Dependence and Association Concepts through Copulas

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Abstract. In this paper, dependence concepts such as affiliation, left-tail decreasing, right-tail increasing, positively regression dependent, and positively quadrant dependent are studied in terms of copulas. Relationships among these dependent concepts are obtained. An affiliation is a notion of dependence between two positively dependent random variables and some measures of it are provided. It has been shown that the affiliation property is preserved using bilinear extensions of subcopula. As an application, the affiliation property of skew-normal copula is investigated. For illustration of dependent concepts and their relationships, several examples are given.

1 Introduction

With the rapid development of mathematical finance and risk management in the last two decades, more and more attention has been paid to creating some practical statistical models beyond normal settings to improve competitive performance in finance and insurance fields. The copula is one of the most important models used in mathematical finance. Specifically, copulas, introduced in [25], are used to model multivariate data as they account for the dependence structure and provide a flexible representation of the multivariate distribution. Copulas are multivariate

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distributions with $[0,1]$-uniform marginal, which contain the most multivariate dependence structure properties and do not depend on the marginals. For references, see [10], [7], [20], and [22].

In analysis of auction theory, valuations of different bidders (modeled as random variables) could be affiliated. In similar situations in econometrics, when dependence of random variables is a concern, the theory of affiliated copulas, which will be defined in next section, offers an appropriate approach. Recently, Rinotta and Scarsini studied the total positivity order for multivariate normal distributions in [20]. The importance of the affiliation properties in application of auction theory can be found in [14], [4], [19], [24] and [21].

As an extension of normal settings, multivariate skew normal distributions are widely used in almost all fields for almost three decades. For references on skew normal distributions, see [1], [2], [8], and many other papers listed in the website of Azzalini [3]. The concept of affiliation on the class of multivariate skew normal family has not been investigated in the literature.

This paper is organized as follows. Dependence and association concepts as well as their relationships are obtained in Section 2. Bilinear extension method of a two dimensional subcopula together with their affiliation property is studied in Section 3. Average and local measures of affiliation are provided with several examples in Section 4. Conditions under which the bivariate skew-normal copulas are affiliated are discussed in Section 5.

2 Basic Concepts

Following the notions of [25], we have definition of affiliation.

Definition 1. The random variables X and Y are said to be **affiliated** (or **positively likelihood ratio dependent**(PLRD)) if

$$h(x,y^*)h(x^*,y) \leq h(x,y)h(x^*,y^*) \tag{1}$$

holds for all $x^* \leq x$ and $y^* \leq y$, where $h(\cdot,\cdot)$ is the joint density function of (X,Y).

Recall that a copula C is a function $C(\cdot,\cdot) : [0,1]^2 \to [0,1]$ satisfying
(i) $C(u,0) = C(0,v) = 0$, for $u,v \in [0,1]$,
(ii) $C(u,1) = u, C(1,v) = v$, for $u,v \in [0,1]$, and
(iii) For any $(u,v) \leq (u',v')$, $C(u',v') - C(u,v') - C(u',v) + C(u,v) \geq 0$.

Sklar's theorem states that if H is the joint distribution of (X,Y), then there is a copula C such that $H(x,y) = C(F(x),G(y))$ for $(x,y) \in R^2$. Copula characterizes dependence structures and dependence measures which is also independent of marginal distributions. It can be viewed as a joint distribution of two random variables U and V located on $[0,1]$. Motivated by this, we give the corresponding affiliation definition for a copula as follows.

Definition 2. A copula $C(u,v)$ is said to be **affiliated** if

$$c(u,v^*)c(u^*,v) \leq c(u,v)c(u^*,v^*) \tag{2}$$

holds for all $u^* \le u$ and $v^* \le v$, where $c(\cdot,\cdot)$ is the joint density corresponding to copula $C(u,v)$ with $c(u,v) = \frac{\partial^2 C(u,v)}{\partial u \partial v}$.

Remark. It is true that the random variables X and Y are affiliated if and only if their corresponding copula is affiliated. Indeed, suppose X and Y are affiliated. Let $h(x,y)$ and $c(u,v)$ be the corresponding density function and copula density, respectively. Then $h(x,y) = c(F(x),G(y))f(x)g(y)$, where $F(x)$ and $G(y)$ are cumulative distribution functions (CDF) of X and Y, respectively. Since X and Y are affiliated, by definition,

$$h(x,y^*)h(x^*,y) \le h(x^*,y^*)h(x,y), \quad x^* \le x, y^* \le y,$$

which is equivalent to

$$\begin{aligned} &c(F(x),G(y^*))f(x)g(y^*)c(F(x^*),G(y))f(x^*)g(y) \\ &\quad \le c(F(x^*),G(y^*))f(x^*)g(y^*)c(F(x),G(y))f(x)g(y), \end{aligned}$$

which is reduced to

$$c(F(x),G(y^*))c(F(x^*),G(y)) \le c(F(x^*),G(y^*))c(F(x),G(y)).$$

Since both F and G are distribution functions and therefore non-decreasing, for any $u^* \le u, v^* \le v$, let $x^* = F^{-1}(u^*), x = F^{-1}(u)$ and $y^* = G^{-1}(v^*), y = G^{-1}(v)$, where $F^{-1}(u) = \inf\{x \in \mathbb{R} | F(x) \ge u\}$. Therefore

$$c(u,v^*)c(u^*,v) \le c(u^*,v^*)c(u,v).$$

The converse relation can be proved similarly. □

For the connection between affiliation property and positively quadrant dependence (PQD), let us recall the definition of PQD given below.

Definition 3. A copula $C : [0,1] \times [0,1] \mapsto [0,1]$ is said to be **positively quadrant dependent** if $C(u,v) \ge uv$ holds for all u, v.

The following Lemma of [10] will be used in the proof of our next result.

Lemma 1. *Let $H(x,y)$, $F(x)$, and $G(y)$ be the joint, and marginal CDFs of X and Y, respectively. If X and Y are positively quadrant dependent, then*

$$H(x,y) = F(x)G(y) + w(x,y) \quad x, y \in \mathbb{R} \tag{3}$$

with $w(x,y)$ satisfying the following conditions:
(i) $w(x,y) \ge 0$ for all x and y,
(ii) $w(x,\infty) = w(\infty,y) = w(x,-\infty) = w(-\infty,y) = 0$, for all x and y, and
(iii) $\frac{\partial^2 w(x,y)}{\partial x \partial y} \ge 0$.

Recall that a function $w(u,v)$ is **totally positive of order-2** (TP2) if

$$w(u',v)w(u,v') \leq w(u',v')w(u,v) \qquad \text{for all } u' \leq u, v' \leq v.$$

Also $w(u,v)$ is said to be 2-**increasing** if

$$w(u',v') + w(u,v) - w(u',v) - w(u,v') \geq 0 \qquad \text{for all } u' \leq u, v' \leq v.$$

Using above lemma, it is easy to prove the similar result given below.

Proposition 1. *If a copula $C(u,v)$ can be written as*

$$C(u,v) = uv + W(u,v) \qquad \textit{for all } u \textit{ and } u,$$

where $W(u,v)$ satisfying the following conditions:
(i) $W(u,v) \geq 0$,
(ii) $W(u,1) = W(1,v) = W(u,0) = W(0,v) = 0$,
(iii) $\frac{\partial^2 W(u,v)}{\partial v \partial u} \geq -1$,
(iv) $\frac{\partial^2 W(u,v)}{\partial v \partial u}$ is a function with TP2 property and is 2-increasing, then the copula C is affiliated.

Note that if we let $W(u,v) = C(u,v) - uv$, then by Theorem 1 below, we know that $C(u,v)$ is affiliated implies it is PQD, then by Lemma 1, conditions (i), (ii), and (iii) hold for W, but (iv) does not necessarily hold.

Example 2.1. For the CDF of Farlie-Gumbel-Morgenstern bivariate distribution [8]:

$$F(x,y) = F_X(x)F_Y(y)[1 + \rho(1 - F_X(x))(1 - F_Y(y))], \qquad -1 \leq \rho \leq 1,$$

the corresponding copula is $C(u,v) = uv[1 + \rho(1-u)(1-v)]$, $-1 \leq \rho \leq 1$. By the remark after Definition 2, it is easy to see Farlie-Gumbel-Morgenstern family is affiliated for $0 \leq \rho \leq 1$. □

In order to show that affiliation implies PQD, we need the following definition.

Definition 4. The random variable Y is said to be **positively regression dependent** in X, denoted by $PRD(Y|X)$, if $P(Y \leq y|X = x)$ is non-increasing in x for all y. Y is said to be **left-tail decreasing** in X, denoted by $LTD(Y|X)$, if $P(Y \leq y|X \leq x)$ is non-increasing in x for all y.

Corresponding to copulas, we have the following definition.

Definition 5. The random variable V in C is said to be **positively regression dependent** in U, denoted by $PRD(V|U)$, if $p_u(u,v)$ is non-increasing in u for all v, where $p_u(u,v) = \frac{\partial C(u,v)}{\partial u}$. V in C is said to be **left-tail decreasing** in U, denoted by $LTD(V|U)$, if $\frac{C(u,v)}{u}$ is non-increasing in u for all v.

Proposition 2. *The following result gives the relationship between X and Y and their corresponding U and V.*

(i) The random variables X and Y are positively regression dependent in X if and only if V in the corresponding copula $C(u,v)$ is positively regression dependent in U.

(ii) The random variables X and Y are left-tail decreasing in X if and only if V in the corresponding copula $C(u,v)$ is left-tail decreasing in U.

(iii) The random variables X and Y are PQD if and only if U and V in the corresponding copula $C(u,v)$ are PQD.

Proof. We prove (i) and (ii) only, and the proof of (iii) is trivial. For (i), suppose Y is Positively regression dependent in X. Let $H_{y|x}(y|x)$ be the conditional CDF of Y given $X=x$. By definition, it is non-increasing in x. Also

$$\begin{aligned} H_{y|x}(y|x) &= \int_{-\infty}^{y} h_{y|x}(t|x)dt = \int_{-\infty}^{y} \frac{h(x,t)}{f(x)}dt = \int_{-\infty}^{y} \frac{\partial}{\partial t}\left(\frac{\partial C(F(x),G(t))}{\partial x}\right)\frac{1}{f(x)}dt \\ &= \int_{-\infty}^{y} \frac{\partial}{\partial t} p_u(F(x),G(t))dt = p_u(F(x),F(y)) - p_u(F(x),0) \\ &= p_u(F(x),G(y)). \end{aligned}$$

Since both F,G are distribution functions and therefore non-decreasing, so that, $H_{y|x}(y|x)$ is non-increasing in x if and only if $p_u(u,v)$ is non-increasing in u.

(ii) Let Y is Left-tail decreasing in X, by definition, $P(Y \le y|X \le x) = \frac{H(x,y)}{F(x)}$ is non-increasing in x. Since both F,G are distribution functions and therefore non-decreasing, so that, $LTD(Y|X)$ if and only if $LTD(V|U)$. $\square$

Theorem 1. *Let $C:[0,1]\times[0,1]\mapsto[0,1]$ be a copula, then the following implications are true.*

$$\text{Affiliation} \quad \Rightarrow \quad PRD(V|U) \quad \Rightarrow \quad LTD(V|U) \quad \Rightarrow \quad PQD.$$

Proof. Suppose that U and V are affiliated.

To show it implies $PRD(V|U)$, for any $u^* < u, v^* < v$, we have, by definition,

$$c(u,v^*)c(u^*,v) \le c(u,v)c(u^*,v^*) \quad \Rightarrow \quad \frac{C(v|u)}{c(u|v)} \le \frac{C(v|u^*)}{c(u|v^*)}.$$

Let $G(v|u) = \frac{c(v|u)}{C(v|u)}$, then we have $G(v|u) \ge G(v|u^*)$ for all $u^* < u, v^* < v$. Note that $G(u|v) = \frac{\partial \ln(C(v|u))}{\partial v}$. We obtain

$$1 - \ln(C(v|u)) = \int_{v}^{1} G(t|u)dt \ge \int_{v}^{1} G(t|u^*)dt = 1 - \ln(C(v|u^*)).$$

Thus, $C(v|u^*) \ge C(v|u)$ for $u^* < u$, which implies $PRD(V|U)$.

For $PRD(V|U) \Rightarrow LTD(V|U)$, we need the fact that for any interval $I \subseteq [0,1]$,

$$P(V > v|U \in I) = \frac{\int_I P(V > v|U = u)dP(U \le u)}{P(U \in I)}.$$

$LTD(V|U)$ is equivalent to the $Pr(V > v|U \le u)$ is non-decreasing in u for all v, which, in turn, is equivalent to

$$P(V > v|U \le u) \ge P(V > v|U \le u^*)$$

for $u^* < u$ and all v. This is also equivalent to $P(V > v|u^* < U \le u) \ge P(V > v|U \le u^*)$, for all $u > u^*$. Note that,

$$\begin{aligned} P(V > v|u^* < U \le u) &= \frac{\int_{u^*}^{u} P(V > v|U = u)dP(U \le u)}{P(u^* < U \le u)} \\ &\ge \frac{P(V > v|U = u^*) \int_{u^*}^{u} dP(U \le u)}{P(u^* < U \le u)} = P(V > v|U = u^*) \\ &\ge \frac{\int_{-\infty}^{u^*} P(V > v|U = u)dPr(U \le u)}{P(-\infty < U \le u^*)} = P(V > v|U \le u^*), \end{aligned}$$

which implies $LTD(V|U)$.

For $LTD(V|U) \Rightarrow PQD$, note that

$$P(V \le v|U \le u) \ge P(V \le v|U \le 1) = P(V \le v) = v, \tag{4}$$

which is equivalent to $C(u,v) \ge uv$.

Note that the FGM-copula has properties PRD, LTD, PQD for $0 \le \rho \le 1$. Several counterexamples of the converse relations of Theorem 2.1 can be found in [25] and [15].

3 Invariance of Affiliation of Subcopula through Bilinear Interpolation

In this section, we are going to discuss the affiliation property of copula, which is obtained from a subcopula through the method of bilinear interpolation.

Definition 6. A *two-dimensional* **subcopula** is a function C' with the following properties:

(a) Domain of C' is $S_1 \times S_2$, where S_1 and S_2 are subsets of $[0,1]$ containing 0 and 1,

(b) C' is 2-increasing and $C'(u,0) = C'(0,v) = 0$,

(c) For every u in S_1 and every v in S_2, and

$$C'(u,1) = u \quad \text{and} \quad C'(1,v) = v.$$

Also, any sub-copula can be extended to a copula, but the extension is generally non-unique. Here we introduce one popular method called **bilinear interpolation** [20]:

Definition 7. Let C' be a sub-copula with domain $S_1 \times S_2$, now for any $(a,b) \in [0,1]^2$, let a_1 and a_2 be, respectively, the greatest and least elements of $\overline{S_1}$ that satisfy $a_1 \leq a \leq a_2$; and let b_1 and b_2 be, respectively, the greatest and least elements of $\overline{S_2}$ that satisfy $b_1 \leq b \leq b_2$, where $\overline{S}$ is the closure of set S . Note that if a is in $\overline{S_1}$, then $a_1 = a = a_2$; and if b is in $\overline{S_2}$, then $\overline{b_1} = b = \overline{b_2}$. Now let

$$\lambda = \begin{cases} \frac{a-a_1}{a_2-a_1} & \text{if } a_1 < a_2 \\ 1 & \text{if } a_1 = a_2 \end{cases}$$

and

$$\mu = \begin{cases} \frac{b-b_1}{b_2-b_1} & \text{if } b_1 < b_2 \\ 1 & \text{if } b_1 = b_2. \end{cases}$$

The copula C given by

$$C(a,b) = (1-\lambda)(1-\mu)C'(a_1,b_1) + (1-\lambda)\mu C'(a_1,b_2) + \lambda(1-\mu)C'(a_2,b_1) \\ +\lambda\mu C'(a_2,b_2),$$

is a well defined copula.

The following result shows that the invariance between a subcopula and its bilinear interpolation of affiliation property.

Theorem 2. *Let C' be a sub-copula over $S_1 \times S_2$, and $C : [0,1]^2 \to [0,1]$ be the copula, which is constructed by bilinear interpolation from C'.*

(i) If C' is affiliated, then C is also affiliated. Furthermore, if C' is not affiliated, then C is also not affiliated.

(ii) If C' is PQD, then C is also PQD. Furthermore, if C' is not PQD, then C is also not PQD.

Proof. Let $a < c, b < d$. Suppose $a_1,a_2,b_1,b_2,c_1,c_2,d_1,d_2$ and $\lambda_1,\mu_1,\lambda_2$ and μ_2 are defined according to the method of bilinear interpolation.

Then

$$C(a,b) = \frac{a_2-a}{a_2-a_1}\frac{b_2-b}{b_2-b_1}C'(a_1,b_1) + \frac{a_2-a}{a_2-a_1}\frac{b-b_2}{b_2-b_1}C'(a_1,b_2) + \frac{a-a_2}{a_2-a_1}\frac{b_2-b}{b_2-b_1}C'(a_2,b_1) \\ + \frac{a-a_2}{a_2-a_1}\frac{b-b_2}{b_2-b_1}C'(a_1,b_1).$$

Then

$$c(a,b) = \frac{\partial^2 C(a,b)}{\partial a \partial b} = \frac{C'(a_1,b_1) - C'(a_1,b_2) - C'(a_2,b_1) + C'(a_2,b_2)}{(a_2-a_1)(b_2-b_1)}.$$

To show (i), we shall consider the following cases,

Case 1. suppose $a_1 = c_1$, $a_2 = c_2$, $b_1 = d_1$ and $b_2 = d_2$, in this case, $c(a,b) = c(c,d) = c(a,d) = c(c,b)$. Thus $c(a,b)c(c,d) \geq c(a,d)c(c,b)$ holds.

Case 2. suppose $a_1 < c_1$, $a_2 < c_2$, $b_1 = d_1$ and $b_2 = d_2$. then we have $c(a,b) = c(c,b)$ and $c(a,d) = c(c,d)$. Thus $c(a,b)c(c,d) \geq c(a,d)c(c,b)$ holds.

Case 3. suppose $a_1 = c_1$, $a_2 = c_2$, $b_1 < d_1$ and $b_2 < d_2$, the proof follows from Case 1 and Case 2.

Case 4. suppose $a_1 < c_1$, $a_2 < c_2$, $b_1 < d_1$ and $b_2 < d_2$, then,

$$\begin{aligned}
c(a,b)c(c,d) &= \frac{C'(a_1,b_1) - C'(a_1,b_2) - C'(a_2,b_1) + C'(a_2,b_2)}{(a_2-a_1)(b_2-b_1)} \\
&\quad \times \frac{C'(c_1,d_1) - C'(c_1,d_2) - C'(c_2,d_1) + C'(c_2,d_2)}{(c_2-c_1)(d_2-d_1)} \\
&= \frac{c'(a_2,b_2)}{(a_2-a_1)(b_2-b_1)} \frac{c'(c_2,d_2)}{(c_2-c_1)(d_2-d_1)} \\
&\geq \frac{c'(a_2,d_2)}{(a_2-a_1)(b_2-b_1)} \frac{c'(c_2,b_2)}{(c_2-c_1)(d_2-d_1)} \\
&= \frac{C'(a_1,d_1) - C'(a_1,d_2) - C'(a_2,d_1) + C'(a_2,d_2)}{(a_2-a_1)(b_2-b_1)} \\
&\quad \times \frac{C'(c_1,b_1) - C'(c_1,b_2) - C'(c_2,b_1) + C'(c_2,b_2)}{(c_2-c_1)(d_2-d_1)} = c(a,d)c(c,b),
\end{aligned}$$

Note that the inequality above holds because affiliation property of $c'(u,v)$. Therefore, $c(a,b)c(c,d) \geq c(a,d)c(c,b)$ holds. This completes the proof of (i).

For (ii), assume that C' is PQD, and for any $a,b \in [0,1]$,

$$\begin{aligned}
C(a,b) = &\ (1-\lambda)(1-\mu)C'(a_1,b_1) + (1-\lambda)\mu C'(a_1,b_2) + \lambda(1-\mu)C'(a_2,b_1) + \lambda\mu C'(a_2,b_2) \\
&\geq (1-\lambda)(1-\mu)a_1b_1 + (1-\lambda)\mu a_1 b_2 + \lambda(1-\mu)a_2b_1 + \lambda\mu a_2 b_2 \\
&= ab,
\end{aligned}$$

The last equality hold since $(1-\lambda)a_1 + \lambda a_2 = a$ and $(1-\mu)b_1 + \mu b_2 = b$, therefore, C is PQD as desired. □

Example 1. For the subcopula C and its mass function c given below:

U \ V	1/3	2/3	1
1/3	1/3	1/3	1/3
2/3	1/3	2/3	2/3
1	1/3	2/3	1

U \ V	1/3	2/3	1
1/3	1/3	0	0
2/3	0	1/3	0
1	0	0	1/3

it is easy to see that C is affiliated so that the corresponding copula constructed by the bilinear extension is also affiliated.

4 Average and Local Measures of Affiliations

Copula characterizes dependence structures and dependence measures. For example, random variables X and Y are independent if and only if their corresponding copula $C(u,v) = uv$. A measure of dependence indicates in some particular manner how closely the random variables X and Y are related; Hence a variety of measures are needed to reveal the nature of affiliation dependence. We review measures of an affiliation discussed in [11]. Let T denote the average measure of the affiliation for $-\infty < x_1 < x_2 < \infty$ and $-\infty < y_1 < y_2 < \infty$, that is,

$$T = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{y_2}\int_{-\infty}^{x_2} [h(x_2,y_2)h(x_1,y_1) - h(x_1,y_2)h(x_2,y_1)]dx_1dy_1dx_2dy_2.$$

Also, it could be defined as average measure for affiliation of copula,

$$T_C = \int_0^1\int_0^1\int_0^{v_2}\int_0^{u_2} [c(u_2,v_2)c(u_1,v_1) - c(u_1,v_2)c(u_2,v_1)]du_1dv_1du_2dv_2.$$

After some calculation, we can get $\frac{1}{2}\tau = T$, where τ is Kendall's τ(See [17]).

For discrete copula, we give the following discrete average measure,

$$T_C = \sum_{i=0}^{n}\sum_{j=0}^{m}\sum_{k=0}^{i}\sum_{l=0}^{j}[c(u_k,v_l)c(u_i,v_j) - c(u_i,v_l)c(u_k,v_j)]. \tag{5}$$

Holland and Wang[5, 6] defined the local dependence index for affiliation as

$$\gamma(x,y) = \frac{\partial^2 \log h(x,y)}{\partial x \partial y}.$$

Also, it can be defined for copula

$$\gamma(u,v) = \frac{\partial^2 \log c(u,v)}{\partial u \partial v}.$$

We list several properties of this local measure of affiliation:

(i) $-\infty < \gamma(u,v) < \infty$.

(ii) $\gamma(u,v) = 0$ for all u,v if and only if U and V are independent.

(iii) If X and Y have a bivariate normal distribution with correlation coefficient ρ, then $\gamma(x,y) = \frac{\rho}{1-\rho^2}$, a constant.

Example 2. Consider the experiment of tossing an unbalanced coin 3 times with success rate p. Let X be the total number of heads observed and Y be the number of heads on the second toss. Then the joint density of X and Y is given by

$Y \backslash X$	0	1	2	3
0	$(1-p)^3$	$2(1-p)^2p$	$(1-p)p^2$	0
1	0	$(1-p)^2p$	$2p^2(1-p)$	p^3

The corresponding copula density of U and V is

$V \backslash U$	$(1-p)^3$	$(1-p)^2(1+2p)$	$1-p^3$	1
$1-p$	$(1-p)^3$	$2(1-p)^2p$	$(1-p)p^2$	0
1	0	$(1-p)^2p$	$2p^2(1-p)$	p^3

Note that U and V are affiliated, its bilinear interpolation is also affiliation. By (5), the average measure for this discrete copula is $T = (1-p)^5p + 8p^3(1-p)^3 + 2(1-p)^2p^4 + (1-p)p^5$. Note that if the coin is balanced then $T = 3/16$.

Example 3. Consider the experiment of tossing a unbalanced coin with success rate p. Let X be value 2^K, where K is number of tosses until the first head occurs, and Y be the number of heads in the first toss. Note that $E(X)$ does not exists for $p < 1/2$. For $p \in [0,1]$, the joint distribution of X and Y is

$Y \backslash X$	2^0	2^1	...	2^n	...
0	0	$p(1-p)$	...	$p(1-p)^{n-1}$	...
1	p	0	...	0	...

and its corresponding copula is,

$V \backslash U$	$p,\ p(1-p)+p,\ ...,\ (1-p)(1-(1-p)^{n-1})+p,\ ...,\ 1$
p	$0,\ \ p(1-p),\ \ ...,\ \ p(1-p)^{n-1},\ \ ...,\ p$
1	$p,\ p(1-p)+p,\ ...,\ \ p(1-p)^{n-1}+p,\ \ ...,\ 1$

This is a discrete copula which is not PQD. The average measure for this discrete copula $T = -p^2(1-p) - p^2(1-p)^2 - \cdots - p^2(1-p)^n - \cdots = -p(1-p) < 0$.

5 Conditions on Affiliation in the Bivariate Skew Normal Family

We first recall the definition of the multivariate skew-normal distribution which are given in [2]. A k-dimensional random variable Z is said to have a multivariate skew-normal distribution if it is continuous with density function

$$2\phi_k(\mathbf{z};\Sigma)\Phi(\alpha^T\mathbf{z}), \qquad \mathbf{z}\in\mathbb{R}^k,$$

where $\phi_k(\mathbf{z};\Sigma)$ is the k-dimensional normal density with zero mean and correlation matrix Σ, $\Phi(\cdot)$ is the CDF of $N(0,1)$, and α is a k-dimensional vector. Here we only consider the case where $k=2$. The density of (X,Y) is given by

$$h(x,y) = 2\phi_\rho(x,y)\Phi(\alpha_1 x+\alpha_2 y), \tag{6}$$

where

$$\phi_\rho(x,y) = (2\pi)^{-1}(1-\rho^2)^{-1/2}\exp\left\{\frac{1}{2(1-\rho^2)}(x^2-2\rho xy+y^2)\right\},$$

and α_1 and $\alpha_2\in\mathbb{R}$ are skewness parameters.

Theorem 3. *Consider the bivariate skew normal random vector (X,Y) with density given by (6). Then X and Y are affiliated if and only if $\rho\ge 0$ and $\alpha_1\alpha_2\le 0$.*

Proof. For the "*if*" part, assume that $\rho\ge 0$ and $\alpha_1\alpha_2\le 0$. By Lemma 3.5 of Rinott and Scarsini [20], we know that $\rho\ge 0$ implies that bivariate normal density is affiliated. That is

$$\phi_\rho(x',y)\phi_\rho(x,y') \le \phi_\rho(x',y')\phi_\rho(x,y) \qquad \text{for all } x'<x \text{ and } y'<y. \tag{7}$$

Now it is sufficient to show that X and Y in $\Phi(\alpha_1 x+\alpha_2 y)$ are affiliated. Without loss of generality, we assume that $\alpha_1<0, \alpha_2>0$. For any $x'<x, y'<y$, we have

$$\alpha_1 x+\alpha_2 y' \le \alpha_1 x'+\alpha_2 y' \le \alpha_1 x'+\alpha_2 y$$

and

$$\alpha_1 x+\alpha_2 y' \le \alpha_1 x+\alpha_2 y \le \alpha_1 x'+\alpha_2 y.$$

Since Φ is log concave, we have

$$\frac{\log\Phi(\alpha_1 x'+\alpha_2 y)-\log\Phi(\alpha_1 x'+\alpha_2 y')}{\alpha_2(y-y')} \le \frac{\log\Phi(\alpha_1 x+\alpha_2 y)-\log\Phi(\alpha_1 x+\alpha_2 y')}{\alpha_2(y-y')},$$

which implies

$$\log\Phi(\alpha_1 x'+\alpha_2 y)-\log\Phi(\alpha_1 x'+\alpha_2 y') \le \log\Phi(\alpha_1 x+\alpha_2 y)-\log\Phi(\alpha_1 x+\alpha_2 y').$$

Thus

$$\log\left[\Phi(\alpha_1 x' + \alpha_2 y)\Phi(\alpha_1 x + \alpha_2 y')\right] \leq \log\left[\Phi(\alpha_1 x + \alpha_2 y)\Phi(\alpha_1 x' + \alpha_2 y')\right],$$

which is reduced to

$$\Phi(\alpha_1 x' + \alpha_2 y)\Phi(\alpha_1 x + \alpha_2 y') \leq \Phi(\alpha_1 x + \alpha_2 y)\Phi(\alpha_1 x' + \alpha_2 y'). \tag{8}$$

Combining (7) and (8), we obtain

$$\phi_\rho(x', y)\Phi(\alpha_1 x' + \alpha_2 y)\phi_\rho(x, y')\Phi(\alpha_1 x + \alpha_2 y') \leq \phi_\rho(x, y)\Phi(\alpha_1 x + \alpha_2 y)\phi_\rho(x', y')\Phi(\alpha_1 x' + \alpha_2 y'),$$

so that X and Y are affiliated.

For the "*onlyif*" part, assume that X and Y are affiliated. It is suffices to show that if conditions $\rho \geq 0$ and $\alpha_1\alpha_2 < 0$ are not satisfied, then there exist $x' < x$, $y' < y$ such that

$$\phi_\rho(x, y)\Phi(\alpha_1 x + \alpha_2 y)\phi_\rho(x', y')\Phi(\alpha_1 x' + \alpha_2 y') < \phi_\rho(x', y)\Phi(\alpha_1 x' + \alpha_2 y)\phi_\rho(x, y')\Phi(\alpha_1 x + \alpha_2 y')$$

which is equivalent to

$$\frac{\rho}{1-\rho^2}(x - x')(y - y') + \log\left[\frac{\Phi(\alpha_1 x + \alpha_2 y)\Phi(\alpha_1 x' + \alpha_2 y')}{\Phi(\alpha_1 x' + \alpha_2 y)\Phi(\alpha_1 x + \alpha_2 y')}\right] < 0. \tag{9}$$

Now consider the following cases.

Case 1. For $\rho < 0$ and $\alpha_1\alpha_2 \leq 0$, without loss of generality, we assume that $\alpha_1 \leq 0, \alpha_2 \geq 0$, if we pick $x' = y' = 0$, and $y = \exp(x)$, then (9) is reduced to

$$\begin{aligned}
&\frac{\rho}{1-\rho^2} x\exp(x) + \log\left[\frac{\frac{1}{2}\Phi(\alpha_1 x + \alpha_2 \exp(x))}{\Phi(\alpha_2 \exp(x))\Phi(\alpha_1 x)}\right] \\
&\quad = \frac{\rho}{1-\rho^2} x\exp(x) - \log(\Phi(\alpha_1 x)) + \log\left[\frac{\frac{1}{2}\Phi(\alpha_1 x + \alpha_2 \exp(x))}{\Phi(\alpha_2 \exp(x))}\right],
\end{aligned}$$

which goes to $-\infty$ as x tends to ∞.

Case 2. For $\rho > 0$ and $\alpha_1\alpha_2 > 0$, without loss of generality, we assume that $\alpha_1 > 0, \alpha_2 > 0$, if we pick $x' = y' = 0$, and $y = 1$, then (9) is reduced to

$$\frac{\rho}{1-\rho^2} x + \log\left[\frac{\frac{1}{2}\Phi(\alpha_1 x + \alpha_2)}{\Phi(\alpha_2)\Phi(\alpha_1 x)}\right],$$

which goes to $-\infty$ as x tends to ∞.

Case 3. For $\rho < 0$ and $\alpha_1\alpha_2 > 0$, then the first part and second part of (9) are all strictly negative, therefore, the desired result follows. □

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