A Delay Fractioning Approach to Global Synchronization of Delayed Complex Networks with Neutral-Type Coupling

Hongli Wu, Ya-peng Zhao, Huan-huan Mai and Zheng-xia Wang

Abstract The global issues of synchronization of complex networks with neutral-type coupling are investigated in this chapter, which is not adequately considered in existing literatures. Based on these new complex models, we derive asymptotical and exponential criterions via delay fraction approach. Numerical examples are then given to illustrate the effectiveness of our scheme and to compare with the recent proposals. We also make (some) attempts to explore the relationship between delay fraction numbers and the conservatism of our criterions.

Keywords Synchronization · Complex networks · Neutral-type coupling · Delay fractioning

1 Introduction

Synchronization is a ubiquitous and interesting phenomenon in nature. For example, how dose thousands of neurons or fireflies or crickets suddenly fall into step with one another, all firing or flashing or chirping at the same time, without

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any leader or signal from the environment [1]? A complex network is a large set of interconnected nodes, in which each node is a fundamental unit with specific state. Recently, many attempts devoting to a better understanding of synchronization take advantage of the topology of complex networks, and they also contribute to the understanding of general emergent properties of networked systems. In fact, synchronization of complex networks has allured much attention as an interdisciplinary subject and provoked various applications in lots of fields such as neuroscience, engineering, computer science, economy, and social sciences.

It is well known that the way we connecting nodes plays an important role in the efficiency of synchronization in large networks. Li et al. studied the global synchronization of complex networks without delays [2–4]. Many researchers believed that there must be some time delays in spreading and responding due to the finite speed of transmission as well as traffic congestions, so delayed coupling should be modeled in order to simulate more realistic networks [5–10]. Moreover, it is natural and important to consider the neutral-type coupling delay in complex dynamical networks. Dai gave an example on it that when complex dynamical networks are used to model a stock transaction system, each node's state is defined as a behavior of the agent such as buying, selling, or holding. And the stock transaction system dynamically in terms of the current and historical fluctuating rate records [11]. On the other hand, the neutral-type coupling may be essential in specific applications such as secure communication. Solís-Perales founded that the derivative term under certain network topology leads to the chaotic synchronous behavior, whereas the standard coupling network reaches the equilibrium or a limit cycle [12]. To the authors' best knowledge, there are only two literatures [11, 12] that introduced the derivative term into the coupling of complex networks. By numerical simulations, [12] illustrated that the derivative terms in coupling have a significant influence on the synchronization. However, the impact of time delay is curtly neglected. While [11] considered the asymptotically synchronization of the neutral-type delay coupling complex networks by classical Lyapunov method.

In this chapter, we will introduce the synchronization of the new complex models first. Combining several techniques such as delay fraction method, the free-weighting matrices approach, Lyapunov–Krasovskii functional, and linear matrix inequality (LMI), we then study the asymptotically synchronization conditions and exponentially synchronization conditions of them. Finally, some simulations will be exercised to demonstrate the effectiveness and applicability of the proposed criterions together with some attempts to discuss the fraction number's influence on the criterions.

2 Model Description and Preliminaries

Notations: R^n denotes the n dimensional Euclidean space, and $R^{m\times n}$ is the set of all $m \times n$ real matrices. $||\cdot||$ denotes the Euclidean norm in R^n or $R^{m\times n}$. Let the Euclidian norm be $||\phi||_\tau = \sup\limits_{-\tau\leq\theta\leq 0} ||x(\theta)||$, $||\phi^*||_\tau = \sup\limits_{-\tau\leq\theta\leq 0} ||\dot{x}(\theta)||$ for a given continuous function. A^T denotes the transpose of matrix A, $\lambda_{\mathrm{Max}}(M)$ denotes the maximum eigenvalue of M, and $\lambda_{\mathrm{Min}}(M)$ denotes the minimum eigenvalue of M.

Consider the following continuous-time complex dynamical network with neutral-type coupling.

$$\dot{x}_i(t) = f(x_i(t), t) + c\sum_{j=1}^{N} G_{ij}\big(Ax_j(t-h) + B\dot{x}_j(t-\tau)\big), \quad i = 1,\ldots,N. \tag{1}$$

With the initial condition $x(t) = \phi(t)$, $t \in [-h_{\max}, 0], h_{\max} = \max\{h, \tau\}$ where $f : R^n \to R^n$ is continuously $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in R^n$ are the state variables of node i, the constant $c > 0$ is the coupling strength, h, τ are the retarded delay and the neutral delay, respectively. $A = (a_{ij}) \in R^{n\times n}$ is a constant inner-coupling matrix of the nodes about the retarded delay, and $B = (b_{ij}) \in R^{n\times n}$ regarding to the neutral one. $G = G_{ij} \in R^{N\times N}$ is the outer-coupling configuration matrix of the network, in which G_{ij} is defined by:

$$G_{ii} = -\sum_{j=1, j\neq i}^{N} G_{ij} = -\sum_{j=1, j\neq i}^{N} G_{ji} \quad i = 1,\ldots,N.$$

Suppose that the network (1) is connected in the sense that there are no isolated clusters. That is, G is an irreducible matrix.

Definition 1 The synchronized state of the entire networks is denoted by $s(t) \in R^n$ is a solution of an isolate node, namely $\dot{s}(t) = f(s(t))$, $s(t)$ may be a limit cycle, or a chaotic orbit in the phase space.

Definition 2 The dynamical network (1) is said to achieve asymptotic synchronization if

$$x_1(t) = x_2(t) = \ldots = x_N(t) = s(t) \text{ as } t \to \infty, \tag{2}$$

Definition 3 The dynamical network (1) is said to be globally exponentially synchronized if, for any solution $x(t)$, if there exist constants $\mu > 0$ and $\alpha > 0$ such that

$$\lim_{t\to\infty} ||x_i(t) - s(t)|| \leq \mu e^{-\alpha t}\max\{||\phi||_\tau, ||\phi^*||_\tau\}, \quad i = 1, 2, \ldots, n. \tag{3}$$

Lemma 1 ([5]) *Suppose that an irreducible matrix* $G = (G_{ij})_{N\times N}$ *satisfies the above conditions. Then,* 0 *is an eigenvalue of matrix* G*, associated with eigenvector* $(1, 1, \ldots, 1)^T$;

All the other eigenvalues of G are real-valued and are strictly negative.

Let $\lambda_i,\ i = 1, 2, \ldots, N$ be the nonzero eigenvalues of G. Lemma 1 is, without loss of generality, all the eigenvalues of G are real numbers and ordered as

$$0 = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$$

Lemma 2 ([13]) *For scalar $r > 0$, let $M \in R^{m\times m}$ be a positive semi-definite matrix and $\rho : [0, r] \to R^m$ be a vector function. If the interactions concerned are well defined, then the following inequity holds*

$$r\int_0^r \rho^T(s)M\rho(s)ds \geq \left(\int_0^r \rho(s)ds\right)^T M\left(\int_0^r \rho(s)ds\right) \tag{4}$$

Lemma 3 ([14]) *The following LMI* $\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0$ *where $Q(x) = Q^T(x)$, $R(x) = R^T(x)$, and $S(x)$ depend affinely on x, is equivalent to $R(x) > 0$, $Q(x) - S(x)R^{-1}(x)S^T(x) > 0$.*

Lemma 4 *If the following $(N-1)\times n$ dimensional neutral-type delay differential equations $\dot{w}_i(t) = J(t)w_i(t) + c\lambda_i(Aw_i(t-h) + B\dot{w}_i(t-\tau))$ $i = 2,\ldots N$ where $J(t)$ is the Jacobian of $f(x(t), t)$ at synchronized state $s(t)$ are asymptotically stable about their zero solution, then the synchronization states are asymptotically stable for the complex networks.*

Lemma 5 *If the following $(N-1)\times n$ dimensional neutral-type delay differential equations*

$$\dot{w}_i(t) = J(t)w_i(t) + c\lambda_i(Aw_i(t-h) + B\dot{w}_i(t-\tau)) \quad i = 2,\ldots,N \tag{5}$$

where $J(t)$ is the Jacobian of $f(x(t), t)$ *at synchronized state $s(t)$ are exponentially stable about their zero solution, then the synchronization states are exponentially stable for the complex networks* (1).

Proof In order to investigate the stability of the synchronized states (2),
Set

$$x_i(t) = s(t) + e_i(t), \quad i = 1, 2, \ldots, N.$$

Substituting (5) into (1), we have

$$\dot{e}_i(t) = f(s(t) + e_i(t)) - f(s(t)) + c\sum_{j=1}^{N} G_{ij}\big(Ae_j(t-h) + B\dot{e}_j(t-\tau)\big),$$
$$i = 1, \ldots, N.$$

Since f is continuous differentiable, then we obtain

$$\dot{e}_i(t) = J(t)e_i(t) + cAe_i(t-h)G^T + cB\dot{e}_i(t-\tau)G^T,$$

there exists a nonsingular matrix, $\varphi = (\phi_1, \ldots, \phi_N) \in R^{N\times N}$ such that $G^T\Lambda = \varphi\Lambda$ with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. Using the nonsingular transform $w(t) =$

$(w_1(t), \ldots, w_N(t)) \in R^{n \times N}$, we have the following matrix equation: $\dot{w}(t) = J(t)w(t) + c(Aw(t-h) + B\dot{w}(t-\tau))\Lambda \quad i = 2, \ldots, N$.

Namely,

$$\dot{w}_i(t) = J(t)w_i(t) + c\lambda_i(Aw_i(t-h) + B\dot{w}_i(t-\tau)) \quad i = 2, \ldots, N.$$

Then, the global exponential synchronization problem of the dynamical networks (1) is equivalent to the problem of global exponential stabilization of the error dynamical system (5).

3 Main Results

Theorem 1 *The synchronous state* $s(t)$ *of complex networks with neutral-type coupling* (1) *is globally asymptotically stable if there exist matrices* O_i, $i = 1, 2, 3, 4$ *and symmetrical positive definite matrices* P, *positive definite matrices* Z, R_i, $i = 1, 2, 3$, *and* $\Omega_1 \in R^{(m+4)\times(m+4)}$ *such that*

$$\Omega_1 = \left[\begin{array}{cccccc} \begin{array}{c} 2PJ(t) + R_1 + R_2 - mZ \\ +2O_1J(t) + J^T(t)R_3J(t) \end{array} & mZ & mZ & \ldots & mZ & c\lambda_i PA2 + c\lambda_i J^T(t)R_3A + c\lambda_i O_1A + J^T(t)O_3^T \\ mZ^T & -2mZ & 0 & \cdots & 0 & 0 \\ mZ^T & 0 & -2mZ & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ mZ^T & 0 & 0 & 0 & -2mZ & 0 \\ 0 & 0 & 0 & 0 & 0 & (c\lambda_i)^2A^TR_3A - mZ - R_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ O_2J(t) - O_1^T & 0 & 0 & 0 & 0 & 0 \\ \begin{array}{c} c\lambda_i B^TP^T + O_4J(t) \\ +2c\lambda_i B^TR_3J(t)c\lambda_i O_1B \end{array} & 0 & 0 & 0 & 0 & 2(c\lambda_i)^2B^TR_3^TA + B^TO_3^T + O_4A \end{array}\right.$$

$$\left.\begin{array}{ccc} 0 & J^T(t)O_2^T - O_1 & 2c\lambda_i J^T(t)R_3Bc\lambda_i O_1B + J^T(t)O_4^T + c\lambda_i PB \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 0 & 0 & 2(c\lambda_i)^2A^TR_3B + O_3B + A^TO_4^T \\ -R_2 & 0 & 0 \\ 0 & h^2Z - 2O_2 & 0 \\ 0 & 0 & (c\lambda_i)^2B^TR_3B - R_3 + 2c\lambda_i O_4B \end{array}\right] < 0. \tag{6}$$

Proof Select a Lyapunov–Krasovskii functional as

$$V_i(w_i(t)) = V_{i1}(w_i(t)) + V_{i2}(w_i(t)) + V_{i3}(w_i(t)) + V_{i4}(w_i(t)) + V_{i5}(w_i(t)), \tag{7}$$

where

$$V_{i1}(w_i(t)) = w_i^T(t)Pw_i(t),$$

$$V_{i2}(w_i(t)) = \int_{-h}^{0} w_i^T(t+\xi)R_1 w_i(t+\xi)d\xi,$$

$$V_{i3}(w_i(t)) = \int_{-\tau}^{0} w_i^T(t+\xi)R_2 w_i(t+\xi)d\xi,$$

$$V_{i4}(w_i(t)) = \int_{-\tau}^{0} \dot{w}_i^T(t+\xi)R_3 \dot{w}_i(t+\xi)d\xi,$$

$$\begin{aligned} V_{i5}(w_i(t)) = {} & h\int_{-h}^{-\frac{(m-1)h}{m}} \int_{t+\theta}^{t} \dot{w}_i^T(\xi)Z\dot{w}_i(\xi)d\xi d\theta \\ & + h\int_{-\frac{(m-1)h}{m}}^{-\frac{(m-2)h}{m}} \int_{t+\theta}^{t} \dot{w}_i^T(\xi)Z\dot{w}_i(\xi)d\xi d\theta \\ & + h\int_{-\frac{(m-2)h}{m}}^{-\frac{(m-3)h}{m}} \int_{t+\theta}^{t} \dot{w}_i^T(\xi)Z\dot{w}_i(\xi)d\xi d\theta + \cdots \\ & + h\int_{-\frac{2h}{m}}^{-\frac{h}{m}} \int_{t+\theta}^{t} \dot{w}_i^T(\xi)Z\dot{w}_i(\xi)d\xi d\theta \\ & + h\int_{-\frac{h}{m}}^{0} \int_{t+\theta}^{t} \dot{w}_i^T(\xi)Z\dot{w}_i(\xi)d\xi d\theta, \end{aligned}$$

The derivative of $V_{i1}(w(t))$ along the solution of the dynamic system (5)

$$\begin{aligned} \dot{V}_{i1}(w(t)) &= 2w_i^T(t)P\dot{w}_i(t) \\ &= 2w_i^T(t)P(J(t)w_i(t) + c\lambda_i Aw_i(t-h) + c\lambda_i B\dot{w}_i(t-\tau)), \end{aligned} \tag{8}$$

$$\dot{V}_{i2}(w_i(t)) = w_i^T(t)R_1 w_i(t) - w_i^T(t-h)R_1 w_i(t-h) \tag{9}$$

$$\dot{V}_{i3}(w_i(t)) = w_i^T(t)R_2 w_i(t) - w_i^T(t-\tau)R_2 w_i(t-\tau), \tag{10}$$

$$
\begin{aligned}
\dot{V}_{i4}(w_i(t)) &= \dot{w}_i^T(t)R_3\dot{w}_i(t) - \dot{w}_i^T(t-\tau)R_3\dot{w}_i(t-\tau) \\
&= [J(t)w_i(t) + c\lambda_i A w_i(t-h) + c\lambda_i B\dot{w}_i(t-\tau)]^T \\
&\qquad R_3[J(t)w_i(t) + c\lambda_i A w_i(t-h) + c\lambda_i B\dot{w}_i(t-\tau)] \\
&\quad - \dot{w}_i^T(t-\tau)R_3\dot{w}_i(t-\tau) \\
&= w_i^T(t)J^T(t)R_3J(t)w_i(t) \\
&\quad + (c\lambda_i)^2 w_i^T(t-h)A^TR_3Aw_i(t-h) \\
&\quad + \dot{w}_i^T(t-\tau)\Big((c\lambda_i)^2B^TR_3B - R_3\Big)\dot{w}_i(t-\tau) \\
&\quad + 2c\lambda_i w_i^T(t)J^T(t)R_3Aw_i(t-h) \\
&\quad + 2c\lambda_i w_i^T(t)J^T(t)R_3B\dot{w}_i(t-\tau) \\
&\quad + 2(c\lambda_i)^2 w_i^T(t-h)A^TR_3B\dot{w}_i(t-\tau),
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\dot{V}_{i5}(w_i(t)) &= h\dot{w}_i^T(t)Z\dot{w}_i(t) - h\int_{-h_M}^{-\frac{(m-1)h}{m}} \dot{w}_i^T(t+\xi)Z\dot{w}_i(t+\xi)\mathrm{d}\xi \\
&\quad - h\int_{-\frac{(m-1)h}{m}}^{-\frac{(m-2)h}{m}} \dot{w}_i^T(t+\xi)Z\dot{w}_i(t+\xi)\mathrm{d}\xi \\
&\quad - h\int_{-\frac{(m-2)h}{m}}^{-\frac{(m-3)h}{m}} \dot{w}_i^T(t+\xi)Z\dot{w}_i(t+\xi)\mathrm{d}\xi - \cdots \\
&\quad - h\int_{-\frac{2h}{m}}^{-\frac{h}{m}} \dot{w}_i^T(t+\xi)Z\dot{w}_i(t+\xi)\mathrm{d}\xi \\
&\quad - h\int_{-\frac{h}{m}}^{0} \dot{w}_i^T(t+\xi)Z\dot{w}_i(t+\xi)\mathrm{d}\xi,
\end{aligned}
$$

According to Lemma 2, we immediately get

$$
\begin{aligned}
&-h\int_{-\frac{(i+1)h}{m}}^{-\frac{ih}{m}} \dot{w}_i^T(t+\xi)Z\dot{w}_i(t+\xi)\mathrm{d}\xi \le -m\left(\int_{-\frac{(i+1)h}{m}}^{-\frac{ih}{m}} \dot{w}_i^T(t+\xi)\mathrm{d}\xi\right)Z\left(\int_{-\frac{(i+1)h}{m}}^{-\frac{ih}{m}} \dot{w}_i(t+\xi)\mathrm{d}\xi\right) \\
&\le -m\Big[w_i\big(t-\tfrac{ih}{m}\big) - w_i\Big(t-\tfrac{(i+1)h}{m}\Big)\Big]^T Z\Big[w_i\big(t-\tfrac{ih}{m}\big) - w_i\Big(t-\tfrac{(i+1)h}{m}\Big)\Big],
\end{aligned}
$$

then

$$
\begin{aligned}
\dot{V}_{i5}(w_i(t)) \leq{} & h^2\dot{w}_i^T(t)Z\dot{w}_i(t) - m\left[w_i\left(t-\frac{(m-1)h}{m}\right)-w_i(t-h)\right]^T \\
& \times \quad Z\left[w_i\left(t-\frac{(m-1)h}{m}\right)-w_i(t-h)\right] \\
& - m\left[w_i\left(t-\frac{(m-2)h}{m}\right)-w_i\left(t-\frac{(m-1)h}{m}\right)\right]^T \\
& \times \left[w_i\left(t-\frac{(m-2)h}{m}\right)-w_i\left(t-\frac{(m-1)h}{m}\right)\right] \\
& - m\left[w_i\left(t-\frac{(m-3)h}{m}\right)-w_i\left(t-\frac{(m-2)h}{m}\right)\right]^T \\
& \times \left[w_i\left(t-\frac{(m-3)h}{m}\right)-w_i\left(t-\frac{(m-2)h}{m}\right)\right] \\
& - \cdots - m\left[w_i\left(t-\frac{h}{m}\right)-w_i\left(t-\frac{2h}{m}\right)\right]^T \\
& \times \left[w_i\left(t-\frac{h}{m}\right)-w_i\left(t-\frac{2h}{m}\right)\right] \\
& - m\left[w_i(t)-w_i\left(t-\frac{h}{m}\right)\right]^T \\
& \left[w_i(t)-w_i\left(t-\frac{h}{m}\right)\right] \\
\leq{} & h^2\dot{w}_i^T(t)Z\dot{w}_i(t) - mw_i^T(t-h)Zw_i(t-h) \\
& - 2mw_i^T\left(t-\frac{(m-1)h}{m}\right)Zw_i\left(t-\frac{(m-1)h}{m}\right) \\
& - 2mw_i^T\left(t-\frac{(m-2)h}{m}\right)Zw_i\left(t-\frac{(m-2)h}{m}\right) \\
& - 2mw_i^T\left(t-\frac{(m-3)h}{m}\right)Zw_i\left(t-\frac{(m-3)h}{m}\right) - \cdots \\
& - 2mw_i^T\left(t-\frac{2h}{m}\right)Zw_i\left(t-\frac{2h}{m}\right) \\
& - 2mw_i^T\left(t-\frac{h}{m}\right)Zw_i\left(t-\frac{h}{m}\right) - mw_i^T(t)Zw_i(t) \\
& + 2mw_i^T(t-h)Zw_i\left(t-\frac{(m-1)h}{m}\right) \\
& + 2mw_i^T\left(t-\frac{(m-1)h}{m}\right)Zw_i\left(t-\frac{(m-2)h}{m}\right) \\
& + 2mw_i^T\left(t-\frac{(m-2)h}{m}\right)Zw_i\left(t-\frac{(m-3)h}{m}\right) + \cdots \\
& + 2mw_i^T\left(t-\frac{h}{m}\right)Zw_i\left(t-\frac{2h}{m}\right) \\
& + 2mw_i^T(t)Zw_i\left(t-\frac{h}{m}\right)
\end{aligned} \tag{12}
$$

For any real matrices $O_i,\ i = 1,2,3,4$ with compatible dimensions

$$\begin{aligned} &2\begin{bmatrix} w_i^T(t) & \dot{w}_i^T(t) & w_i^T(t-h) & \dot{w}_i^T(t-\tau) \end{bmatrix} \begin{bmatrix} O_1 \\ O_2 \\ O_3 \\ O_4 \end{bmatrix} \\ &(-\dot{w}_i(t) + J(t)w_i(t) + c\lambda_i(Aw_i(t-h) + B\dot{w}_i(t-\tau))) \\ &\quad = -2y^T(t)O_1J(t)\dot{y}(t) \\ &\qquad + 2w_i^T(t)\left(J^T(t)O_2^T - O_1\right)\dot{w}_i(t) - 2\dot{w}_i^T(t)O_2\dot{w}_i(t) \\ &\qquad + 2w_i^T(t)\left(c\lambda_iO_1A + J^T(t)O_3^T\right)w_i(t-h) \\ &\qquad + 2w_i^T(t)\left(c\lambda_iO_1B + J^T(t)O_4^T\right)\dot{w}_i(t-\tau) \\ &\qquad + 2\dot{w}_i^T(t)\left(c\lambda_iO_2A - O_3^T\right)w_i(t-h) \\ &\qquad + 2\dot{w}_i^T(t)\left(c\lambda_iO_2B - O_4^T\right)\dot{w}_i(t-\tau) \\ &\qquad + 2c\lambda_iw_i^T(t-h)O_3Aw_i(t-h) \\ &\qquad + 2c\lambda_iw_i^T(t-h)\left(O_3B + A^TO_4^T\right)\dot{w}_i(t-\tau) \\ &\qquad + 2c\lambda_i\dot{w}_i^T(t-\tau)O_4B\dot{w}_i(t-\tau) \end{aligned} \tag{13}$$

Adding (8)–(13), by Lemma 3, we immediately obtain

$$\dot{V}(y(t)) \le \eta_1^T(t)\Omega_1\eta_1(t),$$

where

$$\begin{aligned} \eta_1^T(t) = \Bigg[&w_i^T(t), w_i^T\left(t - \frac{h}{m}\right), w_i^T\left(t - \frac{2h}{m}\right), w_i^T\left(t - \frac{3h}{m}\right), \ldots, \\ &w_i^T\left(t - \frac{(m-1)h}{m}\right), w_i^T(t-h), w_i^T(t-\tau), \dot{w}_i^T(t), \dot{w}_i^T(t-\tau)\Bigg]. \end{aligned}$$

If Ω_1 holds, then $\dot{V}_i(w_i(t)) \le 0$, $V_i(w_i(t)) \le V_i(w_i(0))$, and it implies the global asymptotic stability of the system (5). So by Lemma 4, the synchronized states (2) of network (1) are asymptotically stable. The proof is thus completed.

Theorem 2 The synchronous state $s(t)$ of complex networks with neutral-type coupling (1) is globally exponentially stable and has the exponential synchronization rake α if there exist matrices Q,N_i, $i = 1,2,3,\ldots,m$, O_i, $i = 1,2$, symmetrical positive definite matrices P, positive definite matrices Z, M_i, $i = 1,2,3,\ldots,m$, R_i, $i = 1,2,3$, where m is to be determined, and

$$\Omega_2 \in R^{(2m+3)\times(2m+3)}, \Omega_3 \in R^{(m+!)\times(m+!)}, X_i \in R^{m+1},\ i = 1,2,\ldots,m+2;$$

such that

$$\Omega_2 = \begin{bmatrix} \Omega_3 & X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & \cdots & X_{m+2} \\ X_1^T & -e^{-2\beta\tau}R_2 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ X_2^T & 0 & -2O_2 & c\lambda_i O_2 B & 0 & 0 & 0 & 0 & \cdots & 0 \\ X_3^T & 0 & 0 & \begin{matrix} (c\lambda_i)^2 B^T R_3 B - e^{-2\beta\tau} R_3 \\ + (c\lambda_i)^2 B^T Z B \end{matrix} & 0 & 0 & 0 & 0 & \cdots & 0 \\ X_4^T & 0 & 0 & 0 & -Z & 0 & 0 & 0 & \cdots & 0 \\ X_5^T & 0 & 0 & 0 & 0 & -Z & 0 & 0 & \cdots & 0 \\ X_5^T & 0 & 0 & 0 & 0 & 0 & -Z & 0 & \cdots & 0 \\ X_7^T & 0 & 0 & 0 & 0 & 0 & 0 & -Z & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ X_{m+2}^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -Z \end{bmatrix} < 0, \tag{14}$$

where

$$X_1 = \begin{bmatrix} c\lambda_i PB \\ 0 \\ 0 \\ \vdots \\ 0 \\ c\lambda_i O_2 A \end{bmatrix}, X_2 = \begin{bmatrix} J^T(t) O_2^T - O_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} c\lambda_i PB + c\lambda_i J^T(t) R_3 B + c\lambda_i J^T(t) ZB + c\lambda_i O_1 B \\ 0 \\ 0 \\ \vdots \\ 0 \\ (c\lambda_i)^2 A^T R_3 B + (c\lambda_i)^2 A^T ZB \end{bmatrix},$$

$$X_4 = \begin{bmatrix} \sqrt{h(e^{2\beta h} - 1)}Q \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, X_5 = \begin{bmatrix} \sqrt{h\left(e^{2\beta h} - e^{2\beta\frac{(m-1)h}{m}}\right)}N_1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$X_6 = \begin{bmatrix} \sqrt{h\left(e^{2\beta\frac{(m-1)h}{m}} - e^{2\beta\frac{(m-2)h}{m}}\right)}N_2 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

$$X_7 = \begin{bmatrix} \sqrt{h\left(e^{2\beta\frac{(m-2)h}{m}} - e^{2\beta\frac{(m-3)h}{m}}\right)}N_3 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} X_{m+2} = \begin{bmatrix} \sqrt{h\left(e^{2\beta\frac{h}{m}} - 1\right)}N_m \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix},$$

and

$$\Omega_3 = \left[\begin{array}{cccccc} \begin{array}{c} 2\beta P + 2PJ(t) + R_1 + R_2 \\ +J^T(t)R_3J(t) + hM_m + hQ + 2\sqrt{2\beta} \\ h(N_m - Q) + 2h^2J^T(t)ZJ(t) + 2O_1J(t) \end{array} & \sqrt{2\beta}h(N_{m-1} - N_m) & \sqrt{2\beta}h(N_{m-2} - N_{m-1}) & \cdots & \sqrt{2\beta}h(N_1 - N_2) & \begin{array}{r} c\lambda_iPA + \sqrt{2\beta}h(Q - N_1) \\ +c\lambda_iJ^T(t)R_3A \\ +c\lambda_iJ^T(t)ZA + c\lambda_iO_1A \end{array} \\ 2\sqrt{2\beta}h\left(N_{m-1}^T - N_m^T\right) & he^{-2\beta\frac{h}{m}}(M_{m-1} - M_m) & 0 & \cdots & 0 & 0 \\ 2\sqrt{2\beta}h\left(N_{m-2}^T - N_{m-1}^T\right) & 0 & he^{-2\beta\frac{2h}{m}}(M_{m-2} - M_{m-1}) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2\sqrt{2\beta}h\left(N_1^T - N_2^T\right) & 0 & 0 & \cdots & he^{-2\beta\frac{(m-1)h}{m}}(M_1 - M_2) & 0 \\ c\lambda_iA^TR_3J(t) + \sqrt{2\beta}h\left(Q^T - N_1^T\right) & 0 & 0 & \cdots & 0 & \begin{array}{r} -R_1e^{-2\beta h} \\ +(c\lambda_i)^2A^TR_3A \\ -he^{-2\beta h}(M_1 + Q) \\ +(c\lambda_i)^2A^TZA \end{array} \end{array}\right]$$

Proof Select a Lyapunov–Krasovskii functional as

$$\begin{aligned} V_i(w_i(t)) = {} & V_{i1}(w_i(t)) + V_{i2}(w_i(t)) + V_{i3}(w_i(t)) \\ & + V_{i4}(w_i(t)) + V_{i5}(w_i(t)) + V_{i6}(w_i(t)), \end{aligned} \tag{15}$$

where

$$V_{i1}(w_i(t)) = e^{2\beta t} w_i^T(t) P w_i(t),$$
$$V_{i2}(w_i(t)) = \int_{-\tau}^{0} e^{2\beta(t+\xi)} w_i^T(t+\xi) R_1 w_i(t+\xi) \mathrm{d}\xi,$$
$$V_{i3}(w_i(t)) = \int_{-h}^{0} e^{2\beta(t+\xi)} w_i^T(t+\xi) R_2 w_i(t+\xi) \mathrm{d}\xi,$$
$$V_{i4}(w_i(t)) = \int_{-\tau}^{0} e^{2\beta(t+\xi)} \dot{w}_i^T(t+\xi) R_3 \dot{w}_i(t+\xi) \mathrm{d}\xi,$$

$$\begin{aligned}
V_{i5}(w_i(t)) = {} & h \int_{-h}^{-\frac{(m-1)h}{m}} e^{2\beta(t+\xi)} w_i^T(t+\xi) M_1 w_i(t+\xi) \mathrm{d}\xi \\
& + h \int_{-\frac{(m-1)h}{m}}^{-\frac{(m-2)h}{m}} e^{2\beta(t+\xi)} w_i^T(t+\xi) M_2 w_i(t+\xi) \mathrm{d}\xi \\
& + h \int_{-\frac{(m-2)h}{m}}^{-\frac{(m-3)h}{m}} e^{2\beta(t+\xi)} w_i^T(t+\xi) M_3 w_i(t+\xi) \mathrm{d}\xi \\
& + \cdots + h \int_{-\frac{2h}{m}}^{-\frac{h}{m}} e^{2\beta(t+\xi)} w_i^T(t+\xi) M_{m-1} w_i(t+\xi) \mathrm{d}\xi \\
& + h \int_{-\frac{h}{m}}^{0} e^{2\beta(t+\xi)} w_i^T(t+\xi) M_m w_i(t+\xi) \mathrm{d}\xi \\
& + h \int_{-h}^{0} e^{2\beta(t+\xi)} w_i^T(t+\xi) Q w_i(t+\xi) \mathrm{d}\xi,
\end{aligned}$$

$$\begin{aligned}
V_{i6}(w_i(t)) = {} & h \int_{-h}^{-\frac{(m-1)h}{m}} \int_{t+\theta}^{t} e^{2\beta\xi} \dot{w}_i^T(\xi) Z \dot{w}_i(\xi) \mathrm{d}\xi \mathrm{d}\theta \\
& + h \int_{-\frac{(m-1)h}{m}}^{-\frac{(m-2)h}{m}} \int_{t+\theta}^{t} e^{2\beta\xi} \dot{w}_i^T(\xi) Z \dot{w}_i(\xi) \mathrm{d}\xi \mathrm{d}\theta \\
& + h \int_{-\frac{(m-2)h}{m}}^{-\frac{(m-3)h}{m}} \int_{t+\theta}^{t} e^{2\beta\xi} \dot{w}_i^T(\xi) Z \dot{w}_i(\xi) \mathrm{d}\xi \mathrm{d}\theta + \cdots \\
& + h \int_{-\frac{2h}{m}}^{-\frac{h}{m}} \int_{t+\theta}^{t} e^{2\beta\xi} \dot{w}_i^T(\xi) Z \dot{w}_i(\xi) \mathrm{d}\xi \mathrm{d}\theta \\
& + h \int_{-\frac{h}{m}}^{0} \int_{t+\theta}^{t} e^{2\beta\xi} \dot{w}_i^T(\xi) Z \dot{w}_i(\xi) \mathrm{d}\xi \mathrm{d}\theta,
\end{aligned}$$

the time derivative of Lyapunov–Krasovskii functional along the trajectories of system (5)

$$\begin{aligned}\dot{V}_{i1}(w_i(t)) &= 2\beta e^{2\beta t} w_i(t)^T P w_i(t) + 2e^{2\beta t} w_i(t)^T P(J(t)w_i(t) \\ &\quad + c\lambda_i A w_i(t-h) + c\lambda_i B\dot{w}_i(t-\tau)) \\ &= e^{2\beta t} w_i(t)^T (2\beta P + 2PJ(t))w_i(t) + 2c\lambda_i e^{2\beta t} w_i(t)^T PA w_i(t-h) \\ &\quad + 2c\lambda_i e^{2\beta t} w_i(t)^T PB\dot{w}_i(t-\tau)\end{aligned} \tag{16}$$

$$\dot{V}_{i2}(w_i(t)) = e^{2\beta t} w_i^T(t) R_1 w_i(t) - e^{2\beta(t-h)} w_i^T(t-h) R_1 w_i(t-h) \tag{17}$$

$$\dot{V}_{i3}(w_i(t)) = e^{2\beta t} w_i^T(t) R_2 w_i(t) - e^{2\beta(t-\tau)} w_i^T(t-\tau) R_2 w_i(t-\tau) \tag{18}$$

$$\begin{aligned}\dot{V}_{i4}(w_i(t)) &= e^{2\beta t} \dot{w}_i^T(t) R_3 \dot{w}_i(t) - e^{2\beta(t-\tau)} \dot{w}_i^T(t-\tau) R_3 \dot{w}_i(t-\tau) \\ &= e^{2\beta t}[J(t)w_i(t) + c\lambda_i A w_i(t-h) + c\lambda_i B\dot{w}_i(t-\tau)]^T \\ &\qquad R_3[J(t)w_i(t) + c\lambda_i A w_i(t-h) + c\lambda_i B\dot{w}_i(t-\tau)] \\ &\quad - e^{2\beta(t-\tau)} \dot{w}_i^T(t-\tau) R_3 \dot{w}_i(t-\tau) \\ &= e^{2\beta t} w_i^T(t) J^T(t) R_3 J(t) w_i(t) \\ &\quad + e^{2\beta t}(c\lambda_i)^2 w_i^T(t-h) A^T R_3 A w_i(t-h) \\ &\quad + \dot{w}_i^T(t-\tau)\left((c\lambda_i)^2 e^{2\beta t} B^T R_3 B - e^{2\beta(t-\tau)} R_3\right) \\ &\qquad \dot{w}_i(t-\tau) + 2c\lambda_i e^{2\beta t} w_i^T(t) J^T(t) R_3 A w_i(t-h) \\ &\quad + 2c\lambda_i e^{2\beta t} w_i^T(t) J^T(t) R_3 B\dot{w}_i(t-\tau) \\ &\quad + 2(c\lambda_i)^2 e^{2\beta t} w_i^T(t-h) A^T R_3 B\dot{w}_i(t-\tau)\end{aligned} \tag{19}$$

$$\begin{aligned}\dot{V}_{i5}(w_i(t)) = h\Bigg[& e^{2\beta\left(t-\frac{(m-1)h}{m}\right)} w_i^T\left(t-\frac{(m-1)h}{m}\right)(M_1 - M_2) w_i\left(t-\frac{(m-1)h}{m}\right) \\ &+ e^{2\beta\left(t-\frac{(m-2)h}{m}\right)} w_i^T\left(t-\frac{(m-2)h}{m}\right)(M_2 - M_3) w_i\left(t-\frac{(m-2)h}{m}\right) \\ &+ \cdots + e^{2\beta\left(t-\frac{h}{m}\right)} w_i^T\left(t-\frac{h}{m}\right)(M_{m-1} - M_m) w_i\left(t-\frac{h}{m}\right) \\ &+ e^{2\beta t} w_i^T(t)(M_m + Q) w_i(t) \\ &- e^{2\beta(t-h)} w_i^T(t-h)(M_1 + Q) w_i(t-h)\Bigg],\end{aligned} \tag{20}$$

The popular way of introducing free-weighting matrices is to denote the relationship between the items in the Leibniz–Newton formula. Here, we introduced $Q, N_i,\ i = 1, 2, 3, \ldots, m$, to less comparatively conservativeness condition.

$$
\begin{aligned}
\dot{V}_{i6}(w_i(t)) = {} & \dot{V}_{i6}(w_i(t)) \\
& + 2h\sqrt{2\beta}e^{2\beta t}w_i^T(t)N_1\left[w_i\left(t-\frac{(m-1)h}{m}\right)\right. \\
& \left. -w_i(t-h) - \int_{-h_M}^{-\frac{(m-1)h}{m}} \dot{w}_i^T(t+\xi)\mathrm{d}\xi\right] \\
& + 2h\sqrt{2\beta}e^{2\beta t}w_i^T(t)N_2\left[w_i\left(t-\frac{(m-2)h}{m}\right)\right. \\
& \left. -w_i\left(t-\frac{(m-1)h}{m}\right) - \int_{-\frac{(m-1)h}{m}}^{-\frac{(m-2)h}{m}} \dot{w}_i^T(t+\xi)\mathrm{d}\xi\right] \\
& + 2h\sqrt{2\beta}e^{2\beta t}w_i^T(t)N_3\left[w_i\left(t-\frac{(m-3)h}{m}\right)\right. \\
& \left. -w_i\left(t-\frac{(m-2)h}{m}\right) - \int_{-\frac{(m-2)h}{m}}^{-\frac{(m-3)h}{m}} \dot{w}_i^T(t+\xi)\mathrm{d}\xi\right] \\
& + \cdots + 2h\sqrt{2\beta}e^{2\beta t}w_i^T(t)N_{m-1}\left[w_i\left(t-\frac{h}{m}\right)\right. \\
& \left. -w_i\left(t-\frac{2h}{m}\right) - \int_{-\frac{2h}{m}}^{-\frac{h}{m}} \dot{w}_i^T(t+\xi)\mathrm{d}\xi\right] \\
& + 2h\sqrt{2\beta}e^{2\beta t}w_i^T(t)N_m[w_i(t) \\
& \left. -w_i\left(t-\frac{h}{m}\right) - \int_{-\frac{h}{m}}^{0} \dot{w}_i^T(t+\xi)\mathrm{d}\xi\right] \\
& - 2h\sqrt{2\beta}e^{2\beta t}w_i^T(t)Q[w_i(t) \\
& \left. -w_i(t-h) - \int_{-h}^{0} \dot{w}_i^T(t+\xi)\mathrm{d}\xi\right],
\end{aligned}
$$

then

$$
\begin{aligned}
\dot{V}_{i6}\big(w_i^T(t)\big) \le{} & he^{2\beta t} w_i^T(t)\Big[\Big(e^{2\beta h} - e^{2\beta\frac{(m-1)h}{m}}\Big)N_1^T Z^{-1} N_1 \\
& + \Big(e^{2\beta\frac{(m-1)h}{m}} - e^{2\beta\frac{(m-2)h}{m}}\Big)N_2^T Z^{-1} N_2 \\
& + \Big(e^{2\beta\frac{(m-2)h}{m}} - e^{2\beta\frac{(m-3)h}{m}}\Big)N_3^T Z^{-1} N_3 + \cdots \\
& + \Big(e^{2\beta\frac{2h}{m}} - e^{2\beta\frac{h}{m}}\Big)N_{m-1}^T Z^{-1} N_{m-1} \\
& + \Big(e^{2\beta\frac{h}{m}} - 1\Big)N_m^T Z^{-1} N_m \\
& + \big(e^{2\beta h_M} - 1\big)N^T Z^{-1} N\Big] w_i(t) \\
& + 2h^2 e^{2\beta t}[J(t)w_i(t) + c\lambda_i A w_i(t-h) + c\lambda_i B \dot{w}_i(t-\tau)]^T \\
& \quad Z[J(t)w_i(t) + c\lambda_i A w_i(t-h) + c\lambda_i B \dot{w}_i(t-\tau)] \\
& + 2\sqrt{2\beta} h e^{2\beta t} w_i^T(t)(N_1 - N_2) w_i\left(t - \frac{(m-1)h}{m}\right) \\
& + 2\sqrt{2\beta} h e^{2\beta t} w_i^T(t)(N_2 - N_3) w_i\left(t - \frac{(m-2)h}{m}\right) \\
& + 2\sqrt{2\beta} h e^{2\beta t} w_i^T(t)(N_3 - N_4) w_i\left(t - \frac{(m-3)h}{m}\right) \\
& + \cdots + 2\sqrt{2\beta} h e^{2\beta t} w_i^T(t)(N_{m-1} - N_m) w_i\left(t - \frac{h}{m}\right) \\
& + 2\sqrt{2\beta} h e^{2\beta t} w_i^T(t)(N_m - Q) w_i(t) \\
& + 2\sqrt{2\beta} h e^{2\beta t} w_i^T(t)(Q - N_1) w_i(t-h) \\
& - e^{2\beta t} h \int_{-h}^{-\frac{(m-1)h}{m}} \Big[\sqrt{2\beta} e^{-\beta\xi} w_i^T(t) N_1 + e^{\beta\xi} \dot{w}_i^T(t+\xi) Z\Big] \\
& \quad Z^{-1}\Big[\sqrt{2\beta} e^{-\beta\xi} w_i^T(t) N_1 + e^{\beta\xi} \dot{w}_i^T(t+\xi) Z\Big]^T \mathrm{d}\xi \\
& - e^{2\beta t} h \int_{-\frac{(m-1)h}{m}}^{-\frac{(m-2)h}{m}} \Big[\sqrt{2\beta} e^{-\beta\xi} w_i^T(t) N_2 + e^{\beta\xi} \dot{w}_i^T(t+\xi) Z\Big] \\
& \quad Z^{-1}\Big[\sqrt{2\beta} e^{-\beta\xi} w_i^T(t) N_2 + e^{\beta\xi} \dot{w}_i^T(t+\xi) Z\Big]^T \mathrm{d}\xi \\
& - \cdots - e^{2\beta t} h \int_{-\frac{2h}{m}}^{-\frac{h}{m}} \Big[\sqrt{2\beta} e^{-\beta\xi} w_i^T(t) N_{m-1} + e^{\beta\xi} \dot{w}_i^T(t+\xi) Z\Big] \\
& \quad Z^{-1}\Big[\sqrt{2\beta} e^{-\beta\xi} w_i^T(t) N_{m-1} + e^{\beta\xi} \dot{w}_i^T(t+\xi) Z\Big]^T \mathrm{d}\xi \\
& - e^{2\beta t} h \int_{-\frac{h}{m}}^{0} \Big[\sqrt{2\beta} e^{-\beta\xi} w_i^T(t) N_m + e^{\beta\xi} \dot{w}_i^T(t+\xi) Z\Big] \\
& \quad Z^{-1}\Big[\sqrt{2\beta} e^{-\beta\xi} w_i^T(t) N_m + e^{\beta\xi} \dot{w}_i^T(t+\xi) Z\Big]^T \mathrm{d}\xi \\
& - e^{2\beta t} h \int_{-h}^{0} \Big[\sqrt{2\beta} e^{-\beta\xi} w_i^T(t) Q - e^{\beta\xi} \dot{w}_i^T(t+\xi) Z\Big] \\
& \quad Z^{-1}\Big[\sqrt{2\beta} e^{-\beta\xi} w_i^T(t) Q - e^{\beta\xi} \dot{w}_i^T(t+\xi) Z\Big]^T \mathrm{d}\xi,
\end{aligned}
\tag{21}
$$

Since $Z > 0$, the last $m+1$ parts are less than 0. We can omit them here for the LMI's simplicity, although it may bring more conservatism.

$$\begin{aligned}
\dot{V}_{i6}(w_i(t)) \leq{} & he^{2\beta t} w_i^T(t)\Big[\Big(e^{2\beta h} - e^{2\beta\frac{(m-1)h}{m}}\Big)N_1^T Z^{-1} N_1 + \Big(e^{2\beta\frac{(m-1)h}{m}} - e^{2\beta\frac{(m-2)h}{m}}\Big)N_2^T Z^{-1} N_2 \\
& + \Big(e^{2\beta\frac{(m-2)h}{m}} - e^{2\beta\frac{(m-3)h}{m}}\Big)N_3^T Z^{-1} N_3 + \cdots + \Big(e^{2\beta\frac{2h}{m}} - e^{2\beta\frac{h}{m}}\Big)N_{m-1}^T Z^{-1} N_{m-1} \\
& + \Big(e^{2\beta\frac{h}{m}} - 1\Big)N_m^T Z^{-1} N_m + (e^{2\beta h} - 1)Q^T Z^{-1} Q\Big] w_i(t) \\
& + 2\sqrt{2\beta} h e^{2\beta t} w_i^T(t)(N_1 - N_2) w_i\Big(t - \frac{(m-1)h}{m}\Big) \\
& + 2\sqrt{2\beta} h e^{2\beta t} w_i^T(t)(N_2 - N_3) w_i\Big(t - \frac{(m-2)h}{m}\Big) \\
& + 2\sqrt{2\beta} h e^{2\beta t} w_i^T(t)(N_3 - N_4) w_i\Big(t - \frac{(m-3)h}{m}\Big) \\
& + \cdots + 2\sqrt{2\beta} h e^{2\beta t} w_i^T(t)(N_{m-1} - N_m) w_i\Big(t - \frac{h}{m}\Big) \\
& + 2\sqrt{2\beta} h e^{2\beta t} w_i^T(t)(N_m - Q) w_i(t) + 2\sqrt{2\beta} h e^{2\beta t} w_i^T(t)(Q - N_1) w_i(t-h) \\
& + 2h^2 e^{2\beta t}\Big[w_i^T(t) J^T(t) Z J(t) w_i(t) + (c\lambda_i)^2 w_i^T(t-h) A^T Z A w_i(t-h) \\
& + (c\lambda_i)^2 \dot{w}_i^T(t-\tau) B^T Z B \dot{w}_i(t-\tau) + 2c\lambda_i w_i^T(t) J^T(t) Z A w_i(t-h) \\
& + 2c\lambda_i w_i^T(t) J^T(t) Z B \dot{w}_i(t-\tau) + 2(c\lambda_i)^2 w_i^T(t-h) A^T Z B \dot{w}_i(t-\tau)\Big]
\end{aligned} \tag{22}$$

The another popular way of introducing free-weighting matrices is to denote the relationship between the items in the dynamic systems.

Here, we introduced $O_i, i = 1, 2,$

$$\begin{aligned}
& 2 \times e^{2\beta t}\begin{bmatrix} w_i^T(t) & \dot{w}_i^T(t)\end{bmatrix}\begin{bmatrix} O_1 \\ O_2 \end{bmatrix}(-\dot{w}_i(t) + J(t)w_i(t) + c\lambda_i A w_i(t-h) + c\lambda_i B \dot{w}_i(t-\tau)) \\
&\quad = 2 \times e^{2\beta t}\big(w_i^T(t) O_1 + \dot{w}_i^T(t) O_2\big)(-\dot{w}_i(t) + J(t) w_i(t) + c\lambda_i A w_i(t-h) + c\lambda_i B \dot{w}_i(t-\tau)\big) \\
&\quad = 2e^{2\beta t} w_i^T(t) O_1 J(t) w_i(t) + 2e^{2\beta t} w_i^T(t)\big(J^T(t) O_2^T - O_1\big)\dot{w}_i(t) - 2e^{2\beta t}\dot{w}_i^T(t) O_2 \dot{w}_i(t) \\
&\quad\quad + 2c\lambda_i e^{2\beta t} w_i^T(t) O_1 A w_i(t-h) + 2c\lambda_i e^{2\beta t} w_i^T(t) O_1 B \dot{w}_i(t-\tau) \\
&\quad\quad + 2c\lambda_i e^{2\beta t}\dot{w}_i^T(t) O_2 A w_i(t-h) + 2c\lambda_i e^{2\beta t}\dot{w}_i^T(t) O_2 B \dot{w}_i(t-\tau).
\end{aligned} \tag{23}$$

Adding (16)–(20), (22)–(23), by Lemma 1, we give

$$\dot{V}_i(w(t)) \leq e^{2\beta t}\eta_2^T(t)\Omega_3\eta_2(t),$$

where

$$\begin{aligned}
\eta_2^T(t) = {} & \Big[w_i^T(t), w_i^T\Big(t - \frac{h}{m}\Big), w_i^T\Big(t - \frac{2h}{m}\Big), w_i^T\Big(t - \frac{3h}{m}\Big), \ldots, w_i^T\Big(t - \frac{(m-1)h}{m}\Big), w_i^T(t-h), \\
& w_i^T(t-\tau), \dot{w}_i^T(t), \dot{w}_i^T(t-\tau), w_i^T(t) \times 1, w_i^T(t) \times 1^2, \ldots, w_i^T(t) \times 1^{m+1}\Big].
\end{aligned}$$

However

$$
\begin{aligned}
V_i(w_i(0)) &= V_{i1}(w_i(0)) + V_{i2}(w_i(0)) + V_{i3}(w_i(0)) + V_{i4}(w_i(0)) + V_{i5}(w_i(0)) + V_{i6}(w_i(0)) \\
&= w_i^T(0)Pw_i(0) + \int_{-\tau}^{0} e^{2\beta\xi} w_i^T(\xi)R_1 w_i(\xi)d\xi + \int_{-h}^{0} e^{2\beta\xi} w_i^T(\xi)R_2 w_i(\xi)d\xi \\
&\quad + \int_{-\tau}^{0} e^{2\beta\xi} \dot{w}_i^T(\xi)R_3 \dot{w}_i(\xi)d\xi + h\int_{-h}^{-\frac{(m-1)h}{m}} e^{2\beta\xi} w_i^T(\xi)M_1 w_i(\xi)d\xi \\
&\quad + h\int_{-\frac{(m-1)h}{m}}^{-\frac{(m-2)h}{m}} e^{2\beta\xi} w_i^T(\xi)M_2 w_i(\xi)d\xi \\
&\quad + h\int_{-\frac{(m-2)h}{m}}^{-\frac{(m-3)h}{m}} e^{2\beta\xi} w_i^T(\xi)M_3 w_i(\xi)d\xi + \cdots + h\int_{-\frac{2h}{m}}^{-\frac{h}{m}} e^{2\beta\xi} w_i^T(\xi)M_{m-1} w_i(\xi)d\xi \\
&\quad + h\int_{-\frac{h}{m}}^{0} e^{2\beta\xi} w_i^T(\xi)M_m w_i(\xi)d\xi + h\int_{-h}^{0} e^{2\beta\xi} w_i^T(\xi)Q w_i(\xi)d\xi \\
&\quad + h\int_{-h}^{-\frac{(m-1)h}{m}}\int_{\theta}^{0} e^{2\beta\xi} \dot{w}_i^T(\xi)Z\dot{w}_i(\xi)d\xi d\theta + h\int_{-\frac{(m-1)h}{m}}^{-\frac{(m-2)h}{m}}\int_{\theta}^{0} e^{2\beta\xi} \dot{w}_i^T(\xi)Z\dot{w}_i(\xi)d\xi d\theta \\
&\quad + h\int_{-\frac{(m-2)h}{m}}^{-\frac{(m-3)h}{m}}\int_{\theta}^{0} e^{2\beta\xi} \dot{w}_i^T(\xi)Z\dot{w}_i(\xi)d\xi d\theta + \cdots + h\int_{-\frac{2h}{m}}^{-\frac{h}{m}}\int_{\theta}^{0} e^{2\beta\xi} \dot{w}_i^T(\xi)Z\dot{w}_i(\xi)d\xi d\theta \\
&\quad + h\int_{-\frac{h}{m}}^{0}\int_{\theta}^{0} e^{2\beta\xi} \dot{w}_i^T(\xi)Z\dot{w}_i(\xi)d\xi d\theta \\
&\le \Bigg[\lambda_{\text{Max}}(P) + \lambda_{\text{Max}}(R_1)\int_{-\tau}^{0} e^{2\beta\xi}d\xi + \lambda_{\text{Max}}(R_2)\int_{-h}^{0} e^{2\beta\xi}d\xi + h\lambda_{\text{Max}}(M_{\text{Max}})\int_{-h}^{0} e^{2\beta\xi}d\xi \\
&\quad + h\lambda_{\text{Max}}(Q)\int_{-h}^{0} e^{2\beta\xi}d\xi\Bigg]||\phi||_{\tau}^2 + 2\Bigg[\lambda_{\text{Max}}(R_3)\int_{-\tau}^{0} e^{2\beta\xi}d\xi + \lambda_{\text{Max}}(Z)\int_{-\tau}^{0}\int_{\theta}^{0} e^{2\beta\xi}d\xi d\theta\Bigg]||\phi^*||_{\tau}^2 \\
&= \left[\lambda_{\text{Max}}(P) + \frac{1-e^{-2\beta\tau}}{2\beta}(\lambda_{\text{Max}}(R_1) + \lambda_{\text{Max}}(R_2) + h\lambda_{\text{Max}}(M_{\text{Max}}) + h\lambda_{\text{Max}}(Q))\right]||\phi||_{\tau}^2 \\
&\quad + \left[2\lambda_{\text{Max}}(Z)\frac{2\beta\tau - 1 + e^{-2\beta\tau}}{4\beta^2} + \frac{1-e^{-2\beta\tau}}{2\beta}(\lambda_{\text{Max}}(R_3)\right]||\phi^*||_{\tau}.
\end{aligned}
$$

Since

$$
e^{2\beta t}\lambda_{\text{Min}}(P)||w_i(t)||^2 \le V_i(w_i(t)),
$$

then, we immediately obtain that

$$w_i(t) \le \frac{1}{\sqrt{\lambda_{\text{Min}}(P)}}\left[\lambda_{\text{Max}}(P) + \frac{1-e^{-2\beta\tau}}{2\beta}(\lambda_{\text{Max}}(R_1) + \lambda_{\text{Max}}(R_2) + h\lambda_{\text{Max}}(M_{\text{Max}}) + h\lambda_{\text{Max}}(Q))\right.$$
$$\left. + 2\lambda_{\text{Max}}(Z)\frac{2\beta\tau - 1 + e^{-2\beta\tau}}{4\beta^2}\right]^{\frac{1}{2}} \max\{||\phi||_\tau, ||\phi^*||_\tau\}e^{-\beta t}.$$

And

$$\lim_{t\to\infty} ||x_i(t) - s(t)|| \le \mu e^{-\alpha t} \max\{||\phi||_\tau, ||\phi^*||_\tau\}, \quad i = 1, 2, \ldots, n. \tag{24}$$

where

$$\mu = \frac{1}{\sqrt{\lambda_{Min}(P)}}\left[\lambda_{Max}(P) + \frac{1-e^{-2\beta\tau}}{2\beta}(\lambda_{Max}(R_1) + \lambda_{Max}(R_2) + h\lambda_{Max}(M_{Max})\right.$$
$$\left. + h\lambda_{Max}(Q)) + 2\lambda_{Max}(Z)\frac{2\beta\tau - 1 + e^{-2\beta\tau}}{4\beta^2}\right]^{\frac{1}{2}},$$
$$\alpha = \beta$$

Finally, by Definition 3 and (24), it is obvious that the globally exponential stability of the system (5). So by Lemma 4, the synchronized states of network (1) are asymptotically stable. The proof is thus completed.

Remark 1 Both Theorem 1 and Theorem 2's assumptions are in forms of LMI. The conditions 7 are linear to O_i, $i = 1, 2, 3, 4$ P, Z, R_i, $i = 1, 2, 3$, and conditions 15 are linear to $Q, N_i, i = 1, 2, 3, \ldots, m, O_i, i = 1, 2$, P, Z, M_i, $i = 1, 2, 3, \ldots, m$, R_i, $i = 1, 2, 3$.

When we use the matlab LMI toolbox, we always assume that the m and α is to be a specific value.

Remark 2 Both theorem 1 and theorem 2 use the delay fraction method. In Theorem 2, the number of the free-weighting matrices that denote the relationship between the items in the Leibniz–Newton formula is not any equal to fraction number m, but also is equal to the missed number of negative part

$$-\cdots - e^{2\beta t}h \int_{-\frac{(i+!)h}{m}}^{-\frac{ih}{m}} \left[\sqrt{2\beta}e^{-\beta\xi}w^T(t)N_{m-i} + e^{\beta\xi}\dot{w}^T(t+\xi)Z\right]$$
$$Z^{-1}\left[\sqrt{2\beta}e^{-\beta\xi}w^T(t)N_{m-i} + e^{\beta\xi}\dot{w}^T(t+\xi)Z\right]^T d\xi$$

We are not sure that the bigger value of m will lead less conservative results.

We want to depend on experiments to find the right m so that we can get the better results.

4 A Numerical Example

Consider a three-dimensional stable linear system described by Dai et al. [11]

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -x_1 \\ -2x_2 \\ -3x_3 \end{bmatrix}$$

which is asymptotically stable at $s(t) = 0$, and its Jacobian is

$$J = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Case 1 Assume that the inner-coupling matrices are

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & -0.1 & 0.5 \\ -0.3 & 0.09 & -0.15 \\ 0.3 & 0.1 & 0.2 \end{bmatrix}$$

The outer-coupling matrix is

$$G_1 = \begin{bmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -3 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 1 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Obviously, G_1 is an irreducible symmetrical matrix. The eigenvalues of G_1 are $\lambda_i = 0, -1.382, -2.382, -3.618, -4.618$.

Case 2 Assume that the inner-coupling matrices are

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 & -0.1 & 0.5 \\ -0.3 & 0.09 & -0.15 \\ 0.3 & 0.1 & 0.2 \end{bmatrix}$$

The outer-coupling matrix is

$$G_2 = \begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & -4 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -4 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -4 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -4 \end{bmatrix}$$

Table 1 Simulation result for $c = 0.3$ with the outer-coupling matrix G_1

h	τ	[11]	Theorem 1
0–1	0.1	S	S
1.1	0.1	U	S
0.1	0–∞	S	S
0.15	0.15	S	S
0.15	0.19	S	S
0.15	0.20	U	S
0.16	0.19	U	S

Table 2 Simulation result for $c = 0.2$ with the outer-coupling matrix G_2

h	τ	[11]	Theorem 1
0.2	0–∞	S	S
0.22	0–∞	S	S
0.23	0.7	S	S
0.23	0.8	U	S
0.3	0.3	S	S
0.3	0.31	U	S
0.34	0.3	S	S
0.35	0.3	U	S

Table 3 Simulation result of theorem 1

	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
Case 1 with $c = 0.3$	$h = \tau = 0.3173$	$h = \tau = 0.3173$	$h = \tau = 0.3173$	$h = \tau = 0.3235$	$h = \tau = 0.3235$
Case 2 with $c = 0.2$	$h = \tau = 0.6849$	$h = \tau = 0.6849$	$h = \tau = 0.6849$	$h = \tau = 0.6869$	$h = \tau = 0.6869$

The eigenvalues of G_1 are

$$\lambda_i = 0, -1.7639, -1.7639, -4, -5, -5, -5, -5, -6.2361, -6.2361.$$

The results of Theorems in this letter and those in [11] are listed in Table 1, where "S" means that the criterion is applicable to the corresponding case and "U" means that the criterion is not applicable to the corresponding case.

Obviously, both Tables 1 and 2 illustrated the correctness and efficiency of our results. Furthermore, if we assume that the $h = \tau$, the maximum bound of the delays obtained by Theorem 1 and Theorem 2 are listed as in the Tables 3 and 4, respectively.

Form these two tables, we can see that the $m = 5, 6$ in Theorem 1 and $m = 2$ in Theorem 2 are better choices. It may be concluded that in Theorem 1 which omitted nothing, the bigger value of fraction number leads better results, whereas it isn't work in Theorem 2.

Table 4 Simulation result of theorem 2 with $\alpha = \beta = 0.01$

	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$
Case 1 with $c = 0.3$	$h = \tau = 0.0658$	$h = \tau = 0.0656$	$h = \tau = 0.0656$	$h = \tau = 0.0656$	$h = \tau = 0.0656$
Case 2 with $c = 0.2$	$h = \tau = 0.2338$	$h = \tau = 0.2337$	$h = \tau = 0.2337$	$h = \tau = 0.2337$	$h = \tau = 0.2337$

5 Conclusion

In this chapter, we have investigated the globally asymptotically synchronization and the globally exponentially synchronization of complex networks with neutral-type coupling by combining several techniques such as delay fraction method, the free-weighting matrices approach, Lyapunov–Krasovskii functional, and LMI. Numerical examples are given to show their effectiveness and advantages over others.

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