplicity* and *composition* play the role of eliciting an affective response, they are the aesthetically relevant dimensions. The contemplation of the intentional Euler's identity involves the awareness of its properties of simplicity and composition. Simplicity and composition are attractive, appealing, properties, we are prone to like them rather than to dislike them. That is, we are prone to affectively classify objects that possess these properties on the appealing side of the valence polarity; we experience an "I like it!" feeling when we are presented with such properties. Now, not all mathematical formulae are simple, nor do they involve the most important constants in mathematics; Euler's formula is and does. These facts are encapsulated in the properties of simplicity and composition and when we contemplate them, when we make Euler's identity our object of attention, we respond affectively to it. This contemplation does not involve further activities, since making it our object of attention (in the aesthetic sense) consists precisely in realizing its simplicity and composition.

In the case of Euler's identity, all we need for an affective response is passive contemplation (in the sense that our attention does not shift from item to item), and thus its experience involves a *basic* appreciation response.

The conception of basic response embraced here offers two advantages: first, it is clearly related to the affective response associated with having preferences for certain items. Affective responses play a central role in our aesthetic theory. Second, it differentiates aesthetic responses from emotions, but, at the same time, it allows us to establish a connection between them.

7.5.2 Non "Inductive" Response

Basic appreciation responses are characteristically mathematical because the intentional objects involved in them are characteristically mathematical. If we obviate this fact and think in terms of a broader class of intentional objects, music, narratives, poems, or other cognitive objects of attention, we can learn something about them.

The affective responses in basic aesthetic appreciation responses are connected with biologically conditioned responses. The existence of this kind of responses is evident in our preference for sweetness or aversion to bitterness. One of the features of basic experiences is that the *response* is non-cognitive; the response associated with the cognitive content—a theorem or a result—is an affective response. Another characteristic of these experiences is that, in the general case, even if the responses are elementary and non-cognitive; the inputs that trigger those responses are not necessarily elementary or non-cognitive. A green dot is simple in comparison with the sight of a natural landscape with green trees and grass; however, the landscape is more likely to trigger an affective response of "I like it!" than the mere green dot. The *basic* in basic appreciation response does not refer to the intentional object; it is the *mechanism* that triggers the experiences that is referred to as basic. In mathematics, the experience in our basic response must be triggered by a complex cognitive input; for example, an intentional object containing the mathematical item $e^{i\pi} + 1 = 0$.

By using a basic biologically conditioned mechanism to characterize basic aesthetic appreciation responses, my approach contrasts, again, with McAllister's. In McAllister's view, aesthetic preferences are formed through a history of experiences with certain properties via the aesthetic induction. Basic appreciation responses do not require a history of experiences, since the preferences involved are readily available, since they are biologically determined. Past experiences are not necessary to explain the preferences associated with basic appreciation phenomena. This further highlights that there are preferences that depend on a history of experiences and preferences that do not. My characterization of aesthetic experience takes this into account, as we shall see in the following sections.

7.5.3 Performative Appreciation Responses

In performative aesthetic appreciation responses the affective response is elicited by an *active* content. The main difference from basic responses is that in performative responses the affective response is elicited as the result of performing *intellectual activities* involving the content of attention. We have already discussed and explored mathematical proofs as instances of "active" mathematical items. That discussion can be generalized to mathematical items that involve shifts of attention and intellectual engagement similar to the ones found in mathematical proofs.

We have learnt that a significant role in eliciting enjoyment is played by a person's active mental engagement. Two factors contribute to elicit enjoyment: first, the mental activities performed can be pleasing (or displeasing) in themselves. Second, performing certain activities can modify an intentional object and the resulting object may possess new aesthetic properties; that is, our mental activities help us uncover (or, rather, construct) aesthetic properties not exhibited by the original intentional object.

Now, active engagement is also the source of enjoyment in literature or film. For example, people enjoy the act of anticipating the unfolding of events in the plot of a novel or film, and then witnessing their actual development. But in order to "see" the plot of a story one needs to know the entire set of individual events that constitute the plot. We need to "construct" the plot by mentally assembling it with the individual events presented to us. In music, performative mental engagement plays an even more central role. Peter Kivy argues that seeking hidden patterns and motifs is the main source of pleasure in listening to purely instrumental music [39, 41, 43, 44].

Mathematical proofs or derivations are strings of discrete statements which we have to follow to arrive to a conclusion, I exploited this fact in the narrative analogy in Chap. 3. In following a proof, we actively connect those discrete items. Appreciating a proof is not merely contemplating properties but it involves further mental activities. If the *active* part of the experience elicits an affective response we say we have a *performative* aesthetic appreciation response.

Let us examine an example. We have already presented the derivation of De Moivre's theorem. I now present another version of the theorem involving only integer exponents; I shall refer to it as *De Moivre's formula*:

$$(\cos x + \sin x)^n = \cos nx + i \sin nx$$

Its derivation from Euler's formula serves to illustrate the active content of experience:
Euler's formula:

$$e^{ix} = \cos x + i \sin x$$

The exponential law states:

$$(e^{ix})^n = (e^{inx})$$

Rewriting e^{inx}:

$$e^{inx} = e^{i(nx)}$$

Substituting in the first expression above yields:

$$e^{i(nx)} = \cos nx + i \sin nx$$

In this simple example our attention is focused not on a single item. Rather, it successively shifts from one item to another. We begin by focusing our attention on Euler's formula, then on the exponential law, then on associativity and finally on De Moivre's formula. Performing these activities is necessary to understand how De Moivre's formula is related to Euler's formula and thus to understand the derivation as a derivation. Theorems or formulae establish certain states of affairs concerning certain mathematical objects. Euler's identity establishes the identity between complex exponents and trigonometric functions. But a derivation, or a proof, for that matter, establishes the logical relations between different results. Now, the derivation not only establishes logical relations, it also helps us understand the results themselves: once we "see" the steps that takes us from Euler's identity to De Moivre's formula we see that, for example, the appearing of the term n in different places in the formula—once as an exponent and once as an argument of trigonometric functions—is not arbitrary: we understand why the term appears in both places, we understand their connection. Now, in order to gain this understanding, we need to actively follow the steps of the derivation, we must understand what each step does, we must check that each step makes sense, that there are no contradictions, that there are no errors, etc. The nature of derivations and proofs involves an active engagement of our attention.

Active engagement, of course, turns our experience of the mathematical item into an active experience. But aesthetic experience is characterized by an affective response. It is the affective response elicited by performing mental activities what makes performative experiences characteristically aesthetic. For example, the derivation of De Moivre's theorem from Wessel's geometric interpretation[7] of complex numbers is qualified as elegant by authors like Nahin. Let us remember that the derivation requires only the geometric interpretation of complex numbers; starting with a unit radius vector of direction angle $\frac{\theta}{m}$. It follows that

$$\{1\angle\frac{\theta}{m}\}^m = \{\cos\frac{\theta}{m} + i\,\sin\frac{\theta}{m}\}^m = 1\angle\theta = \cos\theta + i\,\sin\theta$$

Or turning this statement around by taking the mth root,

$$\{\cos\theta + i\,\sin\theta\}^{\frac{1}{m}} = \cos\frac{\theta}{m} + i\,\sin\frac{\theta}{m}$$

Something we immediately notice is that in this derivation our point of departure is more fundamental. While Euler's formula already establishes complicated and non-obvious relations, the geometric interpretation establishes a simple way to deal with complex numbers. Interestingly enough, this simple definition of complex numbers results in a shorter derivation of De Moivre's theorem. Now, noticing the simplicity of the geometric interpretation can be done only after we have gone through the entire set of steps of the derivation. The simplicity of the derivation is not a property of any of the individual steps, it is a property that emerges after seeing the derivation as a whole. And it is the simplicity of the derivation as a whole that results in an affective response. The simplicity in the derivation as a whole is of a different type than the simplicity in Euler's formula, furthermore, the affective response to the derivation-simplicity is not elicited by the merely passive contemplation of the definition or of any of the items involved in the derivation, but rather by facts related to our active engagement in following it: each of the steps makes simple assumptions or is simple in itself, and there are only few step in the derivation.

[7]Eulers' formula was proved in 1714 by Roger Cotes, and published in its current form by Euler in 1748. Wessel introduced his interpretation in 1799 in the *Royal Danish Academy of Sciences and Letters* but it remained obscure for some time.

This example thus exhibits two differences from the example of Euler's identity. First, the object of attention is of a different type: while in Euler's formula we have a single object of attention, in this derivation our attention focuses successively on the geometric interpretation, on the unit radius, on the first line of the derivation, and so forth. There is no single intentional object on which our attention focuses; the derivation is rather composed of multiple objects that are linked by logic, and, in our inner experience, by attention shifts. Second, we appreciate the brevity of the derivation, but this brevity does not appear as a property of any of the individual objects of our attention—the steps of the derivation, it is rather a property of the collection of those objects. Since the properties involved in eliciting an affective response are related to active engagements of our attention, the affective response cannot be elicited by mere passive contemplation.

Active attention is central to the appreciation of derivations, proofs and other mathematical items. We have seen that mathematical derivations or proofs resemble some aspects of narratives: they consist of a collection of individual events that develop in a coherent way to arrive to a closure. Now, just as not all stories are good, not all mathematical derivations or proofs are beautiful; this fact is many times related to the quality of their narratives. Only derivations or proofs whose narratives are able to elicit positive affective responses can be properly qualified as elegant or beautiful. The derivation of De Moivre's formula based on the geometric interpretation seems simpler than the one based on Euler's identity: it is easier to follow and it has a more fundamental and simpler premise than the derivation based on Euler's formula. Its narrative is thus more suitable to be qualified as pleasingly simple yet effective; that is, as *elegant*.

This elegance is related to the way the derivation "tells" its story; that is, to the way the derivation shifts our attention (the "plot" of the derivation) and how it reaches its conclusion (the "resolution" of the plot). It is not only the mere presence of an intentional object that is liable for the enjoyment of this experience, but rather the activities involved in following the "story".

Now, I have proposed two types of possible operations in phenomenological spaces: implicit and meta-intentional operations. These operations can model the intellectual activities that result in an affective response, or that allow us to "see" new properties that result in affective responses. The operations allow us to see properties in active experiences because they allow us to construct new intentional objects in a phenomenological space. For example, the derivation of DeMoivre's theorem from the geometric interpretation presented above is qualified as elegant due in part to its simplicity. But that simplicity is a property of the derivation as a whole. In order to take the effect of this simplicity into account, we need to interpret the derivation as a composite intentional object consisting of the collection of steps of the derivation. We can use, for example, the notion of logical consequence to link each new step to the previous one. This is permitted in our phenomenological space since the most basic type of implicit operation consists of the rules of logic. By including logical transformation operations, our experience consists not only of successive individual intentional objects, the steps of the derivation, but also of the object resulting from connecting all these steps by logical consequence.

We can attribute properties, like simplicity, to this newly constructed object that we cannot attribute to any of the individual steps. We can, for example, consider the influence of the number of steps involved in the derivation. We can introduce the phenomenological space dimension of *step-parsimony* defined as the property of consisting of very few steps. Of course, none of the individual steps can be qualified as being step-parsimonious, since they do not themselves have steps. But the composed intentional object resulting from the logical consequence operation can be qualified as step-parsimonious. *Step-parsimony* is one of the reasons why our derivation is qualified as elegant.

Our active experience of the derivation of the De Moivre's theorem occurs in a phenomenological space that allows the construction of a composed intentional object via the logical consequence implicit operation. This composed object, due to the property of *step-parsimony*, elicits an affective response. Our new phenomenological space thus includes the following dimensions—or properties visible in our experience: *algebra and analytic geometry*, as background understanding; *conceptual-simplicity*, defined, for example, as being understandable by a single mental act and without ad hoc concepts; and *step-parsimony*. It also includes the operation *logical-consequence*. Such a phenomenological space contains both the single intentional objects for each step and the resulting composed object for the whole derivation. In this case simplicity and step-parsimony together elicit an affective response. It must be noted that our appreciation of step-parsimony depends on our active construction of the composed intentional object; contemplating the derivation is not merely passive contemplation.

Just as there are affective responses resulting from the passive contemplation of intentional objects, there are affective responses associated with performing intellectual activities: just as we like or dislike certain stimuli, we like or dislike performing certain activities. Peter Kivy even argues that this is the source of our appreciation of purely instrumental music. We derive amusement and pleasure (or frustration and displeasure) from performing tasks such as seeking patterns and variation of patterns and motifs in a piece of music [39,41,43,44]. Thus, in addition to the passive enjoyment associated with the mere presence of an intentional object, there is an active enjoyment associated with the activities performed in an aesthetic experience and with new properties resulting from those activities.

We can now characterize the second way content and enjoyment relate to each other in aesthetic experience. In performative aesthetic appreciation responses the objects of attention involved in an active experience elicit an affective response, or the performance of the activities results in eliciting an affective response. In other words :

Definition 2. An aesthetic experience is constituted by a *performative* aesthetic appreciation response if and only if (1) The resulting content—an intentional object constructed in our inner experience—can be classified by means of the appealing/unapealling valence, or (2) The intellectual activities involved in the experience can be classified by means of the appealing/unapealling valence.

Table 7.1 Performative combinations

Combination no.	Passive content response	Active content response
1	Pleasure	Pleasure
2	Pleasure	Displeasure
3	Pleasure	None
4	Displeasure	Pleasure
5	Displeasure	Displeasure
6	Displeasure	None
7	None	Pleasure
8	None	Displeasure
9	(Non valid)	(Non valid)

Relation Content-Enjoyment of Performative Responses

There are two possible sources of enjoyment in active experiences: a passive, contemplative, source, resulting from the presence of an intentional object in our inner experience, and an active, performative, source, resulting from the constructed intentional objects or from performing the constructive activities. The relation between content and enjoyment is thus more complex in performative responses than in basic responses. In both cases the intentional object elicits an affective response, but in active experiences the content is active and the performed *activities* themselves can be a source of enjoyment. That is, in basic responses the only source of enjoyment is the object; in performative response both object and mental activities can be responsible for the enjoyment.

Objects and mental activities in our inner experience are irreducible to each other, just as physical objects and activities are irreducible to each other. The enjoyment derived from the objects and activities are conditioned by the features of objects and activities, and are thus also irreducible to each other. Even if constructed objects in performative response could be reduced to basic objects, there is a distinctive enjoyment associated with *performing* the activities that cannot be reduced to enjoyment of objects. The enjoyment derived from objects and activities are thus different. This yields a distinctive relation between the content and the enjoyment in performative appreciation responses.

As in basic appreciation responses, the active content of the experience is accompanied by a corresponding affective response of pleasure or displeasure. However, since I have used an inclusive-or in my definition, in performative responses we must consider the cases in which one of the elements of content does not elicit a response at all. This means that for the passive content the possible responses can be the eliciting of pleasure, displeasure or none. The same is true for active content. This results in the possible combinations shown in Table 7.1.

Response 9, no affective response whatsoever, is not actually possible, since that would amount to a non-affective, hence non-aesthetic, experience. The rest of the combinations are composed responses. Composed responses can be illustrated by the derivation of DeMoivre's theorem. In this derivation we experience a response of pleasure caused mainly by the parsimony of the derivation. This parsimony is not a property of any single step of the derivation, but of the derivation as a whole; it is a property of our constructed intentional object. Of course, there is an active element in the content, related to following the derivation, but these activities are not necessarily the source of our affective response, for the sake of argument we can assume that this active element results in no response. This situation corresponds to combination 3.

The best pleasure eliciting combination is, of course, combination 1; it is a "full pleasure" combination in which both the passive contemplation and the active element are pleasing. The relevant issue here is that the possible responses in performative responses constitute a more complex set of combinations than the mere set of pleasure and displeasure in basic responses.

7.5.4 Adaptive Appreciation Response

The third class of aesthetic experience in mathematics is characterized by adaptive aesthetic appreciation responses. In this type of responses we must take into account the fact that preferences change over time and that this change is influenced by a history of experiences. In adaptive aesthetic appreciation responses (or adaptive responses, for short) the passive or active character of the content is less relevant: the distinctive feature of adaptive aesthetic appreciation response is the *mechanism* liable for forming the preferences involved in eliciting the affective response. In adaptive responses the affective responses are the result of *acquired* preferences. As we know, the eliciting of our responses is affected by our histories of previous experiences.

Basic and performative responses are characterized by the fact that their content is able to elicit an affective response; the passive or active content of the experience triggers a response, a feeling of pleasure or displeasure. In this type of circumstances I shall say that the content of the experience *invokes* an affective response. In adaptive responses, the response is not elicited as a result of a readily available preference, but as a result of a preference we have *acquired*; that is, the intentional object possesses properties to which we have adapted to like or dislike. This is an acquired eliciting of enjoyment. In these circumstances, I shall say that the content of the experience *evokes* an affective response. It must be noted that an acquired responses may become strongly internalized. In fact, acquired responses (of fear, for instance) exhibit the same patterns of physiological arousal as biologically conditioned responses [28, 54, 76].

As McAllister pointed out, there is abundant evidence of evolving preferences in mathematics. Complex analysis offers interesting examples. Imaginary numbers were not fully understood until the sixteenth century. Its introduction was plagued with suspicion, caution and even aesthetic revulsion [68, pp. 16–17]. But imaginary and complex numbers eventually became part of the basics of mathematics and, as we have seen, a source of many "elegant calculations" [68, pp. 48–55]. Now, two factors played a role in changing our appreciation of complex numbers. First, complex numbers allow us to achieve shorter, more easily understandable mathematical derivations and proofs. That is precisely the case with the introduction of the geometric interpretation that led to a more elegant derivation of De Moivre's theorem discussed above. Second, mathematicians developed a familiarity with complex numbers and the mathematical items (proofs, derivations, theorems, etc.) that involve them. This familiarity, as is evident in the mere-exposure effect, eventually resulted in a change in our preferences. In this respect, an accurate model of preference evolution is relevant here. For example, as we know, the aesthetic induction or even the mere-exposure effect are insufficient to account for some patterns of evolution. Familiarity in mathematics exhibits one of those patterns. Familiarity with complex numbers may not result in an increase in preference. Even worse, the properties and interpretation of complex numbers can become so familiar that we may end up finding them unremarkable. For example, Le Lionnais bears witness to the fact that not all mathematicians find the "most beautiful formula of mathematics", Euler's Identity, so remarkable:

> Euler's formula [...] establishes what appeared in its time to be a fantastic connection between the most important numbers in mathematics, 1, $\sqrt{-1}$, π, and e. It was generally considered "the most beautiful formula of mathematics." The brilliance of this expression is due to the nearly perfect elimination of every element foreign to the three numbers just cited. Today the intrinsic reason for this compatibility has become so obvious that the same formula now seems, if not insipid, at least entirely natural [58, p. 128].

David Wells' readers, who ranked the formula as number one in the list of the most beautiful, obviously disagree with calling the formula insipid. Furthermore, Wells himself employs Le Lionnais' opinion to show that aesthetic preferences in mathematics change over time [94, pp. 38–39]. This disagreement also illustrates the variety of adaptive aesthetic responses. Le Lionnais' opinion is a case of negative acquired preference: according to him, the initial attractive composition of Euler's identity lost its appeal as we accumulated experiences with the items and principles involved in the formula. In these circumstances, our response is not elicited via a readily available response, but via a preference shaped by an evolution mechanism that involves past experiences and acquaintances with similar or related items.

To stress the contrast in the way the content of an experience elicits enjoyment, I introduced the terms 'invoking' for readily available eliciting, and 'evoking' for acquired eliciting. Evoked enjoyment is thus closely related to the mechanism that drives preference evolution. We have seen that some preferences vary depending on the recurrent presence of certain properties in empirically adequate theories. Some other preferences tend to remain stable; these preferences can be seen as closely related to invoked enjoyment.

If we concentrate on preferences which change driven by the "inductive" element in the evolution mechanism—the ones possessing a low degree of robustness, as discussed in Chap. 5—we can characterize the evolution of evoked enjoyment in a simple way: the exposure to certain stimuli or certain intellectual activities can induce a change in the elicitation of our feeling of pleasure (or displeasure). For example, in the case of Le Lionnais' response to Euler's formula, his response has been shaped by the familiarity he has with the formula and the results that underlie it, as he recognizes himself. But in the absence of familiarity, "in its time" as Le Lionnais puts it, the response was very enthusiastic; the formula was considered, as he states, the most beautiful formula of mathematics. This is a case in which familiarity has an adverse effect.

The effect of familiarity is well known in psychology. But using familiarity to explain preferences is not alien to aesthetics either. For example, Romantic approaches to music explain its emotional impact by arguing that the development of music resembles the development of emotions, or the development of life itself. More contemporarily, Jenefer Robinson's theory of expression asserts that an object—a pictorial representation, for example—expresses a certain emotion if it holds appropriate similarities to the way the world appears to a person experiencing the emotion.

We can now characterize the third way in which content and enjoyment relate to each other in aesthetic experiences: Adaptive aesthetic appreciation responses occur when the passive or active content of the experience evokes an affective response.

Definition 3. An aesthetic experience is characterized by a *adaptive* aesthetic appreciation response if and only if (1) The content of the experience can evoke an affective response of pleasure or displeasure, or (2) The mental activities involved in the experience can evoke an affective response of pleasure or displeasure.

For example, Euler's identity seems to elicit a positive affective response, except when one is too familiar with it. We have seen that the intentional Euler's identity is located in a phenomenological space with *complex-analysis*, *simplicity* and *composition* as dimensions. All these dimensions should be part of Le Lionnais' experience, since neither our background understanding nor the complexity (simplicity) or the components (composition) of the formula have changed. The properties of the intentional object are the same; the object is thus the same: Wells and Le Lionnais experience a similar intentional Euler's identity in a similar phenomenological space. But due to Le Lionnais' familiarity with the formula, it appears to him unremarkable and even insipid. The difference is not in the passive or active content of the experience, the difference is in how past experiences with the content have changed the effect of the properties of the object for Le Lionnais. Whereas for Wells and his readers the simplicity and composition of Euler's formula are remarkable, for Le Lionnais they are too natural and too obvious. In Le Lionnais's case, his acquired preferences play the central role in constituting his experience. This is a case of adaptive appreciation response, as the passive contemplation of the formula evokes, rather than invokes, the response of insipidness.

Adaptive Content-Response Relation

The relation between content and enjoyment in adaptive response is a little more complicated than the relation in performative responses. We can start by

Table 7.2 Adaptive combinations

Combination no.	Passive content evokes	Active content evokes	Passive content invokes	Active content invokes
1	Pleasure Pleasure	X	X	X
2	Pleasure	Displeasure	X	X
3	Pleasure	None	X	None
4	Pleasure	None	X	Pleasure
5	Pleasure	None	X	Displeasure
6	Displeasure	Pleasure	X	X
7	Displeasure	Displeasure	X	X
8	Displeasure	None	X	None
9	Displeasure	None	X	Pleasure
10	Displeasure	None	X	Displeasure
11	None	Pleasure	None	X
12	None	Pleasure	Pleasure	X
13	None	Pleasure	Displeasure	X
14	None	Displeasure	None	X
15	None	Displeasure	Pleasure	X
16	None	Displeasure	Displeasure	X
Non valid	(None)	(None)	X	X

establishing an analogy between the way the content elicits pleasure or displeasure in performative response and the way the content in adaptive response elicits a response just by replacing 'invoking' with 'evoking' an affective response. We can have purely passive content; the experience of Euler's formula, for example, does not involve active content. We can also have active content, in proofs or derivations, for instance. But from the inclusive-or in Definition 3, it follows there are cases in which the passive or active component that does not evoke a response still can invoke a response. In principle, we can have a case in which the passive content evokes a response and the active content does not evoke a response, but this active component still can invoke a response. Taking this into account, the possible combinations that constitute the relation between content and enjoyment in adaptive response can be summarized as shown in Table 7.2.

Note that the option of invoking a response for a given active or passive content is only available if there is no response evoked by that same content, otherwise the response should be considered as performative; I have employed the symbol 'X' to represent that the respective response in the table is not possible. For example, a combination of four pleasure responses is absent from Table 7.2, for the two invoked pleasure responses entail the two evoked responses. Although such a combination is possible in principle, it is better classified as a performative, rather than a adaptive combination.

Adaptive responses are more complex than performative responses. They not only have more possible combinations, they also include what I call *confusing* combinations. In the case of performative responses I pointed out that the full-pleasure combination (Pleasure, Pleasure) seems to render a clear pleasure response even for complex active experiences like proofs. However, in adaptive responses we have three different cases—1, 4 and 12—that render a similar full-pleasure response. The same phenomenon occurs with full-displeasure responses (Displeasure, Displeasure)—combinations 7, 10 and 16. This fact illustrates the complexity of aesthetic experience itself and it has consequences in other elements of aesthetic-processes particularly concerning aesthetic terms and aesthetic judgement, as we shall discuss later, in the respective chapters.

7.6 The Pleasure-Relation

Basic, performative and adaptive appreciation responses constitute the ways in which content and enjoyment relate in mathematical aesthetic experience. We have seen that in the case of basic responses the relation between the content of the experience and its affective response is very simple: an object of attention elicits either a pleasure or a displeasure response. In performative responses, due to its passive and active components, the possible response consists of eight possible response combinations. In adaptive responses we have sixteen possible response combinations. We can use these facts to model aesthetic pleasure.

It is easy to see that content and enjoyment in aesthetic experience have very different constraints. The content of experience is relatively independent of the experience itself, since an intentional object is determined by features in the original mathematical item and subjective dispositions already present in the observer. Pleasure, by contrast, is triggered by the passive presence of intentional objects or by performing of mental activities. In other words, the enjoyment depends on the content. We have also determined the possible ways in which pleasure is related to the content of experience. All this information can be summarized by saying that the content of an experience is an independent variable, and that pleasure is a variable that depends on the content. There is a dependence relation between pleasure and the content of experience.

The relation between the intentional object and the affective response resembles a function in the sense that it expresses the dependence between two entities. However, there is an important difference: a function associates a single output with an input, but in the case of the pleasure-relation, an intentional object, the input, can result in different affective responses, the outputs, depending on the context—the same object, a beautiful proof, for instance, can be involved in performative or adaptive responses. Several components of an aesthetic-process—the change of value over time, for instance—play a role in determining the affective response. Despite this large-scale dependence, the local dependence of affective response on intentional objects still captures some important features—the multiplicity and complexity of the possibilities of response, for example—that later on shall help us clarify some issues related to the production of aesthetic judgements. For this reason, I devote this section to modelling the local features of the pleasure response.

Pleasure—an affective response—is the result of the content of experience. The content can be seen as a variable with two components: the passive and active content. Pleasure, the affective response, can be seen as a variable with two components corresponding to the passive and active responses. We can define aesthetic pleasure as a relation that maps the content of experience into a set of ordered pairs that represent the possible combinations of responses. As before, we can call these responses or combinations of responses simply *enjoyment*.

We can thus define enjoyment simply as the set *ENJOYMENT* of all possible combinations of affective responses. This includes the 2 possibilities pleasure and displeasure for basic response; the 8 combinations for performative response; and the 16 for adaptive response. Aesthetic pleasure in mathematical aesthetic experiences can be informally characterized as follows[8]:

$$f: \textit{content} \rightarrow \textit{ENJOYMENT}$$

7.6.1 Formalization

Let us now formalize the idea of the pleasure-relation, starting by defining our vocabulary.

$$PC = \{Pas | Pas \text{ is an Aesthetic Mathematical Intentional Object}\}$$

$$AC = \{Act | Act \text{ is a mental activity related to an Aesthetic Mathematical Intentional Object}\}$$

$$PR = AR = \{P, D, Ep, Ed, N\}$$

where:

Pas: the variable for the passive content of the experience. The range of Pas consists of all possible Aesthetic Mathematical Intentional Objects; including no content at all, that is, no object of attention. In this context the symbol $\varnothing$ represents empty content, no object of attention.

Act: the variable for the active content of the experience. The range of Act consists of all possible intellectual activities performed by our attention, including no activity at all. In this context the symbol $\varnothing$ represents no activity.

Rp: the variable for the passive affective response.

Ra: the variable for the active affective response. The range of Rp and Ra is the set $PR = AR = \{P, D, Ep, Ed, N\}$

[8] Although the notation f is usually employed to refer to proper functions, I shall retain it instead of r, for example, in order to avoid confusion with other occurrences of ‘r’ in the discussion.

where:

P: an invoked pleasure response.
D: an invoked displeasure response.
Ep: an evoked pleasure response.
Ed: an evoked displeasure response.
N: no affective response.

We can now define all the ways of relating possible contents of experience to possible responses. This relation includes contents of attention that are completely empty $\langle \varnothing, \varnothing \rangle$ and responses that have no affective response $\langle N, N \rangle$. These cases cannot be categorized as *aesthetic*, since they amount to either an empty experience (in that case we cannot talk about aesthetic *experience*, since there is no content at all), or no affective response (if there is an actual content but no affective response, we have a kind of experience that is not *aesthetic*). There are many ways in which our attention can become engaged without arousing any affective response, but these are just episodes of *attention*, not episodes of aesthetic experience. Experiences with no content or with no affective response do not participate in aesthetic-processes. However, the cases of non-aesthetic experience can be considered limiting cases of "attention experience". Since the relation that admits aesthetic as well as non-aesthetic episodes of attention does not characterize aesthetic experiences but rather all kinds of episodes of attention, I call this relation *attention-relation*. The attention-relation contains all ordered pairs $\langle content, response \rangle$, including the limiting non-aesthetic and empty cases.

The attention-relation is defined as follows:

$$Attention\colon (PC \times AC) \times (PR \times AR)$$

This relation comprises all possibilities, including basic, performative and adaptive aesthetic appreciation responses. Appreciation responses are thus proper subsets of the attention-relation.

Now, we can define a general pleasure-relation as the subset of the attention-relation such that its elements consist of the ordered pairs in which the first coordinate is not empty and the second coordinate is an actual affective response. A non-empty first coordinate is any possible ordered pair $\langle Pass, Act \rangle$ in which at least one coordinate is a non-empty content; in other words any possible pair except $\langle \varnothing, \varnothing \rangle$. An actual affective response is any possible ordered pair $\langle Ra, Rp \rangle$ in which one of the coordinates is an actual response; in other words any possible pair except $\langle N, N \rangle$. In this way, we guarantee that our general pleasure-relation contains only non-empty experiences that do elicit affective responses. Experiences with solely an active content $\langle \varnothing, Act \rangle$ cannot be characterized as mathematical in our model and must also be excluded. Finally, we must exclude cases in which an affective response has no associated content. For example, if the content of attention is just an aesthetic mathematical intentional object and we contemplate it passively—that is, there is no active content—the associated response cannot have an active response.

Thus we must exclude combinations like $\langle\langle Pas, \varnothing\rangle, \langle P, P\rangle\rangle$. We can use a material implication ($\Rightarrow$) to express this condition as follows:

$$(Pas = \varnothing \Rightarrow Rp = \varnothing) \wedge (Act = \varnothing \Rightarrow Ra = \varnothing)$$

I label this condition the *Causal Condition*. In order to simplify the notation I write the symbol CC to stand for (or, being equivalent to, $\equiv$) the causal condition stated above:

$$CC \equiv \forall Pas, Act, Rp, Ra((Pas = \varnothing \Rightarrow Rp = \varnothing) \wedge (Act = \varnothing \Rightarrow Ra = \varnothing))$$

We can thus define the general pleasure-relation as follows:

$$\begin{aligned} \mathit{Pleasure} = \{\langle x, y\rangle | \langle x, y\rangle \in \mathit{Attention} \\ \wedge\, x \neq \langle\varnothing, \varnothing\rangle \wedge y \neq \langle N, N\rangle \wedge \forall Act(x \neq \langle\varnothing, Act\rangle) \wedge CC\} \end{aligned}$$

We can now define pleasure-relations for our three kinds of appreciation responses. I label them basic, performative and adaptive pleasure-relations.

The basic pleasure-relation consists of the ordered pairs whose first coordinate contains no active content, and the second coordinate contains no evoked response:

$$\begin{aligned} \mathit{BasicPleasure} = \{\langle x, y\rangle | \langle x, y\rangle \in \mathit{Pleasure} \wedge \exists Pas(x = \langle Pas, \varnothing\rangle) \\ \wedge\, \exists Rp(y = \langle Rp, \varnothing\rangle) \wedge Rp \notin \{Ep, Ed\}\} \end{aligned}$$

The performative pleasure-relation consists of the ordered pairs whose first coordinate contains an active content, and the second coordinate contains no evoked response:

$$\begin{aligned} \mathit{PerformativePleasure} = \{\langle x, y\rangle | \langle x, y\rangle \in \mathit{Pleasure} \wedge \forall Pas(x \neq \langle Pas, \varnothing\rangle) \\ \wedge\, \forall Pas(x \neq \langle\varnothing, Pas\rangle) \\ \wedge\, \exists Rp, Ra(y = \langle Rp, Ra\rangle \wedge Rp, Ra \notin \{Ep, Ed\})\} \end{aligned}$$

The adaptive pleasure-relation consists of the ordered pairs whose second coordinate contains an evoked response:

$$\begin{aligned} \mathit{AdaptivePleasure} = \{\langle x, y\rangle | \langle x, y\rangle \in \mathit{Pleasure} \\ \wedge\, \exists Rp, Ra(y = \langle Rp, Ra\rangle \\ \wedge\, (\exists Rp, Ra(Rp \in \{Ep, Ed, N\} \\ \vee\, Ra \in \{Ep, Ed, N\}))))\} \end{aligned}$$

It should be noted that the general pleasure-relation does not characterize a mathematical aesthetic experience, since there are many ways of characterizing the relation between content and responses. Only the pleasure-relations defined above characterize mathematical aesthetic experience.

We can characterize the modality of mathematical aesthetic experience as follows:

An aesthetic experience is a *mathematical aesthetic experience* if and only if $Pas \in \{O|O$ is an Aesthetic Mathematical Intentional Object$\}$ and there is a relation between the content of experience ce and its affective response ar such that:

$$\langle ce, ar \rangle \in BasicPleasure \cup PerformativePleasure \cup AdaptivePleasure$$

We can now reformulate our pleasure-relation. Let us define:

$$Content = \{x|\exists x(\langle x, y \rangle \in BasicPleasure \cup PerformativePleasure \cup AdaptivePleasure)\}$$

$$ENJOYMENT = \{y|\exists x(\langle x, y \rangle \in BasicPleasure \cup PerformativePleasure \cup AdaptivePleasure)\}$$

A pleasure-relation P_r for mathematical aesthetic experience is defined by:

$$P_r \subseteq Content \times ENJOYMENT$$

Pleasure and Aesthetic Terms

The interpretation of aesthetic pleasure above yields an interesting insight. One of the arguments Rota presents for reinterpreting mathematical beauty is that 'beauty' is a concept that does not admit degrees [78]. This idea, however, is discredited by facts such as the existence of comparatives of the type "A is more beautiful than B". Now, the idea of measuring the degree of beauty might sound strange. Fortunately, my interpretation of the pleasure-relation allows us to conceptualize the degree of beauty without the need for a measure. In the characterization, we can see that, in the general case, there are several possibilities for affective response. We certainly have cases where a pleasure-response does not seem to admit degrees—perhaps this is why Rota believes beauty does not admit degrees: in the case of experiences of basic response the pleasure-relations renders only pleasure $\langle P, N \rangle$ or displeasure $\langle D, N \rangle$. However, in the general case, the relation renders composed responses—$\langle D, P \rangle$, for instance. It also seems clear that the total enjoyment in full-pleasure combinations $\langle P, P \rangle$ is more enjoyable, so to speak, than combinations in which only one argument renders pleasure. Comparing different combinations can be consistently done by using comparatives, without the need for attributing specific degrees of beauty.

There is another consequence of this interpretation of aesthetic pleasure. If we try to link opposed predicates like ‘beautiful’ and ‘ugly’ to an appropriate pleasure-relation we get an interesting puzzle: As discussed in Chap. 3, the most obvious association we can establish, of course, is to link beauty to pleasure and ugliness to displeasure. This works well for basic responses, since we have only two possibilities. But for performative responses the situation is more complicated. We have eight possible combinations. A compromise here would be to associate beauty with full-pleasure $\langle P, P\rangle$ responses and ugliness with full-displeasure $\langle D, D\rangle$ responses. However, in the case of adaptive responses we have three full-pleasure combinations $\langle E_p, E_p\rangle$, $\langle E_p, P\rangle$, $\langle P, E_p\rangle$, and three full-displeasure combinations $\langle E_d, E_d\rangle$, $\langle E_d, D\rangle$, $\langle D, E_d\rangle$. Now, the problem is not how to assign a predicate to a response in the relation. There is nothing that prevents us from making the assignments to an arbitrary full-pleasure response. However, if we try to interpret what it means that the predicate ‘beautiful’ is the opposite of ‘ugly’ in terms of its associated pleasure-relation, some questions arise: what do the non-assigned possibilities in the relation mean? How should we interpret the negation of the predicate beauty; should we interpret it as one, several, or all the possible responses of the relation, or just as the complementary set of responses? Furthermore, do we need an inverse relation for ugliness? What is the relation between “mirror” responses (responses that have the opposite position for pleasure and displeasure) of the relation? Now, I believe this problem does not arise from the interpretation of the predicate in terms of the pleasure-relation, but rather from the assumption that the relation between beauty and ugliness is just that of their being opposites. I believe there is a significant relation between those predicates, but conceptualizing them merely as opposites is insufficient. In addition, there remains the fact that even if we assign terms like ‘beautiful’ or ‘ugly’ to particular responses, there is a whole range of unassigned combinations (the remaining possible responses) in the pleasure-relation. I think all this can be construed as showing that the conceptualization of aesthetic experiences in terms of predicates is open to many interpretations. I shall further explore these ideas later on, in the chapters on aesthetic terms and judgement. These issues also show that aesthetic issues are intimately connected with each other, but for the sake of organization, I shall stop the discussion of aesthetic experience here and further it in the coming chapters when necessary.

Chapter 8
Aesthetic Value

In this chapter I develop an extensional notion of aesthetic value. To this end, I shall use McAllister's findings, my revision of them, and some of Goldman's ideas.

Let us remember that Goldman conceives aesthetic value as a relation between objective properties in an object and subjective responses in an observer. More specifically, he interprets *evaluative* aesthetic properties as constituted by relations between non-evaluative properties and reactions in observers [29, p. 45]. One reason why Goldman's approach to aesthetic value is attractive to this project is that his proposal captures a tenet that aesthetic theories have assumed ever since it was introduced by Kant: that evaluations of objects in aesthetic terms are grounded on *subjective* matters. A second reason why Goldman's view is attractive is that it is consistent with the idea endorsed here that affective responses are involved in aesthetic experiences. Affective responses offer a way of connecting the subjectivity involved in aesthetic experience with the one in aesthetic value. Thus, I shall further use affective responses to elaborate the concept of aesthetic value.

Now, affective responses, as we have discussed in the previous chapters, can be interpreted as non-cognitive evaluations of stimuli. However, in order to more accurately characterize the role of affective evaluations in propositional evaluations such as "Euler's identity is the most beautiful theorem in mathematics" we need to address affective responses in a different way. Fortunately, James McAllister's conception of aesthetic criteria provides us with an effective way of dealing with evaluations in a concrete cognitive manner. An aesthetic criterion expresses a person's or community's preference for a property P in the form of a propositional rule. For example "if a theory has P, attach more aesthetic value to it than, if other circumstances are equal, it did not". Since a person's preferences are often actualized as affective responses towards objects or stimuli, an aesthetic criterion can be seen as expressing the objective—in the sense of independent of the subject—conditions for eliciting responses. The advantage of aesthetic criteria is that they are explicit and they are not necessarily limited to a specific person; they can also be used to express the tendencies of a community. Aesthetic criteria characterize a normative link between preferences, objects and values. In other

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_8,

words, aesthetic criteria express normative relations between objective states of affairs and subjective responses. Since evaluative aesthetic properties are conceived as relations between objects and responses, aesthetic criteria as characterized above, are instances of evaluative aesthetic properties. For the sake of brevity, I label them *evaluative-instances*. Now, the extension of the concept of aesthetic value comprises the collection of all things it applies to; that is, the exhaustive collection of all possible evaluative-instances. Thus, the extension of the concept of value can be expressed as a set of aesthetic criteria. Conceiving aesthetic value in this extensional way allows us to use set theoretical tools and techniques to gain insight into aesthetic value, which shall be useful below.

Now, to fully characterize aesthetic value in the context of an aesthetic-process, we need to determine two things: the content of the concept and, perhaps more importantly, the role it plays in an aesthetic-process. Here, we shall deal with the *content of the concept* in an extensional manner. The *role* of value in an aesthetic-process shall be interpreted as *to provide the norms that are actualized in evaluations*. There are two ways in which an aesthetic value can be actualized: as a non-cognitive affective evaluation, or as a propositional evaluation expressed in an aesthetic judgement. Aesthetic preferences embody the affective side of a value, whereas propositional aesthetic criteria embody the cognitive side. In this sense, preferences, aesthetic criteria and evaluative-instances covary; that is, they can only change together.

We have seen that an approach focused merely on aesthetic properties is insufficient to account for instances of beauty like the beauty of mathematical proofs. In our analysis of aesthetic experience, we addressed the necessity of incorporating mental activities and mentally constructed objects as sources of affective responses, in addition to properties. Those considerations must be taken into account in developing our notion of aesthetic value. A way to do that is to generalize Goldman's ideas. For Goldman, evaluative properties are relations between properties of objects and reactions in the observer. I shall embrace a generalization of this idea which includes properties, mental activities and *sets* of properties and mental activities, including sets with both properties and activities (property/activity sets, for short), and their associated affective reactions to define the extension of the concept of aesthetic value as follows:

8.1 Extension of Aesthetic Value

The extension of aesthetic value comprises all relations between properties, mental activities and sets of properties and mental activities associated with an object; and their resultant positive or negative reactions in the observer.

Including sets of properties and activities in our conception of aesthetic value allows us to establish a connection between aesthetic values and the intentional objects in the phenomenological space of our aesthetic experience in appreciating objects. Of course, the introduction of property/activity sets involves some complications. For example, for any set S associated with an affective response, we can ask about the aesthetic value of its subsets, supersets and partially overlapping sets. The approach formulated above involves an exhaustive collection of evaluative-instances; that collection should include the subsets, supersets and overlapping sets as elements only if they themselves have an associated affective response.[1] That is also one of the reasons why in order to avoid paradoxes or other inconsistencies the collection of value-instances is construed as a set in the technical set-theoretical sense. I label the set of all possible evaluative-instances the *Value Set*. It must also be noted that my characterization of aesthetic value is restricted to sets of properties associated with actual affective responses. My approach is thus descriptive rather than normative regarding which objects or qualities possess aesthetic value.

This description of aesthetic value, along with the role it plays in a typical aesthetic-process, can be summarized by interpreting aesthetic value as a *repository of all actual norms of evaluation involved in an aesthetic-process*. More formally, aesthetic value is the set of ordered pairs $\langle S, r \rangle$, where S is a set of properties and mental activities associated with an object, and r its associated response. S is not a single property but a set itself. More specifically, S is an element of the power set of the set T of all descriptive properties and mental activities of objects. That is:

$$S \in \wp(T)$$

where: S is a property/activity set; and T is the set of all descriptive properties of objects and all mental activities associated with contemplating those same objects.

As for the second component of the ordered pair we have:

$$r \in ENJOYMENT$$

where:

$$\begin{aligned} ENJOYMENT = \{y | \exists \langle x, y \rangle \in & BasicPleasure \\ & \cup PerformativePleasure \\ & \cup AdaptivePleasure\} \end{aligned}$$

The value-set can be simply expressed as:

$$V = \{\langle S, r \rangle\}$$

The values we held to are different for different types of aesthetic experience: in the appreciation of painting, for example, the properties and mental activities involved are different from the ones involved in the appreciation of music. We can envisage different value repositories for different aesthetic-processes depending on

[1] This approach is consistent with the view that no set of non-aesthetic properties determines an aesthetic property [82].

the type of aesthetic experience involved. In the case of mathematics the properties, mental activities and responses involved in aesthetic value are the same as those involved in mathematical aesthetic experiences. Those properties, activities and responses determine the content of the value repository for aesthetic-processes in mathematics. For example, the property of simplicity has appeared in our experience of the derivation of De Moivre's theorem. Our notion of value tells us that our repository should include the ordered pairs that associate the positive reaction to this derivation with the set $\{simplicity\}$, as follows:

$$\langle\{simplicity\}, \langle P, N\rangle\rangle \in V_M$$

where V_M is the value repository for *mathematical* aesthetic value.

The fact that a value repository like V_M depends on the type of aesthetic experience tells us that there are different types of value. We can think of the general notion of aesthetic value as comprising different types of value, corresponding to different types of experience.

The division of value into different types of repositories can be understood as modelling different types of aesthetic value; musical value, mathematical aesthetic value, negative aesthetic value, etc. The application of predicates to the concept of value can thus be interpreted as taking a subset from the general value set. That subset is the repository governing aesthetic experiences qualified by the same predicate: musical experience has an associated musical value repository, negative aesthetic experience has a negative value repository, and so forth.

We can also interpret the individual elements of a value repository—its evaluative-instances—as the particular preferences that are actualized in an individual aesthetic-process. For example, in the case of Euler's identity, simplicity plays a role in eliciting a positive response. This means that the value repository at work in mathematical appreciation (the mathematical positive value repository) includes $\langle\{simplicity\}, \langle P, N\rangle\rangle$.[2] This pair models the particular preference actualized in our affective response, and in the public description of Euler's identity as beautiful.

Among the different types of value repositories there is a very salient distinction between positive and negative value. There are negative as well as positive aesthetic evaluations. There is bad music, bad painting, as well as ugly pieces of mathematics—let us remember that most theories on mathematical beauty are somehow unsatisfactory when it comes to addressing mathematical ugliness. We can envisage two value repositories characterized by their positive or negative associated responses. Positive and negative value repositories are particularly important when considering the dynamical aspect of preferences and values. Positive and negative preferences played the central role in revising McAllister's aesthetic induction.

[2] In addition to this individual value, our value repository includes pairs associated with the simplicity of different kinds of experience, for example simplicity in the derivation of De Moivre's theorem's; $\langle\{simplicity, parsimony\}, \langle P, N\rangle\rangle$.

The evolution pattern of positive constant preferences might be consistent with McAllister's model, but negative ones are not. An examination of the dynamics of these two repositories is thus in order.

We have discussed how a person's or a community's history of experiences affect to different degrees the evolution of their preferences. Whereas McAllister's model of the aesthetic induction seems to capture some important patterns of evolution documented historically, it also exhibits significant anomalies. The model advanced in Chap. 5 addressed those anomalies. The preferences of a person or a community are most of the time not explicitly available, but rather only implicitly held. Now, aesthetic criteria can be interpreted as explicit, hence public in principle, expressions of preferences. We can thus utilize explicit aesthetic criteria to monitor, to track, implicit preferences. Utilizing explicit aesthetic criteria has the further advantage that the evolution of preferences can be modelled by attaching to each individual criterion a weighting proportional to the degree of intensity of the preference it tracks. In this regard, the discussion encompassing Chaps. 2, 4 and 5 shall now pay off again. In Chap. 5 we saw that by interpreting sets of aesthetic criteria as systems, the different models of evolution differed only in the evolution rule that determines the dynamics of the system. Moreover, that evolution rule was expressed in terms of the parameters attached to a property P; its weighting, critical adequacy and robustness. None of those parameters depend on whether P is a single property or a more complex set of properties or other items. The Naturalized Evolution Rule proposed in Chap. 5 is committed only to a description of the change in the weightings and not to a particular explanation of it; for that very reason, it is neither committed to a particular kind of objects to which the parameters are attached. The introduction of property/activity sets as the entities to which we attach the parameters of weighting, critical adequacy and robustness does not affect the evolution rule itself. Thus, we can use a variation of the Naturalistic Evolution Rule to model the evolution of aesthetic value, just as we used it to model the evolution of the aesthetic canon. To accomplish this we need to introduce the appropriate concepts. First, aesthetic criteria.

8.2 Typical Positive Aesthetic Criterion

If there exists a set S of descriptive properties and mental activities associated with contemplating an object O and those properties and activities are conjunctively responsible for eliciting an affective response, then more aesthetic value is associated with O.

For example, the property of simplicity plays a central role in the aesthetic experience of the derivation of De Moivre's Theorem or Euler's identity. In these cases our property/activity set is $\{$*simplicity*$\}$. The aesthetic criterion at work in our evaluations is:

If the simplicity of a theorem results in an affective response, then more aesthetic value is associated with $\{$*simplicity*$\}$

Our preferences for simplicity are actualized in the derivation of De Moivre's Theorem and in Euler's identity by the eliciting of a positive affective response. The criterion states that such actualization of the preference implies that the derivation and the identity possess aesthetic value.

The above definition of aesthetic criterion generalizes McAllister's conception by taking into account mental activities and constructions. It takes into account not only single properties, but also the mental activities involved in appreciating things like mathematical proofs and derivations—and narratives and music, for that matter. This allows us to account for a wider class of preferences involved in aesthetic experiences. In my approach to aesthetic experience, affective responses are elicited due to the presence of aesthetically relevant properties or the performing of certain mental activities. These properties and activities are the same as those that constitute the property/activity sets S involved in the evaluative-instances $\langle S, r \rangle$. Although neither inner experiences nor personal values are publicly available, we can easily infer the existence of certain aesthetic criteria from the available historical and behavioural evidence. The public reaction of a person to his engaging in contemplating an object is evidence that the person has made an evaluation based on his values. Thus, if a public aesthetic judgement on an object is passed by a person, we can assume that person has made the evaluation based on his values. Even if the process of evaluation and the values involved are not accessible, the resulting judgement *is accessible* to us. Once an aesthetic judgement is available, we have enough information to infer the existence of an aesthetic criterion, and thus to track its associated aesthetic value. This fact has been exploited by McAllister, Kuipers and myself to model the dynamics of preferences and we can thus use it now to model the dynamics of value.

8.3 Dynamics

In order to deal with the dynamics of value we need to model the intensity with which a set of properties and mental activities is able to actualize affective and behavioural tendencies—to utter a public judgement, for example. Thus, our model must incorporate a weighting expressing the intensity of the strength of the preference tracked by an aesthetic criterion. We must simply introduce a weighting W gauging the strength of the relation of S to r. This, of course, is the same idea as the one in our model of preference evolution and shall allow us to recourse to that model to examine the dynamics of aesthetic value.

8.3.1 Value and Aesthetic Canon

Attaching a weighting to each evaluative-instance shall help us to model the changes in preference intensity. But in addition to changes in intensity, the value set undergoes a second type of change: its extension may change. Which evaluative-instances are in the value set—or in a specific repository, for that matter—changes depending on whether or not they are able to elicit an affective response. Some evaluative-instances that were formerly elements of the value set can eventually stop being elements, whereas new instances can become elements. For example, Le Lionnais' view on Euler's identity is it that what was once regarded as the most beautiful formula in mathematics, but ended up being unremarkable or even insipid [58, p. 128]. Thus, in Le Lionnais's view, the intensity of the preference associated with the identity changed from a high degree of intensity to a very low or even nil degree. If Euler's identity is really unremarkable and fails to elicit any response, then its associated evaluative-instance cannot be in the value-set. In this case, the extension of aesthetic value has changed, since a former element is now missing.

In this sense, there is an important difference between an aesthetic canon and a value-set. An aesthetic canon comprises all properties of theories regardless of their associated responses. The value-set is a subset of an aesthetic canon, since it comprises only elements with an evaluative component; that is, with an associated affective response. This means that the evolution of aesthetic value amounts to the evolution of a subset of an aesthetic canon. In this sense, to model the dynamics of value, we can model the evolution of an aesthetic canon and then simply focus on the value subset in which we are interested. Now, the notion of aesthetic canon as proposed by McAllister and even the revision proposed in Chap. 5 is not compatible with our conception of value, since they comprise only properties of objects. But a notion of aesthetic canon adequate for our purposes can be trivially defined as a follows:

Generalized Aesthetic Canon A person is moved to experience an affective response or pass an aesthetic judgement on an object as a consequence of his holding to one or more aesthetic criteria (as define above) which attach aesthetic value to the object.

In principle, there might be as many aesthetic criteria as sets of properties and mental activities, including the ones which do not elicit an actual response. As in Chap. 5, we shall interpret the collection of all those criteria in a set theoretical manner to take advantage of systems theoretical tools. Thus, the set of all possible aesthetic criteria to which a person or community might hold to constitutes his or its generalized aesthetic canon. To each aesthetic criterion there is attached a weighting W gauging the intensity with which the criterion is held. In this way, the evolution of an aesthetic canon can be modelled simply as changes in the weightings. A fully expressed aesthetic criterion, including its intensity, is represented by three items of information: the set S of properties and mental activities, its associated response r, and its associated weighting W_S.

An aesthetic canon is thus the set:

$$C = \{\langle S, r_S, W_S\rangle | S \in \wp(T)\}$$

where:

S: a set of properties and mental activities.
T: the set of *all possible* properties and mental activities.
$\wp(T)$: is the power set of T.

It must be noted that the responses can be a mixture of positive, negative, and *indifferent* (no response whatsoever) and have a passive (by mere observation of properties) or active (by performing activities) source; and the weightings can have a *zero* value.

Ideally, an aesthetic canon comprises an infinite number of elements: one for each set of properties and activities to which aesthetic value could conceivably be attributed. In the case of any given person, the majority of those elements shall carry a weighting of zero, since we typically attach an aesthetic value to only a few properties or mental activities and are indifferent to the rest.

Now, the evolution of an aesthetic canon amounts simply to changes in the weightings W. Changes in preferences must be modelled in accord with historical evidence and empirical findings, as discussed in Chap. 5. Since none of the modifications introduced here concern the mechanism that modifies preferences we can resort to the model advanced there. The evolution of our generalized aesthetic canon is thus governed by a variation of the mechanism I labelled constrained aesthetic induction.

8.4 Aesthetic Canon Evolution

To address the dynamic character of the aesthetic canon, we can envisage that its compilation is carried out as follows:

A community compiles its aesthetic canon $C = \{\langle S, r, W_S(t)\rangle\}$ *at a certain time* t *by attaching to all sets* S *of properties and mental activities a response* r *and a weighting* $W_S(t)$ *determined by the Naturalized Evolution Rule II, defined below.*

8.4.1 Naturalistic Evolution Rule II (NERII)

$$W_S(t) = (1 - R_S)gA_S + R_S W_S(t-1)$$

Where:

$W_S(t)$ is the weighting associated with S at time t, resulting from the evolution of the canon.

$W_S(t-1)$ is the original weighting at a prior time $t-1$, before the evolution of the aesthetic canon.
A_S is the degree of critical adequacy of S with range [0, 1].
R_S is the degree of robustness of S with range [0, 1].
g is a constant that gauges the ratio between the weightings and the degrees of critical adequacy. It expresses the global rate of change in a given aesthetic canon in absence of robustness.

As before, if the robustness is low, the function models classic aesthetic induction; if it is high the function models the tendencies of historical constants. Of course, we need to define critical adequacy and robustness. Critical adequacy is defined as follows:

Critical Adequacy

An object O is critically adequate (or inadequate) if and only if there is a set S of properties of O and mental activities associated with contemplating O that guarantees that an average person with the appropriate experience will pass a positive (or negative) aesthetic judgement on O.

This notion of critical adequacy captures the fact that pleasing properties or activities motivates the eliciting of aesthetic evaluations. But as before, a notion that admits degrees is better suited to be interpreted as a parameter in our evolution rule. Consider thus the following definition:

Degree of Critical Adequacy

An object O has a high degree of critical adequacy (or inadequacy) if and only if there is a set S of properties of O and mental activities associated with contemplating O whose presence makes very probable that an average person with the appropriate experience will pass a positive (or negative) aesthetic judgement on O.

As we know, robustness of critical adequacy is necessary for an anomaly-free model.

Robustness of Critical Adequacy

The critical adequacy of a set S of properties of O and mental activities associated with contemplating O is robust if and only if the properties and activities in S are able to motivate the same affective response despite changes in the history of experiences with those properties and activities.

To incorporate robustness as a parameter into an evolution rule consider:

Degree of Robustness of Critical Adequacy

The critical adequacy of a set S of properties of O and mental activities associated with contemplating O is robust in a high degree if and only if in most cases the properties and activities S are able to motivate the same affective response despite changes in the history of experiences with those properties and activities.

Robustness helps us to model the tendencies of properties like simplicity or complexity to maintain their degree of critical adequacy, despite the fact that a history of experiences with such properties builds up over time.

As mentioned above, the repositories for positive and negative value play a very relevant role in an accurate depiction of the evolution of preferences. As we know from our discussion of aesthetic experience, the affective response elicited by contemplating properties is independent from the response elicited by performing mental activities and thus the experience can involve a mixture of positive and negative responses. To address the possibility of mixed responses, I shall use two independent evolution rules to model the evolution of positive and negative components of the response. In this way we shall be able to model, in principle, a much wider and more complex range of evolution patterns.

8.4.2 Positive Value

We only need special versions of the concepts formulated above to independently model the dynamics of positive and negative value repositories. The notions needed to model the dynamics of positive and negative value can be trivially obtained from the general definitions, as follows: let $+$ denote an affective response with at least one positive component; $-$ one with at least one negative component; and $\varnothing$ no response whatsoever. The rage of the variable for affective response in the aesthetic canon r is thus $\{+, -, \varnothing\}$, or $r \in \{+, -, \varnothing\}$. Notice that a $+$ response does not exclude a $-$ response, I have chosen this characteristic in order to allow aesthetic experiences with mixed responses to be involved in two different patterns of evolution, one induced by the positive component and another by the negative one. These patterns of evolution are induced by positive and negative evolution rules; defined as follows:

Partially Positive Aesthetic Criterion

If there exists a set S of descriptive properties and mental activities associated with contemplating an object O and those properties and activities are conjunctively responsible for eliciting any kind of *positive* affective response, then more positive aesthetic value is associated with O.

A typical evaluative instance in the positive value set has the form:

$$\langle S, +, W_S(t) \rangle$$

Positive aesthetic value can be extensionally defined as:

Positive Aesthetic Value

$$V_+ = \{\langle S, +, W_S(t) \rangle | \langle S, +, W_S \rangle \in C\}$$

The evaluative-instances in V_+ are modulated by the Naturalistic Evolution Rule II in which the parameters model the following notions:

Degree of Positive Critical Adequacy

An object O has a high degree of *positive* critical adequacy if and only if there is a set S of properties of O and mental activities associated with contemplating O whose presence makes very probable that an average person with the appropriate experience will pass a *positive* aesthetic judgement on O.

Degree of Robustness of Positive Critical Adequacy

The positive critical adequacy of a set S of properties of O and mental activities associated with contemplating O is robust in a high degree if and only if in most cases the properties and activities S are able to motivate the same positive affective response despite changes in the history of experiences with those properties and activities.

The evolution rule for positive value is:

Positive Value Naturalistic Evolution Rule (PVNER)

$$W_S(t) = (1 - R_S) g A_S + W_S(t-1) R_S$$

where:

$W_S(t)$ is the weighting associated with S at time t, resulting from the evolution of the positive value set.

$W_S(t-1)$ is the original weighting at a prior time $t-1$, before the evolution of the positive value set.

A_S is the degree of positive critical adequacy of S with range [0, 1].
R_S is the degree of robustness of positive critical adequacy S with range [0, 1].
g is a constant that gauges the ratio between the weightings and the degrees of positive critical adequacy.

8.4.3 Negative Value

When we discussed McAllister model of the evolution of aesthetic preferences we found out that the evolution of negative historical constant was very problematic. The model developed in Chap. 5 solved that problem. With some trivial modifications the Naturalistic Evolution Rule II can now model the dynamics of negative values. The definitions are as follows:

Partially Negative Aesthetic Criterion

If there exists a set S of descriptive properties and mental activities associated with contemplating an object O and those properties and activities are conjunctively responsible for eliciting any kind of *negative* affective response, then more *negative* aesthetic value is associated with O.

A typical evaluative instance in the negative value set has the form:

$$\langle S, -, W_S(t) \rangle$$

Negative aesthetic value is extensionally defined as:

Negative Aesthetic Value

$$V_- = \{\langle S, -, W_S(t) \rangle | \langle S, -, W_S \rangle \in C\}$$

The notions for the parameters in the evolution rule are:

Degree of Negative Critical Adequacy

An object O has a high degree of *negative* critical adequacy if and only if there is a set S of properties of O and mental activities associated with contemplating O whose presence makes very probable that an average person with the appropriate experience will pass a *negative* aesthetic judgement on O.

Degree of Robustness of Negative Critical Adequacy

The negative critical adequacy of a set S of properties of O and mental activities associated with contemplating O is robust in a high degree if and only if in most cases the properties and activities S are able to motivate the same negative affective response despite changes in the history of experiences with those properties and activities.

The evolution rule for negative value is:

Negative Value Naturalistic Evolution Rule (NVNER)

$$W'_S(t) = (1 - R'_S)g'A'_S + W'_S(t-1)R'_S$$

where:

$W'_S(t)$ is the weighting associated with S at time t, resulting from the evolution of the negative value set.
$W'_S(t-1)$ is the original weighting at a prior time $t-1$, before the evolution of the negative value set.
A'_S is the degree of negative critical adequacy of S with range $[0, 1]$.
R'_S is the degree of robustness of negative critical adequacy S with range $[0, 1]$.
g' is a constant that gauges the ratio between the weightings and the degrees of negative critical adequacy.

8.4.4 Aesthetic Experience and the Application of Value Repositories

The conditions under which positive or negative notions are applied depend on the type of value repository that is actualized in a given situation. One of the ways in which aesthetic values are actualized is in affective responses. Since affective responses are accessible only in our inner experience, there is an inherent link between aesthetic experience and values. We can exploit this fact as a way to decide which particular repository should be used in a given situation, and thus which model of evolution is appropriate. We can use the different types of aesthetic experience to determine the application of the positive or negative value repositories by following these rules:

1. Positive and Negative Value Naturalistic Evolution Rules govern positive and negative value subsets respectively.
2. If the set S of properties and activities associated with object O is a superset of the set of dimensions of its corresponding intentional object (the object in

our inner experience) and the enjoyment associated with that intentional object consists of pleasing affective responses (including mixed responses in which any response is positive), then S is involved in *positive* aesthetic criteria, value set, critical adequacy and robustness.

3. If the set S of properties and activities associated with object O is a superset of the set of dimensions of its corresponding intentional object and the enjoyment associated with that intentional object consists of displeasure responses (including mixed responses in which any response is negative), then S is involved in *negative* aesthetic criteria, value set, critical adequacy and robustness.

It is trivial now to illustrate the evolution of value modelled above by recalling Le Lionnais' view on Euler's identity. We started with a positive preference for Euler's identity, but as our experiences (or at least Le Lionnais's) influenced that preference, it changed until the preference finally ended up turning into an irrelevant or even a negative one. In this case, an aesthetic criterion *If simplicity and composition are properties of a mathematical item, then more positive aesthetic value is associated with the item*, started with a high weighting, which eventually faded away as the properties lost their capacity to elicit a response. Although the criterion started with a strong influence, it was not robust enough to remain stable. The *negative* naturalized evolution rule kicked in changing its strength until the criterion was no longer aesthetically relevant. Now, if we regard Le Lionnais's judgement as negative rather than neutral—as in, for example, "Euler's formula is insipid" rather than the more polite "Euler's formula is fairly unremarkable"—we can further say that the negative evolution pushed further, and that the now negative criterion *if simplicity and composition are properties of a mathematical item, then more negative aesthetic value is associated with the item* gained strength.

Note that in cases of mixed responses, positive as well as negative evolution rules govern the change in the weightings. This is consistent with the approach to aesthetic experience endorsed here, since we established that there is no clear cut division between positive and negative experiences, but rather a mixture of positive and negative experiences. This is also consistent with the facts that aesthetic terms possess rather fuzzy meanings, that they seem to admit degrees, and that there are no fixed rules that determine their correct application. According to the definitions above, positive and negative value sets overlap each other, when we discuss aesthetic terms it will be clear that this phenomenon is not only consistent with the way aesthetic judgements are made, but it actually can help us to explain some characteristics of aesthetic judgement. In the next chapters, we shall see that positive and negative evaluations are not simply each other's opposites; but rather that their relation is more like that of the members of *families*.

Chapter 9
Aesthetic Judgement I: Concept

The need for a consistent interpretation of the *term* mathematical beauty has been the leitmotif driving the discussion in this book. Now, aesthetic *terms* and aesthetic *judgements* are very closely related topics. In this and the following chapter, I advance notions of aesthetic terms and judgements that shall enable us to give a sophisticated depiction of aesthetic terms in mathematics, and to complete a consistent interpretation of mathematical beauty.

As in most topics in aesthetics, there are different theories about the nature of aesthetic terms; some of them even contradicting each other. We shall survey some of those theories in order to gain insights that shall be used to devise our own inconsistency-free approach.

9.1 On Aesthetic Terms

Aesthetic descriptions are different from mundane, so to speak, non-aesthetic descriptions: things like musical pieces, paintings, narrations or sculptures move us in a special way, but it is often difficult to describe those things in a manner that does the way they move us justice. The same piece of music can be mundanely described as being *six minutes and thirty seconds long*, or it can be described as being *melancholic*. The second way of describing it is an *aesthetic description*. The nature of such descriptions is closely connected with the nature of the predicates involved in them, that is, with the nature of *aesthetic terms*. The complex nature of aesthetic terms can be illustrated by surveying the approaches of Frank Sibley, Isabel Hungerland, Peter Kivy, Rafael DeClercq and Nick Zangwill.

9.1.1 Sibley

Frank Sibley's seminal work *Aesthetic Concepts* [82] was very influential in twentieth Century aesthetics. Sibley's main tenet is that aesthetic terms are not

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_9,

rule-governed in the sense that we cannot establish definitions, conditions or rules that determine the presence of aesthetic properties—being *delicate*, for instance—in terms of non-aesthetic properties—being thin or curved, for instance. Sibley starts by identifying two groups of descriptions that can be given about works of art: on the one hand, descriptions that can be given by "anyone with normal eyes, ears, and intelligence" [82, p. 421]. For example, that "a novel has a great number of characters and deals with life in a manufacturing town" or "that a painting uses pale colours, predominantly blues and greens, and has kneeling figures in the foreground" [82, p. 421]. On the other hand, there are descriptions that require "the exercise of taste, perceptiveness, or sensitivity, of aesthetic discrimination or appreciation" [82, p. 421]. One may say, for example, "that a poem is tightly-knit or deeply moving" or "that a picture lacks balance" [82, p. 421].

Sibley's view on aesthetic terms is as follows: "when a word or expression is such that taste or perceptiveness is required in order to apply it, I shall call it an aesthetic term or expression [. . .]" [82, p. 421]. According to Sibley, the application of aesthetic terms is governed by "taste". Now, he also offers a further insight; there is a link between aesthetic terms and metaphors:

> Clearly, when we employ words as aesthetic terms we are often making and using metaphors, pressing into service words which do not primarily function in this manner. Certainly also, many words have come to be aesthetic terms by some kind of metaphorical transference. This is so with those like 'dynamic', 'melancholy', 'balanced', 'tightly-knit' which, except in artistic and critical writings, are not normally aesthetic terms [82, p. 422].

Sibley adds some important qualifications: the most commonly used aesthetic terms—such as 'lovely', 'pretty', 'beautiful', 'dainty', 'graceful', 'elegant'—are not metaphorically used since their primary or only use is as aesthetic terms. Furthermore, the aesthetic terms that seem to be metaphorical are not completely metaphorical:

> [. . .] expressions like "dynamic," "balanced," and so forth have come by a metaphorical shift to be aesthetic terms, their employment in criticism can scarcely be said to be more than quasi-metaphorical. Having entered the language of art description and criticism as metaphors they are now standard vocabulary in that language [82, pp. 422–423].

Sibley argues that the language utilized in art criticism must be interpreted from its own particular standpoint, the standpoint of "making aesthetic observations" [82, p. 422]. From that standpoint the usage of aesthetic terms is not a metaphor. By realizing that such terms are aesthetic terms and not metaphors the descriptions offered by a critic should be interpreted as directing our attention to the feature of an object that is aesthetically relevant. Sibley sees the art critic as a guiding person; as someone capable of focusing our sensitivity on the key features of artworks so that aesthetic qualities become apparent to us. The recurrent exposition to correct applications of aesthetic terms improves our own ability to apply those terms. The final result is that what at first seemed metaphorical becomes the natural expression of aesthetic sensitivity. Despite the fact that there are no rules linking aesthetic terms with non-aesthetic terms, with effort, patience, exemplification, repetition, and trial and error we can come to master the use of aesthetic terms.

I must point out that although Sibley claims that the usage of aesthetic terms involves aesthetic sensitivity, the way he describes how we refine their use resembles the learning of much more mundane skills. Learning a language, a craft, or a sporting skill, for example, is also achieved by means of exemplification, correction, repetition and so forth. The approach to be developed here avoids postulating the existence of a special "aesthetic" skill, resorting only to mundane skills like the use of language.

9.1.2 Hungerland

The need for an special faculty of aesthetic taste or sensitivity is also avoided by Isabel Hungerland's account [35]. Hungerland focuses on the conditions of application of aesthetic terms. She argues that aesthetic terms are part of a distinctive class of terms that clearly differs from non-aesthetic terms in a crucial aspect:

Non-aesthetic terms, such as 'strong', can always be meaningfully used in sentences of the following two types: (1) 'John is strong' and (2) 'John looks strong but he is not'. In contrast, aesthetic terms, such as 'elegant', *cannot* be meaningfully used in type (2) sentences. For example, there is little or no difference in using the term 'elegant' in either way: 'John is elegant' or 'John looks elegant'. But sentences like 'John looks elegant but he is not' do not even make sense [35, pp. 50–52].

According to Hungerland, for any non-aesthetic term N there is a difference between *really being* N and *just looking* N [35, pp. 52–54]. Something may look N but not really be N. In contrast, there is no *really is/only looks* distinction (*is/looks* distinction, hereafter) for aesthetic terms. This is so, Hungerland argues, because aesthetic terms are devised to talk about how things may look to the observer under normal circumstances; that is, the application criteria for aesthetic terms depend entirely on the internal, or subjective, experience of the person who uses them [35, pp. 63–65]. In contrast, in the application of a non-aesthetic term like 'strong' there are several external or objective criteria that can be used to correct its application. For example, let us imagine that we have just stated that John was strong—perhaps he appeared strong to us. Imagine that afterwards, someone tells us that John is very ill, or we find out ourselves that John is actually a weak person. We should, after the corroboration of the weakness of John, correct our initial statement to the new statement 'John is not really strong', or 'John looks strong, but he is not'. In principle, there is nothing that prevents us from correcting from 'John is strong' to 'John looks strong but he is not.' This correction is possible because the application of the term 'strong' is governed by objective criteria. The correct use of both sentences that include the term strong—'John is strong' and 'John looks strong but he is not'—depends on external, objective criteria. In contrast, there are no external or objective criteria for correcting the application of aesthetic terms. If John looks elegant to us, and we say that John is elegant, no external criteria can induce us to say that John looks elegant but he is not. This last sentence does not even make sense. In summary, aesthetic terms are characterized by the fact that the distinction *is/looks* does not apply to them, since the correct application of aesthetic terms is not governed by objective criteria.

9.1.3 Kivy

Unlike Sibley and Hungerland, Peter Kivy [38] is sceptical about the existence of a distinct class of aesthetic terms. He argues that there is no definite list of aesthetic terms, but rather, in the appropriate context, any ordinary term can be applied as an aesthetic term:

> [I]t is probably true that any of the term's I have called 'non- aesthetic' can, in the appropriate context, be 'aesthetic' terms. Thus, for example, I am sure we can imagine 'heavy' being used not to refer to the weight of Michelangelo's David, but to some 'aesthetic' property of a work of art [38, p. 198].

Rather than a distinction between aesthetic and non-aesthetic terms, there is a contextual function that aesthetic terms perform and that makes aesthetic terms aesthetic.

> [W]e cannot distinguish aesthetic terms from other terms simply by enumeration. We cannot make a list of aesthetic terms because aesthetic terms are not a distinct subgroup of the terms in ordinary language; rather, they are terms in ordinary language which at times and in certain contexts we call 'aesthetic' [38, p. 198].

Kivy recognizes, however, that certain terms—such as 'beautiful' or 'elegant'—are "always and only aesthetic terms" [38, p. 198]. Kivy argues that what characterizes the contextual function of aesthetic terms is their appearing in *terminal* descriptions, that is, statements that lead to no further conclusion, action or change in attitude.

> Aesthetic terms, then, are those that, characteristically, occur in descriptions which seem to be ends rather than beginnings. They are not, as many seem to think, terms that we can supply no logically compelling criteria for applying. But they are terms that do not provide the reasons for anything else [38, p. 211].

I label Kivy's approach the *terminal approach* to aesthetic terms, and the property he points out the property of *terminality*. For Kivy, the key characteristic of aesthetic terms lies on the type of role the terms play in terminal descriptions: non-aesthetic terms, when used in a description, serve as a "prelude to something else: the premise of an argument or a call to action" [38, p. 210]. Kivy clarifies terminality by stressing the contrast between moral and aesthetic terms:

> [T]he fact that aesthetic descriptions are "terminal," that they lead nowhere, distinguishes them sharply from moral descriptions, which often are preludes to action. To conclude that a course of action is right is to provide some reason for pursuing that course of action in the future. To conclude that a man is greedy is to provide some reason for future actions, or attitudes towards that man. To conclude that a novel is "unified," a painting "garish," a poem "sentimental," a symphonic movement "sad," however, provides no reason for anything except continued contemplation, or an end to it [38, p. 211].

In short, Kivy characterizes aesthetic terms in terms of terminality, that is, in terms of their role in descriptions which, unlike moral or objective descriptions, have no further consequence.

9.1.4 De Clercq

We have mentioned Rafael De Clercq's ideas [17] when discussing non-literal interpretations of mathematical beauty. Like Sibley, De Clercq gives metaphor an important role in characterizing aesthetic terms. He argues that "aesthetic terms cannot be turned into metaphors" [17, p. 27]. It makes no sense to say that something is beautiful, elegant, harmonious or sublime "metaphorically speaking". The reason for this metaphoric resilience is, De Clercq argues, that aesthetic terms are universally applicable, in the sense that they can be applied to any domain without incurring in a category mistake.

> Aesthetic terms do not have a particular area of application associated with them. There is not a particular kind of object to which they are to be applied. As a result, it is not possible to commit something like a 'category mistake' with respect to such terms. By contrast, terms for animal species such as 'elephant' and 'crocodile' can be applied only within the animal kingdom: to apply them outside this area is to commit a 'category mistake' [17, pp. 27–28].

De Clercq, however, points out two significant nuances in his account: some aesthetic terms, 'balanced', for instance, are already metaphors; and some others, 'garish', for instance, are not universally applicable. De Clercq argues that his characterization still applies in the case of already metaphorical terms, since those terms cannot be turned into metaphors. There is no such thing, he argues, as a second order metaphor. As for not universally applicable aesthetic terms, he suggests we should regard them as "semi-aesthetic" terms [17, pp. 28–29].

9.1.5 Zangwill

Nick Zangwill's approach [101] deals with aesthetic terms in music, nonetheless his ideas are illuminating. Zangwill argues that experiences of aesthetic properties of music are ineffable, just like the experience of pain or the smell of coffee. Such experiences cannot be described in a literal manner [101, p. 5]. He defends what he calls the Essential Metaphor Thesis, which is

> the thesis that, beyond very simple terms, the aesthetic properties of music cannot be literally described; they must be described metaphorically (or by other nonliteral devices). It contrasts with the Aesthetic Metaphor Thesis, which is the weaker thesis that we generally do describe music in metaphorical terms. The Essential Metaphor Thesis is that we must do so, in anything other than a superficial description of it [101, p. 1].

Zangwill claims that interesting descriptions of music—such as describing a piece of music as sad or melancholic—necessarily utilize metaphors. Only very simple descriptions—such as describing a piece of music as beautiful—are literal [101, p. 2]. Zangwill argues that his thesis explains the prevalence of metaphor in descriptions of music. Now, the Essential Metaphor Thesis certainly exemplifies how topics like the role of metaphor in aesthetic descriptions is approached in different ways by different authors: Sibley recognizes that metaphor and aesthetic terms are closely linked. For De Clercq aesthetic terms cannot be metaphorical, and the cases that seem to be metaphorical are only semi-aesthetic. For Zangwill, metaphor is necessary for interesting aesthetic descriptions (at least in music).

9.2 A New Model of Aesthetic Terms

We now have a wide variety of views on aesthetic terms. They are characterized by Sibley in terms of aesthetic sensitivity, by Hungerland in terms of the *is/looks* distinction, by Kivy in terms of terminality, and by De Clercq in terms of metaphor resilience. Unfortunately, there are clear incompatibilities among them. For Sibley and Hungerland there is a distinction between aesthetic and non-aesthetic terms. For Kivy there is no such a distinction, but rather a contextual role that aesthetic terms play. For Sibley and Zangwill, metaphor and aesthetic terms are closely linked, but for De Clercq, aesthetic terms characteristically resist metaphor. All this seems to pose a problem rather than offer us a viable approach for our purposes here. I believe that the best way of looking at these conflicting approaches is as providing different perspectives on a complex subject, and as providing us with the opportunity of piecing together a more sophisticated picture of it.

Subjectivity is one of the main tenets of modern aesthetics ever since Kant. It should be no surprise that subjectivity figures prominently in Sibley's and Hungerland's views. The subjective use of aesthetic terms can be utilized as point of departure for our new depiction of aesthetic terms. Hungerland, in particular, gives subjectivity a key role by characterizing aesthetic terms in terms of the non-objective conditions that govern their application. The experience a person undergoes is always subjective and, thus, very likely involved in our application of aesthetic terms. Aesthetic experience is thus an element that must be considered in shaping our view on aesthetic terms. Hungerland's conditions of application can be seen as connected with the subjective responses involved in aesthetic experience. In this sense, the inner affective states of an individual undergoing an aesthetic experience determine whether or not the use of a predicate as an aesthetic term is correct.

Now, the application of aesthetic terms may be characteristically subjective, but there is more to it. Kivy points out that there is no fixed list of aesthetic terms; everyday mundane terms such as 'balanced' or 'unified' can be used as aesthetic terms in descriptions like "The Prelude in B-flat minor from Book I of Bach's Well-tempered Clavier is unified" [38, p. 197], or "Beethoven's fifth symphony is a unified work." Thus, in addition to determining whether a term is correctly applied or not, we need to determine *what* term to apply. For example, Euler's identity is regarded as very beautiful by the Mathematical Intelligencer's readers, but as unremarkable or insipid by Le Lionnais. There is no *objective* conflict here since applying the terms 'beautiful' or 'insipid' depends on the *subjective* states of the *Intelligencer*'s readers or Le Lionnais, respectively. However, the choice among the different terms 'beautiful', 'unremarkable' or 'insipid' clearly seems to involve something else. In this regard, Sibley, De Clercq and Zangwill have pointed out that in using everyday terms in an aesthetic way, metaphor seems to play a significant role. This is an interesting avenue to explore. We can gain an important insight on metaphor by surveying Nelson Goodman's theory of metaphor. We shall see that choosing what term to apply depends not only on whether the term is associated with a certain subjective state, but also on the term's connections with other terms.

9.2.1 Metaphor

Nelson Goodman [30] claims that, in metaphor, a term—or label, as Goodman calls them—does not work in isolation; rather, it works as a member of a family of terms: "a label functions not in isolation but as belonging to a family" [30, p. 71]. Goodman calls these families of terms *schemata*. Schemata are sets of terms that became interrelated by means of context and habit. Schemata are linked to a specific domain—or realm, as Goodman calls them—of referents; the referential domains consist of all the things that each term in the schema denotes. The schema $\{blue, red, green, \ldots\}$, for example, is the schema of terms for colours. Its referential domain consists of all coloured—blue, red, green, and so forth—things.

We have metaphorical reference when we use a term to refer to something that does not belong to the domain normally associated with the term's schema. For example, the family of terms 'red', 'blue', 'green', etc., is appropriate to describe the domain of colours. 'Red' refers to the colour red (red objects) and 'blue' refers to the colour blue (blue objects). A metaphor consists in using a term in the colour schema to refer to any colourless things. We can apply the term 'blue' to refer to sadness, and 'red' to refer to anger, 'yellow' to envy, and so forth. Goodman himself gives the example of calling a painting sad, which he claims is metaphorical because 'sad' is a predicate that normally refers to individuals in certain emotional states, not to inanimate objects.

In Goodman's approach—and this is very important for us, metaphors *reorganize* the new domain in which the metaphorical term is applied. For example, the use of 'sad' in the domain of paintings reorganizes the domain of paintings in a way such that the schema of emotional terms

$$\{sad, cheerful, angry, \ldots\}$$

forces the structure of its original domain of emotions onto the new domain of paintings. Thus, calling a particular painting sad entails that, in principle, other paintings can be called cheerful, if the relations between ‘sad’ and ‘cheerful’ people are also held between sad and cheerful paintings. In general, the metaphor reorganizes the new domain of application by forcing the structure of the schema’s original domain onto the alien domain to which the term is applied. In a sense, metaphors change the perspective with which we perceive the new domain. Because of this reorganization, the use of a metaphorical term depends not so much on the meaning of the isolated term, but rather on the *relations* it holds to their closely related terms.

9.2.2 Metaphorical Terms and Aesthetic Terms

We have identified two important aspects of aesthetic terms: first, their application is governed by subjective matters; associating them to subjective states explain why conflicting descriptions are possible, since they do not entail objective conflicts. Second, the use of non-aesthetic terms in aesthetic descriptions seem to be closely related to metaphor, and, from the foregoing discussion of metaphor, it is reasonable to assume that in using such aesthetic terms their connections with a family of terms is as relevant as its connection with subjective states. The simultaneous connection among subjective states, individual terms, and those terms’ families shall be the backbone of the theory of aesthetic terms proposed here, as we shall soon see.

Now, although Kivy recognizes that there is a class of terms that are “always and only” aesthetic terms, he also draws our attention to the fact that non-aesthetic terms are regularly used to make aesthetic descriptions. We just learned that applying certain term to an alien domain results in a metaphor. This fact explains why some aesthetic terms clearly appear to be metaphorical, as recognized to different extents by Sibley, De Clercq and Zangwill. But there is a difference between metaphorical and aesthetic usages of a term. Sibley argues that using aesthetic terms involve the faculty of sensitivity or taste. De Clercq argues that it involves metaphor-resilience. And Zangwill explains the use of metaphor as the necessary result of the ineffability of aesthetic experience. Zangwill’s insight is significant here. To interpret the usage of aesthetic terms as a merely metaphorical usage disregards the fact that the application of aesthetic terms is grounded on subjective matters. I believe that the connection between aesthetic experience, aesthetic terms and metaphor should be exploited to refine our view on aesthetic terms. I propose to interpret the usage of aesthetic terms as similar to metaphorical usage in some aspects, but with the additional characteristic that this usage *also serves to capture or express subjective states*.

The aesthetic application of non-aesthetic mundane terms like ‘balanced’, ‘unified’ or ‘sad’ should certainly be interpreted as metaphorical. In these cases, the term does not work in isolation but as a member of family of terms—*schema*, hereafter. Their application thus entails that the structures of their schemata’s domains are forced onto the new domain of objects of appreciation, the domain of paintings, for instance. In other words, the aesthetic application of non-aesthetic terms results in a reorganization of the domain of objects of appreciation, in a change in our perspective; a change in the way in which we see the domain of, say, paintings. As in metaphor, this change of perspective helps to place on the foreground certain features of our subjective experience of the objects of attention, which allows us to communicate those features even if our experience is ineffable by literal means. For example, the term ‘balanced’ literally refers to the *even distribution of mass*, but in the metaphor ‘a balanced picture’, the term ‘balanced’ places on the foreground the notion of *even distribution of something*, thus directing our attention to the even distribution of, say, shapes or colours in the “balanced” picture.

The additional characteristic of aesthetic terms, that distinguishes them from metaphorical terms, is that they are connected with our subjective affective responses. The usage of an aesthetic term, in addition to bringing about a metaphor-like change of perspective, depends on the existence of a connection between affective responses and families of terms which allows the expression of subjective states. We qualify an object of aesthetic appreciation as beautiful or ugly, or as balanced or unbalanced, not only to communicate some quality of the object, but also to express our response to that object. Since, in the most general case, an aesthetic term does not work alone but as a member of a schema, of a family of terms, the expressing of subjective states is possible only if different affective responses are associated with different terms. In this respect, our discussion of aesthetic experience shall be very useful.

9.2.3 Experience and Response Spaces

When we discussed aesthetic experience, the relevant states of a person were characterized by a content—intentional object plus mental activities—and an associated affective response. Since the content involves a set of the properties of the observed object, the content is closely connected with the concrete properties of the objects. But the affective response is clearly subjective. This subjective side, the affective responses, is the most obvious candidate to be connected with the application of aesthetic terms.

Recall that pleasure-relations were introduced to model the relation between content and affective responses in aesthetic experiences. Pleasure-relations are sets of ordered pairs in which the first coordinate in the pair is a set of properties and mental activities, and the second coordinate is an affective response—a composed affective response, in the general case. The responses in a pleasure-relation are subjective affective states in an individual, and the set of all responses in the pleasure-relation comprises all the possible subjective states in which an individual can be. That set thus constitutes the space of responses a person can experience; I shall refer to the set of all pleasure-relation responses for the experience of an object as the *response space* of that object.

Individual responses in a response space can be utilized to determine whether the use of an aesthetic term is correct. Thus, response spaces can be used to establish a connection between pleasure-relations and *families* of terms, and, from the foregoing discussion, this connection can be used to express subjective states. If we map the elements in a response space into the elements in a certain schema in a way such that to each possible response in our response space corresponds a term in our schema, then the use of different terms in aesthetic descriptions of an object can be seen as expressing different responses to the object. Which specific term of a schema we deploy to describe the object depends on what response the object elicits in us. For example, we know that there are several ways in which different people describe Euler's identity—as beautiful, unremarkable, insipid, etc. There is no objective conflict in that, since different people simply express *different subjective responses* to Euler's identity by utilizing *different terms*. This illustrates that a family of terms is better suited to express the multiple possibilities of response in any aesthetic experience than an isolated term. A family of terms, being associated with the structure of its referential domain, also offers a structure which can reorganize the domain of objects of appreciation, in a manner similar to metaphor. This reorganization helps to focus the attention on features relevant to the appreciation of the described object. It thus makes sense to model the usage of aesthetic terms as involving the simultaneous occurrence of the expression of subjective states, and a change of perspective—by means of a metaphor-like reorganization of the domain of objects of appreciation—which allows the communication of the qualities responsible for that subjective state. This is the key characteristic of my conception of aesthetic terms below.

9.2.4 A New Model of Aesthetic Terms

I interpret the use of a term as an *aesthetic* term in descriptions of an object as involving two simultaneous occurrences: first, the carrying out, mostly unconsciously, of a mapping from our response-space for the described object into the term's schema, to allow expression of subjective states. Second, the carrying out of a change of perspective resulting from the application of the term's schema to the domain of the described object, which allows the communication of some of the qualities of the object responsible for our subjective state by means of placing some qualities on the attention's foreground.

This view of aesthetic terms can be summarized in two characteristic conditions that the application of an aesthetic term must comply simultaneously, which I label the *Expressive Mapping* Condition and the *Communicative Reorganization* Condition.

(1) Expressive Mapping Condition: for an object of appreciation that can elicit any of the possible affective response in the response space R, a term A in the schema S can be said to be used to express our subjective state if and only if there is an appropriate mapping from R into S.

(2) Communicative Reorganization Condition: a term A in the schema S applied to refer to an object in an *alien* domain D of objects of appreciation can be said to be communicatively reorganizing if and only if by applying A to D the inner perspective of a person changes in a way such that he perceives that the domain structure associated to S is forced onto D—or S reorganizes D—and this reorganization highlights (communicates) a property of the object relevant in inducing in a person an affective response of the response space R.

To apply a term in an aesthetic description requires the simultaneous existence of an expressive mapping and a communicative reorganization (we shall see examples in the next section). If we denote an expressive mapping from R into S as $R \mapsto S$ and a communicative reorganization as $S \approx D$, the above model of aesthetic terms tells us that the aesthetic use of a term requiring the simultaneous existence of $R \mapsto S$ and $S \approx D$. For the sake of brevity we can merge these expressions as: $R \mapsto S \approx D$. The model can be simplified to:
A is applied as an aesthetic term if and only if

$$\exists S, R, D(T \in S \wedge R \mapsto S \approx D)$$

I shall refer to this approach as the *RSD model of aesthetic terms*. I must emphasize that the RSD model is not a two stage model, since both conditions—mapping and reorganization—must be complied simultaneously. The choice of an aesthetic term to be applied in a description is carried out by taking into account that the term's schema must provide both an expressive mapping and a communicative domain reorganization at the same time, and not by first establishing a mapping and then carrying out a domain-reorganization or vice versa. Choosing an adequate aesthetic term is thus a sort of balancing act in which we try to comply with two simultaneous conditions using the same family of terms. In general, this balancing act is not simple and, more importantly, is not arbitrary, as we shall see below.

9.2.5 Genuine Aesthetic Terms

Although the application of an aesthetic term is subjective, it is nonetheless constrained by the existence of appropriate expressive mappings and communicative reorganizations. There are mappings that do not express our response space in an adequate way, and there are reorganizations that do not capture the qualities responsible for our subjective states. Thus, the use of aesthetic terms is not arbitrary. As a matter of fact, there are several possibilities for the use of a mundane term permitted by the RSD model; for example literal usage (no mapping nor reorganization), metaphorical usage (no mapping, but reorganization), genuine aesthetic usage (adequate mapping and reorganization), and what we may call "counterfeit" aesthetic usage (inadequate mapping or reorganization).

Affective responses are connected with whether an aesthetic term is applied correctly. The fact that the existence of complex space responses—like performative and adaptive space responses—matches the existence of multiple terms in a schema constrains the possible choices of terms (or, rather, schemata) we can make. Our term choice must allow the mapping from our subjective states into the term's schema. From our discussion of aesthetic experience we know that affective responses admit composed responses and that there is a range of possibilities for those composed responses. In the example of Euler's identity, the existence of different possible responses accounts for the fact that there exists also a collection of terms—'beautiful', 'unremarkable', 'insipid', etc.—we may employ to evaluate Euler's identity. But *which* term one uses in an aesthetic description depend not only on one's subjective state, but also on how one deals with the collections of possible responses and terms. The use of a particular term depends on the way one maps subjective responses into families of terms. The mapping must appropriately assign particular terms to particular responses. In some cases, it is simple to assign a term to a specific response. For example, recall that in a basic pleasure relation we only have two possible responses—pleasure and displeasure; the basic response space has only those two possibilities. The assignments beauty/pleasure and ugliness/displeasure seem to be rather natural with this basic space response. However, problems quickly emerge for the more complicated performative and adaptive pleasure-relations, since their response space are more complicated. We can have up to 16 possible responses for a single type of experience. Matters get even worse if we remember that within those possible responses there are responses that are similar to each other: the responses we labelled "confusing". This poses the question of what all the non-obviously assignable responses mean in terms of choosing terms. In this respect, illustrations of genuine and counterfeit aesthetic terms shall be illuminating.

Terms such as 'balanced' or 'unified' in their everyday use refer to mundane things like mechanical or other physical properties. They usually refer to things in the physical domain. But in using those terms to make aesthetic descriptions the terms are being applied to an alien domain. In such circumstances the application of the terms follows Goodman's metaphor principles. This means that a domain reorganization is carried out in the domain of objects of appreciation—paintings or musical works, for instance—so that that domain resembles the physical domains of mechanically balanced or physically unified things. Moreover, the application of a non-aesthetic term in an aesthetic way implies that the *whole* schema to which the term belongs is applicable to the new domain of objects of appreciation. Now, this reorganization amounts only to a *metaphorical* use of the term. For an *aesthetic* use of the term we also require an expressive mapping. Here is where complex response spaces and confusing responses play a role.

The different responses in a response space are closely related to each other. The different terms in a schema are also closely related to each other. If we assign a term to a particular response in a response space, the remaining elements in the response space constitute a family of responses to which the remaining unassigned terms in the family of terms can be assigned. Unassigned responses are like "empty slots" to which terms can be eventually mapped. In general, the existence of multiple responses makes explicit the fact that in order to correctly use an aesthetic predicate we need to map collections of terms, rather than just isolated terms.

Thus, if we intend a *correct* aesthetic usage of a term A in describing an object O, assigning A to a specific element in a response space—a particular response, a particular combination among the possibilities of responses—of O is accompanied, in principle, by a mapping between A's schema—the family of labels to which A belongs—and the collection of the possible affective responses in the pleasure-relation associated with O.

Now, the mapping of pleasure-relation responses onto a schema is constrained by the structure of the schema's original domain, since this original structure reorganizes the new domain of objects—domains of paintings, musical pieces, mathematical items, etc.—in specific ways. This is why the application of an aesthetic term, although grounded on subjective matters, is not arbitrary, but rather it is constrained by how suitable is the term's schema to provide a reorganization of the domain of objects of attention, and a mapping between the schema and the response space. The way in which these reorganizations and mappings are carried out determines much of the quality with which an aesthetic term works.

For example, the term 'balanced' has everyday mundane applications governed by objective conditions, but in occurrences like "Did you observe the exquisite balance in all his pictures?" [82, p. 438], the application of the term is aesthetic. The aesthetic application of 'balanced' does not depend on the mundane conditions of application of the term 'balanced', but rather on the relations the term holds to the schema $\{balanced, unbalanced\}$ which allow an adequate response space mapping, and on how well the domain structure associated to this schema reorganizes the domain of pictures.

In other words, the affective response a person experiences is the non-cognitive evaluation that grounds an aesthetic judgement, but how well these experiences are expressed by a certain schema, $\{balanced, unbalanced\}$, for instance, depends on the relations the schema holds to the possible responses a person might experience. In this example, the schema $\{balanced, unbalanced\}$ provides terms that represent opposite polar extremes in the semantic spectrum of the family of terms. In general, the response space for an object includes opposite affective polar extremes in the form of the responses of pleasure and displeasure, or full-pleasure and full-displeasure in response spaces with composed responses. If we map positive affective extremes into positive semantic extremes and negative affective extremes into negative semantic extremes, we have a way to express our subjective states in a manner that is coherent with the way we usually deal with our affective states—we see pleasure as positive and displeasure as negative—and with the schema—'balanced' is seen as positive and 'unbalanced' as negative. If, for example, the experience of the picture is of a basic type, we can assign 'balanced' to pleasure and 'unbalanced' to displeasure. Now, although a mapping from positive affective extremes to negative semantic extremes, and vice versa, is in principle possible, such inverted mapping is rather confusing. An *expressive* mapping intends to express our subjective state, and since and inverted mapping goes against the way we usually deal with affective and semantic extremes, an inverted mapping hinders the expression of our inner subjective states. The best choice is thus to associate the term 'balanced' to a positive affective extreme and 'unbalanced' to a negative one. This mapping associates subjective states and terms in a natural way given the structures of the response space and the schema, allowing a coherent expression of subjective states. Now, when the term 'balanced' is applied in an aesthetic description of a picture, it also occurs that the domain of pictures is reorganized by forcing the domain structure associated with the schema $\{balanced, unbalanced\}$ onto the domain of pictures. The original domain of the schema $\{balanced, unbalanced\}$ is the domain of objects with certain mechanical characteristic—with or without the property of having its mass distributed regularly around its centre of gravity, for instance. The term 'balanced' can be applied in its mundane sense to refer to a picture p, as in 'p is balanced', meaning that the mass of the picture is regularly distributed. But the interesting usage of this term is not when it refers to a mechanical feature, but when it refers to an aesthetic feature. In referring to an aesthetic feature, the term is applied to an alien domain, the domain of pictures as objects of attention and not as objects with mass. In that case, in a manner similar to a metaphor, a partition into balanced and unbalanced pictures is forced onto the domain. The aesthetic usage of the term 'balanced' results thus in the reorganization of the domain of paintings in such a fashion that it resembles the domain of mechanically balanced and unbalanced objects, placing the "balanced" property on the foreground of our attention, and it also results, simultaneously, in the response space for the picture being mapped into the schema $\{balanced, unbalanced\}$, allowing a coherent expression of our subjective state. Since this schema allows appropriate mappings and reorganizations, this instance of application of the term 'balanced' is genuinely aesthetic.

Now, there are usages of terms that are similar to aesthetic usage but are not genuinely aesthetic; usages of "counterfeit" aesthetic terms, as we have labelled them. In those cases, there is either no adequate mapping or reorganization. The use of colour terms to describe music can serve as an illustration. We can certainly apply colour terms to describe music, for example, by calling a certain musical piece "blue". As we know, that act implies applying the schema $\{red, blue, green, \ldots\}$ to the domain of musical pieces and thus that the domain of musical works undergoes a reorganization to fit in the structure of the colour schema. This reorganization is rather arbitrary but, according to Goodman, there is nothing wrong with that, since the adequacy of a schema depends on context and habit [30, p. 71]. Most domain reorganizations in metaphor are to some extent arbitrary. The usage of colour terms up to this point amounts to mere metaphor, and its arbitrariness is not an issue. But if we want to apply the term as a genuine aesthetic term, then, in addition to a metaphor-like reorganization, we need to map our response space into the colour schema in such a way that it allows us to coherently express our subjective states. In contrast to the reorganization, the mapping is never completely arbitrary, since response spaces always have at least a pair of opposite affective poles—full-pleasure and full-displeasure responses. The schema $\{red, blue, green, \ldots\}$ has very few points of reference to allow us to identify obvious semantic opposite polar extremes to be associated with the affective extremes. In these circumstances, any mapping into the colour schema is more or less arbitrary, since any mapping of affective responses into colour labels is as good (or as bad) as any other. There is no way to tell that one mapping is better than another. For example, how can we decide whether red or orange or yellow should be assigned to a positive affective response? The usual way in which we deal with the terms red, orange and yellow does not seem to offer hints as to how to relate, in a non-arbitrary manner,[1] the connections among those terms with the structure of our affective responses, which does possess affective opposite poles. Although a mapping is possible, the arbitrariness of such a mapping hinders the expression of subjective states, and it makes the colour schema a poor choice to coherently express our subjective state. This makes clear that the mere existence of mappings and reorganization is not sufficient for a genuine use of aesthetic terms. The mapping and reorganization must be adequate to carry out a coherent expression of our subjective states and, like in metaphor, an effective communication of the characteristics of the described object. If we stress the relevance of the adequacy of the mappings and reorganizations, the RSD model can be employed to characterize genuine aesthetic terms and differentiate them from non-aesthetic terms as well as counterfeit aesthetic terms.

9.2.6 Definitions of Aesthetic Term, Description and Judgement

After the foregoing discussion, characterizing the notions of aesthetic terms, description and judgments is simple. Consider the following definitions.

Definition: Aesthetic Term

A term A in a schema S that refers to an object O in the domain D is an aesthetic term if and only if (1) there is a response space R for O; and, (2) there is a reorganization of D in terms of the structure of S such that it allows communicating a feature of O; and, (3) R can be mapped into S in a manner that allows the coherent expression of the affective responses in R.

Definition: Aesthetic Description

An aesthetic description is a sentence of the type 'O is A', where O is an object being qualified and A is an aesthetic term as defined above.

[1]Of course there exists the convention of labelling sadness blue, anger red, envy yellow, and so forth. But that *convention* is completely arbitrary in respect of response spaces.

Definition: Aesthetic Judgement

An aesthetic judgement consists in an aesthetic description that expresses the subjective state of an individual resulting from the evaluations involved in his aesthetic experience.[2]

Now, in the aesthetic as process theory endorsed here, even more important than characterizing aesthetic terms and judgements is to understand their function (or functions) in aesthetic-processes. Those functions are discussed in the following chapter. But before that, it is important to address the issue of "always and only" aesthetic terms.

9.2.7 *Always and Only Aesthetic Terms; The Spectrum of Aesthetic Terms*

The above discussion has focused on aesthetic terms that seem to involve a sort of metaphorical mechanism, but the characterization advanced there is valid in general. As Kivy pointed out, some aesthetic terms (beautiful, ugly, elegant, lovely, etc.) seem to work always and only as aesthetic terms, and, as De Clercq pointed out, they even seem to be resilient to metaphor. By contrast, some other aesthetic terms (the interesting ones, according to Zangwill) have mundane applications (balanced, unified, sad, etc.) and acquire their aesthetic character through their use in aesthetic descriptions. This issue also prompts the question of whether there exists a clear distinction between aesthetic and non-aesthetic terms. Let us address these issues with the RSD model.

Regarding the aesthetic/nonaesthetic terms distinction, Sibley and Hungerland assume its existence, Kivy is rather sceptical about it, and De Clercq even postulates a category of semi-aesthetic terms. Now, in the RSD model, there is no fixed list of terms that comply with the definitions formulated above. Rather, a wide range of natural language terms is suitable to be used aesthetically, depending on the context. The aesthetic use of mundane terms such as 'balanced' or 'heavy' is closely related to their metaphorical use. Now, metaphorical terms are not restricted to any particular collection of terms, since metaphors are to some extent arbitrary.

[2]This definition of aesthetic judgement seems to entail that we cannot have false aesthetic judgements. I believe that the role of a genuine aesthetic judgement is to encourage the clarification or elucidation of subjective states (we shall discuss this in the next chapter). In this sense, there is no way to adequately characterize a false aesthetic judgement other than as a mistakenly applied aesthetic term, or, as in the case of counterfeit aesthetic terms, as terms that do not really clarify subjective states—the example of colours illustrated this, since a non-coherent mappings hinders expression.

The RSD model inherits this feature; aesthetic terms are not restricted to any particular collection of terms. This explains Kivy's view that there is no fixed list of aesthetic terms.

The RSD model exploits the metaphor-like characteristics of aesthetic terms. According to the RSD model, aesthetic terms reorganize the domain of objects of appreciation in the same fashion as metaphors. But the model can explain both the metaphor-like and "always and only" aesthetic terms. Always and only aesthetic terms like 'beautiful' or 'elegant' comply with the RSD model in an interesting way. As pointed out by De Clercq, they are literally applicable in every domain. In a sense, any domain is the original domain of the schema consisting of terms like beautiful, ugly or elegant. Thus, in the case of the schema of always and only aesthetic terms, a communicative reorganization of the domain is always trivially available. The strongest RSD-model condition at work in these cases is the existence of a coherent expressive mapping. Coherent mappings are also trivially available, since the literal meanings of always and only aesthetic terms usually involve references to affective reactions. For example, one of the senses in the entry in The Macmillan Dictionary defines 'beautiful' as "very pleasant". This provides obvious points of reference to map a response space into a schema like $\{beautiful, ugly, \ldots\}$. Furthermore, unlike the mappings involved in mundane metaphor-like aesthetic terms, such mappings are completely independent of the domain of application, since a communicative reorganization is always available. Terms like 'beautiful' or 'elegant' are thus characterized by the fact that their correct application as aesthetic terms does not depend on the domain of application. Since the only element in determining the correct application of these terms is the existence of an appropriate mapping, we can say that these terms are characterized by the fact that the adequacy of their mappings is *domain invariant*. In this way, we can consistently explain both the feature of metaphoric resilience of some aesthetic terms, as pointed out by De Clercq, and the existence of aesthetic terms that seem to be metaphoric, as pointed out by Sibley, or even necessarily metaphoric as argued by Zangwill. The RSD model does not need ad hoc hypotheses like the existence of a semi-aesthetic class of terms. We only need to argue that metaphor-resilient always and only aesthetic terms are a limiting case of aesthetic terms: terms for which the adequacy of their mappings is domain invariant. The extension of the set of terms in this limiting case include the terms 'beautiful', 'lovely', 'pretty', 'elegant', 'ugly', 'horrible', and all other "metaphor-resilient" aesthetic terms. In general, the RSD model allows for a rich spectrum of cases, not only the limiting "always and only aesthetic" cases, depending on how well any given schema is able to cover domains and facilitate mappings. Now, it is possible to exemplify the set domain invariant limiting cases, as we did above, by listing some of "always and only" aesthetic terms, but it is hard to do such a listing with non-domain-invariant terms, since their functioning as aesthetic terms depend on how they are applied in a context. Examples of non-domain-invariant aesthetic terms are 'balanced', 'tightly-knit', or 'unified', but in order to see that they are genuine aesthetic terms we need to know the context in which they are applied. This is evident in the fact that other non-domain-invariant terms like ‘red’ or ‘blue’ are much harder to interpret as aesthetic terms.[3]

Thus, although the RSD model does not entail the existence of a clear-cut distinction between aesthetic and non-aesthetic terms, it allows the existence of limiting cases exhibiting domain invariance. Metaphoric resilience and universal applicability are features of terms with domain-invariant mappings. Those features are special cases of compliance permitted by the RSD model, although they are not intrinsic characteristics of all aesthetic terms. In this way, both the acceptance and rejection of the aesthetic/nonaesthetic terms distinction are both justified to some extent: there is no principled division into two classes of aesthetic and non-aesthetic terms; as pointed out by Kivy. But the existence of a limiting “always and only” class of domain invariant aesthetic terms is allowed by the RSD model, which to some extent explains why Sibley or Hungerland embrace the aesthetic/nonaesthetic distinction.

[3]As we have seen, colour terms can acquire the conventional emotional meaning blue-sad, red-anger, and so forth. So, it is possible that they may acquire a conventional aesthetic meaning, but that only stresses the fact that, as aesthetic terms, they are highly context dependent.

Chapter 10
Aesthetic Judgement II: Functions

In the previous chapter we characterized the concept of aesthetic terms in terms of a mapping/reorganization among response spaces, schemata, and domains—hence the "RSD" in *RSD model of aesthetic terms*. Even more important than characterizing their concept is to understand the function aesthetic terms play in aesthetic-processes. In this chapter we shall explore that function, or, more precisely, functions.

Aesthetic judgements, even characterized by the RSD model, share a feature with other kinds of judgements: they deliver evaluations. The evaluations involved in an aesthetic-process can be interpreted non-cognitively—as affective responses—or cognitively—as a propositional result of applying aesthetic criteria. The obvious function of aesthetic terms and judgements seems to be to make explicit in a propositional manner an ineffable inner experience. My claim here is that in conducting that function, aesthetic terms and judgements actually fulfil the additional and more fundamental function of changing the constitution of the experience itself. My goal here is to explicate that an aesthetic judgement is the *result* of a process that articulates and summarizes evaluations in our inner experience, but that process transforms the experience itself.

10.1 Concept and Function

As pointed out in Chap. 6, aesthetic judgements figure in node 5 of our rough description of an aesthetic-process. Unlike aesthetic experience—which I interpreted as a subprocess—or aesthetic value—which I interpreted as a relation—aesthetic judgements *present or express a particular state in the aesthetic-process*; aesthetic judgements are outputs, so to speak, of aesthetic-processes. Aesthetic judgements, however, do not present an objective event in an aesthetic-process; but rather a subjective state in a person engaged in an aesthetic-process.

As in the case of aesthetic experience and value, aesthetic judgement must be understood not only by characterizing its concept, but also by grasping the role

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_10,

it plays in an aesthetic-process. We shall see that aesthetic judgements have two functions, which I label *articulation* and *broadcasting*. We have seen how aesthetic terms can be characterized in terms of semantic relations that allow them to express subjective states and to communicate qualities by metaphorical means. We shall see now that aesthetic judgements are not mere linguistic vehicles that capture qualities involved in an aesthetic experience, they modify the constitution of the experience itself.

10.1.1 Emphases and Perspectives

The differences among the views on aesthetic terms discussed in the previous chapter show that the issue of aesthetic terms is rather complex. Authors like Hungerland and Kivy even appear to contradict each other on topics like the existence of a fixed set of aesthetic terms. I believe that some of the authors' differences can be explained as the result of their different emphases and perspectives. The differences between Hungerland and Kivy illustrate this. It is not hard to see that Hungerland tends to focus on subjective conditions of application, whereas Kivy tends to focus on objective semantic functions. If we consider this, Hungerland and Kivy complement rather than contradict each other. But this also illustrates my point that to understand aesthetic terms, in addition to characterizing them, we need to understand the role they play in the series of events occurring when an individual engages in an aesthetic transaction with the world.

Thus, instead of dwelling on the tensions between the different approaches, I shall try broadening our focus. In addressing aesthetic terms we may concentrate solely on their nature, or we can also address the reasons why we pass aesthetic judgements. And the ideas advanced in the previous chapter can help us to gain insight on why we pass aesthetic judgements. My argument is that *the act of passing aesthetic judgements encourages clarifying our internal affective states*. That clarification is achieved through the subjective changes that accompany the change of perspective induced by the metaphor-like reorganization and the mapping between response spaces and schemata. I shall refer to this clarification function as *articulation*. In a sense, the use of aesthetic terms refines our aesthetic experience. Now, in addition to the clarifying function, aesthetic descriptions, like any other description, has the more obvious function of conveying information from one individual to another. I refer to this function as the *broadcasting* function. Let us examine articulation first.

10.2 Articulation

We have seen that the differences between always and only aesthetic terms and mundane metaphor-like aesthetic terms do not represent a problem. However, for the sake of clarity, I address first the mundane aesthetic terms, as their use in aesthetic judgements illustrates more clearly their functions in aesthetic-processes.

Using a metaphor involves reorganizing a referential domain. Since aesthetic experiences are ineffable, as pointed out by Zangwill, metaphorical domain reorganizations helps us to focus the attention of our interlocutor on some relevant feature of the object being metaphorically described. For example, the use of the term 'balanced' to describe a picture redirects attention to the even distribution of shapes, or colours. Now, this redirection of attention occurs not only in the interlocutor, but also in the speaker himself. This change in attention, of course, is only a subjective change. Metaphorical reorganizations have no influence on the physical objective world. What changes is the perspective from which we contemplate that world. Thus, the act of using mundane metaphor-like aesthetic terms involves a subjective change in both the speaker and the interlocutor. For example, when Kivy uses the term 'balanced' to qualify certain picture the first thing we realize is that he does not intend to literally point out a mechanical feature of the picture. We come to understand that Kivy is pointing out an aesthetic feature some pictures possess. Implicit in this realization is the understanding that some pictures are balanced and others are not. That is, our perspective changes in the sense that we realize there is a balanced/unbalanced partition in the domain of pictures. We have changed the way we perceive the world of pictures. Now, the way we perceive the world determine what properties we discern in it, and, as we discussed in Chap. 7, those properties are constitutive of the aesthetic experience. That is, the act of using an aesthetic term influences the very constitution of our aesthetic experience.

Moreover, simultaneous to the reorganization, the act of using an aesthetic term associates particular terms of a schema with corresponding possible affective responses. Our perspective changes; and that change occurs in a way that also gives us clarity about our own affective inner world.

The kind of clarification described above is what I call articulation. It occurs as part of the events involved in setting an appropriate mapping/reorganization as described by the RSD model. The clarification cannot be carried out by means of non-aesthetic descriptions like "this painting uses pale colours, predominantly blues and greens, and has kneeling figures in the foreground", since this type of descriptions are intended to convey information publicly accessible and thus there is nothing special within us to realize. The clarification can neither be carried out by descriptions like "John's aesthetic experience was two minutes long", since this type of descriptions only summarize surface characteristics of the experience. By contrast, the appropriate use of aesthetic terms sets relations among the responses, terms and domain through the subjective events involved in the articulation. By setting the RSD model relations, a change in our subjective state occurs; a change in the way we experience our own experience, so to speak.

Articulation can thus help us explain why we pass aesthetic judgements: passing aesthetic judgements turns the vague affective state of a person into something more definite, something clearer even to the person himself. This occurs with metaphor-like as wells as always and only aesthetic terms. Qualifying an object as beautiful, for example, establishes that there is a certain relation between the presence of the object and our affective response to it. At the same time, it implies that the use of the schema to which the term 'beautiful' belongs adequately covers the possibilities of our responses, since it coherently maps our possible responses to the terms in the schema. Finally, the use of the term 'beautiful' in a description tells us the location of our specific response in the space of affective responses, it tells us where we are in the response space, so to speak.

10.2.1 Definition of Articulation

The articulation of aesthetic experience can be defined in terms of the conditions of expressive mapping and communicative reorganization. The process of articulation can be seen as the series of subjective events and changes associated with fulfilling, mostly unconsciously, the conditions for correctly applying genuine aesthetic terms.

The condition for correctly applying aesthetic terms is the simultaneous existence of an adequate expressive mapping and communicative reorganization. When this condition is fulfilled a series of events occurs within the person. A crucial event is identifying one's subjective state, since to successfully pass an aesthetic judgement one needs to be minimally aware of the affective state to be expressed in the judgement. In a sense, mapping a response space gives us a general chart of our possible subjective states in terms of a family of terms. But we need not only the chart of our inner world, we also need to determine our current location on that chart. The act of determining that location involves choosing adequate schemata, mappings and reorganizations, and, more importantly, the particular subjective clarification by change of perspective that accompanies that act. For convenience, I label the process of clarification and subjective change by choosing schemata, mappings and reorganizations a process of *subjective articulation*. Let us characterize it.

Subjective Articulation

Subjective articulation is the process by which the conditions of application of an aesthetic term—in order to express the inner state of a person engaged in contemplation—are actualized. It consists in performing the actual mappings and reorganizations required by the expressive mapping and communicative reorganization conditions. Actualizing those conditions is accompanied by a series of events that change the constitution of the individual's aesthetic experience. Subjective articulation thus include a clarification, by means of a change of perspective, of the subjective state of the individual.

Due to the existence of confusing responses, the nature of response spaces seems to need matching complex schemata to provide different terms to express the different possibilities of our inner experiences. We have seen that choosing the right schema to perform a mapping/reorganization is a rather subtle balancing act. There is no non-arbitrary way to choose among different alternatives of schema and their corresponding mapping/reorganization. Apart from the existence of clearly opposite combinations (full-pleasure and full-displeasure) in a response space, there are few constraints within the structures of the response spaces and the schemata that can be objectively considered to decide in favour of one mapping/reorganization over another. These finer, relatively arbitrary, and mostly unconscious decisions on which schema to choose are part of the subjective articulation.

Now, the result of a process of subjective articulation is that the individual becomes aware of the specificity of the experience in which he is engaged. After clarifying our inner experience, we get, so to speak, the coordinates of that experience. In this sense, we pass an aesthetic judgement not to advance an objective statement of the state of affairs, but to elucidate our subjective state. In a sense, using aesthetic terms contributes to making us aware of the aesthetic character of our experience. The subjective state of a person—unlike aesthetic experience, which is a process, or aesthetic value, which is a relation—is one of the many individual events in an aesthetic-process. Since the different elements of the aesthetic-process are interdependent, the articulation occurring in the experience also results in a sort of articulation of the aesthetic-process in general. Hence, using an aesthetic term in a judgement serves also to articulate an *aesthetic-process*, to make an aesthetic-process more definite. In the wider context of aesthetic-processes, the main role of aesthetic judgements is to encourage the articulation of aesthetic-processes. The aesthetic-process is determined by a multitude of events that correlate with each other in an systemic manner; but none of these events by itself is enough to characterize and individualize the process. However, passing an aesthetic judgement about the object involved in the aesthetic experience encourages the person engaged in the aesthetic process to become aware of the character and individuality of the experience. Aesthetic terms thus play a very central role in aesthetic-processes.

We can define *process articulation* as the process of organizing and charting the different aspects of an *aesthetic-process*. The aspects involved in process-articulation are the same as the ones in subjective articulation: a domain of objects, a family of terms that organize that domain, and a response space.

Aesthetic Articulation

Process-articulation is the process that sets the specific relations among the objects of appreciation, the terms that describes them, and the pleasure-relations (recall that responses spaces are constituted by the second coordinate of the elements in pleasure-relations) necessary to locate us in a concrete spot in relation to the events involved in the experience.

For example, the process articulation involved in passing a judgement like ‘this picture is balanced’ includes the decision to employ the schema {*balanced*, *unbalanced*}, rather than, for instance, the colour schema. This decision depends in part on realizing that the first schema is more appropriate, as its structure places in the foreground the “balanced” property—perhaps drawing attention to the analogy between even distribution of mass and even distribution of shapes and colours—and it is also suitable to express the possibilities (pleasure, displeasure) of our subjective experience. The schema choice must provide an appropriate mapping/reorganization. Since the suitability of schemata depend on context, habit and often on personal and subjective circumstances (since the decisions are made by an individual with some particular skills, knowledge, familiarities, experiences, and so forth), the ability to choose schemata and setting appropriate mapping/reorganizations is a skill that improves with experience and practice.

Now, by the time an individual has passed a judgement his perspective would have changed, he would have achieved a mapping of his response space, and located himself utilizing that mapping: he would have elucidated and structured his experience. As we discussed, the perspective change induces a change in the properties we can discern in the world. The constitution of the experience itself is different after the judgement has been passed. Furthermore, the inner subjective state at the core of the experience is no longer merely a vague feeling of 'I like it!'; our decisions and changes of perspective have clarified the experience in such a way that it can be expressed in a complex propositional way; as an aesthetic judgement. In other words, by attempting to express the non-cognitive evaluation at the core of our experience we turn that crude evaluation into something with a much richer structure and cognitive content.

10.3 Broadcasting Function

Aesthetic terms fulfil the function of conveying information among different individuals. But they do so in a particular way, which we shall explore here.

As we have seen, authors like Hungerland and Kivy have pointed out different peculiarities of aesthetic terms. For Hungerland, aesthetic terms describe things in an idiosyncratic manner; as things look to the individual making the description. For Kivy, aesthetic terms play a function in terminal judgements—judgements that do not lead to further consequences or actions. Aesthetic terms certainly convey information, but, in the model presented here, the way aesthetic terms broadcast information is connected with their articulation function.

Aesthetic judgements function as outputs that distribute information to other parts of aesthetic-processes and to the exterior of the process. This is their broadcasting function. It is not difficult to see that broadcasting is not independent of articulation. An aesthetic judgement identifies a core element in an aesthetic process; the subjective state of the individual engaged in appreciating an object. Publicly expressing the state, however, must be carried out by conceptual and linguistic means. The need for a conceptual or linguistic summary of our subjective state prompts us to look for the best choices to clarify and communicate the process, that is, to conduct a process of subjective articulation. Identifying and expressing our subjective states occurs by describing the appreciated objects with an aesthetic term. In simple terms, we need to clarify our subjective state in an aesthetic episode before we can share it with someone else.

Furthermore, the function of articulation can explain the peculiar features pointed out by Hungerland and Kivy. The act of uttering an aesthetic judgement is an event that expresses a subjective state through subjective events (all events involved in subjective articulation are subjective). As we discussed in the previous chapter, always and only aesthetic terms are domain invariant, that means that, for instance, the term elegant in 'John is elegant', can be applied to any domain. That also means that the only way to challenge a judgement like 'John is elegant' is to incur in an inadequate response space mapping. But such mappings are completely subjective, so there is no way in which objective conditions can challenge this kind of judgements. Consistently with Hungerland's approach, we would say that always and only aesthetic terms are meant to express subjective perspectives. On the other hand, articulation in mundane metaphor-like terms explains Kivy's view. Recall that in order to make sense of descriptions like 'this picture is balanced' we need to realize that the sentence does no refer literally to a mechanical property. We need to interpret the sentence not only to realize what the interlocutor intends to communicate with the metaphor, but also whether the judgement is a positive or a negative one, and, therefore, what subjective state prompted him to use the term. Interpreting someone else's aesthetic descriptions thus involves a sort of mirroring subjective articulation in which we need to guess the mapping/reorganization implicit in the speaker's sentence. Kivy claims that aesthetic judgements are *terminal*, but a weaker notion of terminality may be more useful here. After all, there is evidence that actual terminality in aesthetic terms and judgements is weaker than initially proposed by Kivy. Some aesthetic descriptions, in art criticism, for instance, are not terminal in Kivy's sense, since they can lead to further actions or changes in attitude. A positive review of a painting by a prestigious critic may prompt us to, for example, go to a museum and see the picture. A bad review may induce us to change our opinion about the picture's author. Thus, let us define a constrained notion of terminality in terms of our aesthetic as process theory: I say that the processes of subjective-articulation and process-articulation are terminal if they cannot lead to anything else but events *in further aesthetic-processes*. Although aesthetic description may lead to actions like contemplating a painting in a museum, or to changes in attitude like depreciating the work of a formerly appreciated artist, those events ultimately become part of new aesthetic-processes. To contemplate a painting in a museum is part of the process of aesthetic experience, and to change our attitude towards an artist's work amounts to a change in preferences and values. These events can be accounted for in terms of the aesthetic as process theory. A further advantage is that these events are in accord with the dynamic character of aesthetic value.

The weaker notion of terminality can help us to address with more detail the broadcasting function. As we know, the events in subjective and process articulation are subjective, and their resulting function—clarifying subjective states—is also strictly subjective. But an aesthetic judgement performs an objective broadcasting function when its aesthetic description makes information available to external agents that can eventually participate in further aesthetic processes. Aesthetic descriptions have a propositional content that allows the linguistic delivering of evaluations to other members of a community, including from its past—via the judgements we read in a book, for example—and to its future members. The process of articulation makes rich and clear cognitive evaluations available to ourselves and, by summarizing these relations in a public description, other members of the community. An aesthetic judgement articulates a private state, but by making it public it also broadcasts the perspective from which an object should be seen. It publicly places the object in a system of relations between subjective states, families of terms, and referential domains that connects the object with past aesthetic judgements, since those judgements share the same articulation mechanism. An aesthetic judgement publicly relates the object to aesthetic responses, thus situating the objects in the realm of the aesthetically relevant. It gives the object a place in the community's framework of the aesthetically relevant. Thus, in addition to articulating the process, the term associated with a subjective state enables us to perform a quick classification of the experience as belonging to the same class as past aesthetic experiences, while it also enables us to share this classification with other people by linguistic means. The aesthetic term is thus terminal in the sense defined above, as it only leads to other aesthetic-process events.

10.3.1 Locally Terminal Aesthetic Terms

By uttering an aesthetic description like 'O is A', the speaker turns public a private experience. The speaker is not only carrying the information contained in 'O is A' from one person to another; he is also making himself and his interlocutor participate in many of the events involved in articulation; they become somehow aware of the speaker's subjective state, and its associated expressive mapping and communicative reorganization. Schemata in communicative reorganizations offer more flexible mappings and are thus better suited to sharing these implicit events. This is the reason why mundane terms are used in aesthetic descriptions: metaphors are utilized to facilitate presenting an efficient summary of the subjective relations among objects, terms and experience to our interlocutor.

Now, Kivy's proposal of terminality does not refer to the content of the judgements or its conditions of application, but rather to what aesthetic judgements do, or, rather, what they do not do. Analogously, I have used the functions of aesthetic judgements to characterize them. But, of course, there is a very salient difference between Kivy's approach and mine. Kivy characterizes the function of aesthetic judgements in a negative way; by referring to what they do not do—they do not lead to further actions or arguments. The model presented here offers a function-based positive characterization of aesthetic judgements: aesthetic judgements do articulate individual aesthetic-processes and they do broadcast those states, such broadcasting is terminal but in a weaker sense than the one proposed by Kivy.

Further elucidation on the nature of aesthetic judgements can be achieved in a positive manner by summarizing some facts related to their functions. First, expressing a subjective state consists of clarifying such a state and presenting it in such a way that it agrees with the appropriate conditions of application of aesthetic terms and with our actual experience. Second, since the application of aesthetic judgements depends on the application of schemata, which is constrained only by the existence of adequate reorganizations and mappings, there is no set of "characteristically" aesthetic terms but only more or less suitable schemata. Third, aesthetic descriptions have the primary function of making ourselves or someone else aware of our subjective state and articulation process; this is the reason why they are terminal—in the weak sense. Fourth, aesthetic descriptions, like 'O is A', convey public information that becomes part of the personal and collective aesthetic experience; it becomes part of the culture. This also means that aesthetic judgements can play a role in other persons' eventual aesthetic-processes. This aspect show the importance of distinguishing between Kivy's notion of terminality and my weaker notion. Aesthetic terms must be seen as being locally—constrained to aesthetic processes—terminal, so to speak.

10.4 Unifying Approaches

We are now in position to go back to our initial discussion on the various views on aesthetic terms and show that the RSD model dissipates the inconsistencies and unify the diverse views. In the previous chapter, we discussed the issue of the aesthetic/nonaesthetic distinction. Since the conditions of use of aesthetic terms stipulated by the RSD model allow a wide range of variation in conducting successful mapping/reorganizations, there is a wide spectrum of terms that can be used as aesthetic terms. According to Kivy, those terms show that there is no aesthetic/nonaesthetic distinction. The characteristics of mundane aesthetic terms resemble metaphors, but there is a class of limiting-case aesthetic terms whose mappings are domain invariant; those terms are always and only aesthetic terms—beautiful, ugly, elegant, etc.—and they seem to justify Sibley and Hungerland's view that there is an aesthetic/nonaesthetic distinction. The RSD model can reconcile distinctionists and antidistinctionists, since their positions can be seen as the result of concentrating on different shades in the spectrum of aesthetic terms: on the one hand, there is no fixed list of aesthetic terms but a whole range of possibilities. But on the other hand, there is a distinctive class of aesthetic terms, a limiting case in the spectrum, that shows that distinctionits have a point. The RSD model has plenty of room for both views. The RSD model does not entail the existence of a clear-cut distinction between aesthetic and non-aesthetic terms but it allows the existence of limiting cases. Metaphoric resilience and universal applicability, in accord with De Clercq's view, are features of terms with domain-invariant mappings, but such features are a special case of compliance permitted by the RSD model and not intrinsic characteristics of all aesthetic terms. In this way, both the acceptance and denial of the distinction between aesthetic and non-aesthetic terms are justified to some extent.

We can also account for Sibley's description of how metaphor and art criticism helps us to refine our use of aesthetic terms. Sibley believes that the exercise of aesthetic sensitivity is what characterizes aesthetic terms and that we gain proficiency in applying aesthetic terms through several special methods that help us develop such sensitivity. The RSD model does not need to resort to any special faculty or sensitivity to account for our progressive gaining of proficiency in applying aesthetic terms. The ways by which we gain proficiency in using aesthetic terms, as described by Sibley, can be seen as the ways in which we acquire proficiency in choosing schemata, mappings and reorganizations; that is, proficiency in articulating our aesthetic experience. As we have seen, choosing appropriate terms involves a delicate balancing act in which we must mind not only what schema is appropriate to express our subjective states, but also that the schema must force an appropriate reorganization onto the domain of objects. Since the balancing act is not easy, it makes sense to envisage that people progressively gain the skills to perform it. We become proficient in this balancing act through a process of learning that can be described in the same terms as Sibley describes how the critic's guidance helps us to learn the correct use of aesthetic terms: we learn by familiarization with paradigmatic cases of the usage of aesthetic terms, that is by paradigmatic cases of articulation; by diverse sorts of exemplification and by trial and error, that is by contrasting good examples of articulation with poor ones; and by reiteration. All these ways of learning can be interpreted simply as a progressive development of the skills involved in using and choosing schemata/mappings/reorganizations in the RSD model. It must be emphasized that these skills do not involve any special faculties (like a faculty of taste) since the events in processes of subjective and process articulation involve only our regular cognitive, linguistic and introspective capabilities. The role of art criticism is thus to show examples of "good" articulations, that is, examples of appropriate schemata/mapping/reorganization choices, which serve as guides, as controlled contexts to develop proficiency in articulating aesthetic experiences.

In order to account for Hungerland's characterization of aesthetic terms—in terms of the distinction *looks/is*, it is necessary to bring aesthetic descriptions into the picture. As discussed above, in the RSD model the events involved are strictly subjective; this is consistent with Hungerland's idea that there is no objective conditions that can challenge an aesthetic description. More specifically, Hungerland's view is that aesthetic terms do not admit the *is/looks* distinction because they are devised to describe how things may look to a person under certain circumstances. In the RSD model, one of the characteristics of aesthetic terms is that they make us see things in a particular way; a way determined by how a schema reorganizes the domain of objects of appreciation. This is true for metaphorical aesthetic terms, but not for terms such as beautiful or elegant. The absence of an *is/looks* distinction in these cases results from domain invariance: if the subjective conditions for the correct application of a term like 'elegant' are fulfilled, the only possible source of error in applying the term would be to make a categorical mistake, but since these kind of terms are domain invariant there is no possibility to establish an objective distinction between something really *being* elegant and just *looking* elegant. In this way, the RSD model accounts for Hungerland's ideas in metaphorical as well as literal cases of aesthetic terms.

Finally, the RSD model can account for Kivy's terminality approach, with some constraints. In the RSD model the usage of an aesthetic term articulates an aesthetic experience. The circumstances that ground the passing of an aesthetic judgement begin and end in the subjective. Even when an aesthetic judgement is made public, we can only agree or disagree with the description by carrying out our own processes of articulation, which is also a subjective event. This explains why aesthetic descriptions do not lead to arguments on objective matters, actions or changes in attitude (or at least actions and attitudes intended to modify objective circumstances). In a sense, aesthetic descriptions are merely "aesthetic-process terminal" because the role of passing aesthetic judgements involves a sort of "closed loop" that starts with subjective experiences and ends with an enhancement (or an expression) of them. But the loop never reaches the realm of the objective. Aesthetic episodes are terminal in the context of aesthetic-processes. For example, an aesthetic description can be a description made by an art critic to explain an abstract work of art that would otherwise be inaccessible. This critical description makes the artwork available to its audience, and thus it can lead to changes of attitude towards the work and to further aesthetic-processes. This aesthetic description becomes engaged in attitude changes and further aesthetic-processes. Now, aesthetic descriptions and the articulation that leads to them are subjective and they always remain in the realm of the subjective. They may even become part of our value repository and thus participate in further aesthetic-processes. But their influence is still terminal, in the constrained sense, since their role begins and ends in an aesthetic-process: they are terminal in the local context of aesthetic-processes.

Aesthetic terms are terminal in the sense of being aesthetic-process closed. Now, when an aesthetic judgement is made public, it conveys information about certain quality of the object that can be picked up by other people. This information can affect objective circumstances in different ways, but the most relevant of those ways consist in ending up in new aesthetic experiences or even in new processes of articulation of aesthetic experiences. This further supports the idea that a better characterization of aesthetic terms is in terms of aesthetic-process *local* terminality, since the information conveyed by an aesthetic judgement may result in further arguments or actions related to aesthetic experiences; that is, in larger "loops" involving aspects such as aesthetic experiences or the developing of aesthetic values. We have actually described such loops as long term feedback pathways in our description of aesthetic process in Chap. 6.

Chapter 11
Mathematical Aesthetic Judgements

I endorse a literal approach to mathematical beauty in this book. I thus endorse that mathematical aesthetic judgements are not particularly different from other aesthetic judgements. The notion of aesthetic judgements I have advanced in the previous chapters shall allow us to show that mathematical aesthetic judgements can be characterized in the same terms as other aesthetic judgements. In this brief chapter, I use the aesthetic as process theory to finally account for the leitmotif of our discussion: the term 'mathematical beauty', and mathematical aesthetic judgements in general.

Let us recall Le Lionnais's judgement on Euler's Identity, describing it as insipid. The judgement should be seen as expressing Le Lionnais's subjective state in the context of his aesthetic-process, leading hence to no objective conflict with the mathematicians who judge Euler's Identity the most beautiful theorem in mathematics. For many of the readers of *The Mathematical Intelligencer* the contemplation of Euler's formula elicits a response of pleasure. Since we have assumed that the experience is of the basic type, its pleasure-relation has two possibilities: pleasure and displeasure. The positive affective response is articulated by mapping these possibilities into the schema $\{beautiful, ugly, \ldots\}$, or, even better, $\{beautiful, lovely, ugly, unremarkable, insipid, \ldots\}$ and by reorganizing the domain of mathematical formulae, dividing them into beautiful, ugly, insipid, and so forth. A person who experiences a high degree of pleasure expresses his state by means of the term 'beautiful'. But Le Lionnais, who experiences no response, or even a slightly negative affective response, expresses his state with the term 'insipid'. The object of attention and the modality of the experience in this example is characteristically mathematical. But having characteristic objects and experience modalities is something that is also the case for almost any other type of aesthetic experience. Music, painting or poetry are all very different in terms of the objects with which they present us and the way they engage our attention. In general, aesthetic experience depends on the specifics of each discipline. This is also consistent with my interpretation of aesthetic value, since in that interpretation the value set consist of a wide range of value repositories. In this sense, the idea of a specific

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_11,

mathematical value repository makes perfect sense. My conception of aesthetic judgement trivially yields a unified depiction of mathematical aesthetic judgements and the rest of aesthetic judgements, since the process of articulation and the functions of the judgement are the same for mathematical and more traditional aesthetic judgements: mathematical aesthetic judgements express a particular kind of subjective states of mathematicians: the affective evaluation in contemplating a mathematical item.

Now, one of the issues of mathematical beauty is that mathematical aesthetic judgements seem puzzling to a mathematically lay person, since at first sight this person cannot understand how mathematical aesthetic judgements relate to regular, everyday aesthetic judgements. This problem can be easily addressed: the puzzling character of mathematical aesthetic judgements merely manifests the fact that the aesthetic experience in mathematics has a peculiar modality; one that is very dependent on background knowledge. As correctly pointed out by Rota, a great deal of knowledge is necessary to appreciate any piece of mathematics. Now, the modalities of music or painting are also unique in their own way, and, more importantly, they also depend on knowledge: if we know more about music, we hear more things in music; and if we know more about painting we see more things in paintings. In my model, this fact is interpreted as a requirement of the space of mathematical aesthetic intentional objects: in order to be able to appreciate a mathematical item we first must be able to "see" it, that is, we must be able to turn the item into an intentional object. This "seeing" is possible only if we understand the mathematical item. Similarly, in order to enjoy a written poem we must first be able to read it. Mathematical aesthetic judgements are not exceptional: the contents of the experience and the values associated with them are just characteristically mathematical. Mathematical aesthetic judgements constitute one among many classes of aesthetic judgement.

The typical aesthetic-process that grounds the passing of a mathematical aesthetic judgement can be summarized as follows: a literal interpretation of a mathematical aesthetic term A in a mathematical aesthetic judgement 'M is A', entails that the mathematical item M appears in a locally terminal stage of an aesthetic-process. Such an aesthetic-process is characterized by an experience sub-process in which the content is a mathematical intentional object whose aesthetically relevant dimensions consist of a set of properties "seen" in M. The passive and active content of the experience result in an affective response, and, depending on how this occurs, the experience can be categorized as basic, performative or adaptive. The affective response is an affective evaluation that is also involved in a judgement sub-process—described in the previous two chapters. The result of the judgement sub-process is an aesthetic description that expresses the state of the aesthetic-process and that, simultaneously, results in a clarification of the experience—subjective articulation—and the aesthetic-process—process articulation. In addition, mathematical aesthetic judgements carry information that can be used in non-terminal ways, by directly participating in, or encouraging other aesthetic-processes. Mathematical aesthetic terms are terms that participate in encouraging subjective and process articulations in mathematics.

Mathematical aesthetic descriptions are locally terminal, they are bridges between the private and the public aspects of aesthetic-processes. An aesthetic description carries information that can eventually be incorporated in a value repository, thus influencing mathematical aesthetic criteria, future evaluations, and even the work of mathematicians.

We can now give very simple answers to the questions posed at the beginning of this book. Recall that the mathematically lay person is entitled to ask: isn't truth, and not beauty, the goal of mathematics? Or, what is the difference between beautiful and ugly mathematics?

First, truth is the goal of mathematics; beauty comes as an extra, although an important extra that motivates mathematical development. Propositional mathematical knowledge comprises beliefs whose truth is justified by logical means. Truth is a precondition of mathematical knowledge, and this knowledge is in turn a precondition (in the form of a phenomenological space's background-understanding dimension) of mathematical aesthetic experience. A mathematician struggling for beauty, aims at achieving truth *and beyond*, so to speak. Second, the difference between mathematical beauty and ugliness is not simple, as implicit in the existence of composed responses, which leads to the existence of diverse and even overlapping value repositories. This is also manifest in the fact that we need families of terms, rather than isolated terms, to express and clarify aesthetic experiences. However, beauty and ugliness are closely linked in the sense that their expression is often grounded on the fact that they have opposite locations in the same phenomenological space dimension (we like simplicity and dislike complexity, we like harmony and dislike disharmony, and so forth). So, in a sense, the intuitive notion that beauty and ugliness are opposites is valid to a certain extent, even for mathematical beauty and ugliness. Finally, we saw above the reason why the lay person finds strange the use of aesthetic terms by mathematicians: technical knowledge enhances aesthetic experiences in general, but in mathematics knowledge is precondition to even contemplate the object under evaluation.

Now, mathematical aesthetic judgements are intrinsically tied to aesthetic-processes. This means that if we want to analyse a case of mathematical beauty we must be able to produce the appropriate analyses of, at least, experience, value and judgements for mathematical items. The next chapters shall be devoted to present, with some detail, such analyses. I shall apply the theory developed in the previous chapters to three concrete examples of mathematical aesthetic judgements that illustrate a wide enough range of aesthetic terms.

Part III
Applications

Chapter 12
Case Analysis I: Beauty

In this and the following chapters, I present detailed applications of the aesthetic as process theory advanced in the previous ones. This shall also serve to further discuss and clarify some specific issues, like the role of knowledge or of properties like unexpectedness or the shortness of steps in proofs. I analyse three cases of aesthetic judgement in mathematics, beauty, elegance and ugliness, which cover the most used aesthetic terms in mathematics. After presenting the analyses I shall revisit some of the issues introduced discussed in the First Part to conclude with a final assessment of the theory.

Before turning to our first analysis, a brief summary of the ideas developed so far is in order. The basic tenet in the theory formulated in this book is that aesthetic episodes always occur as interrelated elements of a system I called *aesthetic-process*. The notion of *the aesthetic*—that is, the mark that characterizes aesthetic kinds of things—can be understood in terms of aesthetic-processes: something is *aesthetic* if it plays a relevant, non-contingent role in an aesthetic-process. I call this interpretation *the aesthetic as process theory*.

Aesthetic experience is a process in which a person interacts affectively and cognitively with a stimulus—which can be a mental construct—and his environment. In the case of mathematics, the process is characterized by the presence of aesthetic mathematical intentional objects. The role of these intentional objects is to unify the experience by serving as focus of attention and inducing affective responses.

Aesthetic value is a relation between sets of properties and mental activities associated to objects and the subjective responses in the observer. The role of value is to regulate evaluations. Value is inherently dynamic and its dynamics, mirroring the dynamics of aesthetic preferences and criteria, is governed by the mechanism I labelled constrained aesthetic induction.

Aesthetic judgements are expressions of subjective states. Their role is to encourage the articulation—and, eventually, to allow the broadcasting—of experiences in the aesthetic-process.

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_12,

Finally, since things like judgements, pleasure, experience, etc. can be qualified as aesthetic only if they participate as necessary elements in an aesthetic-process, they are characterized by their specific features *and their role* in aesthetic-processes.

12.1 Knowledge, Beauty and "Isolated Pearls"

Before beginning our analyses of mathematical aesthetic judgements, it may be wise to mind, once again, Gian-Carlo Rota, who issued a caveat against presenting instances of mathematical beauty as "isolated pearls":

> [. . .] despite the fact that most proofs are long, despite our awareness of the need for an extensive background in order to appreciate a beautiful theorem, we think back to instances of mathematical beauty as if they had been perceived by an instantaneous realization, in a moment of truth, like a light-bulb suddenly being lit [. . .]
>
> [. . .] Following this mistaken conviction, several attempts have been made to string together beautiful mathematical results, and to present them in the form of books bearing such attractive titles as "The one hundred most beautiful theorems of mathematics". Such anthologies are seldom to be found on any mathematician's bookshelf. The fact is that the beauty of a mathematical theorem is best observed when the theorem is presented as the crown jewel within a context of results of a theory. When instead mathematical theorems from disparate areas of mathematics are strung together and presented as "pearls", then they lose their relevance, and are likely to be appreciated only by those who are already familiar with them [78, pp. 179–180].

Rota points out that a great deal of the appreciation of mathematical beauty is about mathematical understanding. I agree with him, however, whereas he claims that mathematical beauty is *merely* about understanding, I claim that understanding is just part—albeit a fundamental part—of the modality of aesthetic experience in mathematics. Mathematical knowledge is a dimension of phenomenological spaces of mathematical aesthetic objects. Mathematical understanding is precondition for appreciation, something without which we cannot "see" the object of appreciation. Just as a deaf person cannot listen to music, or an illiterate person cannot read a poem, a person unfamiliar with mathematics cannot appreciate mathematical items. Knowledge in appreciation of mathematics is much more fundamental than in other disciplines. Rota correctly points out that the theoretical context is necessary to understand any piece of mathematics, and thus to see its beauty. However, here I do not intend to present instances of mathematical beauty so that the reader can appreciate them. I intend to account for the events that ground their aesthetic character. I shall certainly present isolated instances of mathematical aesthetic judgements; however, my goal is not to make the reader *see* the beauty, elegance or ugliness of certain mathematical items, but rather to *explicate the process that grounds the issuing of the judgements* of that beauty, elegance or ugliness. This circumvents Rota's issue with "isolated pearls" of mathematical beauty, which presentation, I agree, requires a broader theoretical context to be appreciated, and which is also why Rota dismisses "The One Hundred most Beautiful Mathematical

Formulas"-style works. From the beginning I made clear this book had a theoretical focus. Works devoted to the presentation of beautiful mathematical results fail to clarify what is mathematical beauty, as they are merely anecdotal presentations and do not deal with its theoretical foundations. In my approach, by interpreting mathematical understanding as an appreciation condition, once we are provided with a theoretical framework, those anecdotes can serve as starting points to offer a more substantive theoretical interpretation.

12.2 Case 1, Beauty: $y = e^x$

We start by addressing the simplest type of aesthetic experience, basic aesthetic experience. We shall build up from this in the next chapters. A very elementary instance of mathematical beauty should exemplify basic aesthetic experience. In this respect, Le Lionnais suggests the function $y = e^x$ which, according to him, exemplifies classical mathematical beauty:

> Who has not been amazed to learn that the function $y = e^x$, like a phoenix rising again from its own ashes, is its own derivative? [58, p. 126].

12.2.1 *Experience*

In the aesthetic as process theory, aesthetic experience is a sub-process that comprises a shift in the focus of attention, a focusing on aesthetically relevant properties, and a resulting affective response. In basic aesthetic experiences, according to our definition, mental activities do not play a significant role in eliciting affective responses; which, as we shall see, is the case in this example. Now, the process is started by an input that has the qualities needed to focus our attention in an aesthetically relevant way. The function $y = e^x$ is the object of attention. According to Le Lionnais, the fact that its derivative, y', is the same function, $y' = y$, is what makes it beautiful. I shall use the expression $y' = y = e^x$ to emphasize this fact.

The most relevant feature of $y = e^x$ here is not that the mathematical item refers to a set of properties of the object e^x, but rather that it manages to engage our attention in a process that leads to an affective response. As we know from our discussion of intentional objects, not all and only properties of the object are responsible for keeping our attention focused and eliciting a response. It is a collection of properties associated with the object and some properties resulting from of the subject's idiosyncrasies that manages to do so. Our object of attention is located in a phenomenological space whose dimensions are those properties. We established that there are two kinds of dimensions: background understanding and aesthetically relevant dimensions. Here the necessary background knowledge is mathematical analysis. Analysis provides us with the understanding that allows us to "see" the function. However, the aesthetic significance of the function depends not only on seeing a mathematical object, but also in affectively evaluating it. The function possesses features we appreciate. Of course, the reasons why we appreciate certain properties can be practical or of another kind, but among those reasons there is one that is aesthetically significant: the object possesses a type of properties that tend to elicit affective responses. As we have seen, a class of properties that includes simplicity, symmetry, harmony and so forth is often used to explain beauty, as in Hutchenson or Shaftesbury, who explained beauty based on the properties of simplicity and order. Explanations based on this kind of properties cohere with my model since these properties become part of the experience content as dimensions of its phenomenological space. Symmetry, for example, can be seen as an aesthetic principle that explains why we have positive responses to symmetric things. However, single-property principled explanations cannot account for *aesthetic experience*. In the aesthetic as process theory, aesthetic experience is a process in which aesthetic properties play a central role, but experience itself is constituted by all the events and relations in the process. Aesthetic principles are useful to justify our appreciation, but not to account for aesthetic experience: aesthetic episodes are complex systems of relations among individuals, their natural propensities, cultural influence, social interaction and objects. Aesthetic experiences must be explained in the context of these relations.

Now, the very first event in our aesthetic-process is the occurrence of a triggering input, $y = e^x$, to which Le Lionnais draws our attention by pointing out that $y = e^x$ is its own derivative—$y' = y = e^x$. This last fact further draws our attention, helps us to focus it and, eventually, contributes to elicit a response. The focus of our attention is not an instance of the formula, but rather some features of it. In order to understand the interesting property that the derivative of e^x is itself e^x, it is necessary to understand its theoretical context. We need to understand notions like function, derivative, e, and so forth. All these facts can be addressed as elements of an aesthetic experience: the background understanding of analysis is the basic, enabling, dimension of the phenomenological space in which our intentional object is located. We can set this dimension in place by introducing the interpretation of our background understanding dimension:

> *Background-Understanding-Analysis* = *the feature of being understandable only if analysis has been understood*

Background knowledge enables us to "see" the intentional object, and, furthermore, it largely determines which properties are visible in the space. For example, a result that is simple if we understand analysis is not necessarily simple if we understand only arithmetic. Now, just as seeing colours or hearing sounds are not enough for aesthetic appreciation of paintings or symphonies, aesthetic appreciation of mathematics requires something extra. This extra consists of some of the properties we find in our object of attention: once we understand that $y' = y = e^x$, we focus on the fact that there is a repetition of the term e^x, "like a phoenix rising again from its own ashes". Furthermore, as we understand that the term e^x is the result of applying the derivative operation (once), we also know it will appear again as the result of the derivative operation, regardless of how many times we apply it.[1] Now, there is a relevant phenomenon to which we immediately relate repetition: symmetry. Visual symmetry, bilateral symmetry, for instance, consists in the repetition of the visual properties of one side of the object on the other side in such a way that both sides would match if the object were folded. The important fact about symmetry here is that we like symmetric objects. We experience a positive response to symmetrical objects; we unconsciously classify symmetric objects as appealing; with a crude "I like it!" assessment. Now, symmetry occurs not only at the perceptual level. There are more abstract forms of symmetry. Palindromes, for example, can take forms that are not recognizable at first sight: "A man, a plan, a canal, Panama" reads the same when read backwards. To address these more abstract cases of symmetry a more general definition can be employed. For example, an object is symmetric with respect to an operation, if the object remains unchanged after the application of the operation [60, 65]. We can use this idea to represent the kind of symmetry we intuitively see in the repetition of e^x.

In our aesthetic-process, our attention shifts from attending the general properties of the function $y = e^x$ to an attention focused on its aesthetically relevant properties; in this case a type of symmetry. The dimensions of our phenomenological space are thus *analysis* as background understanding, and *symmetry* with respect to differentiation, defined as:

Symmetry = the feature of remaining unchanged under application of differentiation

With these two dimensions we can now interpret $y = e^x$ as an intentional object that appears in the phenomenological space with dimension *analysis* and *symmetry*. *Analysis* allows us to see the object; *symmetry* makes the experience aesthetic. By changing our attention from $y = e^x$ as a mere mathematical object to an object of attention in which our attention focuses on symmetry, the content of attention becomes an *aesthetic* mathematical intentional object. Our aesthetic mathematical intentional object appears in a bidimensional space with dimensions *analysis* and *symmetry*. Symmetry contributes to elicit an affective response.[2]

The understanding of $y' = y = e^x$ strikes us as pleasurable in a fashion similar to the way visual symmetry strikes us. As Rota speculates, we very likely enjoy symmetry in this case because it comes as an extra in our difficulties with understanding analysis: our response stems in part from the fact that symmetry is

[1]The same is true for the case of the integral operation: the term e^x appears repeatedly. Application of an integral operation yields $e^x + c$, however, since we must add a constant c. Subsequent applications of the integral yields even more polynomial terms, which makes the repetition less "clean" than in the case of the derivative.

[2]Interestingly enough, aesthetic principles like Shaftesbury's order or Hutchenson's unity amidst variety are, I believe, not completely alien to symmetry.

neither necessary nor expected in our formulas or results. Symmetry is an aesthetic property by itself, but the fact that we need to undergo all the difficulties of learning analysis in order to be able to see this kind of symmetry enhances its effect. Compare the case of polynomial functions; the functions $y = x^2 + c$ and $y' = 2x$ hold all the relations that allows us to interpret them as a function and its derivative. They are as hard to understand as $y' = y = e^x$. Knowledge of analysis is necessary to see them, but even if we focus our attention on $y = x^2 + c$ and $y' = 2x$ we shall fail to see any obvious symmetry, as the terms x^2 and c are not preserved under differentiation. Derivative-symmetry is not a general characteristic of derivatives and there is nothing in the notion of derivative that makes it so.

Since our attention on $y = e^x$ is not focused merely on mathematical objects and relations, but rather on qualities relevant in the phenomenological space *analysis-symmetry*, we can label the content of our attention:

$$\textit{analysis-symmetry-AMIO-}y = e^x$$

The process of shifting our attention to this intentional object—to our inner representation of the object, the intentional object itself, the fact that it induces an affective response, and the affective response itself constitute our aesthetic experience.

Now, this example of mathematical beauty has been chosen because it is very simple. It intends to illustrate basic aesthetic experience. The way the content *analysis-symmetry-AMIO-*$y = e^x$ relates to its affective response is similar to the way perceptual symmetry induces an affective response: it is a non-cognitive readily available response. Since the repetition of the term e^x is explicitly "visible" in the derivative, the content of our experience does not involve further mental activities, thus the experience is not performative. And since the content elicits a response through our natural preference for symmetry and not through an acquired preference, the experience is not adaptive.

As for the unfolding of the aesthetic-process, we have so far covered nodes 1–4. Node 1 is the triggering stimulus—although there is no actual perceptual stimulus. Node 2 consists in changing the way our attention is focused. In our interpretation, this corresponds to set the relevant *analysis-symmetry* dimensions for the aesthetic mathematical intentional object and interpreting the mathematical item $y = e^x$ as the intentional object: *analysis-symmetry-AMIO-*$y = e^x$. An affective response, node 3, is elicited by passive contemplation of this intentional object. Since the affective response is positive, our response, $\langle P, N \rangle$, is one of pleasure. That is, we experience an affective evaluation; which takes us to node 4 and into the domains of aesthetic value.

12.2.2 Value

I have defined aesthetic value as a relation between sets of properties and mental activities associated with an object, and the responses elicited in the subject.

The pleasure-relation associated with the experience analysed here can be easily identified. The passive content of our experience is the set of properties associated with the intentional object *analysis-symmetry-AMIO-*$y = e^x$. There is no active content. The content elicits a non-acquired pleasure response. However, according to our formalization, composed responses, the second coordinates of a pleasure-relation, consist of ordered pairs $\langle x, y\rangle$ where x and y are individual non-reducible affective responses. In the case of basic pleasure-relations the coordinate y is assigned the no-response constant, N, since there is no active content. The composed response, or output for convenience, of our pleasure-relation can thus be expressed as[3]:

$$f(\text{analysis-symmetry-AMIO-}y = e^x) = \langle P, N\rangle$$

12.3 Dynamics

Multiple factors are at work in the dynamics of value. McAllister's aesthetic induction already involves complex social and historical elements. My approach is even more complicated due to critical adequacy and robustness. Thus, I shall try to keep this interpretation as simple as possible; I shall neglect possible complicating factors, like any relevant aesthetic properties besides symmetry, or the history of the opinion on e^x. By doing so, it is possible to present a nearly ideal case in which a single aesthetic dimension is at work, allowing us to see more clearly how the aesthetic as process theory works.

We have established that symmetry is responsible for our affective response. In terms of value, that means that the pair $\langle\{\textit{symmetry}\}, \langle P, N\rangle\rangle$ consisting of the set containing the property *symmetry* and the pleasure response $\langle P, N\rangle$, is an element of the value set for mathematics: the property set of symmetry possesses a positive aesthetic value. The dynamics of this particular value is modelled by the Naturalistic Evolution Rule II. The only evidence we have to support the existence of experiences or values[4] is the public judgement available on the beauty of $y = e^x$: the text offered by Le Lionnais. The text can be seen as evidence that there is some kind of aesthetic criterion governing the passage of his judgement. Of course, aesthetic criteria are not necessarily held *explicitly* or even consciously by the person who passes the judgement. Rather, aesthetic criteria are useful devices that allow us to use the publicly available information to track the otherwise private and inaccessible values. The criterion at work in this case is something like:

[3]I employ the functional notation $f(x) = y$ to emphasize that in the context of particular aesthetic experience there is an actual assignment of a particular output y (an affective response) to an input x (the intentional object).

[4]The interpretation of experience offered above can be justified only by the evidence of an actual judgement. But that is the case for all kinds of experiences.

If symmetry appears in object O, attach more aesthetic value to O.

The strength of this criterion co-varies with the strength of its associated preference and evaluative-instance. In other words, this criterion is the cognitive side of an affective preference in the person who passes the judgement, and it tracks the evaluative-instance $\langle\{symmetry\}, \langle P, N\rangle\rangle$. Now, in our models of preference and value evolution, two features play a key role: critical adequacy and critical robustness. Critical adequacy represents the, so to speak, inductive character of our judgements. Robustness represents the tendency to remain the same. It is true that appreciation of symmetry can be affected by learning and experience, but symmetry has a rather natural character. Visual symmetry, for example, does not require any special training or a history of previous experiences to be positively appreciated. Furthermore, the historical evidence of positive responses to symmetric items, from art in antiquity to contemporary judgements in mathematics, bear witness that positive judgements regarding symmetry possess a high degree of robustness. This evidence means that the preferences for symmetric items do not necessarily depend on, or are at least relatively immune to, histories of experiences. Furthermore, our judgement on $y = e^x$ certainly needs a good deal of mathematical background to allow us to see the symmetry involved in our criterion, but it does not depend on previous experiences with symmetry. We can conclude that the aesthetic value of *symmetry* shall remain high despite changes in other parts of aesthetic value induced by experience. The preference for symmetry is quite stable; it reflects a readily available preference, and thus its corresponding criterion (ideally) does not seem to depend on a history of experiences. This criterion is also responsible for our evaluation, which takes us to the issue of aesthetic judgement.

12.4 Judgement

For the sake of brevity, I assume that the sentence '$y = e^x$ is beautiful'[5] summarizes the judgement passed by Le Lionnais in his text. I have characterized an aesthetic judgement as an aesthetic description that expresses a subjective state in an aesthetic-process. '$y = e^x$ is beautiful' is such a description. It expresses the state resulting from evaluating and articulating our aesthetic experience in contemplating $y = e^x$. Aesthetic descriptions are sentences that include correctly applied aesthetic terms. As we have seen, there are multiple possibilities to map affective responses (the possible responses in our pleasure-relation, the set *ENJOYMENT*) to families

[5] '$y' = y = e^x$ is beautiful' is perhaps an option for expressing the judgement, but Le Lionnais refers to the function $y = e^x$ as an object that happens to be beautiful, and then explains that its beauty is the result of the fact that $y' = e^x$. In other words $y = e^x$ is the subject matter of our judgement and $y' = e^x$ is the justification of our judgement. I believe that '$y = e^x$ is beautiful' better expresses the idea that our object of attention is a contemplative, non-active content.

of terms (schemata). We have established that the pleasure-relation is of the basic type, thus there are only two possibilities: pleasure, P, and displeasure, D. Our *ENJOYMENT* set is:

$$\mathit{ENJOYMENT} = \{\langle P, N\rangle, \langle D, N\rangle\}$$

The term 'beautiful' can be coherently mapped into the sole pleasure response $\langle P, N\rangle$. Hence, the actual output of our pleasure-relation is $\langle P, N\rangle$. Now, although it is simple to map the pleasure response to the term 'beautiful', that mapping entails the mapping of the whole response space $\{\langle P, N\rangle, \langle D, N\rangle\}$ to an appropriate schema, for example $\{\mathit{beautiful}, \mathit{ugly}\}$. The mapping is accompanied by the reorganization of the extensional domain to which the expression '$y = e^x$' refers; the domain of mathematical functions. This reorganization of the domain is something that occurs subjectively; it is a change in our (or Le Lionnais's) perspective. And it is something we do not necessarily undergo consciously. Despite this subjectivity—and this is quite interesting—the metaphor Le Lionnais employs, comparing the function to a phoenix, testifies to the fact that such a subjective reorganization does occur. Metaphors reorganize the new referential domain where a metaphorical term is applied. In the metaphor that compares $y = e^x$ to a phoenix the domain of mathematical functions is reorganized to match the phoenix's schema—say, the schema of real and imaginary birds or animals. This metaphor should not be interpreted as an explanation of the beauty of $y = e^x$, but rather as a way to place the "rise-again" property on the foreground by reorganizing the domain of mathematical functions. I believe this is also the function of some metaphors in art criticism: metaphors draw our attention to certain features of the observed objects which helps us to articulate the aesthetic experience.

Now, the correct aesthetic use of a term requires more than metaphor. Roughly speaking, it requires three things: (1) expressive mapping, (2) communicative reorganizing, and (3) locating our subjective state. The mapping establishes assignments of our basic response space, $\{\langle P, N\rangle, \langle D, N\rangle\}$, to the schema $\{\mathit{beautiful}, \mathit{ugly}\}$. This mapping can be coherently accomplished by assigning 'beautiful' to pleasure $\langle P, N\rangle$ and 'ugly' to displeasure $\langle D, N\rangle$. The second element consists of reorganizing the referential domain of mathematical functions. The actual reorganization underlying Le Lionnais' judgement seems to consist in having the mathematical functions domain divided into beautiful, ugly and aesthetically-neutral functions. There are several ways of reorganizing the domain, since the adequacy of terms depend on context and habit. To exemplify this, we can think of two alternatives. As we have discussed above, in the domain of mathematical functions some elements exhibit symmetry with respect to differentiation, $y = e^x$; whereas others are non-symmetric, $y = x^2 + c$. By reorganizing the domain of mathematical functions in terms of derivative-symmetry we can have a second way to relate items in this domain to labels in the original schema $\{\mathit{beautiful}, \mathit{ugly}\}$. At the same time, we have an explanation for our affective response. The other alternative is to reorganize the domain of mathematical functions using the phoenix metaphor, in which only one kind of animal is reborn from its ashes, just as only $y = e^x$ remains the same under differentiation.[6] Of the two alternatives, the first has the advantage of offering clearer points of reference which cohere with the way we actually deal with mathematical functions; it facilitates expression. The second schema, as in the case of the colour schema, has a domain whose structure has very few points of reference to reorganize functions other than the phoenix in relation to $y = e^x$: what kind of functions can we associate with birds such as the eagle or the pigeon? These two possible reorganizations show that the process of articulation depends on many factors, and that, in general, choosing a correct reorganization is a skill that must be developed.

Now, the mapping/reorganization provides us with a chart of the possible aesthetic experiences. The third element necessary to articulate our experience is, of course, the coordinates that tell us where on this chart we are. The actual output of our pleasure-relation, $f(\text{AMIO-}y' = y = e^x) = \langle P, N\rangle$, gives us these coordinates: it spots a specific response, which is mapped to the term 'beautiful' (or another one depending on our choices). Hence, the aesthetic judgement '$y = e^x$ is beautiful' is the result of Le Lionnais's subjective articulation.

Now, the need for a linguistic expression is what encouraged articulation in the first place; the necessity of expressing our subjective state $\langle P, N\rangle$ in a linguistic form encourages us to choose a schema, a term, a mapping and to undergo a domain reorganization. It must be emphasised that mundane non-aesthetic judgements like '$y = e^x$ is derivative-symmetric', cannot play the role of encouraging articulation, since they do not express our subjective state and they can be interpreted straightforwardly without the need for any reorganization of the referential domain.

Now, it is quite simple to account for the function of broadcasting. In its broadcasting function the judgement '$y = e^x$ is beautiful' serves to convey information. This function is obviously performed, as is clear from the fact that we can read Le Lionnais's text witnessing the existence of the judgement. The judgement helps us to focus our attention on '$y = e^x$' and then encourages us to undergo the appropriate experience: the judgement thus plays a role in further aesthetic-processes, our own aesthetic-process when we read Le Lionnais. Now, even if this judgement can participate in an aesthetic inductive process in principle, it would probably have little effect in shaping future experiences by means of aesthetic induction, since symmetry seems to be an historical constant. Something peculiar is that Le Lionnais's judgement does not seem not lead to further conclusions or actions. Perhaps we may be entitled to conclude that, in this ideal case, the judgement is not only locally terminal but also terminal in Kivy's strong sense.

[6]Functions like $\sin x$ and $\cos x$ appear again after two iterations of the derivative. In this sense, $y = e^x$ is not completely unique. Furthermore the function $y = ce^x$ has the quality of repeating itself under derivative. However, these kinds of repetition are less appealing than the immediate and more obvious repetition of e^x. In addition, even if the function is not unique in that respect, our focus in this interpretation is on the fact that the affective reaction is automatically elicited by the presence of symmetry; this fact is independent of whether or not e^x is the sole symmetric object. We appreciate visual symmetry because symmetry naturally moves us, not because symmetry is rare. Rarity can enhance our appreciation, but first we need something to be enhanced.

12.5 The Process in Summary

With all these elements we can summarize how this aesthetic-process unfolds:

The expression '$y = e^x$' and Le Lionnais's remarks focus our attention on the function $y = e^x$ (node 1), our attention shifts focus by concentrating on the intentional object labelled *analysis-symmetry-AMIO-*$y = e^x$, instead of on $y = e^x$ simpliciter (node 2). The intentional object elicits a pleasure response, $\langle P, N\rangle$. This response is an affective evaluation (node 4) and is the result of the preference encoded in the criterion *If symmetry appears in object O, attach more aesthetic value to O*, which tracks the evaluative instance $\langle\{\mathit{symmetry}\}, \langle P, N\rangle\rangle$. The resulting subjective state needs to be expressed in a judgement, thus encouraging the articulation of our experience (node 5). This judgement is expressed linguistically as '$y = e^x$ is beautiful' (node 6), and it can result in further aesthetic experiences, but it is not very likely these will alter its aesthetic value, as the historical evidence indicates that symmetry is highly robust.

Chapter 13
Case Analysis II: Elegance

In this chapter Cantor's diagonal proof that the real numbers are uncountable is used to illustrate performative and adaptive aesthetic experiences, and, of course, the usage of the term 'elegant' in mathematics.

We have seen that the appreciation of mathematical proofs usually require a more active engagement from the observer; mental activities are characteristic of performative aesthetic experiences. Cantor's proof shall allow us to illustrate performative experience. But to illustrate adaptive experience, I shall address the technique derived from Cantor's diagonal proof, called *diagonal method* or *diagonal argument*. I approach the same mathematical item, the diagonal proof, by giving two interrelated interpretations of the proof. The first interpretation, focused on the series of individual steps of the proof; allows us to see the proof as a performative experience. The second interpretation, focused on how we construct a new object of attention with each step of the proof, shall allow us to see the proof as a general method and as an instance of adaptive experience.

13.1 Experience

In 1891 Cantor presented his proof by diagonalization that the set of real numbers is non-countable. The following is a translation into English of Cantor's original proof (the signposts for elementary steps D1–D3, etc. are mine).

> "On a property of a set of all real algebraic numbers" (Journ. Math. Bd. 77, S. 258), there appeared, probably for the first time, a proof of the proposition that there is an infinite manifold, which cannot be put into a one-one correlation with the totality (*Gesamtheit*) of all finite whole numbers $1, 2, 3, \ldots, \nu, \ldots$ or, as I am used to saying, which do not have the power (*Machtigkeit*) of the number series $1, 2, 3, \ldots, \nu, \ldots$ From the proposition proved in §2 there follows another, that e.g. the totality (*Gesamtheit*) of all real numbers of an arbitrary interval $(a \ldots b)$ cannot be arranged in the series
>
> $$w_1, w_2, \ldots, w_\nu, \ldots$$

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_13,

However, there is a proof of this proposition that is much simpler, and which does not depend on considering the irrational numbers.

(D1) Namely, let m and n be two different characters, and consider a set (*Inbegriff*) M of elements

$$E = (x_1, x_2, \ldots, x_\nu, \ldots)$$

which depend on infinitely many coordinates $x_1, x_2, \ldots, x_\nu, \ldots$ and where each of the coordinates is either m or w. Let M be the totality (*Gesamtheit*) of all elements E.

(D2) To the elements of M belong e.g. the following three:

$$E^I = (m, m, m, m, \ldots),$$

$$E^{II} = (w, w, w, w, \ldots),$$

$$E^{III} = (m, w, m, w, \ldots).$$

(D3) I maintain now that such a manifold (*Mannigfaltigkeit*) M does not have the power of the series $1, 2, 3, \ldots, \nu, \ldots$

This follows from the following proposition:

(D4) "If $E_1, E_2, \ldots, E_\nu, \ldots$ is any simply infinite (*einfach unendliche*) series of elements of the manifold M, then there always exists an element E_0 of M, which cannot be connected with any element E_ν."

(D5) For proof, let there be

$$E_1 = (a_{1,1}, a_{1,2}, \ldots, a_{1,\nu}, \ldots)$$

$$E_2 = (a_{2,1}, a_{2,2}, \ldots, a_{2,\nu}, \ldots)$$

$$E_u = (a_{u,1}, a_{u,2}, \ldots, a_{u,\nu}, \ldots)$$

$$\vdots$$

where the characters $a_{u,\nu}$ are either m or w. Then there is a series $b_1, b_2, \ldots, b_\nu, \ldots$ defined so that b_ν is also equal to m or w but is *different* from $a_{\nu,\nu}$.

(D6) Thus, if $a_{\nu,\nu} = m$, then $b_\nu = w$.

(D7) Then consider the element

$$E_0 = (b_1, b_2, b_3, \ldots)$$

(D8) Of M, then one sees straight away, that the equation

$$E_0 = E_u$$

cannot be satisfied by any positive integer u, otherwise for that u and for all values of ν.

$$b_\nu = a_{u,\nu}$$

and so we would in particular have

$$b_u = a_{u,u}$$

(D9) Which through the definition of b_v is impossible. From this proposition it follows immediately that the totality of all elements of M cannot be put into the sequence (*Reihenform*): $E_1, E_2, \ldots, E_v, \ldots$ otherwise we would have the contradiction, that a thing (*Ding*) E_0 would be both an element of M, but also not an element of M.

Now, our object of attention here is not the theorem itself, the non-countability of the reals, but the argument that allows us to establish it. Our attention focuses on the different steps presented by Cantor, and on the way they take us from the premise—that there are elements of the type $E = (x_1, x_2, \ldots, x_v, \ldots)$ that can be arranged in a list—to the conclusion—the reals are non countable.

The diagonal method can be identified as the passages labelled D4–D9 above; more specifically the proof of D4:

If $E_1, E_2, \ldots, Ev, \ldots$ is any simply infinite series of elements of the manifold M, then there always exists an element E_0 of M, which cannot be connected with any element E_v.

We can see that Cantor's diagonal proof is remarkably simple. In the following I concentrate only on the diagonal method of proof, and not on the non-countability of the reals.

The proof consists of a sequence of steps that takes us from the premise "$E_1, E_2, \ldots, E_v, \ldots$ is any simply infinite series of elements of the manifold M" to the conclusion "there always exists an element E_0 of M, which cannot be connected with any element E_v". Below it shall be clear that the experience of the diagonal proof can be performative or adaptive, depending on the historical context. In order to differentiate the two types of experience, I shall conduct two different analyses, one focused on the individual steps of the proof, which I label the *step-series interpretation*; and one focused on how the steps are just elements of a general method, which I label the *single-object interpretation*.

Now, in general, the kind of attention a proof requires is different from the kind of attention that formulae or theorems require. Formulae, for example, are single statements, but a proof is an argument that involves the shifting of our attention through a series of statements. As we discussed earlier, the steps of the proof constitute a sort of narrative, an unfolding of events that lead logically to other events and ultimately to a conclusion. The structural properties of the proof's "narrative" are partially liable for our affective response. This is the case of Cantor's proof.

Cantor's proof is aesthetically interesting not because it proves the non-countability of the reals: as a matter of fact, that result had been already proven by Cantor himself in 1874. Cantor's proof is aesthetically interesting thanks to the quality of its narrative. Let us remember that Gian-Carlo Rota was quite sceptical of the role properties like unexpectedness play in mathematical beauty [78, p. 172]; but we shall see below that it is quite plausible that the unexpected plays a role in the beauty of this proof. Now, in general, the conclusion of a proof is already known from the beginning of the proof. Thus, the conclusion of the proof itself is not unexpected. If anything, the unexpected must be located on the procedure, in the narrative of the proof. This is why it is important first to interpret the proof as a series of steps.

13.1.1 Step-Series Interpretation

Let us begin by establishing how the attention is focused. Each step in Cantor's diagonal proof is a single statement, analogous to a formula. Background knowledge is required to see each step. Cantor employed an informal set theory—naive set theory, but we can assume that a basic understanding of set theory (ST) is necessary to understand the proof (the difference between naive set theory and ST is not relevant here). Now, understanding each new step in the proof requires an understanding of the previous steps, thus, in a sense, every previous step in the proof becomes part of the background understanding for the next step. The proof stops when we reach the conclusion we were looking for, in this case that element E_0 is different from any element E_v.

The procedure and the ideas involved in the steps are fairly simple, but nonetheless conclusive. The whole procedure consists of very few steps, D4–D9. We can even reduce it to just two general steps plus one concluding step. The two general steps are, (I) to establish the diagonal in an infinite list of terms consisting of infinite coordinates, and, (II) to construct a new term consisting of the coordinates complementary to the coordinates in the diagonal. The concluding step consists of realizing that the new term is not in the list and thus that there exists a non-countable manifold. Since this last step is specific to this particular proof, I concentrate only on the two general steps and consider that the concluding step can be adapted to the proof at hand. The general steps are very simple operations. The brevity and simplicity of the proof offers the advantage of making it very transparent and reliable, since not many details are involved; it is a very intuitive proof as there is little technical knowledge involved, and there is thus a small probability of something going wrong. As we mentioned earlier, elegance is sometimes defined as the quality of being pleasingly simple yet effective. In this sense, the brevity, simplicity and reliability of Cantor's diagonal proof make it a very elegant proof.

All these properties help to constitute our intentional object. Our phenomenological space is constituted by a set theory, *ST*, background-understanding dimension. The most salient aesthetic property is, of course, *simplicity*. Now, simplicity in this example can be consistently interpreted in two different ways: first, as a "transparency" quality, a quality that facilitates understanding and reduces the possibility of error. We enjoy this simplicity in the same fashion as we find the lack of complications pleasurable. Second, a proof is simpler if it has fewer steps; we can call this type of simplicity step-parsimony. The diagonal method, as I have summarized it, is quite step-parsimonious; it has only two steps. We can introduce these properties with the following definitions:

Simplicity = the feature of facilitating understanding and reducing the possibility of error

Step-Parsimony = the feature of consisting of a small number of steps

Step-parsimony (or simply *parsimony*, for short) shall play a central role in the single-object interpretation, but for now let us concentrate on the individual steps. The steps of the proof appear in a space with the dimensions *ST* (set theoretical background-understanding) and *simplicity*.

Now, a characteristic feature of proofs is that our attention is *actively* involved. For example, the second step of Cantor's proof, constructing the complementary element, depends on the first step. Reaching the proof's conclusion depends on understanding both steps. That is, in order to understand the proof we must understand each step and the way they depend on each other. When our attention shifts to a new step in the proof, the former step becomes a necessary supporting part of the new step. The shifting of attention means that our attention must be constantly changing and performing certain tasks with the object of our attention. For instance, in the first step of Cantor's proof we must establish the diagonal by taking the first coordinate of the first element of the list, the second coordinate of the second element of the list, the nth coordinate of the nth element of the list, and so forth. Our object of attention, the diagonal of the first step of the proof, does not just passively appear in the phenomenological space; it requires *active* attention to become part of the proof. And as it develops into a new step, our attention is developing new background understanding for the next step, since each previous step is needed to understand the new step. The content of our experience in the step-series interpretation consists of a series of interrelated intentional objects in the space *ST-Simplicity*. We can label them *ST-Simplicity-AMIO-Step-I* and *ST-Simplicity-AMIO-Step-II*. Each intentional object requires the performance of some specific intellectual activity, and the attention shifts its focus from the first to the second aesthetic mathematical intentional object.

13.1.2 Single-Object Interpretation

To see the proof as a single object we must construct this object using the steps of the proof as building blocks. My model of aesthetic mathematical intentional object allows the building of new objects in the phenomenological space. The construction of an object is not arbitrary; it is constrained by the dimensions of the space and the conditions of conservatism and relevance of the operations we employ to construct the new object. In the case of Cantor's diagonal proof, the theoretical frameworks of logic and set theory determine which implicit and meta-intentional operations are allowed. A condition for constructing new objects in our phenomenological space is that the resulting object must remain an aesthetic mathematical intentional object. Now, the operation of adding new steps to a proof results in new objects of attention, and in the case of Cantor's proof the steps remain simple even if we establish their connection to other steps. The operation of adding steps is in this case *simplicity*-consistent and *simplicity*-conservative. This means that the object resulting from adding steps in Cantor's proof exists in the same *ST-simplicity* phenomenological space as the steps. We can thus think of logical consequence, which is allowed by logic and set theory, as an implicit operation in our phenomenological space for the orderly connection of steps to each other. For example, the first step of the proof, establishing the diagonal, is a single independent object for which only background understanding is necessary. The second step, constructing E_0, is an object of attention that depends on the same background understanding and on an understanding of the first step of the proof. The proof as a single object of attention is the final object of attention "there is an element E_0" resulting from the previous two steps and the logical consequence operation.

An interesting feature of this new object is that the property of step-parsimony is now visible within it, since we can count the number of steps involved in the object. Since step-parsimony is relevant to our appreciation, the respective *step-parsimony* dimension must now be introduced in our phenomenological space. The new space has the dimensions *ST*, *simplicity* and *step-parsimony*. The resultant single object is a new object that we can label *AMIO-diagonal-method*.

Seeing the proof as a single object also provides us with a way to characterize different mathematical proofs as instances of the application of the diagonal method. Cantor's theorem, and the first of Godel's incompleteness theorems have been proved by diagonalization. The single-object interpretation allows us to incorporate this inter-field application of the method in our aesthetic experience. It must be noticed that this interfield-application of the method does play a role in mathematicians' appreciation of the diagonal method. The reason for this probably has to do with mathematics commitment to a sort of Occam's razor principle; in particular, with the desirability of theoretical and methodological unification. It is desirable to have a common theoretical foundation for mathematics. By the same token, the existence of common mathematical methods is desirable. The diagonal method possesses a very desirable method-unifying power. Interestingly enough, this desirability is not exclusive to mathematics; as we saw in Chap. 1, unity is not unknown in aesthetics. The appreciation of unifying power can be seen as a variation of Hutchenson's aesthetic principle of unity amidst variety.

Now, in order to incorporate this unifying property into our experience we need to incorporate the fact that the method is applicable to other problems. But, in principle, those other problems are not visible in the original phenomenological space of the proof. When we discussed aesthetic experience, I proposed the meta-intentional operations precisely to cope with this kind of problems.

We have established that mathematical intentional objects can be constructed not only by means of implicit operations, but also by external meta-intentional operations. Each instance of application of diagonalization can be seen as a different intentional object existing in their corresponding phenomenological space. In order to see the unifying power of the method within our space we need to introduce a special *unifying-power* dimension. The best way to accomplish this is by allowing a meta-intentional operation, which are operations that are not allowed by the implicit features of the dimensions of the space and that are relevant—in the sense of allowing us to see additional aesthetic features—and conservative with respect to the aesthetic characteristics of our space. In the case of the diagonal method, one such operation is simply the operation:

Apply the same method to a different instance

Let us call this meta-intentional operation the *external reapplication* operation.

For example, let us consider the proof by *reductio ad absurdum*, known to the ancient Greeks, that the square root of 2 is irrational. We can construct an intentional object that possesses all relevant steps and properties of the proof. If we concentrate only on the relevant properties of the method employed, to assume a negated premise and then obtain from it a conclusion that contradicts it, we obtain the method of proof by *reductio ad absurdum*. Now, *reductio ad absurdum* is applicable in many instances. But this applicability to many instances is not a property visible in the proof's original phenomenological space. With the operation introduced above, however, if we apply the operation *Apply the same method to a different instance* we can account for the application of *reductio ad absurdum* in, say, the instance of the proof of the infinity of primes. The resultant object of attention is the same method but employed in a different instance. Now, this new object is different from the original in that it has the property of having been applied to other instances. The property of being applicable to many instances is now visible in our phenomenological space.

Similarly, the diagonal method remains the same when we apply the *external reapplication* operation. And in this case the aesthetically relevant characteristics of the proof, simplicity and parsimony, are preserved; because if we added any new steps or changed the simplicity of the assumptions in the steps, the new method would no longer be an instance of diagonalization. Thus, the *external reapplication* is *simplicity* and *parsimony* conservative. We can apply the *external reapplication* operation to the *AMIO-diagonal-method* to obtain a different intentional object:

reapplied-AMIO-diagonal-method

This new intentional object possesses not only the properties of being transparently and parsimoniously simple, but also of being method unifying, and, thus, it exists in a space constituted by the dimensions *ST*, *simplicity*, *step-parsimony*, and the new dimension of *method-unification*. *Method-unification* can be defined as follows:

Method-unification = *the feature of providing a general procedure that can be applied to multiple problems in different fields*

Once we have established how our attention is focused and what the objects of attention are, we must establish that the experience of those objects is an aesthetic experience; that is, that there is a relation between the presence of the objects and our affective responses. The step-series interpretation allows us to elaborate on the details of the mental activities and the change of attention in the proof. The single object interpretation allows us to take the unifying power of the method into account. As we shall see below, these interpretations also exemplify performative and adaptive aesthetic experiences, respectively.

13.1.3 Affective Response in Step-Series Interpretation

When our attention is focused on an individual step of the proof, there are two ways an affective response can be elicited: passively, by contemplation of properties; or actively by means of the mental activities performed in the step. In the cases of Euler's identity and $y = e^x$, the presence of aesthetic properties, like simplicity or symmetry, elicits affective responses. A type of simplicity is a property of steps I and II (the concluding step can be regarded as trivial) of the diagonal proof, thus their contemplation can elicit a positive affective response. Although passive contemplation does have affective significance, in this proof the most significant part of the affective response comes from the active content: we not only focus our attention on the properties of the steps, we also actively envisage and evaluate what is being presented to us. In step I, for example, we have to figure out that terms E_n can be arranged following the order of natural numbers; that the coordinate of each element can also be arranged in that manner; that we can then take the first coordinate of the first element, the second coordinate of the second, and so forth. These tasks are relatively easy to perform; they are *simple* tasks. In addition to the intellectual activities performed in following the proof, there is the activity of keeping track of the development of the proof. We know that each step is not presented to our attention arbitrarily, but rather that their purpose is to lead us somewhere else, and, ultimately, to the desired conclusion: we see each step as a section of a narrative that develops to reach a conclusion. For example, we actively check for coherence between steps, that is, that one step consistently leads to another. By establishing step I of the proof (the diagonal in the list of elements) we have also presented the basis for step II, since step II consists of using the coordinate n from element n as proposed in step I to build a new element with all its coordinates complementary with respect to the diagonal. We can see that step II naturally develops the narrative introduced in step I.

Now, we also know that the conclusion we are trying to reach is about an infinite manifold. At first sight, we could expect a rather complicated narrative—it is about infinity! after all. The fact that the narrative takes only two very simple and coherent steps to reach the conclusion strikes us as completely unexpected. As I have mentioned before, the *conclusion* of the proof is not unexpected at all; we knew it from the beginning. But the fact that two simple steps are enough to prove something so profound about infinite sets is certainly an unexpected outcome. Now, unexpectedness can naturally elicit affective responses. In emotions, for example, an affective response is triggered in response to *changes* in the environment. Different kinds of change in the environment result in different kinds of affective responses. The kind of change in the diagonal proof is that of unexpected success. This unexpected success, coupled with the simplicity of the performed intellectual activities, results in an affective response associated not with the passive content of an individual step, but with the active performance of mental activities in each step to reach the conclusion. In a sense, the proof is a narrative with a happy ending, but most proofs have happy endings. The happy ending is less relevant than the quality of the narrative, the way the story is told: the most moving element in Cantor's narrative is that we reach the ending as an unexpected success.

Unlike simplicity or parsimony, unexpected success is not a property of the proof; it is not a quality we can see in it. Rather, it is a type of outcome of our actions. This inherent difference is one of the reasons why passive and active affective responses cannot be reduced to each other. Another reason for this is that passive contents elicit responses by means of contemplation of properties. Hutchenson and Shaftesbury were dealing with this kind of pleasure when they proposed their aesthetic principles of order and unity. Active contents, by contrast, elicit responses as the result of performing certain tasks or obtaining certain outcomes. If *unexpected success* were not a type of outcome of our actions but a property, there would be no reason to distinguish between passive and active contents.

Now, even if the passive contemplation of the simplicity of each step of the proof already elicits an affective response, the performance of mental activities involved in the proof, elicits an even more definite affective response.

In summary, the aesthetic experience of the diagonal proof consists of a passive content, the aesthetic mathematical intentional objects corresponding to each step, and an active content, the mental activities performed with those objects, which results in unexpected success. Both components result in a pleasure response; the total, composed response is $\langle P, P \rangle$. Since we have an active content, our experience is performative. The value of the pleasure-relation can be expressed as:

$$f((step1, step2), (\textit{Activities-step1, Activities-step2, Story-Development})) = \langle P, P \rangle$$

An interesting result can be drawn from this example: against Rota's ideas, unexpectedness sometimes plays a role in mathematical beauty. Perhaps the reason why Rota was sceptical about unexpectedness is that he sees mathematical beauty as grounded merely on properties, but, as we have seen, unexpectedness play a role very different from the role played by properties. Unexpectedness, at least in cases like the one analysed here, seems to be primarily suited to items that involve active attention like proofs or derivations, since in mere passive contemplation there is no shifting in our focus of attention and, thus, it cannot result in expected or unexpected developments. Due to the dynamic nature of unexpectedness, its role is more prominent in instances of active attention.

13.1.4 Affective Response in the Single-Object Interpretation

The proof as a single object is the result of a meta-intentional operation. We use the steps of the proof to construct a larger single object. This constructed object can be a further subject of the operator *external-application*.

In the single-object interpretation, the intentional objects and mental activities involved in the proof must be seen in a particular way. The steps of the proof should not be seen as separate objects of attention, but rather as the building blocks of the newly constructed intentional object *AMIO-diagonal-method*. Similarly, the mental activities performed in each step and in the supervision of the proof development must be seen as the activity of internally organizing this new object.

In addition to these activities we have seen that in order to see the property of method-unification we need to apply a meta-intentional operation, *external reapplication*. Applying this operation constitutes another kind of activity. More specifically, it consists in checking that the relevant steps of the diagonal proof are applied in a recognizable way in another instance—to prove Cantor's theorem, for example. The result of these activities is the object:

AMIO-diagonal-method-external-application

This object is the passive content of our experience. The active content consists of the activity of constructing the object and applying the method externally. These activities, except for the external application, are basically the same set of activities as in the step-series interpretation, and thus we can also conclude that they elicit a positive affective response.

There is, however, an important difference between the two interpretations: the passive content in the single-object interpretation has the additional aesthetic property of *method-unification*. While *simplicity* is visible in the step-series interpretation, *method-unification* is not. But both properties are visible in the single-object interpretation. In the single-object interpretation the proof is not only simple and parsimonious, but also method-unifying. *Method-unification* is a property of the proof and thus is part of the passive content of our experience; for this reason the passive content fares better in eliciting pleasure.

Our appreciation of method-unification, however, depends not only on our current experience of the proof, but also on the *different experiences* of application of the proof. If the pleasure associated with method-unification is experience dependent, we have reasons to think that a mechanism of preference evolution can play a role. For the sake of the argument, let us assume it does. The composed response, although still a full-pleasure response as in the step-series interpretation, is different in that it includes an acquired, Ep, pleasure response. The response is $\langle Ep, P\rangle$. The experience must be characterized as an adaptive experience. If we label the set of activities that result in the new intentional object as *ConstructionActivities*, the pleasure-relation can be expressed as follows:

$$f(\textit{diagonal} - \textit{method} - \textit{AMIO}, \textit{ConstructionActivities}) = \langle Ep, P\rangle$$

The Process so Far

Regarding the unfolding of our aesthetic-process, we have so far covered node 1, in which perceptual stimuli play no significant role but cognitive stimuli do. In node 2 we focus our attention on *simplicity* and *step-parsimony*, which establish the dimensions *ST-simplicity-parsimony* for the respective step-intentional objects. We also devote attention to following the proof; that is, we concentrate on performing the activities we are required to carry out the proof, node 3. Affective responses, node 4, are elicited by the presence of intentional objects, due to our readily available responses to simplicity. The active content renders an unexpected-success outcome, which turns our experience into a performative one. In the single-object interpretation, however, the response can be affected by previous experiences and thus the passive response may be construed as acquired, and the experience may thus be considered as adaptive.

13.2 Value

The step-series interpretation can be construed as modelling the experience we undergo when we encounter Cantor's proof for the very first time, with no knowledge of whether this method is applicable to other proofs. The single-object interpretation incorporates the effect that applying the method to other proofs has on our appreciation of it. For the sake of brevity, let us call the experience associated with the step-series interpretation just *diagonal proof experience*, and the experience associated with the single-object interpretation *diagonal method experience*.

In the step-series interpretation the content results in a full pleasure $\langle P, P\rangle$. This response is an affective evaluation, resulting from our preferences for simplicity and from the satisfying feeling of unexpected success.

Simplicity and unexpected success are responsible for the readily available affective response, however, only simplicity appears as a dimension in our space, since unexpected success is not a property but the outcome of a series of activities. In terms of value (the relation between property/activity sets and responses) this means that

$$\langle\{\textit{simplicity}\}, \langle P, P\rangle\rangle \in V_M$$

that is, the evaluative-instance above is an element of the value-set for mathematics. The dynamics of this evaluative instance can be analysed in terms of constrained aesthetic induction. Cantor's proof is often qualified as elegant. The public judgements stating the elegance of the proof constitute the evidence that supports our claim that the ordered pair presented above is in the value set. Mathematicians' judgements thus testify there is some kind of aesthetic criterion governing their use of the description "Cantor's diagonal proof is elegant". These criteria are not necessarily explicitly held by the person who passes the judgement; rather, they are the cognitive expression of implicit preferences. The criterion at work in this example can be (ideally) the following:

If simplicity appears on object O, attach more aesthetic value to O

The criterion's dynamics is governed by constrained aesthetic induction. Constrained aesthetic induction allows us to model the effect of the history of experiences, but also the robustness of simplicity. As we have seen, this robustness is evident throughout history. This means that the preferences for Cantor's diagonal proof depend only to a small degree on experience, especially when we encounter the proof for the very first time. In order to describe the evolution of its aesthetic value let us first concentrate on the diagonal proof as it is encountered for the first time.

Calling the diagonal proof elegant comes as no surprise; elegance is an aesthetic property usually related to the lack of complication, to pleasingly simple yet effective things. Calling the diagonal proof elegant intends to express our subjective view that the proof works in an unexpectedly simple way; just two steps. Even if one has little experience with other mathematical proofs, just two steps is quite parsimonious. This means that step-parsimony plays a significant role in our initial appreciation of the diagonal proof, independently of our previous experiences. In an ideal first encounter with the proof, the aesthetic dimensions of *simplicity*, *step-parsimony*, and our preference for success are at work and are thus responsible for eliciting the full-pleasure response $\langle P, P \rangle$. In our first appreciation of the proof, the aesthetic value of simplicity is thus high and does not seem to depend on previous experiences. The value is not yet influenced by dynamical factors.

The aesthetic criterion involved in the aesthetic judgements is something like:

If simplicity appears on object O, attach more aesthetic value to O

Now, aesthetic value in this example is not static. Although our very first response to the proof reflects a natural preference for simplicity, this preference can be affected by a history of experiences, depending on its degree of robustness. Moreover, in the single object interpretation, we can incorporate *method-unification*. The appreciation of this property depends on our history of experiences. We can have a remarkable inner experience the first time we encounter the diagonal method, but when we discover that it can be used to prove many other results, our initial experience is enhanced. However, I believe that this enhancement can be explained without constrained aesthetic induction. The mechanism of constrained aesthetic induction models the reinforcement effect that the recurrent presence of properties has in our preferences (I am deliberately neglecting robustness). I do not believe we appreciate the diagonal method just because it appears many times in different proofs. Rather, I think we appreciate the fact that, in addition to its initial elegance, the method has the extra quality of being applicable to many problems. Our aesthetic experience is enhanced by a change in the constitution of the experience, as the result of *adding the new appreciable quality of method-unification* to our set of qualities, and not by changing the strength of the preference.

Thus, the diagonal method experience differs from our original diagonal proof first-time experience in two aspects: first, the criterion that expresses our preference in the single-object interpretation is different, namely:

If the set of properties {simplicity, parsimony, MethodUnification}

appear on object O, attach more aesthetic value to O'

Second, the evaluative instance representing the diagonal method's value involves a set of properties which includes *method-unification*, namely:

$$\langle\{\textit{simplicity, parsimony, MethodUnification}\}, \langle Ep, P\rangle\rangle$$

These differences further show that a systemic approach to mathematical aesthetic judgements has advantages over property based approaches like Rota's or even McAllister's, since our approach not only explains the dynamic changes of aesthetic judgements, but it can resort to mechanisms other than the constrained aesthetic induction. In the current example, the aesthetic experience is altered, as a new property appears in the phenomenological space. In genuine aesthetic induction the aesthetic criterion should modify its strength, but in actuality the strength of the criterion remains the same. The change involved in the current example is a large one: we have a new experience, a new criterion and a new value. We are not tracking the same value and thus we cannot claim that the enhancement of our appreciation of the diagonal method is the result of a mechanism of preference or value evolution.

Now, this explanation of the change in our appreciation does not rule out constrained aesthetic induction. The property of method-unification may gain strength as we recursively encounter the diagonal method. In that case, our preference evolution, governed by constrained aesthetic induction, should be added to our account.

Our preference (and thus its associated aesthetic value) for the diagonal method might change with recurrent appearances of the proof, but independently of that, the experience itself changes once we realize the method's generality and power. We have two different aesthetic experiences; one when we meet the proof for the very first time and another when we realize that the method can be applied to different proofs. Thus, even if aesthetic induction can model actual changes of aesthetic value derived from our history of experiences with the method, in order to fully understand the significance of this change we need to consider the changes in the constitution of the experience itself. Our aesthetic theory is quite apt to accomplish this; it allows us to consider both the dynamics resulting from aesthetic induction, and the dynamics resulting from the change in the nature of the experience.

13.3 Judgement

The use of the term 'elegant' to qualify Cantor's proof provides us with an opportunity to further examine some nuances of my idea of aesthetic terms and judgements.

I shall assume that the sentence 'Cantor's diagonal proof is elegant' (or 'DP is elegant', for short) summarizes the judgements that are often used to qualify it.

Let us first address the characteristics of aesthetic judgements. Although the step-series and single-object interpretations illustrate performative and adaptive experiences, respectively, the processes by which they are articulated are very similar. It thus suffices to concentrate on the most general case; the single-object interpretation. The aesthetic term's conditions of application—existence of an expressive mapping and a communicative reorganization—determine whether a term is an aesthetic term. Let us examine whether the usage of the term 'elegant' complies with them.

The application of the term 'elegant' is not an isolated assignment of terms: it entails the use of the whole schema to which the term belongs, the reorganization of the new referential domain, and a mapping from the response space into the schema. As for the mapping, since our example involves an adaptive experience, the pleasure-relation has sixteen possibilities, and there are three different full-pleasure combinations. We need a rich schema to map all those combinations. A suitable schema is, for instance:

$$\{\textit{beautiful, ugly, elegant, inelegant, } \ldots\}$$

Unlike the case of the simpler schema used for $y = e^x$, the terms of the above schema offer more possibilities for assignment to response spaces. Of course, the most coherent way to map these possibilities is by mapping corresponding opposite poles of the adaptive response space into poles in the schema. Although there is no obvious way of mapping all possible responses, the schema gives valuable hints; for example, a full-pleasure response may be mapped to 'beauty', a full-displeasure response to 'ugly'. But a salient problem is that in our adaptive response space there are three different full-pleasure ($\langle Ep, Ep\rangle$, $\langle Ep, P\rangle$, $\langle P, Ep\rangle$) and three full-displeasure ($\langle Ed, Ed\rangle$, $\langle Ed, D\rangle$, $\langle D, Ed\rangle$) responses.

Fortunately, we can resort to the associated domain reorganization to get even more clues on the mapping. Mapping and reorganization must be carried out simultaneously and in a coherent and (relatively) non-arbitrary way. Our new referential domain is the domain of mathematical proofs. Proofs are the kinds of objects to which we now refer to by terms like ‘beautiful, ‘ugly’, ‘elegant’, etc. The domain of mathematical proofs is reorganized according to the organization of the original schema’s domain. The original domain is the domain of aesthetically qualified objects, the domain consisting of beautiful objects, ugly objects, elegant objects, etc. In this domain, an object is elegant if it is pleasingly simple yet effective, for example. An object is beautiful if it is very pleasing regardless of its complexity or simplicity. An object is inelegant if it is complicated and not pleasing. It is baroque if it is complicated but pleasurable, and so forth. With enough terms we can map the entire space in a way that allows to expresses nuances in the responses (and, in addition, nothing prevent us from mapping several responses into the same term). The structure of the schema is forced onto the new domain of mathematical proofs: the domain of proofs is reorganized in such a way that now our objects are structured following the structure of the domain of aesthetically qualified objects. Therefore, some proofs are beautiful; some others are ugly, elegant, inelegant and so forth. This reorganization is not an objective reorganization, it is just a change in our perspective, and thus it can be carried out in many ways. Again, although there is no unique way of reorganizing the domain, we intuitively can see that some reorganizations better suit the mapping of our subjective states. In the current case, reorganizing the domain of proofs in terms of aesthetically qualified objects rather than, for example, in terms of colours, better articulates our experience, since the possibilities of our experience have better referents than in the case of colour. We experience pleasure or displeasure due to the simplicity or the complexity of an object, for example. This fact gives us a reference to associate our experience with objects that elicit pleasure or displeasure; the domain of aesthetically qualified objects offers us such references, but the domain of coloured objects does not. Distinguishing between a beautiful and an ugly proof, for example, can be done by relating them to their respective response of pleasure or displeasure. But what kind of referents can we use to distinguish between a yellow and a green proof? Yellow or green proofs can only be metaphors precisely because the colour schema does not allow a non-arbitrary mapping of our affective responses. Although colour terms do allow a reorganization of the domain, such reorganizations are always trivially possible in metaphors. Aesthetic application of terms requires *both* a mapping of our responses and reorganization of the referential domain. Furthermore, since our experience is no longer a two-possibilities basic experience, we need not only a schema of aesthetic labels, but also a richer schema that offers us different options for mapping the multiple possibilities in performative or adaptive experiences; a schema such as the one presented above.

Let us now address the functions of aesthetic judgement. Our experience is articulated by mapping our adaptive response space into the schema $\{$*beautiful, ugly, elegant, inelegant,* $\dots\}$ and reorganizing the domain of proofs according to the structure of the original domain of this schema. By choosing one of the terms in the schema to express our actual subjective state, we commit to the whole schema and its associated domain structure. For example, beautiful objects are, say, very pleasing regardless of other properties; but elegant objects involve pleasing effective simplicity. Our schema affords more options to choose and compare to better express our subjective states than the simpler schema {*beautiful, ugly*}.

Mathematicians often remark on the simplicity of the diagonal proof. So in this case our experience of Cantor's diagonal proof is more accurately expressed using the more specific term 'elegant' than the more general term 'beautiful'. The application of the term 'beautiful' is possible but we intuitively feel that 'elegant' better expresses our state, since simplicity plays a role in eliciting our affective response. Consider, by contrast, the articulation of the experience of $y = e^x$. '$y = e^x$ is beautiful' appropriately articulates our *basic* experience. But consider the description '$y = e^x$ is elegant'. This description sounds odd, the reason for this is that it fails to incorporate the particularities involved in the relations of the term 'elegant' to the other terms in the schema. The pleasure-relation for $y = e^x$ is of the basic type, which allows few possibilities for domain reorganization. Whereas the use of the term 'beautiful' makes it clear that the reorganization involved depends mainly on the opposite polar extremes beautiful/ugly, the use of 'elegant' raises the question of how exactly the domain should be reorganized. For example, in what sense can a mathematical function, $y = x^2$, for instance, be inelegant? Performative experiences possess more complicated pleasure-relations and they allow us to map more complicated schemata whose domain can, in turn, mirror the qualities of the domain of proofs. This explains why mathematicians describe the diagonal proof as elegant.

The articulation in the use of the term 'elegant' can be summarized in a simple way: when we read the judgement 'DP is elegant' we intuitively realize that if there is an elegant proof, there must also exist inelegant proofs, plainly ugly proofs, and so forth. The conditions of application of the term 'elegant' depend only on subjective matters. That fact is in accord with the fact that the existence of ugly or inelegant proofs is also subjective, since that existence amounts only to a reorganization of the domain of proofs. Our aesthetic judgement is characterized by the aesthetic description 'DP is elegant', which expresses our subjective state. The need for a public description is what encouraged the articulation of our aesthetic experience in the first place.

Let us now examine the second function of aesthetic judgements. In its broadcasting function, the statement 'DP is elegant' conveys certain information. But unlike a more general description like 'DP is beautiful', the use of the term 'elegant' highlights the simplicity involved in the experience of the proof. The aesthetic judgement 'DP is elegant' not only helps us to start new aesthetic-processes by focusing our attention on the aesthetic quality of the proof, it also helps us refine our aesthetic appreciation of it by implicitly pointing out that the term 'elegant', rather than 'beautiful', expresses more accurately our experience of the proof.

As for the terminality of the judgement, we can see that the judgement does lead to further judgements. Perhaps in our first encounter with the proof our natural preference for simplicity plus a welcomed feeling of unexpected success elicited a positive response. But the history of the diagonal method certainly gives us reasons to believe that it can influence further episodes of appreciation. As we experience subsequent encounters and different instances of its application, the experience and value associated with the proof change. Thus, the judgement is not Kivy-terminal but only terminal in the local context of aesthetic processes; *locally* terminal. The judgement can lead to further development of aesthetic values and to further aesthetic-processes in other individuals.

13.4 The Process in Summary

Our aesthetic-process thus unfolds roughly as follows: we start by focusing our attention on Cantor's original 1891 paper; node 1. In its aesthetic appreciation our attention focuses on *ST-simplicity-parsimony-AMIO diagonal method*; node 2. We engage passive and actively with this aesthetic mathematical intentional object; node 3. This results in an affective response $\langle Ep, P \rangle$, which can be interpreted as an affective evaluation; node 4. This response is determined by our preferences—particularly for simplicity—expressed as criteria that track corresponding values; node 7. The proof's value changes due to a constitutive change in our experience and perhaps eventually to a process of constrained aesthetic induction; node 8. The aesthetic experience eventually leads to the judgement 'Cantor's diagonal proof is elegant'; node 6. That judgement articulates the experience; node 5. It also results in further aesthetic experiences and in ulterior changes in aesthetic value; nodes 7 and 8, again.

Chapter 14
Case Analysis III: Ugliness, Revisited

In this chapter we revisit computer assisted proofs, which serves to more conclusively illustrate performative and adaptive experiences, and also the limitations of a property-based model of preference evolution, and the capacity of the aesthetic as process theory to satisfactorily deal with a wide spectrum of mathematical aesthetic judgements—including negative ones. In this chapter I utilize much of the discussion advanced in Chaps. 3 and 4, thus, this chapter can be seen as a sort of refinement and closure of that discussion.

As we have seen, Appel and Haken's computer-assisted proof of the four colour theorem provides us with a conspicuous and interesting case of mathematical ugliness. As Paul Nahin remarks, this proof "is almost always what mathematicians think of when asked 'What is an example of ugly mathematics?' " [69, p. 5]. Although the four-colour theorem itself ranks number nine in David Wells' list of the most beautiful theorems [94], its computer-assisted proof has been poorly welcomed. But McAllister [64] conjectures that aesthetic induction might eventually alter this negative reception. After all, McAllister argues, aesthetic standards in mathematics seem to depend on the acceptability of the proofs, just as scientific beauty depends on the empirical adequacy of the theories. In Chap. 3 we saw that changes in preferences for certain properties are insufficient to explain ugliness in computer-assisted proofs. We are now in position to better explicate that fact.

14.1 The Proof

The following is the original introduction of the proof presented in 1976 by Appel and Haken [4]:

> The following theorem is proved.
>
> **Theorem.** *Every planar map can be colored with at most four colors.*

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_14,

> As has become standard, the four color map problem will be considered in the dual sense as the problem of whether the vertices of every planar graph (without loops) can be colored with at most four colors in such a way that no pair of vertices which lie on a common edge have the same color. The restriction to triangulations with all vertices of degree at least five is a consequence of the work of A. B. Kempe. Over the past 100 years, a number of authors including A. B. Kempe, G. D. Birkhoff, and H. Heesch have developed a theory of reducibility to attack the problem. Simultaneously, a theory of unavoidable sets has been developed and the fusion of these has led to the proof.
>
> A configuration is a subgraph of a planar triangulation consisting of a circuit (called the ring) and its interior. A configuration is called reducible if it can be shown by certain standard methods that it cannot be immersed in a minimal counterexample to the four color conjecture. [. . .] A set of configurations is called unavoidable if every planar triangulation contains some member of the set. From the definitions, it is immediate that the four color theorem is proved if an unavoidable set of reducible configurations is provided.
>
> The most efficient known method of producing unavoidable sets of configurations is called the method of discharging. This method treats the planar triangulation as an electrical network with charge assigned to the vertices. Euler's formula is used to show that the initial charge distribution, giving positive charge to vertices of degree five and negative charge to vertices of degree greater than six, has positive total charge [. . .]

Appel and Haken report that by studying different discharge algorithms, with the help of computer programs, they were able to choose an algorithm that produced a set of fewer than 2,000 configurations (let us assume 2,000, for the sake of brevity); each configuration was proved, with the assistance of a computer program, to be reducible. That is, the theorem was proved by dividing the proof into 2,000 cases.

The proof is an instance of proof by cases, in which one analyses, documents, and proves every instance of an assertion in a case-by-case fashion. Proofs by cases are often qualified as cumbersome, clumsy, inelegant, and ugly. Proofs by cases contrast with elegant, parsimonious proofs, such as Cantor's diagonal method. A multiplicity of cases means lack of simplicity and lack of unity. Despite this fact, the analysis of the experience of a proof by cases is analogous to the analysis of Cantor's proof: the phenomenological space has *simplicity* and *step-parsimony* as dimensions, since these are the relevant properties involved in eliciting an affective response. The difference is that proofs by cases have negative scores on simplicity and parsimony.

Proof by cases can be interpreted as a series of steps, with the peculiarity that each case of the proof is independent of the others and thus each case requires its own proof. This means that there is no narrative connection between cases; rather, each case is an independent story. A proof by cases, as an intentional object, consists of multiple disconnected experiences, each with its own series of steps. A computer-assisted proof further worsens this scenario, since the proof itself only offers a general description of the cases to prove, and we have to trust that the computer has documented the truth of each case. As we saw in Chap. 3, this means that the computer-assisted proof does not even offer a complete intentional object on which we can focus our attention. To see this clearly, let us analyse the experience in more detail.

14.2 Experience

Our aesthetic-process begins by focusing our attention on the four colour theorem, and, more importantly, on its proof. Before we can appreciate the theorem or the proof we must be able to see them. Background understanding is necessary. Let us assume that the appropriate background is graph theory, (*GT*). Our phenomenological space has *GT* as a dimension, defined as follows:

GT = the feature of being understandable only if graph theory has been understood

With this background we actively follow the introductory reasoning presented by the authors, in a fashion similar to a step-series interpretation of the proof. Each step becomes an object of attention. We perform certain activities; we check for logical story development to accept the next step. We eventually arrive at the 2,000 maps to be discharged. These maps constitute the cases necessary to prove the theorem. At this point, a regular proof would involve another 2,000 small proofs. But the authors tell us only that every case was checked by means of a computer. This is peculiar, since this step does not allow us to focus our attention actively—which is characteristic of proofs as intentional objects. Rather, it offers only the passive acceptance of the computer's results. The proof offers only black-box outputs, so to speak.

Now, in Chap. 3 we saw that methodological and epistemological concerns about computer-assisted proofs do not play a significant role in their aesthetic evaluation. Let us assume that the proof is epistemic and methodologically sound. Even so, our experience of the 2,000 steps of the proof is that of mere acceptance of the results. In these last 2,000 steps we are not even offered a passive object of attention. Our experience is occurring in a space whose first dimension is background understanding; but a result from a computer does not need any special understanding: we only need to accept it. The nature of the results generated by a computer does not allow us to interpret them as intentional objects in our phenomenological space. Unlike the steps of a regular proof, the results of computer programs do not require neither the background understanding nor the active attention characteristic of experiences of proofs. The steps do not play the role of engaging us in intellectual activity to elicit pleasure. Now, this fact does not turn our experience into a *negative* experience; it turns it into an *incomplete* experience.

We can rephrase our conclusion in Chap. 3. Mathematicians do not appreciate computer-assisted proofs simply because they do not offer anything (or they offer very little) to be appreciated: even if we know and understand the kind of operation the computer performs, we cannot actually have the experience of following the last 2,000 steps of the proof; we do not perform any intellectual activities ourselves and we do not even have an object to passively appreciate.

Now, one might argue that, in principle, we know the kind of operations and processes a computer performs. This knowledge can be the basis of an experience in the same way graph theory is. But the introduction of knowledge of computer programming is still a problem for two reasons. First, the knowledge involved in knowing what the program does is different from our background understanding. This knowledge is about the working principles of computers, or about programming principles, or even about the exact computer code employed. But it is not our background understanding knowledge *GT*. There is still a kind of interruption of our original mathematical experience. We suddenly jump in the middle of our proof from our *GT* phenomenological space to another, computer-programming space. Second, mathematicians do not see themselves as computer programmers, code writers or debuggers. By introducing computer-programming events, mathematicians, in a sense, are forced into alien territory. It is true that all these activities can be seen as beautiful, but it would be a different kind of beauty; computer-programming beauty. Computer-programming beauty may have its own principles and types of experience in a manner similar to how mathematical beauty has its types of experience and principles; but it is still an alien kind of beauty in relation to our original mathematical beauty. When we suddenly introduce computer-related knowledge in the midst of a traditional mathematical argument we interrupt the original experience and switch to an experience of computer-programming items. Although this is not the same as an incomplete experience, my argument still holds, since the interruption turns the original experience into something else. The introduction of computer-programming events as the focus of attention alienates our mathematical experience. The cases of an incomplete experience and a switch of attention to a computer-programming item experience (we can call it an *alienated* experience) can be understood in similar terms: in both cases we are not provided with experiences that are characteristically mathematical.

In the computer-assisted proof of the four colour theorem the content of the experience is similar to a proof by cases, except for the last 2,000 steps, which are absent from our experience. Our experience of the proof is similar to the experience we would have if someone told us a story that resulted in 2,000 other stories, except the author would not tell us those 2,000 stories; rather, he would ask us to accept that all those stories have happy endings; resulting in a feeling of frustration. Let us recall that not only contemplating properties, but also performing mental activities elicits affective responses. The feeling of frustration does not come from the properties of the proof, but, analogously to Cantor's diagonal, from the outcome of our activities. In summary, the content of experience can be interpreted as an intentional object similar to the aesthetic mathematical intentional object of a proof by cases, except for the fact that the last steps are missing: our intentional object is a crippled, deformed intentional object, and the affective response of displeasure is due to the unsatisfactory way the story's conclusion is reached. Since we have an incomplete object of attention, it is not likely that the passive content of the experience elicits an affective response, so the response of frustration, caused by the active content, plays the most significant role. The output of the pleasure-relation for this proof can be expressed as:

$$f(semi - AMIO - 4colour, incompleteactivities) = (N, D)$$

14.3 Value

In Chaps. 3 and 4 we saw that in McAllister's conception of the aesthetic induction it makes sense to conjecture that computer-assisted proofs might one day be regarded as beautiful, as their acceptance grows. However, in Chaps. 4 and 5, the constrained model of aesthetic induction rendered a different result. As in the case of Cantor's proof, in addition to aesthetic induction, an aesthetic experience can be enhanced by a change in the constitution of the experience. Thus, in our model, we have two possible scenarios for a change in the aesthetic value of computer-assisted proofs: by constrained aesthetic induction and by change in experience. Let us analyse these two scenarios.

14.3.1 Scenario 1: Change by Aesthetic Induction

Let us assume that the acceptance of computer-assisted proofs grows. Mathematicians no longer argue about the soundness of the results generated by computers and are not concerned about any of their epistemic or methodological drawbacks. Our proof of the four-colour theorem would be just a perfectly valid proof by cases. However, as we discussed in Chap. 4, proofs by cases have been regarded as a good and acceptable method since they first appeared, and the method is as old as mathematics itself. There has never been shortage of proofs by cases and every mathematician is very familiar with the method. But this familiarity has not resulted in an increase in mathematician's preference for proofs by cases. It seems thus that the properties associated with proofs by cases exhibit negative robustness: their appreciation tends to stay negative. The set of properties associated with proofs by cases (lack of simplicity, lack of parsimony) exhibits low or negative critical adequacy and a high degree of robustness. All the evidence shows that the method's value is not likely to change. Proofs by cases are not aesthetically appreciated, and that fact seems to remain stable. Now, the assistance of computers does not entail anything that may affect those traits in any relevant way. We thus have reasons to think that proofs by cases shall remain negatively judged, since it seems implausible that, one day, computer-assisted proofs by cases would be judged positively, despite the fact that regular proofs by cases were judged negatively.

14.3.2 Scenario 2: Change in the Nature of Our Experience

We can once again draw an analogy with our discussion of the diagonal method, where we established that our first experience of the proof is different from ulterior experiences, in which we become aware of the method's unification power. Once the diagonal method has become widely applicable, the property of method-unification plays a significant role in enhancing our appreciation. Let us assume that something similar occurs to computer-assisted proofs; they become so powerful and acceptable that they begin to appear in different kinds of proofs. The constitution of the experience changes; the content of our aesthetic experience is no longer the original content associated with our first computer-assisted proof by cases of the four-colour theorem. We now have a new property to consider, method-unification, which may result in improving our judgement of computer-assisted proofs in general. However, the assistance of the computer remains in all instances of experiences of computer-assisted proofs. This means that at least one part of the proof consists in accepting results generated by the computer. This acceptance of the results cannot result in an intentional object (or at least in a non-alienated object), and it also prevents us from performing any (non-alienated) activities in that specific step. This is analogous to having many different ways of telling stories, all of them sharing the feature that at some point we are asked to pretend that an event convenient for concluding the story just happened.

Even if the computer-assisted-proof method is ubiquitous and method-unification is important, it remains the case that our experience's content is an incomplete content (or an alienated content), for that is the very nature of accepting results from a computer: the computer-assisted steps translate into circumventing experience steps (or into shifting to a computer-programming experience): accepting results is equivalent to *bypassing mathematical experience*.

The problem with computer-assisted proofs is not acceptability (as it is in the case of aesthetic induction), but rather that the computer-assisted steps of the proof only give us something to accept and not something to appreciate (at least, not without alienating our mathematical experience). Acceptability can be settled by addressing the relevant epistemic issues of the assistance of computers. If I were trying to establish the validity of a certain theorem, accepting a result is correct as far as there are no epistemic problems with that acceptance. But this acceptance still does not give us something to appreciate (that is, something to contemplate or some task to perform); and having something to appreciate constitutes the very basis of aesthetic experience. Acceptability is not appreciability. For aesthetic evaluation, having something to see is a precondition, and accepting results is just bypassing this condition. Acceptance does not even need background understanding: we do not need to understand anything in order to accept a result. Accepting results is something very different from *experiencing* mathematics.

In the experience and enjoyment of doing mathematics, a mathematician becomes engaged in understanding assertions, or in performing intellectual activities. In aesthetic experience this engagement is further deepened by changing the way our attention is focused and by undergoing affective responses. This is what aesthetic experience is; mathematical engagement plus affective engagement. In any aesthetic experience we want to see something, we want to engage in appreciation. But accepting a result is not seeing or doing; it is avoiding experience. If someone covered my eyes in front of a painting and I was not able to see it, I could still believe the painting is there and it has certain characteristics—if, for instance, I accepted the testimony of someone reliable. But if I were asked about my experience of the painting, I would be able to answer only that I have none, because I cannot see the painting. The same occurs with computer-assisted steps of proofs. Even if a property such as method unification is added to the experience, it does not change the crucial fact that the experience still has deforming narrative gaps. A computer-assisted proof shall always give us an incomplete experience, something we cannot fully appreciate, despite the fact that the proof is a perfectly acceptable and widespread method. Thus, it is not very plausible that we shall come to regard computer-assisted proofs as beautiful.

Part IV
Closing Remarks

Chapter 15
Issues of Mathematical Beauty, Revisited

With the application of the aesthetic as process theory in the previous chapters, we can now recapitulate and further discuss some of the issues and insights on mathematical beauty gained and pointed out throughout this book.

Although a non-literal interpretation of the term 'mathematical beauty' seems to be supported by attitudes like the two cultures divide, we found out that the arts/sciences divide is a cultural contingency. Moreover, we examined historical attempts to interpret mathematical beauty as a genuine aesthetic phenomenon. In addition, there are principled reasons against reinterpreting 'mathematical beauty'; metaphorical uses of terms such as 'beauty' or 'elegant' seem impossible. And, from a pragmatical perspective, literal approaches are in fact more fruitful. The most systematic and fruitful literal approach to beauty in science is McAllister's model of scientific development. However, we identified serious anomalies and theoretical drawbacks in McAllister's approach. We addressed those issues by introducing critical adequacy and robustness in a naturalistic revision of the aesthetic induction. This also led us to propose a matching naturalistic aesthetic theory, the aesthetic as process theory.

Now, the analysis of $y = e^x$ and the step-series interpretation of Cantor's diagonal proof illustrated that changes in aesthetic judgements not necessarily depend on a history of previous experiences, as assumed by McAllsiter's original approach. Our examples showed how judgements based on aesthetic inner experience can be the result of changes in the constitution of the experience itself.

Now, although this book does not endorses Gian-Carlo Rota's non-literal approach, the approach advanced here salutes and to some extent vindicates it, since it provided us with valuable insights: we learned that mathematical beauty is socio-historical, that properties like shortness play a role in mathematical beauty, and that familiarity with mathematics is necessary for appreciating mathematical beauty. These insights found a place in the aesthetic as process theory. For example, the fact that mathematical beauty depends on social and historical context is evident not only in the dynamic character of value, but also in the fact, illustrated in the three proceeding case analyses, that contextual knowledge plays a decisive role

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2_15,

in allowing us to see the object of appreciation. Historical and social context are part of the background knowledge that is required in our interpretation of aesthetic mathematical intentional objects. However, I must emphasize that this dependence on context and history is not a feature that comes along with or results from the aesthetic character of mathematical beauty. Rather, this dependence on context and history is the result of the dependence on context and history of *understanding* in mathematics.

Now, aesthetic principles based on single properties such as order, uniformity or simplicity, also play a role in the aesthetic as process theory. We have seen that the role of properties like simplicity or step-parsimony, which are related to brevity or shortness, is to provide "extra" qualities on which our attention focuses so they are able to elicit affective responses. It must be noted that these properties are not part of the background understanding of mathematical items, but rather something extra; contingent virtues that we perceive in addition to the necessary characteristics of mathematical items. For example, the property we defined as derivative-symmetry is not a property of every function, or of the notion of derivative. The derivative of $y = e^x$ must have all properties of derivatives, but that $y = e^x$ is symmetric with respect to differentiation is an extra quality. Similarly, simplicity is not a necessary characteristic of proofs; that Cantor's diagonal proof is simple is an extra we appreciate, an extra to which we react affectively. The properties of uniformity or unity used by Shaftesbury or Hutchenson can be interpreted as some of these extra properties; as constitutive dimensions of aesthetic experience.

The need for familiarity with mathematics to appreciate mathematical beauty is addressed by the condition of a background-understanding dimension in phenomenological spaces. In this sense, familiarity with mathematics is necessary to locate, to "see", our objects of attention in a phenomenological space. This does not mean that familiarity is trivial for aesthetic response. Rather, the background understanding is what determines the particularity of mathematical aesthetic experience; it is what makes aesthetic experience a *mathematical* aesthetic experience. Just as seeing and hearing makes experiences of painting and music particular, background mathematical knowledge makes our aesthetic experience of mathematics particular. The background knowledge necessary for a mathematical aesthetic experience determines the modality of that experience.

We are now in position to address subtle issues such as the role of properties like shortness of steps in mathematical proofs. The second step of Cantor's diagonal proof (constructing the element complementary to the diagonal) is a good example of a short step in a proof. Shortness, in the context of the aesthetic experience of a proof, an experience that involves *active* content, facilitates the performance of activities and the checking that steps are related to each other. That is the case in Cantor's proof. In addition, properties like simplicity and parsimony allow us to focus our attention more effectively, and to have a more complete picture of an otherwise complex experience. These facts result in a more pleasurable performance of activities in our experience. The function of short steps in proofs can thus be accounted for by the roles played by such steps: they make a proof simpler, and they facilitate the active pleasure response.

Another subtle issue with Rota's view is the lack of a satisfactory explanation of the nature of mathematical ugliness. Rota explained that negative aesthetic judgements of mathematical items like proofs frequently result in further mathematical development: mathematicians keep working, looking for a more aesthetically acceptable proof. In our theory, the usage of aesthetic terms depends on the relations to their family of terms. The use of terms like mathematical beauty is linked to the usage of the entire family of interrelated terms (handsome, pretty, ugly, elegant, etc.), since their correct usage requires articulation. In addition, aesthetic judgements are locally terminal and they can participate in further aesthetic developments. The preferences held by a person or a community become public by uttering or publishing an aesthetic judgement. These aesthetic judgements can serve as a guide for further developments, telling us what is aesthetically meritorious and what not. If we try to develop more elegant proofs, for example, judgements of elegance of other proofs can show us which instances of proof are regarded as elegant. In a sense, mathematical aesthetic judgements can be compared to art criticism: they not only describe states of affairs regarding artworks, but also articulate processes in which personal preferences and social values interact with each other. Judgements of beauty or elegance, due to their term-family interdependence, set in place paradigms of beauty *and* elegance, and, by the same token, they also set corresponding negative paradigms. Thus, in encouraging further mathematical developments, paradigms of beauty (to be followed) are as valuable as paradigms of ugliness (to be avoided). Thus, ugliness as well as beauty and other interrelated aesthetic terms have a heuristic role; they set examples to be followed or avoided. This result is interesting also in the sense that it shows that the RSD model of aesthetic terms not only allows, but actually forces close relations among families of aesthetic terms and thus it entails closely related explanations of aesthetic phenomena. Hence the fact that ugliness as well as beauty have heuristic roles.

Furthermore, the use of different aesthetic terms ('elegant' instead of 'beautiful', for example) also play a role in refining our paradigms of aesthetic evaluation. The aesthetic as process theory permits complex processes of articulation. In the case of Cantor's proof, for instance, we have seen that whereas the use of the term beauty is possible, the term elegant is more accurate. Aesthetic judgements that include more accurate terms provide us with more sophisticated paradigms of articulation and thus with paradigms of more refined uses of terms. And, at another level, the usage of terms like 'elegant' provides us not only with paradigms of elegance, but also with paradigms of aesthetic articulation.

Conclusion

C.P. Snow denouncement of the two cultures is recognized as having been highly influential in public discourse during the second half of the twentieth century. Although Snow is certainly credited with labelling the existing tension between the arts and the sciences, the perception that there is a divide between them precedes and survives it. Even today, at the beginning of the twenty-first century, the prevalent view is that the arts and the sciences are alien to each other. The effects of that view on important social issues such as education and scientific policies are still discussed and they need to be addressed in the most serious way possible.

This book has thus taken a very necessary stand in addressing the two cultures divide: with a historically informed scepticism, which allowed us to tackle the task of determining, in a way as rigorous, serious and technically accurate as possible, the nature of aesthetic phenomena in science, more specifically of the usage of 'beauty' in mathematics. I addressed the fact that, apparently disregarding the two cultures divide, many mathematicians enjoy giving lyric expression to their enthusiasm for mathematics by using terms like 'a beautiful theory' or 'an elegant proof' and that, despite this, very few authors have addressed in a rigorous way the issue of an actual aesthetics of mathematics. As a result, in this book I have advanced a first sketch of an aesthetics of mathematics that incorporates insights from philosophy and empirical science, and addresses problems discussed by a variety of authors.

Following Francois Le Lionnais's call for a serious aesthetic of mathematics and Gian-Carlo Rota's concern with the use of the term 'mathematical beauty', I have interpreted aesthetic judgements in mathematics literally. My general strategy was to formulate an aesthetic theory able to explain such judgements. I proposed the theory I labelled *the aesthetic as process*, in which the concept of aesthetic-process is central. Aesthetic-processes are systems of interrelated events whose unfolding involves the interaction of objects, subjective responses, and historic-cultural contexts. The idea of aesthetic-process provides a general framework for understanding aesthetic events in mathematics as a particular class of aesthetic events. I have focused on aesthetic experience, value and judgements in mathematics. I have interpreted

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2,

aesthetic experience as an embedded sub-process which is unified by a mathematical intentional object (although there are cases of incomplete or alienated experience). The experience process develops by undergoing changes in the passive and active mental contents and by eliciting affective responses associated to those contents. An aesthetic mathematical intentional object not only unifies experience, but also elicits the characteristic affective element of the experience. This intentional object exists in a phenomenological space whose dimensions are properties relevant to our affective responses. In this model, knowledge plays a role analogous to sensory perception, since knowledge enables us to perceive intentional objects.

Aesthetic value has been interpreted as a relation between sets of properties and mental activities and their associated affective responses. Our evaluations are driven by preferences that manifest themselves in two ways: affectively, as affective evaluations, and cognitively, as rule-like aesthetic criteria. These aesthetic criteria can be used to track the evolution of value; the mechanism of this evolution is the constrained aesthetic induction, which is a generalization of McAllister's ideas to which the constraints imposed by the nature of our affective responses have been added. Finally, I have interpreted aesthetic judgements as aesthetic descriptions that express subjective states. Aesthetic descriptions include aesthetic terms whose conditions of application—the existence of mappings/reorganizations among Response spaces, Schemata and referential Domains, in the RSD model—are strictly subjective. Aesthetic judgements perform the functions of articulation and broadcasting. Articulation is the elucidation of our subjective state in terms of the RSD conditions of application of aesthetic terms. Broadcasting consists in making the information conveyed by the judgement publicly available. The need to express subjective states by linguistic means encourages not only the articulation of individual subjective states—subjective articulation, but also the articulation of the aesthetic-process itself—process articulation. The awareness of our internal state further defines and identifies the events involved in the process as aesthetic events, which alters the constitution of the aesthetic experience itself.

With the aesthetic as process theory, it is rather trivial to address mathematical beauty. 'Mathematical beauty' is an aesthetic term. It expresses certain subjective states elicited in the observer by the engagement of his attention on a certain mathematical item. Now, in addition to offering a simple explanation of mathematical beauty the theory offers further advantages. For example, we can account for mathematical elegance and ugliness in the same way as we account for beauty. These terms express differences in our subjective experiences, by matching them to the differences in the family of terms we employ, and by highlighting certain properties taken from the referential domain of the term's schema. Beauty, ugliness and elegance are members of a family, which is manifest in facts such as the heuristic role not only of ugliness, as proposed by Rota, but also of beauty and elegance. Another advantage, perhaps the most important one, is that the theory allows us to distinguish different types of aesthetic experience. The most salient difference in the type of experience is a difference in the modality of the aesthetic experience. The modality is determined by the kind of events that focus our attention (sensory, cognitive, or both), the kind of properties involved in our phenomenological space (for example, our appreciation of representational painting depends less on knowledge than our appreciation of mathematics, and the knowledge involved in different kinds of appreciation is also a different kind of knowledge), the kind of activities involved during the process (some types of experience are more passive or more active), and the way the content relates to affective responses (some experiences depend on readily available responses, whereas some others depend on our history of experiences). Mathematical knowledge, as *background-understanding*, plays a central role in the modality of mathematical experience. But this is not so rare, something similar occurs in music, literature or painting: the amount and quality of technical and even contextual knowledge changes the way we perceive an object of appreciation. In this sense, mathematical beauty is not different from other expressions of beauty, except for the fact that knowledge plays a deeper role in constituting aesthetic experiences.

The aesthetic as process theory has incorporated many individual insights on the nature of mathematical beauty. I have presented a model of aesthetic events that vindicates them, since it allows us to give all those insights a relevant place in a more comprehensive depiction of the aesthetic in mathematics. Peter Kivy's ideas on intentional objects contributed to giving intentional objects a central role in the theory. Naturalistic approaches such as Theo Kuiper's interpretation of the aesthetic induction, or Jenefer Robinson's approach to emotion in art showed the advantages of using empirical results and insights. Shaftesbury and Hutchenson's property-based aesthetic principles find a place in my model as dimension of the phenomenological space, that is, as part of the elements that turn an experience into an *aesthetic* experience, since they are responsible for our affective responses. Gian-Carlo Rota's emphasis on knowledge and understanding finds a central place in my model as the dimension which allows us to see mathematical intentional objects. I have incorporated James McAllister's and Kuipers' ideas on the aesthetic induction by adding the appropriate generalizations and introducing the constraints imposed by historical evidence and empirical results. Frank Sibley, Isabel Hungerland Peter Kivy, Rafael DeClercq and Nick Zangwill's discussions on aesthetic terms, and Nelson Goodman's analysis of metaphor shaped my approach to aesthetic terms and their functions in aesthetic-processes.

Topics for Further Research

There are topics I have not been able to discuss in this book. Among these there are problems that have not been addressed and new issues and questions that arise form the approach presented here. Some issues, like discussing types of experience beyond the three discussed in this book, or identifying the other kinds of aesthetic value involved in mathematics are topics that can be easily addressed in follow-up works. The goal in this book was to propose an articulated theory, and going into too much detail might have disrupted the discussion. But there are issues that are closely related but not necessarily follow-ups, and with an importance in themselves.

I started this book with a very brief historical survey showing how our views of art, science and mathematics have changed. A more thorough account of such change is an enterprise that greatly exceeds the scope of this book, but one with great potential. In this respect there are topics that were left untouched. For example, as many disciplines, in its early stages, mathematics was considered an art in the sense of a craft, a "techne" if you will. An interesting question is what kind of aesthetic judgements—if any—were associated with the ancient mathematical craft and, more importantly, how would they compare to contemporary evaluations. Moreover, one may also examine non-western mathematical cultures like the Indian, Chinese or Arabic ones. Furthering this line of thought, the early period of Modernity and of the emergence of science as an empirical discipline is also a fascinating source of questions. For example, it would be interesting to examine in detail the views of philosopher-mathematicians like Descartes, Newton or Leibniz. I am certain much can be learn from inquiring into all these topics, but the task is so ambitious it should be tackled as an entire different and complementary work, which of course could bring additional evidence and test material for the theory advanced in this book.

Now, beyond historical concerns, most obvious among the non-addressed topics is the relationship between the issue of beauty and other philosophical issues in mathematics: truth, knowledge and the ontology of mathematics being the most salient ones. My proposal is grounded, among other things, on results related to the domain of psychological and neuro-physiological phenomena, which seem rather distant from the traditional approach to the philosophical problems of mathematics. The present work should be seen as devoted to the practice of mathematics. Mathematical practice has repercussions on many aspects of other problems, but not all of them should be addressed from that perspective. Truth, knowledge and ontology are particularly resilient to that treatment.

Mathematical Truth

Truth is relevant to beauty in my theory: truth is a condition for mathematical items to be considered as objects in a phenomenological space. Despite this, the role of beauty seems to be relatively independent of mathematical truth. The methods for securing the truth of a mathematical statement do not depend on empirical information or subjective matters; rather, they are based on logic. But there is an issue in which beauty and truth are more alike: as Rota pointed out, aesthetic considerations sometimes encourage searching for further proofs of a theorem. Mathematical beauty is a property that plays an role in encouraging mathematical work, just as truth does. The search for beauty, just like the search for truth, promotes mathematical work. Both searches are justified in themselves. Truth is certainly a value in itself; it is one of the qualities that define mathematics as a scientific endeavour. But what about beauty? In Rota's words, the search for beauty distinguishes mathematics among the sciences. In this sense mathematics has a particular stance as a cognitive scientific discipline. The idea that beauty is a characteristic value of the mathematical discipline, just as truth is, is worth investigating, but that deserves much more space than the available here.

The subject of truth poses other issues that we can briefly survey. For example, McAllister claims that the relation between truth and beauty must be clarified by empirical means [62, pp. 98–104]. Theo Kuipers [49] further explores the formal relation between truth of empirical theories and aesthetic induction. In those approaches the relation between beauty and truth in science is contingent and should be investigated empirically. My version of aesthetic induction, however, does not allow us to borrow this conclusion easily. My model incorporates a biologically determined component, which is very likely determined by evolutionary adaptation; this introduces a further element to take into account. Furthermore, this element is particularly relevant in the evolution of properties such as symmetry, simplicity or uniformity. Investigating the relation between truth and beauty must be carried out with special attention to investigating the relation between truth and symmetry, simplicity, uniformity, an so forth.

Mathematical Knowledge and Ontology

One of the most important problems of mathematical knowledge is its very possibility [9, 14, 95]. The problem of how mathematical knowledge is possible has to do with the fact that mathematical objects are abstract: they have no spatial or temporal location and they do not interact causally with the physical universe. If there is no way in which mathematical objects can interact with our physical universe, how can we have any knowledge of mathematical objects whatsoever?

Now, the knowledge involved in my approach, background understanding, has to do with the mathematical items as objects of experience. These objects are mental events. They are as problematic as any mental content, but not in the same sense as causally isolated abstract objects. My discussion of beauty does not add much regarding the relation between mental objects and knowledge of causally isolated entities. However, in this respect, it does not fare differently from other approaches to mathematical beauty. Rota's approach, for example, interprets mathematical beauty as a form of knowledge. For Rota the property of being enlightening consists of enabling us to see connections and relations that help us to have a proper grasp of the significance of a mathematical item. But this is not the kind of knowledge that concerns the traditional problem of knowing abstract objects. Now, this of course prompts the question of what is the relation between the traditional problem of knowledge about abstract objects and the more mundane knowledge involved in my or Rota's conception of mathematical beauty? But, of course, this question has consequences far beyond the topics discussed here.

Concerning the ontological issues of mathematics, they are closely linked to the problem of knowledge [14]. Under a realist interpretation of mathematics, mathematical theories refer to abstract objects. I have explicitly differentiated mathematical objects from mathematical items, so it would seem that our discussion has little to contribute. However, the problems related to the existence and properties of abstract objects differ depending on which ontological stance we take. In a realist approach to mathematical objects, mathematical items and objects are relatively independent of each other. In a nominalist approach, like Hartry Field's [25], a closer contact can be established between them. In Field's nominalism, mathematical objects do not literally exist and mathematical theories and statements are true merely in the same sense in which we say that a narrative *fiction* is true. Field's nominalism leaves some questions unanswered, such as what kinds of fictions are mathematical objects? And how exactly do they work in practice? A literal aesthetics of mathematics can surely offer clues to study some qualities of mathematical objects seen as fictions, especially because mathematical practice is driven by aesthetic considerations just as it is driven by a search for a "fictional" truth. According to Field, fictional mathematical objects are merely useful in practice. In this book one of our tenets is that some of those objects can also be enjoyable, therefore intrinsically valuable. A balance between the practically and the intrinsically valuable would seem to suit a nominalist account of mathematics quite well.

To conclude, I believe it is not necessary to further stress the importance of beauty in mathematical practice. There is no shortage of texts, historical or philosophical, emphasizing the importance of mathematical beauty. In addition, it is also true that one usually approaches mathematics motivated by passionate curiosity; and that a need for harmony and beauty is what drives much of the mathematicians' work. My hope is that this book can contribute to rethinking and legitimating that passion.

Bibliography

1. Aaboe, A. (1963). *Episodes from the early history of mathematics*. New York: MAA.
2. Aigner, M., & Ziegler, G. M. (2004). *Proofs from the book*. Berlin/New York, Springer.
3. Alperson, P., & Kivy, P. (2004). The philosophy of music: Formalism and beyond. In P. Kivy (Ed.), *The Blackwell guide to aesthetics* (pp. 254–275). Malden: Blackwell
4. Appel, K., & Haken, W. (1976). Every planar map is four colorable. *Bulletin of the American Mathematical Society, 82*(5), 711–713.
5. Appel, K., & Haken, W. (1977). Every planar map is four colorable. Part I: Discharging. *Illinois Journal of Mathematics, 21*(3), 429–490.
6. Appel, K., & Haken, W. (1986). The four color proof suffices. *The Mathematical Intelligencer, 8*(1), 10–20.
7. Appel, K., Haken, W., & Koch, J. (1977). Every planar map is four colorable. Part II: Reducibility. *Illinois Journal of Mathematics, 21*(3), 491–567.
8. Ax, A. F. (1953). The physiological differentiation between fear and anger in humans. *Psychosomatic Medicine, 15*(5), 433–442.
9. Balaguer, M. (2001). *Platonism and anti-platonism in mathematics*. New York: Oxford University Press.
10. Benacerraf, P. (1973). Mathematical truth. *The Journal of Philosophy, 70*(19), 661–679.
11. Brady, E., & Levinson, J. (2001). *Aesthetic concepts: Essays after Sibley*. New York: Oxford University Press.
12. Budd, M. (1992). *Music and the emotions: The philosophical theories*. Psychology Press. London: Boston.
13. Cohen, T. (1973). Aesthetic/non-aesthetic and the concept of taste: A critique of Sibley's position. *Theoria, 39*(1–3), 113–152.
14. Colyvan, M. (2003). *The indispensability of mathematics*. New York: Oxford University Press.
15. Cooper, A. A. (1964). Third earl of shaftesbury. *Characteristics of Men, Manners, Opinions, Times, 1711*, 397–400.
16. Damasio, A. (2005). *Descartes' error: Emotion, reason, and the human brain*. London/New York: Penguin (Non-Classics).
17. De Clercq, R. (2005). Aesthetic terms, metaphor, and the nature of aesthetic properties. *The Journal of Aesthetics and Art Criticism, 63*(1), 27–32.
18. de Villiers, M. (2004). The role and function of quasi-empirical methods in mathematics. *Canadian Journal of Math, Science & Technology Education, 4*(3), 397–418.
19. Dirac, P. (1980). Why we believe in the Einstein theory? In B. Gruber & R. S. Millman (Eds.), *Symmetries in science*. New York: Plenum.

U. Montano, *Explaining Beauty in Mathematics: An Aesthetic Theory of Mathematics*, Synthese Library 370, DOI 10.1007/978-3-319-03452-2,

20. Dreyfus, H. L., & Wrathall, M. A. (2006). *A companion to phenomenology and existentialism* (Vol. 35). Malden: Wiley-Blackwell.
21. Ekman, P., & Davidson, R. J. (Eds.). (1994). *The nature of emotion: Fundamental questions* (1st ed.). New York: Oxford University Press.
22. Feigl, H. (1970). Beyond peaceful coexistence. In R. H. Stuewer (Ed.), *Historical and philosophical perspectives of science* (Number 5 in Minnesota studies in the philosophy of science, pp. 3–11). Minneapolis: University of Minnesota Press.
23. Feynman, R. P., & Weinberg, S. (1999). *Elementary particles and the laws of physics: The 1986 Dirac memorial lectures*. New York/Cambridge: Cambridge University Press.
24. Feynman, R. P., Leighton, R. B., & Sands, M. (2011). *The Feynman lectures on physics, Vol. I: The new millennium edition: mainly mechanics, radiation, and heat*. New York: Basic Books.
25. Field, H. H. (1980). *Science without numbers: A defence of nominalism*. Princeton: Princeton University Press.
26. Franks, J. (2010). Cantor's other proofs that R is uncountable. *Mathematics Magazine, 83*(4), 283–289.
27. Frege, G. (1980). Frege against the formalists. In M. Black & P. Geach (Eds.), *Philosophical writings: Translations* (3rd ed., pp. 162–213; 2nd ed., 1970, 1969, 1966, 1960; 1st ed. 1952). Oxford: Blackwell.
28. Frijda, N. H. (1986). *The emotions*. Cambridge/New York: Cambridge University Press.
29. Goldman, A. (1995). *Aesthetic value* (Focus series). Boulder: Westview Press.
30. Goodman, N. (1968). *Languages of art: An approach to a theory of symbols*. Indianapolis: The Bobbs-Merrill Company.
31. Gow, J. (2010). *A short history of Greek mathematics*. Cambridge: Cambridge University Press.
32. Guyer, P. (2004). The origins of modern aesthetics: 1711–35. In P. Kivy (Ed.), *The Blackwell guide to aesthetics* (pp. 15–44). Malden: Blackwell.
33. Hardy, G. H. (1992). *A mathematician's apology*. Cambridge/New York: Cambridge University Press.
34. Hillman, D. J. (1962). The measurement of simplicity. *Philosophy of Science, 29*, 225–252.
35. Hungerland, I. C. (1962). The logic of aesthetic concepts. *Proceedings and Addresses of the American Philosophical Association, 36*, 43–66.
36. Hutcheson, F. (1973). *Francis Hutchenson: An inquiry concerning beauty, order, harmony, design* (Vol. 9). The Hague: Martinus Nijhoff Publishing.
37. Kaku, M., & Thompson, J. T. (1997). *Beyond Einstein: The cosmic quest for the theory of the universe*. Oxford: Oxford University Press.
38. Kivy, P. (1975). What makes "Aesthetic" terms aesthetic? *Philosophy and Phenomenological Research, 36*(2), 197–211.
39. Kivy, P. (1980). *The corded shell: Reflections on musical expression*. Princeton: Princeton University Press.
40. Kivy, P. (1989). *Sound sentiment: An essay on the musical emotions, including the complete text of the corded shell*. Philadelphia: Temple University Press.
41. Kivy, P. (1991). *Music alone: Philosophical reflections on the purely musical experience*. Ithaca: Cornell University Press.
42. Kivy, P. (1993). *The fine art of repetition: Essays in the philosophy of music*. Cambridge/New York: Cambridge University Press.
43. Kivy, P. (1997). *Philosophies of arts: An essay in differences*. Cambridge/New York: Cambridge University Press.
44. Kivy, P. (2002). *Introduction to a philosophy of music*. Oxford: Oxford University Press.
45. Kline, M. (1990). *Mathematical thought from ancient to modern times, Vol. 1* (1st ptg. ed.). New York: Oxford University Press.
46. Kline, M. (1990). *Mathematical thought from ancient to modern times, Vol. 2*. New York: Oxford University Press.

47. Kline, M. (1990). *Mathematical thought from ancient to modern times, Vol. 3*. New York: Oxford University Press.
48. Kuipers, T. A. F. (2000). *From instrumentalism to constructive realism: On some relations between confirmation, empirical progress, and truth approximation*. Dordrecht: Springer.
49. Kuipers, T. A. F. (2002). Beauty, a road to the truth. *Synthese, 131*(3), 291–328.
50. Lakatos, I., Worrall, J., & Zahar, E. (Eds.). (1976). *Proofs and refutations: The logic of mathematical discovery*. Cambridge/New York: Cambridge University Press.
51. Langer, S. K. K. (1957). *Philosophy in a new key: A study in the symbolism of reason, rite, and art* (Vol. 17). Cambridge: Harvard University Press.
52. Laudan, L. (1978). *Progress and its problems: Towards a theory of scientific growth*. Berkeley: University of California Press.
53. Lazarus, R. S. (1991). *Emotion and adaptation*. New York: Springer.
54. LeDoux, J. (1996). Emotional networks and motor control: A fearful view. *Progress in Brain Research, 107*, 437–446.
55. LeDoux, J. E. (1998). *The emotional brain: The mysterious underpinnings of emotional life*. New York: Simon & Schuster.
56. LeDoux, J., & Bemporad, J. R. (1997). The emotional brain. *Journal of the American Academy of Psychoanalysis, 25*(3), 525–528.
57. Le Lionnais, F. (1971). *Les grands courants de la pensée mathématique*. Marseille: Cahiers du Sud.
58. Le Lionnais, F. (2004). *Great currents of mathematical thought: Mathematics: Concepts and development* (Vol. 1). Mineola: Dover.
59. Levenson, R. W. (1994). The search for autonomic specificity. In P. Ekman & R. J. Davidson (Eds.), *The nature of emotion: Fundamental questions* (pp. 252–257). New York: Oxford University Press.
60. Livio, M., & Grunwald, E. (2006). The equation that couldn't be solved: How mathematical genius discovered the language of symmetry. *The Mathematical Intelligencer, 28*(4), 63–64.
61. Marenbon, J. (2003). *Boethius*. Oxford/New York: Oxford University Press.
62. McAllister, J. (1996). *Beauty and revolution in science*. Ithaca: Cornell University Press.
63. McAllister, J. (1998). Is beauty a sign of truth in scientific theories? *American Scientist, 86*(2), 174–183.
64. McAllister, J. W. (2005). Mathematical beauty and the evolution of the standards of mathematical proof. In M. Emmer (Ed.), *The visual mind II* (Vol. 2). Cambridge: MIT.
65. McWeeny, R. (2002). *Symmetry: An introduction to group theory and its applications*. Mineola: Dover.
66. Montano, U. (2010). *Beauty in mathematics*. PhD thesis, [s.n.], S.l. University of Groningen, The Netherlands.
67. Montano, U. (2012). Ugly mathematics: Why do mathematicians dislike computer-assisted proofs? *The Mathematical Intelligencer, 34*(4), 21–28.
68. Nahin, P. J. (1998). *An imaginary tale: The Story of i [the square root of minus one]*. Princeton: Princeton University Press.
69. Nahin, P. J. (2006). *Dr. Euler's fabulous formula: Cures many mathematical ills*. Princeton: Princeton University Press.
70. Nahin, P. J. (2011). *Dr. Euler's fabulous formula: Cures many mathematical ills*. Princeton: Princeton University Press.
71. Newton, I. (2003). *Newton's philosophy of nature selections from his writings*. Whitefish, Mont. Kessinger Publishing.
72. Newton-Smith, W. (1981). *The rationality of science*. Boston: Routledge.
73. Ortony, A. (1991). Value and emotion. In G. Mandler, W. Kessen, A. Ortony, & F. I. M. Craik (Eds.), *Memories, thoughts, and emotions: Essays in the honor of George Mandler* (pp. 337–353). Hillsdale: Erlbaum.
74. Ortony, A., Clore, G. L., & Collins, A. (1990). *The cognitive structure of emotions*. Cambridge/New York: Cambridge University Press.
75. Popper, K. R. (2002). *The logic of scientific discovery*. London/New York: Routledge.

76. Robinson, J. (2005). *Deeper than reason: Emotion and its role in literature, music, and art*. New York: Oxford University Press.
77. Robinson, J. (2005). *Deeper than reason: Emotion and its role in music literature and art*. London: Claredon.
78. Rota, G. (1997). The phenomenology of mathematical proof. *Synthese, 111*(2), 183–196.
79. Rota, G.-C. (1997). The phenomenology of mathematical beauty. *Synthese, 111*(2), 171–182.
80. Rota, G. C. (2005). The phenomenology of mathematical beauty. In M. Emmer (Ed.), *The visual mind II* (pp. 2–13). Cambridge: MIT.
81. Russell, B. (2004). *Mysticism and logic*. Mineola: Dover.
82. Sibley, F. (1959). Aesthetic concepts. *The Philosophical Review, 68*(4), 421–450.
83. Snow, C. P. (2012). *The two cultures*. Cambridge/New York: Cambridge University Press.
84. Solomon, R. C. (2005). Subjectivity. In T. Honderich (Ed.), *The Oxford companion to philosophy new edition* (2nd ed.). Oxford/New York: Oxford University Press.
85. Stuewer, R. H. (Ed.). (1970). *Historical and philosophical perspectives of science* (Volume 5 of Minnesota studies in the philosophy of science). Minneapolis: University of Minnesota Press.
86. Swinnerton-Dyer, P. (2005). The justification of mathematical statements. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 363*(1835), 2437–2447.
87. Thagard, P. (2005). Why is beauty a road to the truth? *Poznan Studies in the Philosophy of the Sciences and the Humanities, 84*(1), 365–370.
88. Thomas, R. S. D. (2007). The comparison of mathematics with narrative. In B. Kerkhove & J. P. Bendegem (Eds.), *Perspectives on mathematical practices* (Volume 5 of Logic, epistemology, and the unity of science, pp. 43–59). Dordrecht: Springer.
89. Tymoczko, T. (1979). The four-color problem and its philosophical significance. *The Journal of Philosophy, 76*(2), 57–83.
90. Weber, K. (2009). How syntactic reasoners can develop understanding, evaluate conjectures, and generate counterexamples in advanced mathematics. *The Journal of Mathematical Behavior, 28*(2), 200–208.
91. Weber, K., & Alcock, L. (2004). Semantic and syntactic proof productions. *Educational Studies in Mathematics, 56*(2), 209–234.
92. Weinberg, S. (1994). *Dreams of a final theory*. New York: Vintage Books.
93. Wells, D. (1988). Which is the most beautiful? *The Mathematical Intelligencer, 10*(4), 30–31.
94. Wells, D. (1990). Are these the most beautiful? *The Mathematical Intelligencer, 12*(3), 37–41.
95. Wright, C. (1994). *Truth and objectivity*. Cambridge/London: Harvard University Press.
96. Zajonc, R. B. (1984). On the primacy of affect. *American Psychologist*, 39(2), 117–123.
97. Zajonc, R. B. (1998). Emotions. In D. T. Gilbert, S. T. Fiske, & G. Lindzey (Eds.), *The handbook of social psychology* (Vol. 2, pp. 591–632). Boston: McGraw-Hill
98. Zajonc, R. B. (2001). Mere exposure: A gateway to the subliminal. *Current Directions in Psychological Science, 10*(6), 224–228.
99. Zangwill, N. (2003). Beauty. In J. Levinson (Ed.), *The Oxford handbook of aesthetics*. Oxford/New York: Oxford University Press.
100. Zangwill, N. (2010). Aesthetic judgment. In E. N. Zalta (Ed.), *The Stanford encyclopedia of philosophy* (Fall 2010 ed.). http://plato.stanford.edu/archives/fall2010/entries/aesthetic-judgment/.
101. Zangwill, N. (2011). Music, essential metaphor, and private language. *American Philosophical Quarterly, 48*(1), 1.