# Chapter 4
# On the One Method of Constructing Digital Control System with Minimal Structure

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**Abstract** We consider the linear digital control system with invariable matrix $A$. In this report we introduce one method which permit to obtain the characteristic of completely controllability and construct the matrix of control $B$ with minimal structure without calculation of eigenvalues of matrix $A$.

## 4.1 The Statement of Problem and Some Familiar Results

Let us discuss stationary open discrete system

$$X_{k+1} = AX_k. \tag{4.1}$$

We will find the full rank matrix $B$ of control actions with $n \times p$ size such that the following closed stationary system

$$X_{k+1} = AX_k + BU_k + F_k \tag{4.2}$$

will be completely controllable.

**Definition 4.1** Characteristic of completely controllable for system (4.1) is the minimal number $p \in \mathbb{N}$ such that the system (4.2) can make completely controllable by the choice of full rank matrix $B$ of $n \times p$ size.

On 2010 the article [1] was published in journal Doklady Akademii Nauk. There the structural minimization problem discussed and the following result obtained:

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**Theorem 4.1** *Characteristic of completely controllable of system* (4.1) *is equivalent to the maximal geometric multiplicity of eigenvalues of* $A$.

More over, in this article the method of constructing the matrix $B$ was established for the case where the Jordan canonical form of $A$ was given.

In this article we obtain the method to find the characteristic of completely controllable for (4.1) and constructing the matrix $B$ without evaluation of eigenvalues of $A$.

## 4.2 Definitions and Some Preliminary Transformations

Suppose that $A$ is square matrix with $n \times n$ size, $\lambda_j$ are eigenvalues of $A$, $q_A(x)$ is the minimal polynomial for $A$ and $d_A(x) = \det(xE - A)$. We note that the polynomial $q_A(x)$ can be found without calculations of eigenvalues of $A$; $d_A(x)$ is characteristic polynomial of $A$, multiplied by $(-1)$ powering relevant (so that the leading coefficient equal to 1) hence, this polynomial can be obtained without calculations of eigenvalues of $A$. Let

$$q_A(x) = \prod_{j=1}^{m} (x - \lambda_j)^{k_j}.$$

We denote

$$q(x, A, \geq r) = \prod_{j: k_j \geq r} (x - \lambda_j),$$

$$q(x, A, = r) = \prod_{j: k_j = r} (x - \lambda_j).$$

Let us note that the polynomial $q(x, a, \geq r)$ can be found without factorization of $q_A(x)$. For example,

$$q(x, A, \geq 1) = \frac{q_A(x)}{\text{g.c.d.}(q_A(x), q'_A(x))},$$

and $q(x, A, \geq 2)$ can be obtained by the same formula, where $q_A(x)$ changes by $\text{g.c.d.}(q_A(x), q'_A(x))$ and so forth. We can evaluate the polynomials $q(x, A, = r)$ by the polynomials $q(x, A, \geq r)$:

$$q(x, A, \geq r) = q(x, A, = r) q(x, A, \geq r + 1).$$

Similarly we define $d(x, A, \geq r)$ and $d(x, A, = r)$.

## 4.3 The Method to Obtain the Characteristic of Completely Controllable

Let us denote $A_1 = q(A, A, \geq 1)$. Because the polynomial $d_A(x)$ is the divisor for $(q(x, A, \geq 1))^n$ and $d_A(A) = 0$, the following identity holds: $A_1^n = 0$. More over, from the definition of the eigenvector of $A$ and the Jordan canonical form for the matrix $A_1$ it follows that the eigenvectors $\{v_1, \dots, v_s\}$ of the matrix $A$ form the basis in the kernel of $A_1$.

If $A_1 = 0$ then the matrix $A$ has the basis consists of it's eigenvectors. Hence, the geometric multiplicity of any eigenvalue of the matrix $A$ is equivalent to it's algebraical multiplicity. Thus, $p = \max\{t \mid d(x, A, \geq t) \neq 1\}$ in this case.

Suppose that $A_1 \neq 0$, $\mathrm{Ker}(A_1) = \mathrm{Lin}\{v_1, \dots, v_s\}$. Let us note that we can find vectors of the basis of the kernel of $A_1$ as orthogonal complement of the linear envelope of the set of rows of $A_1$. Suppose that $v_{s+1}, \dots, v_n$ is basis of the set of columns of $A_1^T$. Let $C_1$ be the matrix constructed of vectors $v_1, \dots, v_n$ as columns. Let $j \leq s$ be the fixed index and $e_1, \dots, e_n$ is a basis consists of unit vectors. By virtue of definition of the matrix $C_1$ we have $C_1 e_j \in \mathrm{Lin}\{v_1, \dots, v_s\}$, $AC_1 e_j \in \mathrm{Lin}\{v_1, \dots, v_s\}$, $C_1^{-1} AC_1 e_j \in \mathrm{Lin}\{e_1, \dots, e_s\}$. Hence the matrix $C_1^{-1}AC_1$ is sectional upper triangular:

$$C_1^{-1} AC_1 = \begin{pmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{pmatrix}. \tag{4.3}$$

Further from the arguments given above it follows that there is one-to-one correspondence between the eigenvectors of $A$ and the eigenvectors of $M_{11}$. Hence, the characteristics of completely controllable for matrix $A$ and $M_{11}$ are equivalent. Let us note that $M_{11}$ has the basis consists of eigenvectors.

*Remark 4.1* The set $\mathrm{Ker}(d(A, M_{11}, = t))$ is the linear envelope of all eigenvectors of $A$ such that their correspond eigenvalues has geometric multiplicity of exactly $t$.

## 4.4 Auxiliary Statements

To describe a method of constructing a matrix $B$ without finding eigenvalues of the matrix $A$, we need two lemmas. The proof of the first of them is trivial, and we omit it.

**Theorem 4.2** *Let $\lambda_1, \dots, \lambda_s$—eigenvalues of a matrix $A$, vector $h \in \mathrm{Ker}(\prod_{j=1}^{s}(A - \lambda_j E))$, $h \neq 0$. Then if $q \in \mathbb{N}$ of such that vectors $h,\ Ah,\ A^2h, \dots, A^{q-1}h$ are linearly independent, and vectors $h,\ Ah,\ A^2h, \dots, A^q h$ linearly dependent, there will be eigenvectors $z_1, \dots, z_q \in \mathrm{Ker}(\prod_{j=1}^{s}(A - \lambda_j E))$ such that $h = z_1 + z_2 + \dots + z_q$.*

*Back, if there are eigenvectors* $z_1, \ldots, z_q \in \mathrm{Ker}(\prod_{j=1}^{s}(A - \lambda_j E))$ *such that* $h = z_1 + z_2 + \cdots + z_q$, *vectors* $h,\ Ah,\ A^2h, \ldots, A^{q-1}h$ *are linearly independent, and vectors* $h,\ Ah,\ A^2h, \ldots, A^q h$ *linearly dependent.*

**Theorem 4.3** *Let* $\pi_k(x)$ *is a polynomial of degree* $t_k$ *such that* $\mathrm{Ker}(\pi_k(A))$ *is the linear envelope of the eigenvectors of the matrix A of geometric multiplicity of exactly k. Then, without finding the eigenvalues of the matrix A can be built vectors* $w_1, \ldots, w_{CC}$ *such that* $\mathrm{Ker}(\pi_k(A)) = \mathrm{Lin}\{w_1, Aw_1, \ldots, A^{t_k-1}w_1, \ldots, A^{t_k-1}w_{kk}\}$.

*Proof* We give an algorithm that allows each step reduce one of $k$ or $t_k$ by 1. Let us take an arbitrary non-zero vector $h_1 \in \mathrm{Ker}(\pi_k(A))$. There is a $q \in \mathbb{N}$ such that the vectors $h_1, ACAh_1, \ldots, A^{q-1}h_1$ are linearly independent, and vectors $h_1, ACAh_1, p(A^2h_1, \ldots, A^q h_1$ are linearly dependent. On Lemma 1, we obtain that $q \le t_k$. If $q = t_k$, then $w_1 = h_1$, and, demanding further orthogonality of the vector $h_2$ to all vectors $A^j h_1$, we obtain that $k$ has decreased by 1. If $q < t_k$, then, by Lemma 1, there are eigenvectors $z_1, \ldots, z_q$ such that $h_1 = z_1 + \cdots + z_q$. Add orthogonal vectors $v_{q+1}, \ldots, v_n$ in the system of vectors $v_1 = h_1, v_2 = Ah_1, \ldots, v_q = A^{q-1}h_1$ to obtain the basis of all space. We write the matrix $C_2$, the columns of which are vectors $v_1, \ldots, v_n$. As in the previous section, the matrix $C_2AC_2^{-1}$ is upper triangular. Let us denote by $N_{11}$ its upper the left bloc. Left to note that there exists a polynomial $\tilde{p}(x)$ such that

$$\pi_k(x) = d(x, N_{11}, \ge 1)\tilde{p}(x).$$

Thus, the problem for the polynomial $\pi_k(x)$ is reduced to the problem for polynomials $d(x, N_{11}, \ge 1)$ and $\tilde{p}(x)$, the sum of which degrees is equal to $t_k$. This means that in this case we have managed to reduce the $t_k$ at least by 1.

## 4.5 The Absence of Associated Vectors Case

Let us discuss the method of constructing $B$ in the case when the matrix $A$ has a basis consists of the eigenvalues. In this case we construct polynomials $\pi_k(x) = d(x, A, = k)$ and, using lemma 2, we obtain vectors $w_{1k}, \ldots, w_{kk}$ for any of these polynomials. Let us denote

$$b_j = \sum_{k=j}^{p} w_{jk}.$$

Left to notice that the matrix $B$ with the columns $b_1, \ldots, b_p$ is sought-for matrix.

## 4.6 The Case of General Position

Let $\tau$ is the degree of polynomial $q_A(x)$. Let $V_1 = \mathrm{Ker}(q(A, A, \geq 1))$, $V_2$ are the set of all vectors from $\mathrm{Ker}(q(A, A, \geq 2))$, orthogonal to $V_1$, $V_3$ are the set of all vectors from $\mathrm{Ker}(q(A, A, \geq 3))$, orthogonal to $V_1 + V_2$, etc. Let us notice that for finding basis in any $V_j$ it suffices to use orthogonalization method. Let $W_1$ is the set of orthogonal to $AV_2$ vectors from $V_1$, $W_2$ is the set of orthogonal to $AV_3$ vectors from $V_2$, etc.. The basis in each of the spaces $W_j$ can be found by using orthogonalization method. Let us consider the mapping

$$g\colon \sum_{j=1}^{\tau} W_j \to V_1.$$

By this mapping the vector $gw_j \in V_1$ is associated to the vector $w_j \in W_j$ such that $gw_j \in V_1$ is orthogonal projection of vector $A^{j-1}w_j$ on $V_1$. The mapping $g$ is invertible: it is sufficient to note that $g$ is linear, has zero kernel, and to set the basis in any of $W_j$, and the result of mapping $g$ on this basis.

Let us describe the method of constructing the matrix $B$ in the case of general position. As well as the case of absence of associated vectors, we construct the polynomials $\pi_k(x) = d(x, M_{11}, = k)$ and, using the lemma 2 for each polynomial, we obtain the vectors $w_{1k}, \ldots, w_{kk}$. Further, we put

$$\tilde{b}_j = \sum_{k=j}^{p} w_{jk},$$

$$b_j = g^{-1}\tilde{b}_j.$$

Left to notice that the matrix $B$ with the columns $b_1, \ldots, b_p$ is sought-for matrix.

## Reference

1. Zubov, A.V., Dikusar, V.V., Zubov, N.V.: Kriterii Upravlyaemosti Statsionarnykh Sistem. Doklady Akademii Nauk. **430**(1), 13–14 (2010)