Cooperative Game as Non-Additive Measure

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Abstract. This chapter surveys cooperative game theory as an important application based on non-additive measures. In ordinary cooperative game theory, it is implicitly assumed that all coalitions of N can be formed; however, this is in general not the case. Let us elaborate on this, and distinguish several cases: 1) *Some coalitions may not be meaningful*. 2) *Coalitions may not be "black and white"*. In order to deal with such situations, various generalizations/extensions of the theory have been proposed, e.g., *bi-cooperative games, games on networks, games on combinatorial structures*. We give a survey on values and interaction indices for these extended cooperative game theory.

Keywords: cooperative game, bi-cooperative game, network, combinatorial structure, value, interaction index.

1 Introduction

Measure is one of the most important concepts in mathematics and so is the integral with respect to a measure. They have many applications in economics, engineering, and many other fields, and one of their main characteristics is additivity. This is very effective and convenient, but often too inflexible or too rigid. As a solution to the rigidness problem, several approaches based on non-additive measures have been proposed in various fields. The non-additivity can represent *interaction phenomena* among elements to be measured.

Let N be a finite set and v a set function (non-additive measure) on 2^N. Given a subset $S \subseteq N$, the precise meaning of the quantity $v(S)$ depends on the kind of intended application or domain [22]:

N **is the set of states of nature.** Then $S \subseteq N$ is an *event* in decision under uncertainty or under risk, and $v(S)$ represents the degree of certainty, belief, etc.

N **is a the set of criteria, or attributes.** Then $S \subseteq N$ is a group of criteria (or attributes) in multi-criteria (or multi-attributes) decision making, and $v(S)$ represents the degree of importance of S for making decision.

N **is the set of voters, political parties.** Then $S \subseteq N$ is called a *coalition* in voting situations, and $v(S) = 1$ iff bill passes when coalition S votes in favor of the bill, and $v(S) = 0$ else.

N **is the set of players, agents, companies, etc.** Then $S \subseteq N$ is also called a *coalition* in cooperative game theory, and $v(S)$ is the worth (or payoff,

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or income, etc.) won by S if all members in S agree to cooperate, and the other ones do not.

In the current chapter, we discuss and focus on cooperative games as an application based on non-additive measures.

2 Ordinary Cooperative Game

2.1 Definitions and Several Representations of Cooperative Games

Definition 1 (cooperative game). The function v that assigns to every coalition $S \subseteq N$ its value or worth $v(S)$ is commonly referred to as the *characteristic function*. It is always assumed that $v(\emptyset) = 0$. A pair (N, v) consisting of a player set N and a characteristic function v constitutes a *cooperative game* or *coalitional game*. These games are also referred to as *TU games*, where TU stands for *transferable utility* (We often identify (N, v) with v). Sometimes, we want to focus on only a few of the players involved in a cooperative game (N, v). For a coalition $S \subseteq N$, $v|_S$ denotes the restriction of the characteristic function v to the player set S, i.e., $v|_S(T) = v(T)$ for each coalition $T \subseteq S$. Then, $(S, v|_S)$ is called a *subgame* of the game (N, v).

In order to avoid a heavy notation, we will often omit braces for singletons, e.g., by writing $v(i)$, $N \setminus i$ instead of $v(\{i\})$, $N \setminus \{i\}$. Similarly, for pairs, we will write ij instead of $\{i, j\}$. Furthermore, cardinalities of coalitions $S, T, \ldots,$ will often be denoted by the corresponding lower case letters $s, t, \ldots,$ otherwise by the standard notation $|S|, |T|$, etc.

The set of all cooperative game with player set N will be denoted by $\mathcal{G}^N$. The set $\mathcal{G}^N$ is a $(2^n - 1)$-dimensional linear space.

Definition 2 (unanimity game). For each $T \in 2^N$, the *unanimity game* $u_T \in \mathcal{G}^N$ is defined by

$$u_T(S) := \begin{cases} 1 & \text{if } S \supseteq T, \\ 0 & \text{otherwise.} \end{cases}$$

The set $\{u_T \mid T \in 2^N \setminus \{\emptyset\}\}$ is a basis of the linear space $\mathcal{G}^N$. For any game $v \in \mathcal{G}^N$, we have

$$v(S) = \sum_{\emptyset \neq T \subseteq N} c_T(v)\, u_T(S) \qquad \forall S \in 2^N \setminus \{\emptyset\},$$

where

$$c_T(v) = \sum_{R \subseteq T} (-1)^{|T \setminus R|} v(R) \qquad \forall T \in 2^N \setminus \{\emptyset\}.$$

Then, $\{c_T\}_{T \in 2^N \setminus \{\emptyset\}}$ is called *unanimity coefficients* or *Harsanyi dividends* [33] of v.

Definition 3 (the Möbius transforms [56]). Let $(P, \leq)$ be a poset. For a function $f : P \to \mathbb{R}$, the *Möbius transform* Δ^f of f is the unique solution of the equation:

$$f(x) = \sum_{y \leq x} \Delta^f(y) \quad \forall x \in P,$$

given by

$$\Delta^f(x) = \sum_{y \leq x} \mu(y, x) f(y), \quad x \in P,$$

where μ is the so-called *Möbius function* on P and given by

$$\mu(y, x) = \begin{cases} 1 & \text{if } x = y, \\ -\sum_{y \leq z < x} \mu(y, z) & \text{if } y < x, \\ 0 & \text{otherwise.} \end{cases}$$

Now, considering a pair $(2^N, \subseteq)$ as a poset and a characteristic function $v : 2^N \to \mathbb{R}$ of cooperative game (N, v), then the *Möbius transform* of the game v on the poset $(2^N, \subseteq)$, is obtained as

$$\Delta^v(T) := \sum_{R \subseteq T} (-1)^{|T \setminus R|} v(R) \qquad \forall T \in 2^N.$$

That is, the concept of Möbius transform fits with of Harsanyi dividends in cooperative game theory. i.e., $c_T(v) = \Delta^v(T)$ for any $T \in 2^N \setminus \{\emptyset\}$. Inversely,

$$v(S) = \sum_{T \subseteq S} \Delta^v(T) \qquad \forall S \in 2^N.$$

Here, the Möbius transform or Harsanyi dividends of cooperative games can be interpreted as follows:

> *The Möbius transform is vanishing at the empty set, its worth $v(i)$ for every singleton $i \in N$, while recursively, the Möbius transform of every coalition of at least two players is equal to its worth minus the sum of the Möbius transforms of all its proper subcoalitions. In this sense, the Möbius transform of a coalition S can be interpreted as an extra contribution of cooperation among the players in S that they did not already achieve by smaller coalitions.*

Definition 4 (multilinear extension). Let I^N be the n-dimensional unit hyper cube, i.e.,

$$I^N := \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, \ \ \forall i \in N = \{1, \cdots, n\}\}.$$

The extreme points of I^N are the vectors χ_S, $S \subseteq N$, where

$$(\chi_S)_i = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

So, we notice that $v \in \mathcal{G}^N$ determines a real function $\bar{v}$ on the corners of I^v by

$$\bar{v}(\chi_S) = v(S) \quad \forall S \subseteq N.$$

Hence, $\bar{v}$ may be extended to I^N by

$$\bar{v}(\boldsymbol{x}) = \sum_{T \subseteq N} \left(\prod_{i \in T} x_i \prod_{i \in N \setminus S} (1 - x_i) \right) v(T) \quad \forall \boldsymbol{x} \in I^N.$$

Then, this function $\bar{v} : I^N \to \mathbb{R}$ is called the *multilinear extension* (MLE) of v. The multilinear extension of v can also be represented via the Harsanyi dividends (Möbius transform) as follows [53]:

$$\bar{v}(\boldsymbol{x}) = \sum_{T \subseteq N} \Delta^v(T) \prod_{i \in T} x_i \quad \forall \boldsymbol{x} \in I^N.$$

2.2 Intuitive Representations of Importance and Interaction

In order to intuitively approach the concept of importance of each player and of interaction among players, consider two players i and $j \in N$. Clearly, $v(i)$ is one of representations of importance of $i \in N$. An inequality

$$v(ij) > v(i) + v(j) \quad \text{(resp., <)}$$

seems to model a *positive (resp., negative) interaction* or *complementary (resp., substitutive) effect* between players i and j. However, as discussed in Grabisch and Roubens [27], the intuitive concept of interaction requires a more elaborate definition. We should not only compare $v(i)$, $v(j)$, and $v(ij)$ but also see what happens when i, j, and ij join the other coalitions. That is, we should take into account all coalitions of the form $T \cup i$, $T \cup j$, and $T \cup ij$. For a player i and a coalition $T \not\ni i$,

$$\Delta_i v(T) := v(T \cup i) - v(T) \tag{1}$$

seems to represent an index of importance of i in $T \cup i$. The equation (1) is called the *marginal contribution* of a player i to a coalition T. Then it seems natural to consider that if for T not containing i and j

$$\Delta_i v(T \cup j) > \Delta_i v(T) \quad \text{(resp., <),}$$

then i and j interact positively (resp., negatively) each other in the presence of T since the presence of j increases (resp., decreases) the marginal contribution of i to T. Then

$$\Delta_{ij} v(T) := \Delta_i v(T \cup j) - \Delta_i v(T)$$

is called the *marginal interaction* [28] between i and j in the presence of T. Note that

$$\begin{aligned} \Delta_i v(T \cup j) - \Delta_i v(T) &= v(T \cup ij) - v(T \cup i) - v(T \cup j) + v(T) \\ &= \Delta_j v(T \cup i) - \Delta_j v(T). \end{aligned}$$

For three players $i, j, k \in N$ and a coalition T not containing i, j and k, $\Delta_{\{i,j,k\}} v(T)$ can be naturally defined as

$$\Delta_{\{i,j,k\}} v(T) := \Delta_{ij} v(T \cup k) - \Delta_{ij} v(T).$$

Then we have $\Delta_{ij} v(T \cup k) - \Delta_{ij} v(T) = \Delta_{ik} v(T \cup j) - \Delta_{ik} v(T) = \Delta_{jk} v(T \cup i) - \Delta_{jk} v(T)$. Moreover, for two distinct coalitions S and $T \subseteq N \setminus S$,

$$\Delta_S v(T) := \Delta_{S \setminus i} v(T \cup i) - \Delta_{S \setminus i} v(T)$$

for $i \in S$. Then $\Delta_{S\setminus i}v(T \cup i) - \Delta_{S\setminus i}v(T) = \Delta_{S\setminus j}v(T \cup j) - \Delta_{S\setminus j}v(T)$ for any $i, j \in S$. Similarly, when, for example, $\Delta_S v(T) > 0$ (resp., $<$), we shall consider that players among S interact positively (resp., negatively) each other in the presence of T.

These marginal contributions and interactions can be represented through the following notion, *discrete derivative*.

Definition 5 (discrete derivative [28]). Given a game $v \in \mathcal{G}^N$ and finite coalitions $S, T \subseteq N$, we denote by $\Delta_S v(T)$ the *S-derivative* of v at T, which is recursively defined by

$$\Delta_i v(T) := v(T \cup i) - v(T \setminus i) \qquad \forall i \in N,$$

and

$$\Delta_S v(T) := \Delta_i \left(\Delta_{S\setminus i} v(T)\right) \qquad \forall i \in S,$$

with convention $\Delta_\emptyset v(T) := v(T)$.

Proposition 1 ([18,20,28]). *For any $S \subseteq N$, $T \subseteq N \setminus S$ and $v \in \mathcal{G}^N$, the S-derivative of v at T can be represented as follows:*

$$\Delta_S v(T) = \sum_{L\subseteq S} (-1)^{|S\setminus L|} v(T \cup L) = \sum_{L\subseteq T} \Delta^v(S \cup L),$$

i.e.,

$$\Delta_S v(T) = \sum_{T\subseteq L\subseteq S\cup T} (-1)^{|(S\cup T)\setminus L|} v(L) = \sum_{S\subseteq L\subseteq S\cup T} \Delta^v(L).$$

In particular, $\Delta^v(S) = \Delta_S v(\emptyset)$ for any $S \subseteq N$. Moreover, if σ is a permutation on N such that $S = \{\sigma(1), \cdots, \sigma(|S|)\}$,

$$\Delta_S v(T) = \left. \frac{\partial^{|S|}}{\partial x_{\sigma(1)} \cdots \partial x_{\sigma(|S|)}} \bar{v}(\boldsymbol{x}) \right|_{\boldsymbol{x}=\chi_{S\cup T}.}$$

where $\bar{v}$ is the MLE of v.

Definition 6 (k-monotonic game [10]). Given an integer $k \geq 2$, a game $v \in \mathcal{G}^N$ is said to be *k-order monotone* (for short, *k-monotone*) if and only if, for any (at most) k coalitions $S_1, \cdots, S_k$, we have

$$v\left(\bigcup_{i=1}^{k} S_i\right) \geq \sum_{\substack{J\subseteq\{1,\cdots,k\}\\ J\neq\emptyset}} (-1)^{|J|+1} v\left(\bigcap_{i\in J} S_i\right). \tag{2}$$

It is easy to verify that k-monotonicity ($k \geq 2$) implies l-monotonicity for all integer $2 \leq l \leq k$. By extension, 1-monotonicity (which does not correspond to $k = 1$ in Eq. (2)) is defined as standard monotonicity, i.e.,

$$v(S) \leq v(T) \quad \text{whenever} \quad S \subseteq T \subseteq N.$$

A game $v \in \mathcal{G}^N$ is called *totally monotone* if Eq.(2) holds for any positive integer k. 2-Monotonic games v, i.e.,

$$v(S \cup T) \geq v(S) + v(T) - v(S \cap T) \quad \forall S, T \subseteq N,$$

are also referred to as *convex games.*

The notion of k-monotonicity can be characterized through the use of discrete derivatives as follows.

Proposition 2 ([10,18,20]). *Let v be a game on N (i.e., $v \in \mathcal{G}^N$) and k a positive integer. Then v is k order monotone if and only if*

$$\Delta_S v(T) \geq 0$$

for any $S \subseteq N$ and $T \subseteq N \setminus S$ such as $1 \leq |S| \leq k$.

Note: In evidence theory [58], belief functions $Bel : 2^N \to [0, 1]$ have been introduced as totally monotonic games, i.e., whose Möbius transforms Δ^{Bel}, which are called "*basic probability assignments*", are non-negative for all events (coalitions).

2.3 The Shapley Value as an Acceptable Allocation in Cooperative Games

The players in a cooperative game are eventually interested in what they individually will get out of cooperating with the other players. How will individual players benefit from cooperation? So far, various solutions concepts and allocation rules of benefits have been proposed (see, e.g., [54]). Some of them (e.g., *the core, bargaining set, prekernel, kernel, prenucleolus, nucleolus*) are based on *domination*, and some of them (e.g., *the Shapley value*) are based on *expectation*. This subsection discusses only the Shapley value and relatives from the standpoint of Shapley's statement [59]:

> *"At the foundation of the theory of games is the assumption that the players of a game can evaluate, in their utility scale, every "prospect" that might arise a result of a play. In attempting to apply the theory to any field, one would normally expect to be permitted to include, in the class of "prospects", the prospect of having to play a game. The possibility of evaluating games is therefore of critical importance."*

A *payoff vector* or *allocation* is a vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^N$ that specifies for each player $i \in N$ the profit x_i that this player can expect when he cooperates with the other players. Thus, a payoff vector $x = (x_1, \cdots, x_n)$ such as $\sum_{i \in N} x_i > v(N)$ is not feasible. That is, payoffs $x = (x_1, \cdots, x_n)$ with $\sum_{i \in N} x_i = v(N)$ are the most efficient allocations of $v(N)$. However, not all these efficient allocations will be acceptable to the players. Here, we introduce the Shapley value, which provide a priori evaluations of every cooperative game as an acceptable allocation to each player.

Definition 7 (the Shapley value [59]). The *Shapley value* $\phi : \mathcal{G}^N \to \mathbb{R}^N$ is given by

$$\phi_i(v) = \sum_{S \subseteq N \setminus i} \frac{s! \cdot (n - s - 1)!}{n!} [v(S \cup i) - v(S)]$$

for any $v \in \mathcal{G}^N$ and any $i \in N$, where $\phi_i(v)$ is the i-th component of $\phi(v) \in \mathbb{R}^N$.

The Shapley value is one of the most well-known allocation rule defined as a certain type of expectation of marginal contributions for each player and characterized as the unique allocation rule satisfying the following four properties (axioms): *symmetry, efficiency, null zero,* and *additivity (low of aggregation)* [59].

Definition 8 (symmetry). Let $\Pi(N)$ denote the set of all permutations on N. If $\sigma \in \Pi(N)$, then writing $\sigma(S)$ for the image of $S \subseteq N$ under σ, i.e., $\sigma(S) := \{\sigma(i) \mid i \in S\}$, we may define the game σv by $\sigma v(\sigma(S)) = v(S)$ for all $S \subseteq N$. An allocation rule $F : \mathcal{G}^N \to \mathbb{R}^N$ is said to be *symmetry* if

$$F_{\sigma(i)}(\sigma v) = F_i(v)$$

for any $\sigma \in \Pi(N)$ and $v \in \mathcal{G}^N$, where $F_i(v)$ is the i-th component of $F(v) \in \mathbb{R}^N$.

Under symmetry allocation rules, the names of players play no role in determining the allocation to each player.

Definition 9 (efficiency). An allocation rule $F : \mathcal{G}^N \to \mathbb{R}^N$ is said to be *efficient* if

$$\sum_{i \in N} F_i(v) = v(N)$$

for any $v \in \mathcal{G}^N$, where $F_i(v)$ is the i-th component of $F(v) \in \mathbb{R}^N$.

Under efficient allocation rules, the total worth $v(N)$ is allocated to all the players.

Definition 10 (null-zero). An allocation rule $F : \mathcal{G}^N \to \mathbb{R}^N$ is said to be *null-zero* if

$$F_i(v) = 0$$

for any $v \in \mathcal{G}^N$ and $i \in N$ such that $v(S \cup i) = v(S)\ \forall S \subseteq N$, where $F_i(v)$ is the i-th component of $F(v) \in \mathbb{R}^N$.

Under null-zero allocation rules, a player who adds nothing to the worth of any coalition is allocated nothing.

Definition 11 (additivity). An allocation rule $F : \mathcal{G}^N \to \mathbb{R}^N$ is said to be *additive* if

$$F(v + w) = F(v) + F(w)$$

for any $v, w \in \mathcal{G}^N$.

Under additive allocation rules, if two allocation problems are combined into one by adding the characteristic functions, then for each player the allocation under the combined problem is the sum of the allocations under the two individual problems.

The Shapley value also can be treated as a power and/or importance index in various fields, *e.g., decision making problems [55], voting power in the council [60], etc.,* and represented via the Möbius transform and the multilinear extension of v as follows [54]:

$$\phi_i(v) = \sum_{i \in S \subseteq N} \frac{1}{s} \Delta^v(S).$$

$$\phi_i(v) = \int_0^1 \frac{\partial}{\partial x_i} \bar{v}(t, t, \cdots, t)\, dP_\phi(t),$$

where $\frac{\partial}{\partial x_i} \bar{v}(t, t, \cdots, t) := \frac{\partial}{\partial x_i} \bar{v}(x) \mid_{x=(t,t,\cdots,t)}$, $P_\phi(t) = t$ for any $t \in [0, 1]$ and the integral is to be understood in the sense of Riemann-Stieljes.

Note (the Banzhaf power index): The Banzhaf power index $\beta : \mathcal{G}^N \to \mathbb{R}^N$, defined by

$$\beta_i(v) := \sum_{S \subseteq N \setminus i} \frac{1}{2^{n-1}} \; [v(S \cup i) - v(S)] = \sum_{i \in T \subseteq N} \frac{1}{2^{t-1}} \varDelta^v(T) \quad \forall v \in \mathcal{G}^N,$$

is also a well-known voting power index [2], which is not efficient. The Banzhaf power index also has an integral-representation as follows [13,18]:

$$\beta_i(v) = \int_0^1 \frac{\partial}{\partial x_i} \bar{v}(t, t, \cdots, t) \, dP_\beta(t) = \frac{\partial}{\partial x_i} \bar{v}(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}),$$

where $\frac{\partial}{\partial x_i} \bar{v}(t, t, \cdots, t) := \frac{\partial}{\partial x_i} \; \bar{v}(x) \,|_{x=(t,t,\cdots,t)}$, $P_\beta(t) = \mathbf{1}_{[0.5,1]}$ for any $t \in [0, 1]$.

2.4 Interaction Index in Cooperative Games

The study of the notion of *interaction* among players is relatively recent in the framework of cooperative game theory. The first attempt is probably due to Owen [53] for superadditive games. More developments are due to Murofushi and Soneda [50], Roubens [57], Grabisch and Roubens [27], Marichal and Roubens [49], and Fujimoto et al. [18]. The concept of interaction index, which can be seen as an extension of the notion of value, is fundamental for making it possible to measure the interaction phenomena modeled by a game on a set of players. The expression "*interaction phenomena*" refers to either *complementarity* or *redundancy* effects among players of coalitions resulting from the non additivity of the underlying game. Thus far, the notion of *interaction index* has been primarily applied to multi-criteria decision making in the framework of aggregation by the Choquet integral. In this context, it is used to appraise the overall interaction among criteria (see, e.g., [27,29,40]), thereby giving more insight into the decision problem. Other natural applications concern statistics and data analysis (see, e.g., [21,39]).

An allocation rule $\phi : \mathcal{G}^N \to \mathbb{R}^N$, e.g., the Shapley value, can be regarded as a function $F : \mathcal{G}^N \times N \to \mathbb{R}$ such that

$$F(v, i) = \phi_i(v)$$

for any $v \in \mathcal{G}^N$ and any $i \in N$. Then, setting $\mathcal{N} := 2^N \setminus \{\emptyset\}$, we define an *interaction index* as a function $I : \mathcal{G}^N \times \mathcal{N} \to \mathbb{R}$ to measure the (simultaneous) interaction among players in a cooperative game, i.e., $I(v, S)$ represents the (simultaneous) interaction among players S in playing a game v.

Definition 12 (interaction indices). The (Shapley-type) *interaction index* with respect to $S \in \mathcal{N}$ of v is defined by

$$I(v, S) := \sum_{T \subseteq N \setminus S} \frac{(n - t - s)! \; t!}{(n - s + 1)!} \varDelta_S v(T).$$

This index is an extension of the Shapley value in the sense that $I(v, i)$ coincides with the Shapley value $\phi_i(v)$ of any player i.

The Shapley-type interaction index is a type of expectation of marginal interactions among the players in each coalition and characterized as the unique interaction index satisfying the following six properties (axioms): *symmetry, k-monotone positivity, dummy partnership, reduced partnership consistency, additivity, efficiency* [18].

Definition 13 (additivity). An interaction index $I : \mathcal{G}^N \times \mathcal{N} \to \mathbb{R}$ is said to be *additive* if

$$I(v + w, S) = I(v, S) + I(w, S)$$

for any $v, w \in \mathcal{G}^N$ and any $S \in \mathcal{N}$.

Definition 14 (symmetry). Let $\Pi(N)$ denote the set of all permutations on N. If $\sigma \in \Pi(N)$, then writing $\sigma(S)$ for the image of $S \subseteq N$ under σ, i.e., $\sigma(S) := \{\sigma(i) \mid i \in S\}$, we may define the game σv by $\sigma v(\sigma(S)) = v(S)$ for all $S \subseteq N$. An interaction index $I : \mathcal{G}^N \times \mathcal{N} \to \mathbb{R}$ is said to be *symmetry* if

$$I(\sigma v, \sigma(S)) = I(v, S)$$

for any $\sigma \in \Pi(N)$, any $S \in \mathcal{N}$, and any $v \in \mathcal{G}^N$.

Definition 15 (efficiency). An interaction index $I : \mathcal{G}^N \times \mathcal{N} \to \mathbb{R}$ is said to be *efficient* if

$$\sum_{i \in N} I(v, i) = v(N)$$

for any $v \in \mathcal{G}^N$.

Under efficient interaction indices, the interaction among itself is represented as its allocation for each player.

A coalition $P \in \mathcal{N}$ is said to be a *partnership* [38] in a game (N, v) if

$$v(S \cup T) = v(T)$$

for any $S \subsetneq P$ and any $T \subseteq N \setminus P$. In other words, as long as all the members of a partnership P are not all in coalition, the presence of some of them only leaves unchanged the worth of any coalition not containing elements of P. In particular $v(S) = 0$ for all $S \subsetneq P$. Thus, a partnership behaves like a single hypothetical player $[P]$, that is, the game $v \in \mathcal{G}^N$ and its reduced version $v_{[P]} \in \mathcal{G}^{(N \setminus P) \cup [P]}$, which is defined by

$$v_{[P]}(S) = \begin{cases} v(S) & \text{if } S \subseteq N \setminus P \\ v(S \cup P) & \text{if } S \ni [P] \end{cases},$$

can be considered as equivalent.

Definition 16 (reduced partnership consistency). An interaction index $I : \mathcal{G}^N \times \mathcal{N} \to \mathbb{R}$ is said to satisfy *reduced partnership consistency* property/axiom if

$$I(v, P) = I(v_{[P]}, [P])$$

for any $v \in \mathcal{G}^N$ and any partnership P in (N, v).

Recall that a partnership can be considered as behaving as a single hypothetical player. Furthermore, it is easy to verify that the marginal interaction among the players of a partnership $P \in \mathcal{N}$ in a game (N, v) in the presence of a coalition $T \subseteq N \setminus P$ is equal to the marginal contribution of P to coalition T, i.e.,

$$\varDelta_P v(T) = v(T \cup P) - v(T).$$

In other words, when we measure the interaction among the players of a partnership, it is as if we were measuring the value of a hypothetical player. The *reduced partnership consistency property* then simply states that the interaction among players of a partnership P in a game (N, v) should be regarded as the value of the reduced partnership $[P]$ in the corresponding reduced game $v_{[P]}$.

A player $d \in N$ is said to be *dummy* in a game $v \in \mathcal{G}^N$ if

$$v(S \cup d) = v(S) + v(d)$$

for all $S \subseteq N \setminus d$. In other words, the marginal contribution of a dummy player $d \in N$ to any coalition $S \subseteq N \setminus d$ is simply its worth $v(d)$, i.e., there are no interaction between d and any $S \subseteq N \setminus d$. Similarly, a coalition $D \in \mathcal{N}$ is said to be *dummy* if $v(T \cup D) = v(T) + v(D)$ for any $T \subseteq N \setminus D$.

Definition 17 (dummy partnership). An interaction index $I : \mathcal{G}^N \times \mathcal{N} \to \mathbb{R}$ is said to satisfy *dummy partnership* property/axiom if the following two conditions hold:

(i) $I(v, D) = v(D)$
(ii) $I(v, S \cup D) = 0 \ \forall S (\neq \emptyset) \subseteq N \setminus D$

for any $v \in \mathcal{G}^N$ and any dummy partnership $D \in \mathcal{N}$ in (N, v).

The first part of *dummy partnership property* states that the interaction index of a dummy partnership D in a game (N, v) should be its worth since the marginal interaction among the players in D in the presence of any coalition T not containing elements of D is its worth, that is, $\varDelta_D(T) = v(D)$ for any $T \subseteq N \setminus D$. The second part of the property says that there should be no simultaneous interaction among players of coalitions containing dummy partnerships since dummy partnerships behaves like a single hypothetical dummy player and he does not interact with any outsider coalition (see, [49]).

Definition 18 (*k*-monotone positivity). An interaction index $I : \mathcal{G}^N \times \mathcal{N} \to \mathbb{R}$ is said to be *k-monotone positive* if, for any positive integer k and k-order monotonic game $v \in \mathcal{G}^N$,

$$I(v, S) \geq 0 \ \forall S \in \mathcal{N} \ \text{ whenever } |S| \leq k.$$

As discussed in Subsection 2.2, in a k-monotone game, it seems sensible to consider that there are necessarily complementarity effects among players in coalitions containing (at most) k players. This axiom then simply states that these effects should be represented as positive interactions.

The (Shapley-type) interaction index, as similar to the Shapley value, can be represented via the Möbius transform Δ^v and the multilinear extension $\bar{v}$ of $v \in \mathcal{G}^N$ as follows [18]:

$$I(v, S) = \sum_{T \supseteq S} \frac{1}{t - s + 1} \Delta^v(T).$$

$$I(v, S) = \int_0^1 \frac{\partial^{|S|}}{\partial x_{\sigma(1)} \cdots \partial x_{\sigma(|S|)}} \bar{v}(t, t, \cdots, t)\, dP(t),$$

where σ is a permutation on N such that $S = \{\sigma(1), \cdots, \sigma(|S|)\}$ and $P(t) = t$ for any $t \in [0, 1]$.

Note (other interaction indices): Another Shapley-type interaction index I_{ch} called *chaining interaction index* and the Banzhaf-type interaction index I_B have been proposed and characterized axiomatically (see, e.g., [18,27,49]).

The chaining interaction index $I_{ch} : \mathcal{G}^N \times \mathcal{N} \to \mathbb{R}$ is defined by

$$I_{ch}(v, S) := \sum_{T \subseteq N \setminus S} \frac{s(n - s - t)!(s + t - 1)!}{n!} \Delta_S v(T)$$

for any $v \in \mathcal{G}^N$ and any $S \in \mathcal{N}$, and also has the following representations:

$$I_{ch}(v, S) = \sum_{T \supseteq S} \frac{s}{t} \Delta^v(T).$$

$$I_{ch}(v, S) = \int_0^1 \frac{\partial^{|S|}}{\partial x_{\sigma(1)} \cdots \partial x_{\sigma(|S|)}} \bar{v}(t, t, \cdots, t)\, dP_{ch}(t),$$

where σ is a permutation on N such that $S = \{\sigma(1), \cdots, \sigma(|S|)\}$ and $P_{ch}(t) = t^s \mathbf{1}_{]0,1]}$ for any $t \in [0, 1]$.

The Banzhaf-type interaction index $I_B : \mathcal{G}^N \times \mathcal{N} \to \mathbb{R}$ is defined by

$$I_B(v, S) := \sum_{T \subseteq N \setminus S} \frac{1}{2^{n-s}} \Delta_S v(T)$$

for any $v \in \mathcal{G}^N$ and any $S \in \mathcal{N}$, and also has the following representations:

$$I_B(v, S) = \sum_{T \supseteq S} \frac{1}{2^{t-s}} \Delta^v(T).$$

$$I_B(v, S) = \int_0^1 \frac{\partial^{|S|}}{\partial x_{\sigma(1)} \cdots \partial x_{\sigma(|S|)}} \bar{v}(t, t, \cdots, t)\, dP_B(t),$$

where σ is a permutation on N such that $S = \{\sigma(1), \cdots, \sigma(|S|)\}$ and $P_B(t) = \mathbf{1}_{[0.5,1]}$ for any $t \in [0, 1]$.

3 Bi-cooperative Game

To date, there have been some attempts to define more general concept in cooperative game theory. Aubin [1] has proposed the concept of *generalized coalition* as a function $c : N \to [-1, 1]$ which associates each player i with his/her level of participation $c(i) \in [-1, 1]$. A positive level is interpreted as attraction of the player i for the coalition, and a negative level as repulsion. Later, the concept of bi-cooperative game has been introduced by Bilbao et al. [5] as a generalization of classical cooperative games, where each player can participate positively to the game (defender), negatively (defeater), or do not participate (abstentionist). In a voting situation (simple games), they coincide with ternary voting games, on the set of all *signed coalitions* given by $\{c : N \to \{-1, 0, 1\}\}$, of Felsenthal and Machover [15], where each voter can vote in favor (1), against (-1) or abstain (0). Labreuche and Grabisch [43] give the following example:

Example 1. A set N of farmers raise three kinds of plants called A, B and C (for instance colza, grass and reed) in a given area. Plant A (defeater) needs a lot of pesticide and chemical fertilizers so that it pollutes a lot the local river. Plant B (abstentionist) needs no special treatment and thus no pollution is caused by this plant. Plant C (defender) helps in reducing the pollution since it absorbs some chemicals. The Governor of this area wants to determine the tax for each farmer on the basis of the impact of the farming on the river pollution rate. The bi-cooperative game $v(S, T)$ measures the pollution rate in the river compared to the time when there were only meadows in the area, when farmers S raise plant C, farmers T raise plant A and farmers $N \setminus (S \cup T)$ raise plant B.

3.1 Definitions and Several Representations of Bi-cooperative Games

We will denote $\mathcal{P}(N) := 2^N$ and $\mathcal{Q}(N) := \{(A_1, A_2) \in \mathcal{P}(N) \times \mathcal{P}(N) \mid A_1 \cap A_2 = \emptyset\}$. When equipped with the following order: for $(A_1, A_2), (B_1, B_2) \in \mathcal{Q}(N)$

$$(A_1, A_2) \sqsubseteq (B_1, B_2) \text{ iff } A_1 \subseteq B_1 \text{ and } A_2 \supseteq B_2,$$

$(\mathcal{Q}(N), \sqsubseteq)$ becomes a lattice, which will be defined in Definition 42. The binary operators $\sqcup$ (sup) and $\sqcap$ (inf) are given by

$$(A_1, A_2) \sqcup (B_1, B_2) = (A_1 \cup B_1, A_2 \cap B_2),$$
$$(A_1, A_2) \sqcap (B_1, B_2) = (A_1 \cap B_1, A_2 \cup B_2).$$

Then the top and bottom are respectively $(N, \emptyset)$ and $(\emptyset, N)$.

Definition 19 (irreducible elements [14]). Let $(L, \leq, \vee, \wedge, \top, \bot)$ be a lattice, where $\vee, \wedge, \top, \bot$ denotes sup, inf, the top and bottom element, respectively. An element $x \in L$ is said to be $\vee$-*irreducible* if $x \neq \bot$ and $x = a \vee b$ implies $x = a$ or $x = b$, $\forall a, b \in L$.

Proposition 3 ([24]). *The* $\sqcup$*-irreducible elements of* $\mathcal{Q}(N)$ *are* $(\emptyset, N \setminus i)$ *and* $(i, N \setminus i)$*, for all* $i \in N$*. Moreover, for any* $(A_1, A_2) \in \mathcal{Q}(N)$*,*

$$(A_1, A_2) = \bigsqcup_{i \in A_1} (i, N \setminus i) \;\sqcup \bigsqcup_{j \in N \setminus (A_1 \cup A_2)} (\emptyset, N \setminus j). \tag{3}$$

Eq. (3) is called the *minimal decomposition* of (A_1, A_2) [23].

> *Here,* $\sqcup$*-irreducible elements permit to define* layer *in* $Q(N)$ *as follows [23]:* $(\emptyset, N)$ *is the bottom layer (layer 0) (the black square in Fig. 1), the set of all* $\sqcup$*-irreducible elements forms layer 1 (black circles in Fig. 1), and layer k, for* $k = 2, \dots n$*, consists of all elements whose minimal decomposition contains exactly k* $\sqcup$*-irreducible elements. In other words, layer k consists of all elements* $(A_1, A_2) \in Q(N)$ *such that* $|A_2{}^c| = k$*, for* $k = 2, \dots, n$*. On the other hand, let us consider the Boolean lattice* $(\mathcal{P}(N), \subseteq, \cup, \cap, N, \emptyset)$*. Then, the empty set is the bottom layer; all singletons are* $\cup$*-irreducible elements, (i.e., in layer 1); the set of all* $A \in \mathcal{P}(N)$ *whose cardinality is k, for* $k = 2, \dots, n$*, forms layer k.*

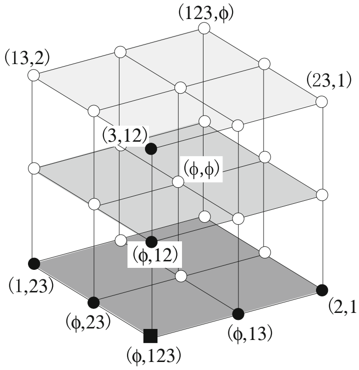

Fig. 1. The lattice $Q(123)$: the element in layer 0 is indicated by a black square and elements in layer 1 black circles

Definition 20 (bi-cooperative game [3,24]). The function v that assigns to every bi-coalition $(S_1, S_2) \in Q(N)$ its value or worth $v(S_1, S_2)$ is commonly referred to as the *bi-characteristic function*. It is always assumed that $v(\emptyset, \emptyset) = 0$. A triplet $(N, v, Q(N))$ consisting of a player set N, a bi-characteristic function v, and a lattice $Q(N)$ constitutes a *bi-cooperative game* or *bi-coalitional game*. (We often identify $(N, v, Q(N))$ with v).

Definition 21 (the Möbius transform of bi-cooperative game). To any bi-cooperative game $v : Q(N) \to \mathbb{R}$, another function $m : Q(N) \to \mathbb{R}$ can be associated by

$$\Delta^v(A_1, A_2) := \sum_{\substack{B_1 \subseteq A_1 \\ A_2 \subseteq B_2 \subseteq N \setminus A_1}} (-1)^{|A_1 \setminus B_1| + |B_2 \setminus A_2|} \, v(B_1, B_2)$$

for $(A_1, A_2) \in Q(N)$. This correspondence proves to be one-to-one, since conversely

$$v(A_1, A_2) = \sum_{(B_1, B_2) \sqsubseteq (A_1, A_2)} \Delta^v(B_1, B_2) \tag{4}$$

for all $(A_1, A_2) \in Q(N)$. The validity of Eq. (4) is proved by Grabisch and Labreuche [23] who call $\Delta^v : Q(N) \to \mathbb{R}$ the *Möbius transform* of v.

Fujimoto and Murofushi [20] have introduced another equivalent representation, the *bipolar Möbius transform*, of a bi-cooperative game as follows:

Definition 22 (the bipolar Möbius transform). To any bi-cooperative game $v : \mathcal{Q}(N) \to \mathbb{R}$, another function $b : \mathcal{Q}(N) \to \mathbb{R}$ can be associated by

$$b^v(A_1, A_2) := \sum_{\substack{B_1 \subseteq A_1 \\ B_2 \subseteq A_2}} (-1)^{|A_1 \setminus B_1| + |A_2 \setminus B_2|} \, v(B_1, B_2) \tag{5}$$

$$= \sum_{(\emptyset, A_2) \sqsubseteq (B_1, B_2) \sqsubseteq (A_1, \emptyset)} (-1)^{|A_1 \setminus B_1| + |A_2 \setminus B_2|} \, v(B_1, B_2)$$

for $(A_1, A_2) \in \mathcal{Q}(N)$. Then, the function defined by Eq. (5) is called the *bipolar Möbius transform* of v.

Proposition 4 ([20]). *Let $v : \mathcal{Q}(N) \to \mathbb{R}$ be a bi-cooperative game, and $b^v : \mathcal{Q}(N) \to \mathbb{R}$ the bipolar Möbius transform of v. Then,*

$$v(A_1, A_2) = \sum_{\substack{B_1 \subseteq A_1 \\ B_2 \subseteq A_2}} b^v(B_1, B_2)$$

for any $(A_1, A_2) \in \mathcal{Q}(N)$.

Definition 23 (piecewise multi linear extension). The set of all bi-cooperative games $\{v : \mathcal{Q}(N) \to \mathbb{R}\}$ is isomorphic to the set of all *ternary pseudo-Boolean functions* $\{f : \{-1, 0, 1\}^N \to \mathbb{R}\}$. Indeed, there exists the isomorphism $\varphi : \mathcal{Q}(N) \to \{-1, 0, 1\}^N$ such that $\varphi(A_1, A_2) = \chi_{(A_1, A_2)}$ for any $(A_1, A_2) \in \mathcal{Q}(N)$, where $\chi_{(A_1, A_2)}$ denotes the characteristic vector of (A_1, A_2), which is the vector of $\{-1, 0, 1\}^N$ whose i-th element is 1 if $i \in A_1$, -1 if $i \in A_2$, and 0 otherwise. Then, for any bi-cooperative game $v : \mathcal{Q}(N) \to \mathbb{R}$ there exists a ternary pseudo-Boolean function f_v (i.e., $f_v : \{-1, 0, 1\}^N \to \mathbb{R}$) corresponding to v. Now, we introduce an equivalent representation, by using a ternary pseudo-Boolean function, of v as follows:

$$f_v(\boldsymbol{x}) = \sum_{(S_1, S_2) \in \mathcal{Q}(N)} b^v(S_1, S_2) \left(\prod_{i \in S_1} x_i^+ \cdot \prod_{j \in S_2} x_j^- \right) \tag{6}$$

for $\boldsymbol{x} \in \{-1, 0, 1\}^N$, where $x^+ = \max\{x, 0\}$ and $x^- = 0 - \min\{x, 0\}$. This correspondence is represented as

$$f_v(\chi_{(A_1, A_2)}) = \sum_{\substack{B_1 \subseteq A_1 \\ B_2 \subseteq A_2}} b^v(B_1, B_2) = v(A_1, A_2) \quad \forall (A_1, A_2) \in \mathcal{Q}(N).$$

Here, Eq. (6) leads to the *piecewise multilinear extension* $g_v : [-1, 1]^N \to \mathbb{R}$, of the ternary pseudo-Boolean function $f_v : \{-1, 0, 1\}^N \to \mathbb{R}$ corresponding to the bi-cooperative game $v : \mathcal{Q}(N) \to \mathbb{R}$, defined by

$$g_v(\boldsymbol{x}) := \sum_{(S_1, S_2) \in \mathcal{Q}(N)} b^v(S_1, S_2) \left(\prod_{i \in S_1} x_i^+ \cdot \prod_{j \in S_2} x_j^- \right)$$

for $\boldsymbol{x} \in [-1, 1]^N$.

3.2 Monotonicity and Derivatives

As seen in Section 2.2 and 2.4, the definition of the derivative is the key concept for the interaction index. Grabisch and Labreuche [23] extended the notion of *discrete derivative* of ordinary cooperative games to that of bi-cooperative games.

Definition 24 **((T_1, T_2)-derivative of bi-cooperative game).** Let $(N, v, \mathcal{Q}(N))$ be a bi-cooperative game. For $(T_1, T_2) \in \mathcal{Q}(N)$, the (T_1, T_2)-*derivative* at a point $(S_1, S_2 \cup T_2) \in \mathcal{Q}(N)$, where $(S_1, S_2) \in \mathcal{Q}(N \setminus (T_1 \cup T_2))$, is denoted as $\Delta_{(T_1,T_2)} v(S_1, S_2 \cup T_2)$ and defined by

$$\begin{aligned}\Delta_{(T_1,T_2)} v(S_1, S_2 \cup T_2) &:= \sum_{\substack{L_1 \subseteq T_1\\ L_2 \subseteq T_2}} (-1)^{|T_1 \setminus L_1| + |L_2|}\, v(S_1 \cup L_1, S_2 \cup L_2) \\ &= \sum_{(S_1, S_2 \cup T_2) \sqsubseteq (A_1, A_2) \sqsubseteq (S_1 \cup T_1, S_2)} (-1)^{|A_1 \setminus S_1| + |A_2 \setminus S_2|}\, v(A_1, A_2).\end{aligned} \tag{7}$$

The formula (7) is led by the following recursive relations [24]:

$$\begin{aligned}\Delta_{(i,\emptyset)} v(S_1, S_2) &:= v(S_1 \cup i, S_2) - v(S_1, S_2), \\ \Delta_{(\emptyset,j)} v(S_1, S_2 \cup j) &:= v(S_1, S_2) - v(S_1, S_2 \cup j), \\ \Delta_{(T_1,T_2)} v(S_1, S_2 \cup T_2) &:= \Delta_{(i,\emptyset)} \left(\Delta_{(T_1 \setminus i, T_2)} v(S_1, S_2 \cup T_2)\right) \\ &= \Delta_{(\emptyset,j)} \left(\Delta_{(T_1, T_2 \setminus j)} v(S_1, (S_2 \cup j) \cup (T_2 \setminus j))\right),\end{aligned}$$

where $i \in T_1, j \in T_2$.

Example 2. Let us consider the $(12, 3)$-derivative at $(\emptyset, 3)$. Then, $\Delta_{(12,3)} v(\emptyset, 3)$ is represented by

$$\Delta_{(12,3)} v(\emptyset, 3) = \Delta_{(1,3)} v(2, 3) - \Delta_{(1,3)} v(\emptyset, 3).$$

The derivatives $\Delta_{(1,3)} v(2, 3)$ and $\Delta_{(1,3)} v(\emptyset, 3)$ are represented by

$$\Delta_{(1,3)} v(2, 3) = \Delta_{(\emptyset,3)} v(12, 3) - \Delta_{(\emptyset,3)} v(2, 3) \text{ and } \Delta_{(1,3)} v(\emptyset, 3) = \Delta_{(\emptyset,3)} v(1, 3) - \Delta_{(\emptyset,3)} v(\emptyset, 3),$$

respectively. The derivatives $\Delta_{(\emptyset,3)} v(12, 3)$, $\Delta_{(\emptyset,3)} v(2, 3)$, $\Delta_{(\emptyset,3)} v(1, 3)$ and $\Delta_{(\emptyset,3)} v(\emptyset, 3)$ are represented by

$$\Delta_{(\emptyset,3)} v(12, 3) = v(12, \emptyset) - v(12, 3), \quad \Delta_{(\emptyset,3)} v(2, 3) = v(2, \emptyset) - v(2, 3),$$

$$\Delta_{(\emptyset,3)} v(1, 3) = v(1, \emptyset) - v(1, 3), \text{ and } \Delta_{(\emptyset,3)} v(\emptyset, 3) = v(\emptyset, \emptyset) - v(\emptyset, 3),$$

respectively. Inversely, first, consider the first order derivatives $\Delta_{(\emptyset,3)}$ at $(\emptyset, 3)$, $(1, 3)$, $(2, 3)$, and $(12, 3)$. These derivatives correspond to thin arrow lines in Fig. 2, respectively. Second, the second order derivatives $\Delta_{(1,3)}$ at $(\emptyset, 3)$ and $(2, 3)$ correspond to thick black arrow lines in Fig. 2, which represent the differences between the first order derivatives represented by thin arrow lines. Finally, the $(12, 3)$-derivative at $(\emptyset, 3)$ corresponds to the thick gray arrow line in Fig. 2.

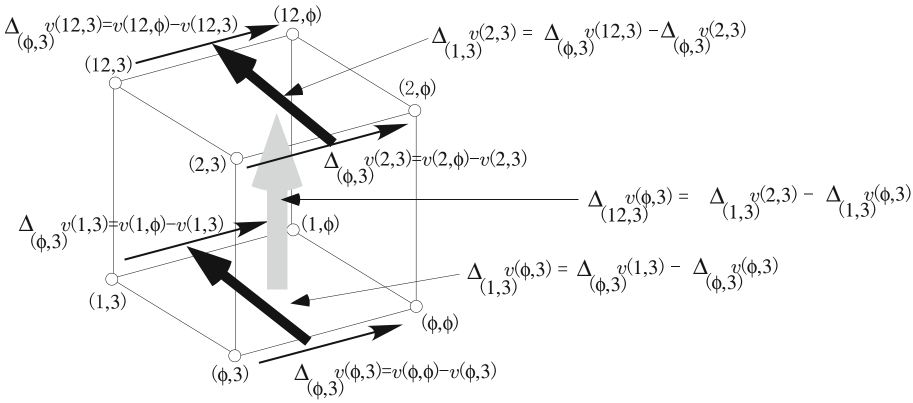

Fig. 2. The (12, 3)-derivative at $(\emptyset, 3)$

Proposition 5 ([20,23]). *For* $(T_1, T_2) \in \mathcal{Q}(N)$,

$$\Delta_{(T_1,T_2)} v(S_1, S_2 \cup T_2) = \sum_{(T_1, N\setminus(T_1\cup T_2)) \sqsubseteq (A_1,A_2) \sqsubseteq (S_1\cup T_1, S_2)} \Delta^v(A_1, A_2)$$
$$= (-1)^{|T_2|} \sum_{\substack{L_1 \subseteq S_1 \\ L_2 \subseteq S_2}} b^v(L_1 \cup T_1, L_2 \cup T_2)$$

for any $(S_1, S_2) \in \mathcal{Q}(N \setminus (T_1 \cup T_2))$. *Thus,*

$$\Delta^v(T_1, T_2) = \Delta_{(T_1,T_2)} v(\emptyset, N \setminus T_1) \quad \forall (T_1, T_2) \in \mathcal{Q}(N).$$

Proposition 6 ([20]). *Let* $v : \mathcal{Q}(N) \to \mathbb{R}$ *be a bi-cooperative game and* $g_v : [-1, 1]^N \to \mathbb{R}$ *the piecewise multilinear extension of the ternary pseudo-Boolean function corresponding to v, and* $(T_1, T_2) := (\{t_1(1), \ldots, t_1(p)\}, \{t_2(1), \ldots, t_2(q)\}) \in \mathcal{Q}(N)$, *where* $|T_1| = p$ *and* $|T_2| = q$. *Then,*

$$\Delta_{(T_1,T_2)} v(S_1, S_2 \cup T_2) = \left. \frac{\partial^{(p+q)}}{\partial x_{t_1(1)} \cdots \partial x_{t_1(p)} \partial x_{t_2(1)} \cdots \partial x_{t_2(q)}} g_v(\boldsymbol{x}) \right|_{\boldsymbol{x} = \chi_{(S_1 \cup T_1, S_2 \cup T_2)}}$$

for all $(S_1, S_2) \in \mathcal{Q}(N \setminus (T_1 \cup T_2))$, *where* $\chi_{(S_1\cup T_1, S_2 \cup T_2)} \in [-1, 1]^N$ *is the characteristic vector of* $(S_1 \cup T_1, S_2 \cup T_2) \in \mathcal{Q}(N)$. *It should be noticed that* $\frac{\partial}{\partial x_i} g_v$ *cannot be defined on* $\{\boldsymbol{x} \in [-1, 1]^N \mid x_i = 0\}$.

Labreuche and Grabisch [42] have proposed the notion of *k-monotonicity* in bi-cooperative games as a bipolar extension of that in ordinary cooperative games.

Definition 25 (*k*-monotonic bi-cooperative game [24,42]). Given an integer $k \geq 2$, a bi-cooperative game $v : \mathcal{Q}(N) \to \mathbb{R}$ is said to be *k-order monotone* (for short, *k-monotone*) if and only if, for any (at most) k bi-coalitions $S_1, \cdots, S_k$, we have

$$v\left(\bigsqcup_{i=1}^{k} S_i \right) \geq \sum_{\substack{J \subseteq \{1,\cdots,k\} \\ J \neq \emptyset}} (-1)^{|J|+1} v\left(\bigsqcap_{j \in J} S_j \right). \tag{8}$$

It is easy to verify that k-monotonicity ($k > 2$) implies l-monotonicity for all integer $2 \le l \le k$. By extension, 1-monotonicity (which does not correspond to $k = 1$ in Eq. (8)) is defined as standard monotonicity, i.e.,

$$v(S) \le v(T) \quad \text{whenever} \quad (\emptyset, N) \sqsubseteq S \sqsubseteq T \sqsubseteq (N, \emptyset).$$

The notion of k-monotonicity of bi-cooperative games can be characterized via discrete derivatives in a similar way to ordinary cases.

Proposition 7 ([18]). *Let $v : \mathcal{Q}(N) \to \mathbb{R}$ be a bi-cooperative game and k a positive integer. Then v is k order monotone if and only if*

$$\Delta_{(T_1,T_2)} v(S_1, S_2 \cup T_2) \ge 0$$

for any $(T_1, T_2) \in \mathcal{Q}(N)$ such as $1 \le |T_1 \cup T_2| \le k$ and any $(S_1, S_2) \in \mathcal{Q}(N \setminus (T_1 \cup T_2))$.

3.3 Value and Interaction Index in Bi-cooperative Games

In this subsection, we introduce the notion of value and power index for bi-cooperative games and ternary simple games, following Labreuche and Grabisch [43], in the spirit of what was done by Shapley [59] for cooperative games, and by Shapley and Shubik [60] for simple games. In ordinary cooperative games, an allocation rule (pre-imputation) is a vector $\boldsymbol{x} \in \mathbb{R}^N$ which represents the share of the total worth of the game $v(N)$ among the players, assuming that all players have decided to join the grand coalition N. For bi-cooperative games, the situation differs since apart from not participating to the game, each player has two possible actions, namely to play in the defender or the defeater part, while he/she has only one in classical (ordinary cooperative game) case. In order to generalize the notion of imputation, the concept of *reference action* or *level* has been introduced. The reference action is the action such that if all players do this action, then the outcome of the game is 0. For ordinary games, the reference action is to "not participate" since $v(\emptyset) = 0$. For bi-cooperative games, it is also the non participation since $v(\emptyset, \emptyset) = 0$. An imputation is defined for each possible action (except the reference one) of a player with respect to the reference action, that is, it represents a kind of average contribution of the player for a given action, compared to the reference action. For bi-cooperative games, the possible actions are: to play in the defender part, or to play in the defeater part. For preserving the meaning of "contribution" (which has a positive sense) and for compatibility with previous works, it requires to consider two values ϕ^+ (defender part) and ϕ^- (defeater part): ϕ^+ is the contribution of "playing in the defender part" instead of "doing nothing", and ϕ^- is the contribution of "doing nothing" instead of "playing in the defeater part". Then, an overall contribution ϕ can be defined as $\phi = \phi^+ + \phi^-$.

Labreuche and Grabisch [43] have proposed the following value for bi-cooperative games, axiomatically.

Definition 26 (value of bi-cooperative game [43]). The *(Shapley-type) value* ϕ^B of a bi-cooperative game $v : \mathcal{Q}(N) \to \mathbb{R}$ is given by

$$\begin{aligned}\phi_i^B(v) &:= \sum_{S \subseteq N \setminus i} \frac{s!\,(n-s-1)!}{n!} \left[v(S \cup i, N \setminus (S \cup i)) - v(S, N \setminus S) \right] \\ &= \sum_{(S,T) \sqsupseteq (\emptyset, N \setminus i)} \frac{1}{n-t} \Delta^v(S,T)\end{aligned}$$

for every $i \in N$, where $\phi_i^B(v)$ is the i-th component of $\phi^B(v) \in \mathbb{R}^N$. The value ϕ^B is decomposable into two values ϕ^+ (defender part) and ϕ^- (defeater part), i.e., $\phi^B = \phi^+ + \phi^-$, as follows:

$$\phi_i^+(v) = \sum_{S \subseteq N \setminus i} \frac{s!\,(n-s-1)!}{n!} \left[v(S \cup i, N \setminus (S \cup i)) - v(S, N \setminus (S \cup i)) \right],$$

$$\phi_i^-(v) = \sum_{S \subseteq N \setminus i} \frac{s!\,(n-s-1)!}{n!} \left[v(S, N \setminus (S \cup i)) - v(S, N \setminus S) \right].$$

The (Shapley-type) value ϕ^B for bi-cooperative game is a kind of generalization of the Shapley value of ordinary cooperative games in the following sense:

> *If a bi-cooperative game $v^B : \mathcal{Q}(N) \to \mathbb{R}$ is given via some ordinary cooperative game $v : 2^N \to \mathbb{R}$ as*
>
> $$v^B(S,T) = \begin{cases} v(S) & \text{if } T = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$
>
> *for any $(S,T) \in \mathcal{Q}(N)$, then*
>
> $$\phi^B(v^B) = \phi^+(v^B) = \phi(v), \quad \phi^-(v^B) = 0,$$
>
> *where $\phi(v)$ is the Shapley value of ordinary game v.*

The *(Shapley-type) interaction index for bi-cooperative games* has been introduced by Grabisch and Labreuche [24,26] and characterized axiomatically by Lange and Grabisch [45], by analogy with that for ordinary cooperative games.

Definition 27 (interaction index for bi-cooperative game). The *(Shapley-type) interaction index* $I^B(v,(S,T))$ with respect to $(S,T) \in \mathcal{Q}(N)$ of a bi-cooperative game $v : \mathcal{Q}(N) \to \mathbb{R}$ is defined by

$$\begin{aligned}I^B(v,(S,T)) &:= \sum_{U \subseteq N \setminus (S \cup T)} \frac{(n-s-t-u)!\,u!}{(n-s-t+1)!} \Delta_{(S,T)}(U, N \setminus (S \cup U)) \\ &= \sum_{\substack{(A,B) \sqsupseteq (S, N \setminus (S \cup T)) \\ (A,B) \sqsubseteq (N \setminus T, \emptyset)}} \frac{1}{n-s-t-b+1} \Delta^v(A,B).\end{aligned}$$

Kojadinovic [41] has proposed another interaction index for bi-cooperative games in the context of aggregation by the bipolar Choquet integral [25], however his solution is not completely axiomatized.

Definition 28 (Kojadinovic's interaction index). The *(Kojadinovic-type) interaction index* $I^K(v,(S_1,S_2))$ with respect to $(S_1,S_2) \in \mathcal{Q}(N)$ of a bi-cooperative game $v : \mathcal{Q}(N) \to \mathbb{R}$ is defined by

$$I^K(v,(S_1,S_2)) := \sum_{(T_1,T_2)\in\mathcal{Q}(N\setminus S)} \frac{1}{2^t}\frac{(n-s-t+1)!\,t!}{(n-s+1)!}\Delta_{(S_1,S_2)}(T_1,T_2\cup S_2),$$

where $T := T_1 \cup T_2$ and $S := S_1 \cup S_2$.

Note (Ternary voting games and its values) : Bi-cooperative games are a generalization of the notion of *ternary voting game* which has been proposed by Felsenthal and Machover [15]. In a play of a ternary voting game, each player can choose between voting *in favor* (yes), *against* (no), or *abstaining*. Formally, a ternary voting game is a function $v : \mathcal{Q}(N) \to \{-1,1\}$, where $v(F,A)$ represents the result of the vote (1 if the bill is passed, -1 if it is defeated) when voters in F vote in favor, voters in A vote against, and the remaining voters abstain. Then, obviously, it should be satisfied that

$$v(\emptyset,N) = -1, \quad v(N,\emptyset) = 1, \text{ and } \quad (F_1,A_1) \sqsubseteq (F_2,A_2) \implies v(F_1,A_1) \le v(F_2,A_2).$$

Definition 29 (ternary roll-call and pivot). A *ternary roll-call* R is a triplet $R = (\sigma_R,F_R,A_R)$ consisting of a permutation σ_R on N, a coalition F_R which contains all voters that are in favor of the bill, and a coalition A_R which contains all voters that are against the bill. Ternary roll-calls are interpreted as follows. The voters are called in order given by $\sigma_R : \sigma_R(1),\ldots,\sigma_R(n)$. When a voter i is called, he/she tells his/her opinion, that is to say *in favor* if $i \in F_R$, *against* if $i \in A_R$, or abstention otherwise. The set of all ternary roll-calls on N is denoted by $\mathcal{T}_N$, whose cardinality is $3^n \cdot n!$.

A *pivot* $Piv(v,R)$ for a ternary voting game $(N,v,\mathcal{Q}(N))$ and a ternary roll-call $R = (\sigma_R,F_R,A_R)$ is the player i, represented by $i = \sigma_R(m)$ for some $m \in \{1,\ldots,n\}$, satisfying the following two conditions:

1. $v(F_R^{m-1},A_R^{m-1}) \neq v(F_R^m,A_R^m)$ if $m \neq 1$.
2. $v(F_R^m,A_R^m) = v(F_R^k,A_R^k)$ for any $k > m$,

where $F_R^k := \bigcup_{j\le k}\{\sigma_R(j)\}\cap F_R$ and $A_R^k := \bigcup_{j\le k}\{\sigma_R(j)\}\cap A_R$. That is, $Piv(v,R)$ is decisive in the result of the vote.

Felsenthal and Machover [15] have proposed a voting power index $\phi^{ter}(v)$ for a ternary voting game $(N,v,\mathcal{Q}(N))$, which is a generalization of the Shapley-Shubik power index, as follows:

$$\phi_i^{ter}(v) := \frac{|\{R \in \mathcal{T}_N \mid i = Piv(v,R)\}|}{|\mathcal{T}_N|}.$$

However, this solution is not completely axiomatized.

4 Cooperative Game and Network

In ordinary cooperative game theory, it is implicitly assumed that all coalitions of N can be formed, this is in general not the case. In order for players to be able to coordinate their actions, they have to be able to communicate. The bilateral communication channels between players in N are described by a *communication network*. Such a network can be represented by an *undirected graph* $G = (N, E)$, which has the set of players as its *nodes* $S \subseteq N$ and in which those players are connected by the set of *edges/links* $E \subseteq \{ij \mid i, j \in N, i \neq j\}$, i.e., players i and j can communicate (directly) with each other if $ij \in E$. To avoid cumbersome notations, we often omit braces for a graph $G = (N, E)$, we denote $E - ij := E \setminus ij$ for any $ij \in E$, $E + ij := E \cup ij$ for any $ij \notin E$, and $G(S) := (S, E(S))$ for any $S \subseteq N$, where $E(S) := \{ij \in E \mid i, j \in S\}$. Then, $G(S)$ is called the subgraph induced from the underlying graph G and the subset S of N.

The number of the nodes adjacent to a node $i \in N$ is said to be the *degree* of i in $G = (N, E)$ and denoted by $a^G(i)$. A sequence of different nodes $(i_1, \ldots, i_m)$ is called a *path*, whose length is $m - 1$, between j and k in a graph (N, E) if $j = i_1$, $k = i_m$, and $\{i_l, i_{l+1}\} \in E$ for any $l \in \{1, \ldots, m-1\}$. If there is a path between j and k in an undirected graph G, then we say that j is *reachable* to k in G and denote $j \sim_G k$. A set of nodes $S \subseteq N$ is called *connected* in an undirected graph $G := (N, E)$ if for any $i, j \in S$, $i \neq j$, there exists a path $(i_1, \ldots, i_m)$ between i and j in G satisfying that all nodes of the path are in S, i.e., $i_k \in S$ for any $k \in \{1, \ldots, m\}$. Notice that, by definition, the empty set and all singletons are *connected*. An undirected graph $G = (N, E)$ is said to be connected if N is connected in G. Clearly, the relation $\sim_G$ is an equivalence relation on N. Hence, the notion of reachableness induces a partition $N/E := N/\sim_G$ of N. Then, for any $S \subseteq N$, $C \in S/E(S) = S/\sim_{G(S)}$ is called a *(connected) component* of S. A *geodesic* (also often called a "shortest" path) between two nodes $i, j \in N$ is a path whose length is the minimum among all paths between i and j. The length of a geodesic between two nodes $i, j \in N$ in G, if i and j are reachable, is called their *geodesic distance*. If i and j are unreachable, then the distance between i and j is infinite. A sequence of nodes $(i_1, \ldots, i_m)$ is called a *cycle* if $i_1 = i_m$ and $(i_1, \ldots, i_{m-1})$ is a path. A graph is *cycle free* if it does not contain any cycle. A connected cycle free graph is called *tree*.

Definition 30 (communication situation). The triplet (N, v, E), which reflects a situation consisting of a game $v \in \mathcal{G}^N$ and a communication network (N, E), is called a *communication situation*. We denote the set of all communication situations on N by CS^N.

Definition 31 (feasible coalition). A coalition $S \subseteq N$ is said to be *feasible* in the communication network $G = (N, E)$ if S is connected in G (i.e., $S/E = \{S\}$).

Example 3. Consider the communication situation (N_1, v, E_1) with $N_1 = \{1, 2, \cdots, 7\}$ and $E_1 = \{12, 15, 26, 37, 47, 56\}$ (Fig.3). Then, all the players in $\{1, 2, 6\}$ can communicate with one another, i.e., the coalition $\{1, 2, 6\}$ is feasible. Hence, they can fully coordinate their actions and obtain the value $v(\{1, 2, 6\})$. On the other hand, in the coalition $\{1, 2, 3, 4\}$, players 1 and 2 are reachable, however, both of players 3 and 4 cannot communicate with any other players in $\{1, 2, 3, 4\}$. Then, feasible subcoalitions of

$\{1,2,3,4\}$ are $\{1,2\}$, $\{3\}$, and $\{4\}$ (i.e., $\{1,2,3,4\}/E_1 = \{12,3,4\}$, thus forming the coalition $\{1,2,3,4\}$ is unfeasible). Hence, the value attainable by the players in $\{1,2,3,4\}$ should be $v(1,2)+v(3)+v(4)$. That is, in general, the value attainable by the players in S under a communication situation (N,v,E) is represented by $\sum_{T\in S/E} v(T)$.

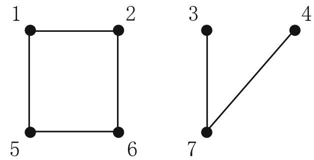

Fig. 3. Communication Network(N_1, E_1)

Definition 32 (network-restricted game [52]). The *network-restricted game* (N, v^E) associated with (N,v,E) is defined by

$$v^E(S) := \sum_{T\in S/E} v(T) \quad \text{for each } S \subseteq N. \tag{9}$$

Note that if (N,E) is the complete graph (i.e., $E = \{ij \mid i,j \in N, i \neq j\}$), the network-restricted game v^E is equal to the original game v.

The network-restricted game evaluates the possible gains from cooperation in a communication situation from the point of view of the players. Next example focuses on the importance of communication channels/links in a communication situation.

Example 4. In the communication network E_1 represented by Fig.3, the value obtainable by the players in the grand coalition N is

$$v^{E_1}(N) = v(\{1,2,5,6\}) + v(\{3,4,7\}),$$

since $N/E_1 = \{\{1,2,5,6\},\{3,4,7\}\}$. If for some reason the communication link between players 4 and 7 is lost, the communication network E_1 turns to a new communication network $E_2 = \{12,15,26,37,56\}$. Then, $N/E_2 = \{\{1,2,5,6\},\{3,7\},\{4\}\}$ and the value obtainable by the players in the grand coalition N turns to

$$v^{E_2}(N) = v(\{1,2,5,6\}) + v(\{3,7\}) + v(\{4\}).$$

Then $v^{E_1}(N) - v^{E_2}(N)$ can be interpreted as a kind of marginal contribution of the link $47 \in E_1$ to the communication network E_1.

Definition 33 (link game [7]). The *link game* associated with (N,v,E) consisting of a zero-normalized game v (i.e., $v(i) = 0$ for any $i \in N$) is a game on E defined by

$$\gamma^v(H) := v^H(N) = \sum_{T\in N/H} v(T) \quad \text{for each } H \subseteq E.$$

The link game $\gamma^v(H)$ represents the worth of communication network $H \subseteq E$ as the worth of the grand coalition in the communication situation (N, v, H) through the network-restricted game v^H. Note that, for an ordinary game v, the link game γ^v is generally not a game on E since $\gamma^v(\emptyset) = \sum_{T \in N/\emptyset} v(T) = \sum_{i \in N} v(i) \neq 0$.

Example 5 (wighted majory voting game). Consider the weighted majority voting situation $(N, [q : s_1, \ldots, s_n])$ with $N = \{1, 2, 3, 4\}$, $s_1 = 35$, $s_2 = 30$, $s_3 = 25$, $s_4 = 10$, and $q = 51$. So, there are 100 members of parliament who are divided among four political parties labeled 1,2,3, and 4, and decisions are made by majority voting. The parties 1,2,3, and 4 have 35, 30, 25, 10 seats, respectively. This situation can be represented by the game v such that

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} s_i \geq q = 51, \text{ i.e., win,} \\ 0 & \text{if } \sum_{i \in S} s_i < q = 51, \text{ i.e., lose.} \end{cases}$$

Then, the winning coalitions are $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$, $\{2, 3, 4\}$, and $\{1, 2, 3, 4\}$. In general, every coalition $S \subseteq N$ could not been formed, due to ideological and policy differences. Suppose that the party 4 cannot form coalitions with any other parties due to ideological differences; the parties 1 and 3 cannot form the coalition $\{1, 3\}$ due to some policy differences but can form a coalition $\{1, 2, 3\}$ through an intermediary, the party 2. Such a situation can be represented by the graph $G := (N, E)$ as shown in **Fig.**4.

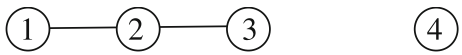

Fig. 4. Relations among political parties

In the network restricted game v^E, the winning coalition are $\{1, 2\}$, $\{2, 3\}$, $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{2, 3, 4\}$, and $\{1, 2, 3, 4\}$; for instance, $v^E(13) = v(1) + v(3) = 0 + 0 = 0$, i.e., lose, while $v^E(124) = v(12) + v(4) = 1 + 0 = 1$, i.e., win. (However, *feasible* winning coalitions are only $\{1, 2\}$, $\{2, 3\}$, and $\{1, 2, 3\}$).

4.1 Allocation Rule in Communication Situation

In this subsection, we will briefly introduce major two existing values (allocation rules), the *Myerson value* [52] and *the position value* [7], for communication situations.

Definition 34 (the Myerson value [52]). The *Myerson value* for a communication situation (N, v, E) is denoted as $\Psi(N, v, E)$ and defined by

$$\Psi(N, v, E) := \phi(v^E),$$

where $\phi(N, v^E)$ is the Shapley value of (N, v^E). Note that the $\Psi(N, v, E) = \phi(v)$ if (N, E) is the complete graph.

The Myerson value is one of the most famous allocation rules, which assigns to every communication situation (N, v, E) the Shapley value of the network-restricted game (N, v^E) and is characterized as the unique allocation rule satisfying the following two properties/axioms, *component efficiency* and *fairness* (see, Myerson [52]).

Definition 35 (componennt efficiency). For any communication situation $(N, v, E) \in CS^N$, it holds that

$$\sum_{i \in S} \Psi_i(N, v, E) = v(S)$$

for any $S \in N/E$, where $\Psi_i(N, v, E)$ is the i-th component of $\Psi(N, v, E)$.

Definition 36 (fairness). For any communication situation $(N, v, E) \in CS^N$, it holds that

$$\Psi_i(N, v, E) - \Psi_i(N, v, E - ij) = \Psi_j(N, v, E) - \Psi_j(N, v, E - ij)$$

for any $ij \in E$, where $\Psi_i(N, v, E)$ is the i-th component of $\Psi(N, v, E)$.

Component efficiency means that the sum of the players' allocations in a component equal to the worth of the component. Fairness means that the two players connected by a link obtain the same change of allocation if the link is deleted.

Definition 37 (position value [7]). The *position value* for a communication situation (N, v, E) consisting of zero-normalized game v is denoted as $\pi(N, v, E)$ and defined by

$$\pi_i(N, v, E) := \frac{1}{2} \sum_{\substack{e \in E \\ e \ni i}} \phi_e(\gamma^v) \quad \text{for each } i \in N.$$

The Shapley value $\phi_e(\gamma^v)$ of a link $e \in E$ can be interpreted as a kind of *expected marginal contribution* of the link (edge) $e \in E$ to all communication networks containing e. Then, the value is divided equally between the two players at the ends of the considered link $e \in E$. The position value of a given player $i \in N$ is obtained as the sum of all these shares.

Example 6. Consider the communication situation in Example 5. Then, the Shapley value of the underlying game (i.e., the Shapley-Shubik index [60]) is $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$; the Myerson value for the communication situation (N, v, E) (i.e., the Shapley value of the network-restricted game (N, v^E)) is $(\frac{1}{6}, \frac{4}{6}, \frac{1}{6}, 0)$; the position value for the communication situation (N, v, E) is $(\frac{1}{4}, \frac{1}{2}, \frac{1}{4}, 0)$.

4.2 Poset Induced by Communication Network

In this subsection, we consider and introduce a subposet of $(2^N, \subseteq)$ induced by a communication network $G := (N, E)$.

For a communication network $G := (N, E)$, the set of all feasible coalitions in G is denoted as $\mathfrak{F}(G)$, i.e.,

$$\mathfrak{F}(G) := \{S \subseteq N \mid S : \text{connected in } G := (N, E),\ \textit{i.e.}, |S/E| = 1\}.$$

The set $\mathfrak{F}(G)$ together with set inclusion $\subseteq$ as an order on $\mathfrak{F}(G)$ is called the *poset induced by a communication network G*.

Example 7. Let $N = \{1, 2, 3\}$, $E_a = \{12, 13, 23\}$, $E_b = \{13, 23\}$, and $E_c = \{12\}$. Then the posets induced by communication networks $G_a := (N, E_a)$, $G_b := (N, E_b)$, and $G_c := (N, E_c)$, as shown in (a) – (c) in Fig. 5, are represented as shown in (a) – (c) in Fig. 6, respectively.

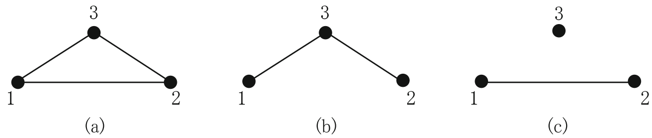

Fig. 5. Communication networks on $N = \{1, 2, 3\}$

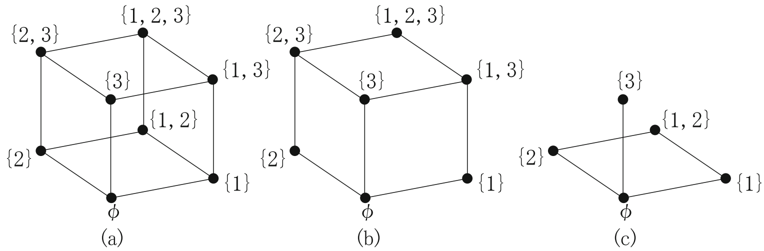

Fig. 6. Posets corresponding to networks in Fig. 5

In a communication situation (N, v, E) consisting of a game (N, v) and a communication network $G := (N, E)$, at least two types of restrictions of $v \in \mathcal{G}^N$ can be considered. One is the network-restricted game v^E defined by Eq. (9). Another is the restriction of $v \in \mathcal{G}^N$, i.e., $v : 2^N \to \mathbb{R}$, to the poset $\mathfrak{F}(G)$ induced by $G = (N, E)$, i.e., $v|_{\mathfrak{F}(G)} : \mathfrak{F}(G) \to \mathbb{R}$. Then it is denoted as $v^{\mathfrak{F}(G)} : \mathfrak{F}(G) \to \mathbb{R}$ and defined by

$$v^{\mathfrak{F}(G)}(S) := v|_{\mathfrak{F}(G)}(S), \ \ i.e., v^{\mathfrak{F}(G)}(S) := v(S), \quad \forall S \in \mathfrak{F}(G). \tag{10}$$

Definition 38 (Möbius transform on poset). Let (N, v, E) be a communication situation consisting of a game (N, v) and a communication network $G := (N, E)$. Then, the *Möbius transform* of $v^{\mathfrak{F}(G)} : \mathfrak{F}(G) \to \mathbb{R}$ on the poset $(\mathfrak{F}(G), \subseteq)$ is denoted by $\Delta^{(N,v,E)}$ and defined through the following equation:

$$v^{\mathfrak{F}(G)}(S) = \sum_{\substack{T \in \mathfrak{F}(G) \\ T \subseteq S}} \Delta^{(N,v,E)}(T) \quad \forall S \in \mathfrak{F}(G).$$

Conversely, $\Delta^{(N,v,E)}$ is explicitly represented by

$$\Delta^{(N,v,E)}(S) = \sum_{\substack{T\in\mathfrak{F}(G)\\ T\subseteq S}} (-1)^{|S\setminus T|}\, v^{\mathfrak{F}(G)}(T) \quad \forall S \in \mathfrak{F}(G).$$

Proposition 8. *Let (N, v, E) be a communication situation consisting of a game (N, v) and a communication network $G := (N, E)$, (N, v^E) denote the network-restricted game defined by Eq. (9), and $v^{\mathfrak{F}(G)}$ the restriction of v to $\mathfrak{F}(G)$ defined by Eq. (10). Then,*

$$\Delta^{(N,v,E)}(S) = \Delta^{v^E}(S) \quad \forall S \in \mathfrak{F}(G),$$
$$\Delta^{v^E}(S) = 0 \quad \forall S \in 2^N \setminus \mathfrak{F}(G).$$

Moreover, the Myerson value $\Psi(N, v, E)$ can be represented as

$$\Psi_i(N, v, E) = \sum_{i\in S\in\mathfrak{F}(G)} \frac{1}{s}\, \Delta^{(N,v,E)}(S).$$

4.3 Harsanyi Power Solution for Communication Situation

In this subsection, we introduce a class of allocation rules, *Harsanyi power solutions*, to which many existing allocation rules for communication situations belong. Briks et al. [9] have introduced the concept of *Harsanyi power solution* for communication situations, which is based on Harsanyi solutions for TU-games. The concept of Harsanyi solution is proposed as a class of solutions for TU-games in Vasil'ev [63,64] (see also Derks et al. [12], where a Harsanyi solution is called a sharing value). The idea behind a Harsanyi solution is that it distributes the Harsanyi dividends over the players in the corresponding coalitions according to a chosen sharing system which assigns to every coalition S a sharing vector specifying for every player in S its share in the dividend $\Delta^v(S)$ of S. The payoff to each player $i \in N$ is thus equal to the sum of its shares in the dividends of all coalitions of which he is a member. A famous Harsanyi solution is the Shapley value.

Now, we consider the case $N = \{1, 2\}$, the Shapley value $\phi_1(N, v)$ of player 1 in a game (N, v) is obtained as

$$\phi_1(N, v) = \frac{1}{1}\Delta^v(1) + \frac{1}{2}\Delta^v(12).$$

This expression, as an allocation rule of *Harsanyi dividends* (i.e., the Möbius transform), has the following (at least two) interpretations:

Interpretation 1 (Egalitarian allocation) : *The Shapley value distributes the dividend of any coalition S equally among the players in S, i.e., $\frac{1}{s}\Delta^v(S)$, (so players outside S do not share in the dividend of S).*

Interpretation 2 (Allocation based on coalition forming process) : *We consider a process to form the coalition* $\{1,2\}$. *Then, there are two shortest paths from* $\emptyset$ *to* $\{1,2\}$ *in Fig. 7* (a). *One is the path* $\emptyset \to \{1\} \to \{1,2\}$*; another is the path* $\emptyset \to \{2\} \to \{1,2\}$. *The path* $\emptyset \to \{1\} \to \{1,2\}$ *can be interpreted as follows: Player 1 makes an offer to player 2 for forming the coalition* $\{1,2\}$. *Player 2 accepts the offer and adds to the coalition* $\{1\}$ *to form the new coalition* $\{1,2\}$. *Among these two paths, the only path that passes through* $\{1\}$, *i.e., the player 1 plays a role of initiator in forming* $\{1,2\}$, *is* $\emptyset \to \{1\} \to \{1,2\}$. *That is, the number of paths from* $\emptyset$ *to* $\{1,2\}$ *is 2, while the number of paths via* $\{1\}$ *is 1. Then player 1 obtains* $\frac{1 \text{ path}}{2 \text{ paths}}$ *of the amount of the Harsanyi dividend* $\Delta^v(12)$ *(i.e.,* $\frac{1}{2}\Delta^v(12)$*). In the same way, player 1 obtains* $\frac{1}{1}\Delta^v(1)$ *and* $\frac{0}{1}\Delta^v(2)$. *The Shapley value of player 1 is obtained as the sum of all these shares, i.e.,* $\frac{1}{1}\Delta^v(1) + \frac{0}{1}\Delta^v(2) + \frac{1}{2}\Delta^v(12)$. *This allocation rule can be extended to the case* $N = \{1,2,3\}$, *e.g., there are six shortest paths from* $\emptyset$ *to* $\{1,2,3\}$ *(see, Fig. 7* (b)*). Among them, two paths,* $\emptyset \to \{1\} \to \{1,2\} \to \{1,2,3\}$ *and* $\emptyset \to \{1\} \to \{1,3\} \to \{1,2,3\}$, *pass through* $\{1\}$. *Then, the following holds:*

$$\phi_1(\{1,2,3\},v) = \frac{1}{1}\,\Delta^v(1) + \frac{1}{2}\,\Delta^v(12) + \frac{1}{2}\,\Delta^v(13) + \frac{0}{2}\,\Delta^v(23) + \frac{2}{6}\,\Delta^v(123).$$

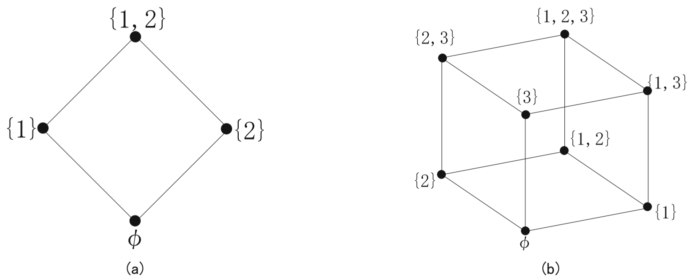

Fig. 7. The Boolean lattice $B(2)$ and $B(3)$

Definition 39 (sharing system [9]). Let (N, v, E) be a communication situation consisting of a game (N, v) and a communication network $G := (N, E)$. A *sharing system* on $\mathfrak{F}(G)$ is a system $p := (p^S)_{S \in \mathfrak{F}(G)}$, where p^S is a s-dimensional vector assigning a non-negative share $p_i^S \geq 0$ to every player $i \in S$ with $\sum_{j \in S} p_j^S = 1$, for any $S \in \mathfrak{F}(G)$.

Definition 40 (the Harsanyi solution [9]). We denote the collection of all sharing systems on $\mathfrak{F}(G)$ by $\mathfrak{S}^G$. For a communication situation $(N, v, E) \in CS^N$ and a sharing system $p \in \mathfrak{S}^G$, the corresponding *Harsanyi payoff vector* is the payoff vector $H^p(N, v, E) \in \mathbb{R}^N$ given by

$$H_i^p(N, v, E) = \sum_{i \in S \in \mathfrak{F}(G)} p_i^S\, \Delta^{\mathfrak{F}(G)}(S) = \sum_{i \in S \in \mathfrak{F}(G)} p_i^S\, \Delta^{v^E}(S) \quad \forall i \in N.$$

A *Harsanyi solution*, as an allocation rule, assigns for a given sharing system $p \in \mathfrak{S}^G$ the Harsanyi payoff vector $H^p(N, v, E)$ to each communication situation (N, v, E). Due to the equality $v(N) = \sum_{S \in \mathfrak{F}(G)} \Delta^{(N,v,E)}(S)$, we have

$$\sum_{i \in N} H_i^p(N, v, E) = v(N),$$

and thus each Harsanyi solution is efficient. The Shapley value is the Harsanyi solution that assigns to any communication situation $(N, v, \{ij \mid i, j \in N, i \neq j\})$ (i.e., to any ordinary cooperative game (N, v)), the Harsanyi payoff vector $H^p(N, v, \{ij \mid i, j \in N, i \neq j\})$ with the sharing system p given by $p_i^S = \frac{1}{s}$ for each $S \in \mathfrak{F}(G)$ containing i.

Definition 41 (Harsanyi power solution [9]). A *power measure* on graph $G = (N, E)$ is a function q that assigns to any subgraph $G(S) = (S, E(S))$, $S \subseteq N$ a non-negative vector $q(S, E(S)) \in \mathbb{R}_+^S$, yielding the non-negative power $q_i(S, E(S))$ of each node $i \in S$ in the graph $G(S)$. Then, given a positive power measure q, we can define the corresponding *Harsanyi power solution*, denoted by $H^{p(q)}(N, v, E)$, through the sharing system $p(q) = (p^S(q))_{S \in \mathfrak{F}(G)}$ induced by the power measure q as

$$p_i^S(q) = \frac{q_i(S, E(S))}{\sum_{j \in S} q_j(S, E(S))}$$

for all $i \in S$ whenever $\sum_{j \in S} q_j(S, E(S)) \neq 0$ and $p_i^S(q) = \frac{1}{s}$ if $\sum_{j \in S} q_j(S, E(S)) = 0$.

A characteristic of the Harsanyi power solutions for communication situations is that we associate a sharing system with some *power measure*, being a function which assigns a non-negative real number to every node in the graph, for the underlying communication networks. These numbers represent the strength or power of those nodes in the graph. Given a power measure we define the corresponding sharing system such that the share vectors for every coalition are proportional to the power measure of the corresponding subgraph.

Social network researchers have considered some fundamental properties of the individuals, that inform us about specific factors such as who is who in the network: who is leader, who is intermediary, who is nearly isolated, who is central, and who is peripheral. Here, we introduce several concepts of *degree* of *centrality* and *peripherality* for a node (position, actor, individual) in a network (undirected graph) as examples of power measures on undirected graphs [16,19,17].

Centrality measures [16,19]. *Centrality* is a sociological concept which is not clearly defined; it is frequently defined only in an undirected manner. For example, the literature presents several alternative definitions for centrality. We review some of these definitions below:

(Dc: degree centrality). It measures the degree to which an actor i can communicate directly with other actors:

$$q_i^{DEG}(S, E(S)) := \sum_{j \in S} a^{G(S)}(i).$$

(Cc: closeness centrality). It measures the degree to which an actor i is close to other actors:

$$q_i^{CLO}(S, E(S)) := \sum_{j \in S \setminus \{i\}} \frac{1}{d_{ij}^{G(S)}} .$$

(Bc: betweenness centrality). It measures the degree to which an actor i lies on the shortest paths between other actors in the network:

$$q_i^{BET}(S, E(S)) := \sum_{j,k \in S \setminus i} \frac{\text{\# of geodesics in } S \text{ from } j \text{ to } k \text{ via } i}{\text{\# of geodesics in } S \text{ from } j \text{ to } k}.$$

(Oc: originator centrality). It measures the degree to which an actor i is required as an initiator/originator in network-forming processes:

$$q_i^{ORI}(S, E(S)) := \frac{||i \to S||}{||\emptyset \to S||},$$

where $||i \to S||$ (resp., $||\emptyset \to S||$) means the number of shortest paths from i (resp., $\emptyset$) to S in the Hasse diagram of the poset induced by a graph $(S, E(S))$.

Peripherality Measure [17]. With regard to networks such as roads, railways, airways, the Internet, and others that use nodes or terminals such as airports and railway stations, etc, terminal cities/nodes benefits far more from direct/indirect access to big cites (important nodes or central hubs) than do big cities receive from connecting to terminal cities/nodes. Indeed, peripheral cities bear a heavier burden than central cities in the construction/extension of highways/railways. Fujimoto [17] has proposed a *peripherality measure* on undirected graphs axiomatically as follows.

$$q_i^{PER}(S, E(S)) := \frac{||\emptyset \to S \setminus i||}{||\emptyset \to S||}.$$

Note (The Myerson and Position Values as Harsanyi Power Solutions): Brinks et al. [9] pointed out and demonstrated that the Myerson and position values are typical Harsanyi power solutions with simple power measures for some types of communication situations.

Let (N, v, E) be a communication situation consisting of a game (N, v) and a communication network $G := (N, E)$. The Myerson value $\Psi(N, v, E)$ is the Harsanyi power solution with the sharing system p induced by the *egalitarian power measure* q^E, e.g.,

$$q_i^E(S, E(S)) = 1 \quad \forall S \in \mathfrak{F}(G),\ \forall i \in S,$$

i.e.,

$$p_i^S = \frac{1}{s} \quad \forall S \in \mathfrak{F}(G),\ \forall i \in S.$$

If (N, E) is cycle free, the position value $\pi(N, v, E)$ is the Harsanyi solution with the sharing system p induced by the *degree centrality measure* $q^{DEG}(S, E(S))$.

All the power measures, q^E, q^{DEG}, q^{CLO}, q^{BET}, q^{ORI}, and q^{PER} induce the Shapley value under complete communication situations.

4.4 Numerical Examples

In this subsection, we make comparisons among the existing five types of Harsanyi power solutions (the Shapley value ϕ, the Myerson value Ψ, the position value π, the Harsanyi power solutions induced by the originate centrality measure Φ^{ORI} and the peripherality measure Φ^{PER}) in some communication situations. The Harsanyi dividend of any coalition which is not contained within any connected component of the communication network in a communication situation is always zero, i.e., $\Delta^{(N,v,E)}(S) = 0$ if $S \nsubseteq C$ for any $C \in N/E$. Therefore, in considering the Harsanyi power solutions for a communication situation (N, v, E), we can assume that the communication network (N, E) is connected without loss of generality. Examples 8, 9, and 10 not only show comparisons of them but also illustrate criticisms against the the Myerson value and/or the position value. Two criticisms are reproduced below (see, e.g., [37] for additional details):

On the Myerson value :

$$\Psi_i(N, u_S, E_1) = \Psi_i(N, u_S, E_2) = \frac{1}{|S|} \quad \forall i \in N$$

whenever S is a feasible coalition in both (N, E_1) and (N, E_2), where u_S is the unanimity game of S. Furthermore, in the communication situation with $E^* = \{ij \subseteq N \mid j \in N \setminus i\}$ (i.e., E^* is a star-shape graph with a central player i), every player receives the same value (see $\Psi(N, v, E_e)$ in Example 10).

On the position value :

Irrelevant null players often have positive values (see Example 9). Recall that a null player $i \in N$ of the game (N, v) is a player satisfying that $v(S \cup i) = v(S)$ for any $S \subseteq N \setminus i$.

Example 8. Consider the communication situation (N, v, E) with $N = \{1, 2, 3\}$, $E = \{13, 23\}$ (Fig. 5 (b)), and

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq 1, \\ 30 & \text{if } |S| = 2, \\ 36 & \text{if } S = N. \end{cases}$$

Then,

$$\phi(N, v) = (12, 12, 12),$$
$$\Psi(N, v, E) = (7, 7, 22),$$
$$\pi(N, v, E) = (9, 9, 18),$$
$$\Phi^{ORI}(N, v, E) = (9, 9, 18),$$
$$\Phi^{PER}(N, v, E) = (3, 3, 30).$$

Example 9. Consider the communication situation (N, v, E) with $N = \{1, 2, 3\}$, $E = \{12, 13, 23\}$ (Fig. 5 (a)), and

$$v(S) = \begin{cases} 12 & \text{if } S \supseteq \{1, 2\}, \\ 0 & \text{otherwise.} \end{cases}$$

That is, the player 3 is a null player. Then,

$$\phi(N, v) = (6, 6, 0),$$
$$\Psi(N, v, E) = (6, 6, 0), \quad \pi(N, v, E) = (5, 5, 2),$$
$$\Phi^{ORI}(N, v, E) = (6, 6, 0), \quad \Phi^{PER}(N, v, E) = (6, 6, 0).$$

Example 10. Consider communication situations (N, u_N, E) with connected graphs in Fig.8 which shows all connected graphs (up to isomorphism) with $2 \leq n \leq 4$ nodes. Then, for any such communication situations,

$$\phi_i(N, u_N, E) = \Psi_i(N, u_N, E) = \frac{1}{n} \quad \forall i \in N.$$

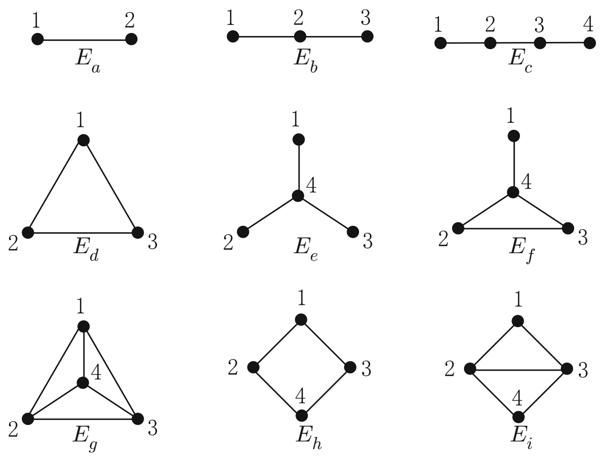

Fig. 8. Graphs with at most four nodes

Table 1 displays the remaining values (i.e., the position value π, the Harsanyi power solutions induced by the originate centrality measure Φ^{ORI} and the peripherality measure Φ^{PER}).

5 Cooperative Game and Combinatorial Structure

In section 4, we considered the following case:

> **Some subsets of** N **may not be meaningful.** *When N is the set of political parties, it means that some coalitions of parties are unlikely to occur, or even impossible (coalition mixing left and right parties); When N is the set of players, for players in order to coordinate their actions, they must be able to communicate [13,22,61].*

Table 1. Comparison of existing values

	π	Φ^{ORI}	Φ^{PER}
E_a	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$	$(\frac{1}{2}, \frac{1}{2})$
E_b	$(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$	$(\frac{1}{2}, 0, \frac{1}{2})$
E_c	$(\frac{1}{6}, \frac{2}{6}, \frac{2}{6}, \frac{1}{6})$	$(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8})$	$(\frac{1}{2}, 0, 0, \frac{1}{2})$
E_d	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$
E_e	$(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{3}{6})$	$(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{3}{6})$	$(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0$
E_f	$(\frac{3}{12}, \frac{2}{12}, \frac{2}{12}, \frac{5}{12})$	$(\frac{2}{14}, \frac{3}{14}, \frac{3}{14}, \frac{6}{14})$	$(\frac{3}{7}, \frac{2}{7}, \frac{2}{7}, 0)$
E_g	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
E_h	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$	$(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$
E_i	$(\frac{13}{60}, \frac{17}{60}, \frac{17}{60}, \frac{13}{60})$	$(\frac{2}{10}, \frac{3}{10}, \frac{3}{10}, \frac{2}{10})$	$(\frac{3}{10}, \frac{2}{10}, \frac{2}{10}, \frac{3}{10})$

In this section, we elaborate on more general cases, including the case discussed in section 3, as follows:

> **Subsets of N may not be "black and white [22] ",** *which means that the membership of an element to N may not be simply a matter of member or non-member. This is the case with multi-criteria decision making when underlying scales are bipolar, i.e., a central value exists on each scale, which is a demarcation between values considered as "good", and as "bad", the central value being neutral; In voting situation, it is convenient to consider that players may also abstain, hence each voter has three possibilities [15]; When N is the set of players, one may consider that each player can play at different level of participation [36].*

5.1 Generalization of Domains of Cooperative Games

Definition 42 (lattice). Let L be a non empty set and $\leq$ a partial order on L (i.e., $(L, \leq)$ is a poset). A poset $(L, \leq)$ is said to be a *lattice* if for $x, y \in L$, the supremum $x \vee y$ and the infimum $x \wedge y$ always exist. $\top$ and $\bot$ are the top (greatest) and bottom (least) elements of L, if they exist. An element $j \in L$ is said to be *join-irreducible* if it is not $\bot$ and cannot be express as a supremum of other elements (i.e., there are no $i, k < j$ such that $j = i \vee k$). The set of all join-irreducible elements of L is denoted by $\mathfrak{J}(L)$. A lattice $(L, \leq)$ is *distributive* if $\vee, \wedge$ obey distributivity. We often identify a lattice $(L, \leq)$ with L or with $(L, \leq, \vee, \wedge, \top, \bot)$.

Definition 43 (cooperative game on lattice). A pair (L, v) consisting of a lattice L and a (characteristic) function $v : L \to \mathbb{R}$ such as $v(\bot) = 0$ constitutes a *cooperative game on a lattice*.

The power set 2^N of N can coincide with the Boolean lattice $B(n)$. Therefore, an ordinary cooperative game (N, v) is regarded as a *cooperative game on a lattice* $((2^N, \subseteq), v)$. Indeed, the infimum (bottom element) in the lattice $(2^N, \subseteq)$ is the empty set $\emptyset$ and $v(\emptyset) = 0$. A communication situation (N, v, E) also can be regarded as a *cooperative game on a lattice*, if the communication network (N, E) is connected, because the poset induced by the connected graph (N, E) is obviously a lattice with $\emptyset$ as the bottom element. However, a bi-cooperative game $(N, v, \mathcal{Q}(N))$ is generally not a *cooperative game on a lattice*. Indeed, the family $\mathcal{Q}(N)$ with $(\mathcal{Q}(N), \sqsubseteq, \sqcup, \sqcap, (N, \emptyset), (\emptyset, N))$ is a lattice with $(\emptyset, N)$ as the bottom element, but $v(\emptyset, N)$ is not always zero.

Proposition 9 ([6]). *Let L be a distributive lattice. Any element $x \in L$ can be written as an irredundant supremum of join-irreducible elements in a unique way. That is, for any $x \in L$ there uniquely exists $\{j_1, \ldots, j_m\} \subseteq \mathfrak{J}(L)$ such that*

$$x = \bigvee_{i=1}^{m} j_i \tag{11}$$

and that if there exists $M \subseteq \mathfrak{J}(L)$ such that $x = \bigvee_{j \in M} j$, then $M \supseteq \{j_1, \ldots, j_m\}$. The equation (11) is called the minimal decomposition *of x and the $\{j_1, \ldots, j_m\}$ is denoted by $\eta^*(x)$. For any $x \in L$, we denote by $\eta(x) := \{j \in \mathfrak{J}(L) \mid j \leq x\}$, then $x = \bigvee_{j \in \eta(x)} j$. For example, in Fig. 9 (b), $\eta(23, 1) = \{(\emptyset, 13), (2, 13), (\emptyset, 12), (3, 12)\}$ and $\eta^*(23, 1) = \{(2, 13), (3, 12)\}$.*

Theorem 1 (Birkhoff's theorem [6]). *For any poset $(P, \leq)$, a subset $Q \subseteq P$ is said to be a* down set *of P if $x \in Q$ and that $y \leq x$ implies $y \in Q$. We denote by $\mathcal{O}(P)$ the set of all downsets of P. One can associate to any poset $(P, \leq)$ a distributive lattice which is $\mathcal{O}(P)$ endowed with inclusion. Then, for any lattice L, the mapping η is an isomorphism of L onto $\mathcal{O}(\mathfrak{J}(L))$.*

5.2 Examples of Generalizations of Games [22]

This subsection shows some examples of *cooperative games on lattices*.

Restricted Domains

Definition 44 (game on convex geometry [3]). Let N be a set of players. A collection $\mathcal{CG}$ of subsets of N is called a *convex geometry* if (i) it contains the empty set, (ii) it is closed under intersection, and (iii) $S \in \mathcal{CG}$, $S \neq N$ implies that there exists $j \in N \setminus S$ such that $S \cup j \in \mathcal{CG}$. A *cooperative game on a convex geometry* $\mathcal{CG}$ is a triplet $(N, v, \mathcal{CG})$ with a function $v : \mathcal{CG} \to \mathbb{R}$ such that $v(\emptyset) = 0$. In addition, several other games on restricted domains (e.g., *union stable systems*, *matroids*, and so on), which are generalization of posets induced by connected graphs, also have been proposed and studied by Bilbao [3].

Extended Domains

Definition 45 (multichoice game [36]). Let N be a set of players. Each player $i \in N$ has a finite number of feasible participation levels whose set we denote by $M_i = \{0, 1, \ldots, m_i\}$ and $\mathcal{M} = \prod_{i \in N} M_i$. Each element $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathcal{M}$ specifies a *participation profile* for players and is referred to as a *multichoice coalition*. So, a multichoice coalition indicates the participation level of each player. A triplet $(N, v, \mathcal{M})$ consisting of a (characteristic) function $v : \mathcal{M} \to \mathbb{R}$ such that $v(\mathbf{0}) = 0$, where $\mathbf{0} = (0, 0, \ldots, 0) \in \mathcal{M}$, constitutes a *multichoice game*.

Definition 46 (game on direct product of distributive lattices [46]). Let $N = \{1, \cdots, n\}$ be a finite set, $\{L_i\}_{i \in N}$ a set of distributive lattices and $\boldsymbol{L} := \prod_{i \in N} L_i$. (Notice that $\boldsymbol{L}$ is also a distributive lattice with the product order induced by $\{L_i\}_{i \in N}$). A triplet $(N, v, \boldsymbol{L})$ consisting of a product lattice $\boldsymbol{L} = \prod_{i \in N} L_i$ and a (characteristic) function $v : \boldsymbol{L} \to \mathbb{R}$ such as $v(\bot_1, \cdots, \bot_n) = 0$, where $\bot_i$ is the bottom element of L_i for each $i \in N$, constitutes a *cooperative game on a direct product of distributive lattices*.

Here, we consider some examples of games on a direct product of distributive lattices (see, also Fig. 9). If L_i is a two-element lattice (i.e., $L_i := \{\bot_i, \top_i\}$) for all players $i \in N$, then we get ordinary games on 2^N (Fig. 9 (a)); If $L_i := \{0, 1, \ldots, m_i\}$ for all players $i \in N$, we obtain multichoice games on $\mathcal{M} = \prod_{i \in N} L_i$ (Fig. 9 (c)); If $L_i := \{\bot_i, x_i, \top_i\}$, $\bot_i < x_i < \top_i$ (e.g., $\{-1, 0, 1\}$) for all players $i \in N$, then the product lattice $\boldsymbol{L} := \prod_{i \in N} L_i$ is isomorphic to $(\mathcal{Q}(N), \sqsubseteq)$ (Fig. 9 (b)). The bottom element $(\bot_1, \cdots, \bot_n)$ in $\boldsymbol{L}$ corresponds to $(\emptyset, N)$ in $\mathcal{Q}(N)$. That is, a bi-cooperative game $(N, v, \mathcal{Q}(N))$ is generally not regarded as a game on a direct product of distributive lattices since bi-cooperative games need not be vanishing at the bottom element $(\emptyset, N)$.

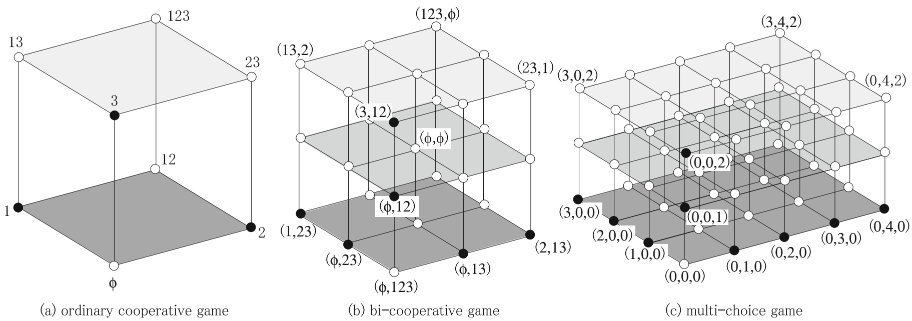

Fig. 9. Examples of direct products of distributive lattices: elements indicated by black circles are join-irreducible

Generally, the Möbius transform Δ^v of a game v on a lattice L can be implicitly defined through Definition 3. As it will be seen in the next section, *derivatives* of games on lattices are a very useful tool, and have been generalized (in particular) for games on *distributive lattices* (see, [26]).

Definition 47 (discrete derivative on distributive lattice). Let L be a distributive lattice. The *first order derivative* of a game $v : L \to \mathbb{R}$ with respect to a join-irreducible element $i \in \mathfrak{J}(L)$ at $x \in L$ is given by

$$\Delta_i v(x) := v(x \vee i) - v(x).$$

The *derivative* of v with respect to $y \in L$ at $x \in L$ is iteratively defined by

$$\Delta_y v(x) := \Delta_{j_1}[\Delta_{j_2}[\cdots \Delta_{j_{m-1}}[\Delta_{j_m} v(x)] \cdots]],$$

where $\eta^*(y) = \{j_1, j_2, \ldots, j_m\} \subseteq \mathfrak{J}(L)$ is the minimal decomposition of y. Note that if $j_k \leq x$ for some k, the derivative is null. Also, $\Delta_y v(x)$ does not depend on the order of the j_k's. The explicit formula is:

$$\Delta_y v(x) = \sum_{S \subseteq \{1,\ldots,m\}} (-1)^{m-s}\, v(x \vee \bigvee_{k \in S} j_k), \tag{12}$$

equivalently,

$$\Delta_y v(x) = \sum_{y \leq z \leq x \vee y} \Delta^v(z).$$

In particular,

$$\Delta_y v(\bot) = \Delta^v(y) \quad \forall y \in L.$$

Example 11. An ordinary game (N, v) can be regarded as a cooperative game on the lattice $L = (2^N, \subseteq)$. The join operator in the lattice is $\cup$ operator. The set of all join-irreducible elements $\mathfrak{J}(L)$ is N, i.e., any $i \in N$ is a join-irreducible element. For any $T \subseteq N$ (i.e., $T \in L$), the minimal decomposition of T is T itself, i.e., $\eta^*(T) = T = \{\sigma_T(1), \ldots, \sigma_T(t)\}$ for some permutation σ_T on N. An order $\{\sigma_T(k)\} \leq U$ coincides with $\sigma_T(k) \in U$. Then, we can easily find that Eq.(12) coincides with the ordinary discrete derivative:

$$\Delta_T v(U) = \sum_{S \subseteq T} (-1)^{|T|-|S|}\, v(U \cup S).$$

Example 12. Considering the lattice $L = (\mathcal{Q}(N), \sqsubseteq)$ with the join operator $\sqcup$. The set of all join-irreducible elements $\mathfrak{J}(L)$ is represented by

$$\{\{(i, N \setminus i)\}_{i \in N}, \{(\emptyset, N \setminus i)\}_{i \in N}\}.$$

For any $(T_1, T_2) \in \mathcal{Q}(N)$ (i.e., $(T_1, T_2) \in L$), the minimal decomposition of (T_1, T_2) is represented by

$$\eta^*(T_1, T_2) = \left\{\{(i, N \setminus i)\}_{i \in T_1}, \{(\emptyset, N \setminus j)\}_{j \in N \setminus (T_1 \cup T_2)}\right\}.$$

However, the discrete derivative in a bi-cooperative game $(N, v, \mathcal{Q}(N))$ (see, Eq. (7) in Definition 24) does not coincide with that in the cooperative game on the lattice $(\mathcal{Q}(N), \sqsubseteq)$. Indeed, for $N = \{1, 2, 3\}$, the $(123, 0)$-derivative at $(\emptyset, 3)$ can be defined in a cooperative game on the lattice $(\mathcal{Q}(\{1, 2, 3\}), \sqsubseteq)$, but cannot in the bi-cooperative game $(\{1, 2, 3\}, v, \mathcal{Q}(\{1, 2, 3\}))$. Now, let $\Delta^B_{(T_1,T_2)}$ denote the (T_1, T_2)-derivative in the sense of bi-cooperative games and $\Delta^L_{(T_1,T_2)}$ the (T_1,T_2)-derivative in the sense of cooperative games on lattices. For $(A_1,A_2),(B_1,B_2) \in \mathcal{Q}(N)$ such as $(A_1,A_2) \sqsubseteq (B_1,B_2)$ and $A_2 \cap B_1 = \emptyset$, we consider the following formula:

$$\Delta^v([(A_1,A_2),(B_1,B_2)]) := \sum_{(A_1,A_2)\sqsubseteq(S_1,S_2)\sqsubseteq(B_1,B_2)} (-1)^{|S_1\setminus A_1|+|S_2\setminus B_2|}\, v(S_1,S_2). \tag{13}$$

Then, for $(T_1,T_2) \in \mathcal{Q}(N)$ and $(S_1,S_2) \in \mathcal{Q}(N \setminus (T_1 \cup T_2))$,

$$\Delta^B_{(T_1,T_2)} v(S_1, S_2 \cup T_2) = \Delta^v([(S_1, S_2 \cup T_2),(S_1 \cup T_1, S_2)]).$$

For $(A_1,A_2),(B_1,B_2) \in \mathcal{Q}(N)$ such as $(A_1,A_2) \sqsubseteq (B_1,B_2)$ and $A_2 \cap B_1 = \emptyset$,

$$\Delta^v([(A_1,A_2),(B_1,B_2)]) = \Delta^B_{(B_1\setminus A_1, A_2\setminus B_2)} v(A_1,A_2) = \Delta^L_y v(A_1,A_2),$$

where $y = \bigsqcup_{(S_1,S_2)\in\eta^*(B_1,B_2)\setminus\eta^*(A_1,A_2)} (S_1,S_2)$. For example (see, Fig. 2),

$$\begin{aligned}\Delta^v([(\emptyset,3),(1,\emptyset)]) &= v(1,\emptyset) - v(1,3) - v(\emptyset,\emptyset) + v(\emptyset,3)\\ &= \Delta^B_{(1,3)} v(\emptyset,3)\\ &= \Delta^L_{(1,2)} v(\emptyset,3).\end{aligned}$$

5.3 Value and Interaction Index in Games on Distributive Lattices

In this subsection, we discuss on a specific type of game on lattice, where the lattice is a direct product of distributive lattices. Let $N := \{1,\dots,n\}$ and $\boldsymbol{L} := L_1 \times \cdots \times L_n$, where $L_1,\dots,L_n$ are finite distributive lattices. Then, $\boldsymbol{L}$ is also a distributive lattice and all join-irreducible elements of $\boldsymbol{L}$ are of the form $(\bot_1,\dots,\bot_{i-1}, j_i, \bot_{i+1},\dots,\bot_n)$ for some $i \in N$ and some $j_i \in \mathfrak{J}(L_i)$. A *vertex* of $\boldsymbol{L}$ is any element whose components are either top or bottom. Vertices of $\boldsymbol{L}$ will be denoted by $\top_Y$, $Y \subseteq N$, whose coordinates are $\top_k$ if $k \in Y$, $\bot_k$ otherwise, for $k \in N$. Each lattice L_i represents the poset of action, choice, or participation level of player $i \in N$ to the game. An ordinary cooperative game (N,v) can be regarded as the following game $v^{\boldsymbol{L}} : \boldsymbol{L} \to \mathbb{R}$:

> Let $L_i := \{0,1\}$ with the ordinary order $\leq$ on integers for all $i \in N$, and $\boldsymbol{L} = \prod_{i\in N} L_i$. Then, $\mathfrak{J}(L_i) = \{1\}$ for all $i \in N$. So, $\bot_i = 0$ and $\top_i = 1$ for all $i \in N$, therefore $\top_S = \chi_S$. Moreover, for any $y \in \boldsymbol{L}$ there uniquely exists $Y \subseteq N$ such that $y = \chi_Y = \top_Y$. Thus,
>
> $$v^{\boldsymbol{L}}(\top^Y) := v(Y) \quad Y \subseteq N$$
>
> is the desired one.

Lange and Grabisch [46] give the following interpretation for games on $\boldsymbol{L}$:

> *We assume that each player $i \in N$ has at her/his disposal a set of* elementary or pure actions $j_1,\cdots,j_{j_i}$. *These elementary actions are partially ordered (e.g., in the sense of benefit caused by the action), forming a partially ordered set*

$(\mathfrak{E}_i, \leq_i)$, $\mathfrak{E}_i = \{j_1, \cdots, j_{j_i}\}$. *Then by Birkhoff's theorem (Theorem 9), the set* $(\mathcal{O}(\mathfrak{E}_i), \subseteq)$ *of downsets of* $\mathfrak{E}_i$ *is a distributive lattice denoted by* L_i*, whose join-irreducible elements correspond to the elementary actions. The bottom action* $\bot_i$ *of* L_i *is the action which amounts to do nothing. Hence, each action in* L_i *is either a pure action* j_k *or a combined action* $j_k \vee j_{k'} \vee j_{k''} \vee \cdots$ *consisting of doing all pure actions* $j_k, j_{k'}, j_{k''} \cdots$ *for player* $i \in N$.

For a given elementary action $j_i \in \mathfrak{J}(L_i) \subseteq L_i$, the importance index (the (Shapley-type) value) of a game v on a direct product lattice $\boldsymbol{L} = \prod_{i \in N} L_i$ of distributive lattices $\{L_i\}_{i \in N}$ is written as a weighted average of the marginal contributions of j_i, taken at vertices of $\boldsymbol{L}$. This important index has been a generalization of the Shapley value in both ordinary games and multichoice games.

Definition 48 (importance index). *Let* $i \in N$ *and* $j_i \in \mathfrak{J}(L_i)$. *The* importance index *with respect to* j_i *of a game* $v : \boldsymbol{L} \to \mathbb{R}$ *is defined by*

$$\phi_{j_i}(v) := \sum_{Y \subseteq N \setminus i} \frac{y!(n-y-1)!}{n!} \Delta_{j_i} v(\top_Y).$$

As an extension of the importance index for every element of $\boldsymbol{L}$ and every game $(N, v, \boldsymbol{L})$, the interaction transform on $\boldsymbol{L}$ has been proposed by Lange and Grabisch [44]. For any $\boldsymbol{x} \in \boldsymbol{L}$, $I_{\boldsymbol{x}}(N, v, \boldsymbol{L})$ expresses the interaction in the game among all elementary actions $\boldsymbol{j}$ of the minimal decomposition $\boldsymbol{x} = \bigvee_{\boldsymbol{j} \in \eta^*(\boldsymbol{x})} \boldsymbol{j}$.

Definition 49 (antecessors). The *antecessor* $\underline{\boldsymbol{x}}$ of $\boldsymbol{x} \in \boldsymbol{L}$ is defined as

$$\underline{\boldsymbol{x}} = \bigvee \{ j \in \eta(\boldsymbol{x}) \mid j \notin \eta^*(\boldsymbol{x}) \}$$

with convention $\underline{\bot} = \bot$ and $\bigvee \emptyset = \bot$. If $\boldsymbol{x}$ is a join-irreducible element (i.e., $\boldsymbol{x} \in \mathfrak{J}(\boldsymbol{L})$), the antecessor of $\boldsymbol{x}$ is obviously its predecessor, in accordance with the notation $\underline{\boldsymbol{x}}$. Note also that the definition $\underline{\boldsymbol{x}}$ is consistent with the structure of each lattices L_i. Indeed, $\underline{\boldsymbol{x}} = (\underline{x_1}, \cdots, \underline{x_n})$.

Definition 50 (interaction transform on product lattices [44]). The *(Shapley-type) interaction transform* $I_{\boldsymbol{x}}(N, v, \boldsymbol{L})$ with respect to $\boldsymbol{x} \in \boldsymbol{L}$ of $v : \boldsymbol{L} \to \mathbb{R}$ is defined by

$$I_{\boldsymbol{x}}(N, v, \boldsymbol{L}) := \sum_{Y \subseteq N \setminus X} \frac{|Y|!\,(n - |X| - |Y|)!}{(n - |X| + 1)!} \Delta_{\boldsymbol{x}} v(\underline{\boldsymbol{x}} \vee \top_Y),$$

where $X = \{ i \in N \mid x_i \neq \bot_i \}$. Equivalently,

$$I_{\boldsymbol{x}}(N, v, \boldsymbol{L}) = \sum_{\boldsymbol{x} \leq \boldsymbol{z} \leq \boldsymbol{x}^\perp} \frac{1}{k(\boldsymbol{z}) - k(\boldsymbol{x}) + 1} \Delta^v(\boldsymbol{z}),$$

where $\boldsymbol{x}^\perp := \top_i$ if $x_i = \bot_i$ and $x^\perp := x_i$ if $x_i \neq \bot_i$, and $k(\boldsymbol{y}) = |\{ i \in N \mid y_i \neq \bot_i \}|$. Recall that any direct product $\boldsymbol{L} = \prod_{i \in N} L_i$ of distributive lattices $\{L_i\}_{i \in N}$ also a distributive lattice. Thus, the Möbius transform $\Delta^v(\boldsymbol{z})$ and the marginal interaction $\Delta_{\boldsymbol{x}}(\boldsymbol{y})$ in a game $(N, v, \boldsymbol{L})$ can be defined via Definition 47.

Each interaction index in ordinary games, and multichoice games is obtained as a special case of this interaction transform.

5.4 An Importance Index of Games on Regular Set Systems

In this subsection, we introduce an important index on a more general combinatorial structure, which is called the *regular set system* proposed by Honda and Grabisch [35] (see, also [47]). The concept of regular set system is induced by the following condition:

> A condition "*if $S \subsetneq N$ is feasible, then it is possible to find a player $i \in N \setminus S$ such that $S \cup i$ is still feasible*" is one of the weakest restrictions on feasible coalitions in a context where the grand coalition N can form. Because, it says that from a given coalition, it is possible to augment it gradually to reach the grand coalition.

Definition 51 (regular set system). Let us consider $\mathfrak{N}$ a set of coalitions, i.e., $\mathfrak{N} \subseteq 2^N$. Then, a pair $(N, \mathfrak{N})$ is said to be a *set system* on N if $\mathfrak{N}$ contains $\emptyset$ and N, i.e. $\emptyset, N \in \mathfrak{N}$. Elements of $\mathfrak{N}$ are called *feasible coalitions*. For any two feasible coalitions $A \subsetneq B$, we say that *A is covered by B*, and write $A \prec B$, if there is no $C \in \mathfrak{N}$ such that $A \subsetneq C \subsetneq B$. A set system $(N, \mathfrak{N})$ is said to be *regular* if $|B \setminus A| = 1$ whenever $A, B \in \mathfrak{N}$ and $A \prec B$.

Definition 52 (game on regular set system). A triplet $(N, v, \mathfrak{N})$ consisting of a regular set system $(N, \mathfrak{N})$ and a (characteristic) function $v : \mathfrak{N} \to \mathbb{R}$ such as $v(\emptyset) = 0$ constitutes a *game on a regular set system.*

Honda and Fujimoto [34] have proposed axiomatically an importance index of a game on a *regular set system* as a generalization of importance indices of all ordinary games, games on convex geometries, and multichoice games.

Definition 53 (maximal chain of regular set system). Let $\mathfrak{N} \subseteq 2^N$ be a regular set system. If a sequence $C = (C_0, \ldots, C_n)$ satisfies that $C_i \in \mathfrak{N}$ for any $i \in \{0, \cdots, n\}$ and $\emptyset = C_0 \prec C_1 \prec \cdots \prec C_n = N$, then C is called a *maximal chain* of $\mathfrak{N}$. The set of all maximal chains of $\mathfrak{N}$ is denoted by $\mathsf{M}(\mathfrak{N})$.

For any maximal chain $C = (C_0, \ldots, C_n)$, there exists a permutation σ_C on N such that

$$C_i = \bigcup_{k \leq i} \{\sigma_C(k)\} \quad \forall i \in \{1, \ldots, n\}. \tag{14}$$

Definition 54 (importance index on regular set system). A *marginal contribution* $\delta_i^v(C)$ of $i \in N$ for a maximal chain $C \in \mathsf{M}(\mathfrak{N})$ in a game $(N, v, \mathfrak{N})$ is defined by

$$\delta_i^v(C) := v(\bigcup_{k \leq i} \{\sigma_C(k)\}) - v(\bigcup_{k < i} \{\sigma_C(k)\})$$

where σ_C is a permutation on N satisfying Eq. (14). The *importance index* $\phi(N, v, \mathfrak{N}) \in \mathbb{R}^N$ with respect to a player $i \in N$ of a game $v : \mathfrak{N} \to \mathbb{R}$ on a regular set system $\mathfrak{N}$ is defined by

$$\phi_i(N, v, \mathfrak{N}) := \frac{1}{|\mathsf{M}(\mathfrak{N})|} \sum_{C \in \mathsf{M}(\mathfrak{N})} \delta_i^v(C)$$

for every $i \in N$, where $\phi_i(N, v, \mathfrak{N})$ is the i-th component of $\phi(N, v, \mathfrak{N}) \in \mathbb{R}^N$.

In a case that a regular set system is the power set of N, i.e., $\mathfrak{N} = 2^N$, any game $(N, v, \mathfrak{N})$ coincides with the ordinary game (N, v). Then, $\phi(N, v, \mathfrak{N})$ also coincides with the ordinary Shapley value $\phi(N, v)$. Moreover, through lattice-isomorphic mappings and Birkhoff's Theorem (Theorem 1), this importance index can be applied to games on distributive lattices as the following way :

Definition 55 (set systems induced by lattices). Let $(L, \leq, \vee, \wedge, \top, \bot)$ be a distributive lattice. Then $(L, \leq, \vee, \wedge, \top, \bot) \cong (\eta(L), \subseteq, \cup, \cap, \mathfrak{J}(L), \emptyset)$ with the lattice isomorphism η, where $\eta(x) = \{y \in \mathfrak{J}(L) \mid y \leq x\}$ for $x \in L$, i.e., $\eta(L) = \bigcup_{x \in L}\{\eta(x)\}$ (see, e.g., [6]). Then $(\mathfrak{J}(L), \eta(L))$ is called the *set system* induced by $(L, \leq)$.

All games discussed in this chapter, except bi-cooperative games, can be regarded as games on lattices. All the set systems induced by these lattices become regular [34]. Notice that the set system induced by $(Q(N), \sqsubseteq)$ is also regular. Therefore, we have another representation of importance indices of these games via η as follows:

$$I_{j_i}(N, v, L) := \phi_{\eta(j_i)}\left(\mathfrak{J}(L), v\eta^{-1}, \eta(L)\right) \quad \forall j_i \in \mathfrak{J}(L).$$

6 Concluding Remarks

This chapter shows cooperative games on various extended or restricted domains. We discussed only the Shapley-type values and interaction indices. However, there are various allocation rules and solution concepts in ordinary cooperative game theory, e.g., *the core, bargaining set, prekernel, kernel, prenucleolus, nucleolus, etc.* These various allocation rules and solution concepts can be seen in the literature [11,54]. The *Core* of cooperative games on various domains also have been studied by several researchers (see, e.g., [4,30]). More information about "cooperative game in combinatorial structures" and "social and economic networks in cooperative games" can be found in the literatures (see, e.g., [3,61]). To our knowledge, the topics "interaction indices of games with networks, and on regular set systems" have not been studied yet.

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