Chapter 3
The Distribution of Values of Arithmetic Functions

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Abstract Let us usual $\tau_k(n)$ denote the number of ways n may be written as a product of k fixed factors. In this chapter there introduce the notation

$$D_k(x) = \sum_{n \le x} \tau_k(n).$$

We show that the asymptotic formula for $D_k(x)$ is changing with growing values of k and present specific values of k, which is a change.

In [1], this author obtained the estimate

$$D_k(x) \le x \sum_{j=0}^{k-1} \binom{k-1}{j} \frac{\ln^j x}{j!}, \tag{3.1}$$

for $D_k(x)$, which is uniform in the parameter k and holds for any real $x \ge 1$ and integer $k \ge 2$.

The value of the quantity $D_k(x)$ equals the number of points in the integer lattice in a domain of the form $1 \le x_1, x_2, \ldots x_k \le x$. Note that if the parameter k grows as $x \to \infty$, then the form of the asymptotic formula for $D_k(x)$ is different from that of the formula for fixed k. In 2001, Pavlov [3] proved the following assertion.

Theorem 3.1 *Suppose that $x \to \infty$, k is an integer, and $C_1(\ln x)^{\beta} < k < C_2(\ln x)^{\alpha}$, where $\alpha < \frac{2}{3}$ and $\beta > 6$ are fixed and C_1 and C_2 are positive constant. Then*

$$D_k(x) = x \frac{(\ln x)^{k-1}}{(k-1)!} e^{\gamma \frac{k^2}{\ln x}} \left(1 + O\left(k^{-\rho_0}\right)\right),$$

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where γ *is the Euler constant and* $\rho_0 > 0$ *is positive and does not depend on* k *and* x.

In this chapter, we obtain more accurate boundary values of the parameter k in Pavlov's theorem. The following assertion is valid.

Theorem 3.2 (Main Theorem) *Suppose that the integer parameter satisfies the condition* $k = k(x) \to \infty$ *as* $x \to \infty$, *and for some fixed* $0 < \rho < \frac{1}{3}$, *the inequality* $k \ll (\ln x)^{\frac{4}{5+\rho}}$ *holds. Then, the asymptotic formula*

$$D_k(x) = x\frac{(\ln x)^{k-1}}{(k-1)!}\exp\{q_k(x)\}\,L_k(x)\left(1 + O\left(\frac{k^{5+\rho}}{\ln^4 x}\right) + O\left(k^{3\rho-1}\right)\right),$$

is valid, in which the functions $q_k(x)$ *and* $L_k(x)$ *are defined by*

$$q_k(x) = \gamma_0\frac{k^2}{\ln x} - \left(\gamma_0^2 + \gamma_1\right)\frac{k^3}{\ln^2 x} + \left(\frac{5}{3}\gamma_0^3 + 3\gamma_0\gamma_1 + \frac{\gamma_2}{2}\right)\frac{k^4}{\ln^3 x}, \tag{3.2}$$

$$L_k(x) = 1 - \frac{k}{\ln x}\left(\gamma_0 + \frac{3}{2}\right) + \frac{k^2}{\ln^2 x}\left(\frac{3\gamma_0^2}{2} + \gamma_1 + 3\gamma_0 + \frac{7}{4}\right) -$$

$$- \frac{k^3}{\ln^3 x}\left(\frac{21}{4}\gamma_0^2 + \frac{5}{2}\gamma_0^3 + 3\gamma_0\gamma_1 + \frac{3}{2}\gamma_1 + \frac{21}{4}\gamma_0 + \frac{15}{8}\right), \tag{3.3}$$

and the Stieltjes constants are defined by

$$\gamma_n = \lim_{m\to\infty}\left(\sum_{k=1}^{m}\frac{(\ln k)^n}{k} - \frac{(\ln m)^{n+1}}{n+1}\right), \tag{3.4}$$

in particular, $\gamma_0 = \gamma$ *is the Euler constant.*

The proof of the *main theorem* is based on the following assertion.

Lemma 3.1 *Suppose that* $\sigma = 1 + \frac{1}{b}$, $b = \gamma_0 + \frac{\ln x}{k}$ *and*

$$I_k(x) = \frac{1}{2\pi i}\int_{\sigma-\frac{i}{2}}^{\sigma+\frac{i}{2}}\zeta^k(s)\frac{x^{s+1}}{s(s+1)}ds.$$

Suppose also that $x \to \infty$ *and* $k \to \infty$ *so that, for some fixed* $0 < \rho < \frac{1}{3}$, *the inequality* $k \ll (\ln x)^{\frac{4}{5+\rho}}$ *holds. Then the asymptotic formula*

$$I_k(x) = \frac{x^2}{2} \cdot \frac{(\ln x)^{k-1}}{(k-1)!} \exp\{q_k(x)\} L_k(x) \left(1 + O\left(\frac{k^{5+\rho}}{\ln^4 x}\right) + O\left(k^{3\rho-1}\right)\right), \tag{3.5}$$

is valid, in which the functions $q_k(x)$ and $L_k(x)$ are determined from (3.2) *and* (3.3)*.*

This lemma sharpens the corresponding lemma from Pavlov's chapter ([3], Lemma 1).

The proof of *lemma* 3.1 uses the Laurent expansion of Riemann's zeta function $\zeta(s)$ in the neighborhood of the pole $s = 1$

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \gamma_n \cdot (s-1)^n,$$

where constants γ_n defined from 3.4.

As is known, for $\Re s > 1$

$$\sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s} = \zeta^k(s),$$

where $\zeta(s)$ is the Riemann zeta function. We have (see [2])

$$\int_1^x D_k(t)dt = \frac{1}{2\pi i} \int_{\sigma-iT}^{\sigma+iT} \zeta^k(s) \frac{x^{s+1}}{s(s+1)} ds + R(x) = J_k(x) + R(x), \tag{3.6}$$

where the parameter σ is the same as in the *lemma* 3.1. Using estimate (3.1), we obtain the following estimate for the remainder:

$$R(x) \ll \frac{x^2}{T} \left(\frac{\ln x}{k}\right)^k \exp\left\{k + (\gamma_0 + 1)\frac{k^2}{\ln x}\right\}$$
$$+ \left(\frac{x^2}{T} + x\frac{\ln T}{\ln x}\right) \sqrt{k} \left(\frac{\ln x}{k}\right)^k \exp\left\{k + \frac{k^2}{\ln x}\right\}. \tag{3.7}$$

For $k \ll (\ln x)^{\frac{5}{7}}$, we deform the interval of integration in $J_k(x)$ as

$$\int_{\sigma-iT}^{\sigma+iT} = \int_{\sigma-iT}^{1-iT} + \int_{1-iT}^{1-\frac{i}{2}} + \int_{1-\frac{i}{2}}^{\sigma-\frac{i}{2}} + \int_{\sigma-\frac{i}{2}}^{\sigma+\frac{i}{2}} + \int_{\sigma+\frac{i}{2}}^{1+\frac{i}{2}} + \int_{1+\frac{i}{2}}^{1+iT} + \int_{1+iT}^{\sigma+iT},$$

by virtue of the estimate $|\zeta(1+it)| \le C \ln^{\frac{2}{3}} |t|$ where $|t| > 2$ and C is some constant, we have

$$J_k(x) = I_k(x) + O\left(C^k x^2 (\ln T)^{\frac{2k}{3}}\right).$$

We can choose the parameter T so that $\frac{k^2}{\ln x} \ll \ln T \ll \left(\frac{\ln x}{k}\right)^{\frac{3}{2}}$ and the remainders in expressions (3.6) and (3.7) do not exceed those in (3.5). Applying the *lemma* 3.1, we obtain

$$\int_1^x D_k(t)dt =$$

$$= \frac{e^k}{\sqrt{2\pi k}} \frac{x^2}{2} \left(\frac{\ln x}{k}\right)^{k-1} \exp\{q_k(x)\} L_k(x) \left(1 + O\left(\frac{k^{5+\rho}}{\ln^4 x}\right) + O\left(k^{3\rho-1}\right)\right). \tag{3.8}$$

In the case of $k \gg (\ln x)^{\frac{2}{3}}$, we decompose the integral $J_k(x)$ into three integrals as

$$J_k(x) = \frac{1}{2\pi i}\int_{\sigma-iT}^{\sigma-\frac{i}{2}} + \frac{1}{2\pi i}\int_{\sigma-\frac{i}{2}}^{\sigma+\frac{i}{2}} + \frac{1}{2\pi i}\int_{\sigma+\frac{i}{2}}^{\sigma+iT} =$$

$$= I_k(x) + O\left(x^2 \left(\frac{\ln x}{k}\right)^k \exp\left\{k + \frac{\gamma_0^2 k^3}{\ln^2 x}\right\}\right),$$

Let $T = x$; then the remainder $R(x)$ does not exceed the remainders in formula (3.5) given in the *lemma* 3.1; therefore, the relation (3.8) again holds.

The function $D_k(x)$ is nondecreasing. We have

$$\frac{1}{h}\int_{x-h}^{x} D_k(t)dt \le D_k(x) \le \frac{1}{h}\int_{x}^{x+h} D_k(t)dt.$$

Applying (3.8) and choosing $h = x\frac{k^{5+\rho}}{\ln^4 x}$, we obtain the assertion of the *Main Theorem*.

References

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