Chapter 21
Distributed Optimal Control in One Non-Self-Adjoint Boundary Value Problem

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Abstract We prove the solvability of the optimal control problem for elliptic equation with nonlocal boundary conditions in a circular sector with terminal quadratic cost functional in the class of distributed controls.

21.1 Introduction

The theory of linear-quadratic optimal control problems for distributed systems is well researched [1, 2]. In many cases the original problem can be decomposed with the help of Fourier method [3–5]. In this chapter we consider the control problem for elliptic equation with non-local boundary conditions in circular sector [6] with terminal quadratic cost functional. This problem does not allow total decomposition and using of L^2-theory. To resolve this problem in the class of distributed controls we use apparatus of specially constructed biorthonormal basis systems of functions [7] and then we analyze the solutions of Fredholm matrix equations.

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21.2 Setting of the Problem

In a circular sector $Q = \{(r, \theta) | r \in (0, 1),\ \theta \in (0, \pi)\}$ we consider the optimal control problem

$$\begin{cases} \Delta y := \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial y}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 y}{\partial \theta^2} = u(r, \theta),\ (r, \theta) \in Q, \\ y(1, \theta) = p(\theta),\ p(0) = 0, \\ y(r, 0) = 0,\ r \in (0, 1), \\ \frac{\partial y}{\partial \theta}(r, 0) = \frac{\partial y}{\partial \theta}(r, \pi),\ r \in (0, 1), \end{cases} \tag{21.1}$$

$$J(y, u) = \|y(\alpha)\|_D^2 dr + \int_0^1 \|u^2(r)\| dr \to \inf, \tag{21.2}$$

where $p \in C^1([0, \pi])$ is given function, $\alpha \in (0, 1)$ is fixed number, $\| \cdot \|_D$ is a norm in $L^2(0, \pi)$, which is equivalent to standard one and is given by the equality

$$\|v\|_D = \left(\sum_{n=1}^{\infty} v_n^2 \right)^{1/2},$$

where $\forall n \geq 1$, $v_n = \int_0^\pi v(\theta)\psi_n(\theta) d\theta$, $\psi_0(\theta) = \frac{2}{\pi^2}$, $\psi_{2n}(\theta) = \frac{4}{\pi^2}(\pi - \theta)\sin 2n\theta$, $\psi_{2n-1}(\theta) = \frac{4}{\pi^2}\cos 2n\theta$.

The aim of this paper is to establish classical solvability of the problem (21.1)–(21.2), that is to find optimal one among admissible processes $\{u, y\} \in C(\bar{Q}) \times \left(C(\bar{Q}) \bigcap C^2(Q)\right)$. For the application of the spectral method we use biorthonormal and complete in $L^2(0, \pi)$ well-known Samarsky-Ionkin systems of functions [7]

$$\Psi = \{\psi_n\}_{n=1}^{\infty} \text{ and}$$

$$\Phi = \{\varphi_0(\theta) = \theta,\ \varphi_{2n}(\theta) = \sin 2n\theta,\ \varphi_{2n-1}(\theta) = \theta \cos 2n\theta\}_{n=1}^{\infty}. \tag{21.3}$$

Then $\forall u \in L^2(Q)$

$$u(r, \theta) = \sum_{n=0}^{\infty} u_n(r) \cdot \varphi_n(\theta), \tag{21.4}$$

where $u_n(r) = \int_0^\pi u(r, \theta)\psi_n(\theta) d\theta$. So, we will seek for the solution of the problem (21.1) in form

$$y(r, \theta) = y_0(r)\theta + \sum_{n=1}^{\infty} \left(y_{2n-1}(r)\theta \cos 2n\theta + y_{2n}(r) \sin 2n\theta\right), \tag{21.5}$$

where the functions $\{y_k(r)\}_{k=0}^{\infty}$ are solutions of the system of ordinary differential equations

$$\frac{d}{dr}(r\frac{dy_0}{dr}) = r \cdot u_0(r),\ y_0(1) = p_0, \tag{21.6}$$

$$r \cdot \frac{d}{dr}\left(r \cdot \frac{dy_{2k-1}}{dr}\right) - (2k)^2 y_{2k-1} = r^2 \cdot u_{2k-1}(r),\ y_{2k-1}(1) = p_{2k-1}, \tag{21.7}$$

$$r\frac{d}{dr}\left(r \cdot \frac{dy_{2k}}{dr}\right) - (2k)^2 y_{2k} - 4k \cdot y_{2k-1} = r^2 \cdot u_{2k}(r),\ y_{2k}(1) = p_{2k}, \tag{21.8}$$

where $p_k = \int_0^{\pi} p(\theta) \cdot \psi_k(\theta) d\theta$.

Thus the original problem (21.1)–(21.2) is reduced to the following one: among admissible pairs $\{u_n(r), y_n(r)\}_{n=0}^{\infty}$ of the problem (21.6)–(21.8) one should minimize the cost functional

$$J(y, u) = y_0^2(\alpha) + \int_0^1 u_0^2(r)dr + \sum_{k=1}^{\infty}(y_{2k-1}^2(\alpha) + y_{2k}^2(\alpha) + \\ + \int_0^1 (u_{2k-1}^2(r) + u_{2k}^2(r))dr) = J_0 + \sum_{k=1}^{\infty} J_k. \tag{21.9}$$

Herewith the optimal process $\{\tilde{u}_n(r), \tilde{y}_n(r)\}_{n=0}^{\infty}$ should be such that the formula (21.4) defines function from $C(\bar{Q})$, and the formula (21.5) defines function from $C(\bar{Q}) \bigcap C^2(Q)$.

21.3 Main Results

A structure of the problem (21.6)–(21.8) allows to reduce it to sequence of the following problems:

On the solutions of (21.6) one should minimize the cost functional

$$J_0 = J_0(u_0), \tag{21.10}$$

on the solutions of (21.7), (21.8) one should minimize the cost functional

$$J_k = J_k(u_{2k-1}, u_{2k}),\ k \geq 1. \tag{21.11}$$

For fixed $\{u_k(r)\}_{k=0}^{\infty} \subset C([0, 1])$ solutions of the problem (21.6)–(21.8) have form

$$y_0(r) = p_0 - \int_r^1 \left(\frac{1}{s}\int_0^s \xi u_0(\xi)d\xi\right) ds = p_0 + \int_0^1 G_0(r,s)u_0(s)ds, \quad (21.12)$$

where

$$G_0(r,s) = \begin{cases} s\ln r, & s\in[0,r], \\ s\ln s, & s\in[r,1], \end{cases}$$

$$y_{2k-1}(r) = p_{2k-1}\cdot r^{2k} + \frac{1}{4k}\int_0^1 s\cdot G_k(r,s)u_{2k-1}(s)ds, \quad (21.13)$$

where

$$G_k(r,s) = \begin{cases} s^{2k}(r^{2k}-r^{-2k}), & s\in[0,r], \\ r^{2k}(s^{2k}-s^{-2k}), & s\in[r,1], \end{cases}$$

$$\begin{aligned} y_{2k}(r) = & p_{2k}\cdot r^{2k} + p_{2k-1}\cdot r^{2k}\cdot\ln r + \frac{1}{4k}\int_0^1 s\cdot G_k(r,s)u_{2k}(s)ds + \\ & + \frac{1}{4k}\int_0^1 s\cdot \bar{G}_k(r,s)u_{2k-1}(s)ds, \end{aligned} \quad (21.14)$$

where

$$\begin{aligned} \bar{G}_k(r,s) &= \int_0^1 p^{-1}\cdot G_k(r,p)G_k(p,s)ds \\ &= \begin{cases} \frac{1}{2k}\left((\frac{s}{r})^{2k}-(rs)^{-2k}\right)+r^{2k}s^{2k}\ln(rs)-(\frac{s}{r})^{2k}\ln(\frac{s}{r}), & s\in[0,r], \\ \frac{1}{2k}\left((\frac{r}{s})^{2k}-(rs)^{-2k}\right)+r^{2k}s^{2k}\ln(rs)-(\frac{r}{s})^{2k}\ln(\frac{r}{s}), & s\in[r,1]. \end{cases} \end{aligned}$$

Lemma 21.1 *For any $k\geq 0$ the formulas (21.12)–(21.14) define the solutions of the problem (21.6)–(21.8) $y_k\in C([0,1])\bigcap C^2(0,1)$.*

Proof Since y_k are the solutions of the problem (21.6)–(21.8), then it suffices to show that $\forall k\geq 0$ $y_k\in C([0,1])$. We denote $\prod=[0,1]\times[0,1]$. Then $G_0\in C(\prod)$, $\max\limits_{\prod}|G_0(r,s)|=e^{-1}$, so, $y_0\in C([0,1])$.

For $k\geq 1$ $G_k\in C(\prod\setminus\{0,0\})$, $\max\limits_{\prod}|G_k(r,s)|\leq 1$, so, $y_{2k-1}\in C([0,1])$. Since $x^k\ln x\in C([0,1])$, $\max\limits_{x\in[0,1]}|x^k\ln x|=e^{-1}\cdot k^{-1}$, then for $\bar{G}_k\in C(\prod\setminus\{0,0\})$ we have: $\max\limits_{\prod}|\bar{G}_k(r,s)|\leq\frac{1}{k}$, so, $y_{2k}\in C([0,1])$. Lemma is proved.

Theorem 21.1 *The problems (21.10), (21.11) have the unique solution $\{\tilde{u}_k\}_{k=0}^{\infty}$, moreover $\forall k\geq 0$ $\tilde{u}_k\in C([0,1])$.*

Proof From the formulas (21.12)–(21.14) it follows that the functionals $J_0 : L^2(0,1)\mapsto\mathbf{R}$, $J_k : L^2(0,1)\times L^2(0,1)\mapsto\mathbf{R}$ are strictly convex, continuous and coercive, which means under [1] that the problems (21.10), (21.11) have the unique solution in the spaces $L^2(0,1)$ and $L^2(0,1)\times L^2(0,1)$ correspondingly.

Equating to zero Frechet derivatives of J_0, J_k, we obtain the following Fredholm integral equations:

$$u_0(s) = -\int_0^1 G_0(\alpha,s)G_0(\alpha,p)u_0(p)dp - p_0\cdot G_0(\alpha,s), \tag{21.15}$$

$$\begin{aligned} u_{2k-1}(s) = &-\frac{1}{2}\cdot\frac{1}{(4k)^2}\int_0^1 [\big(s\cdot G_k(\alpha,s)p\cdot G_k(\alpha,p) + s\cdot \bar{G}_k(\alpha,s)p\cdot\bar{G}_k(\alpha,p)\big)u_{2k-1}(p) + \\ &+ 2s\cdot\bar{G}_k(\alpha,s)p\cdot G_k(\alpha,p)u_{2k}(p)]dp - p_{2k}\alpha^{2k}\frac{1}{4k}s\cdot G_k(\alpha,s) - \\ &- \left(p_{2k}\alpha^{2k} + p_{2k-1}\alpha^{2k}\ln\alpha\right)\frac{1}{4k}\cdot s\cdot\bar{G}_k(\alpha,s), \end{aligned} \tag{21.16}$$

$$\begin{aligned} u_{2k}(s) = &-\frac{1}{2}\cdot\frac{1}{(4k)^2}\int_0^1 [\big(2s\cdot G_k(\alpha,s)p\cdot\bar{G}_k(\alpha,p)\big)\,u_{2k-1}(p) + \\ &+ s\cdot G_k(\alpha,s)p\cdot\bar{G}_k(\alpha,p)u_{2k}(p)]dp - \\ &- \left(p_{2k}\alpha^{2k} + p_{2k-1}\alpha^{2k}\ln\alpha\right)\frac{1}{4k}\cdot s\cdot G_k(\alpha,s). \end{aligned} \tag{21.17}$$

Since $\max\limits_{(p,s)\in\prod}|G_0(\alpha,p)G_0(\alpha,s)| \le e^{-2} < 1$, then the Eq. (21.15) has the unique solution $\tilde{u}_0 \in C([0,1])$.

Put

$$A_k(p,s) = \begin{pmatrix} s\cdot G_k(\alpha,s)p\cdot G_k(\alpha,p) + s\cdot\bar{G}_k(\alpha,s)p\cdot\bar{G}_k(\alpha,p) & 2s\cdot\bar{G}_k(\alpha,s)p\cdot G_k(\alpha,p) \\ 2s\cdot G_k(\alpha,s)p\cdot\bar{G}_k(\alpha,p) & s\cdot G_k(\alpha,s)p\cdot\bar{G}_k(\alpha,p) \end{pmatrix},$$

$$f_k(s) = \begin{pmatrix} -p_{2k}\alpha^{2k}\frac{1}{4k}s\cdot G_k(\alpha,s) - \left(p_{2k}\alpha^{2k} + p_{2k-1}\alpha^{2k}\ln\alpha\right)\frac{1}{4k}\cdot s\cdot\bar{G}_k(\alpha,s) \\ -\left(p_{2k}\alpha^{2k} + p_{2k-1}\alpha^{2k}\ln\alpha\right)\frac{1}{4k}\cdot s\cdot G_k(\alpha,s) \end{pmatrix}.$$

Then from the Eqs. (21.16), (21.17) we have that vector

$$z_k(s) = \begin{pmatrix} u_{2k-1}(s) \\ u_{2k}(s) \end{pmatrix}$$

satisfies the equation

$$z_k(s) = -\frac{1}{2}\cdot\frac{1}{(4k)^2}\int_0^1 A_k(p,s)z_k(p)dp + f_k(s). \tag{21.18}$$

Under estimates from Lemma 21.1 we obtain

$$\max_{\Pi} \|A_k(p,s)\| \le 4, \ \max_{s\in[0,1]} \|f_k(s)\| \le \frac{\alpha^{2k-1}}{2k}\big(|p_2k| + |p_{2k-1}|\big).$$

Then $\forall k \ge 1$ the equation (21.18) has the unique solution

$$\tilde{z}_k(s) = \begin{pmatrix} u_{2k-1}(s) \\ u_{2k}(s) \end{pmatrix} \in C([0,1]),$$

herewith $\forall r \in [0,1]$

$$|u_{2k-1}(r)| \le \frac{\alpha^{2k-1}}{k}\big(|p_{2k-1}| + |p_{2k}|\big), \ |u_{2k}(r)| \le \frac{\alpha^{2k-1}}{k}\big(|p_{2k-1}| + |p_{2k}|\big). \quad (21.19)$$

The theorem is proved.

From the estimates (21.19) it follows that the series $\sum\limits_{n=0}^{\infty} \tilde{u}_n(r)\varphi_n(\theta)$ converges uniformly on $\bar{Q}$ and it defines the function $\tilde{u}(r,\theta) \in C(\bar{Q})$ by the formula (21.4).

Theorem 21.2 *Series*

$$\tilde{y}_0(r)\theta + \sum_{n=1}^{\infty}\big(\tilde{y}_{2n-1}(r)\theta \cdot \cos 2n\theta + \tilde{y}_{2n}(r)\sin 2n\theta\big),$$

defines the function $\tilde{y}(r,\theta) \in C(\bar{Q}) \bigcap C^2(Q)$ *by the formula* (21.5)*, where* $\{\tilde{y}_n\}_{n=0}^{\infty}$ *are the solutions of the system* (21.6)–(21.8) *with controls* $\{\tilde{u}_n\}_{n=1}^{\infty}$.

Proof By the formulas (21.12)–(21.14) desired series has the form

$$p_0 \cdot \theta + \theta \cdot \int_0^1 G_0(r,s)u_0(s)ds + \sum_{n=1}^{\infty}\Big(p_{2k-1} \cdot r^{2k} \cdot \theta \cos 2n\theta +$$

$$+ \big(p_{2k} \cdot r^{2k} + p_{2k-1} \cdot r^{2k} \cdot \ln r\big)\sin 2n\theta\Big) + \sum_{n=1}^{\infty} \theta \cos 2n\theta \cdot \frac{1}{4n}\int_0^1 sG_n(r,s)\tilde{u}_{2n-1}(s)ds +$$

$$+ \sum_{n=1}^{\infty} \sin 2n\theta \Big(\frac{1}{4n}\int_0^1 sG_n(r,s)\tilde{u}_{2n}(s)ds + \frac{1}{4n}\int_0^1 s\bar{G}_n(r,s)\tilde{u}_{2n-1}(s)ds\Big). \quad (21.20)$$

The functions $r^{2n}\sin 2n\theta$ and $r^{2n}(\ln r \cdot \sin 2n\theta + \theta\cos 2n\theta)$ are harmonic, $p \in C^1([0,\pi])$, $p(0) = 0$, so, from [6] the first series in 21.20) is the function from the class $C(\bar{Q}) \bigcap C^2(Q)$.

From Lemma 21.1 and the estimates (21.19) we have under Weierstrass theorem that $\tilde{y} \in C(\bar{Q})$.

On $\forall[a,b]\times[c,d]\subset(0,1)\times(0,\pi)$ it remains to investigate the uniform convergence of the series from the first and second-order derivatives on r,θ of functions

$$B_n(r,\theta)=\frac{1}{4n}\int_0^1 sG_n(r,s)\tilde{u}_{2n-1}(s)ds\cdot\theta\cos 2n\theta=b_n(r)\cdot\theta\cos 2n\theta,$$

$$C_n(r,\theta)=\frac{1}{4n}\int_0^1 sG_n(r,s)\tilde{u}_{2n}(s)ds\cdot\sin 2n\theta=c_n\cdot\sin 2n\theta,$$

$$D_n(r,\theta)=\frac{1}{4n}\int_0^1 s\bar{G}_n(r,s)\tilde{u}_{2n-1}(s)ds\cdot\sin 2n\theta=d_n\cdot\sin 2n\theta.$$

From the estimates (21.19) we obtain that the series from derivatives $\frac{\partial}{\partial\theta}$, $\frac{\partial^2}{\partial\theta^2}$ converge on $\bar{Q}$ uniformly under Weierstrass theorem.

For $\forall r\in[a,b]$, $\forall n>1$

$$b_n(r)=\frac{1}{4n}\Big((r^{2n}-r^{-2n})\int_0^r s^{2n+1}\tilde{u}_{2n-1}(s)ds+r^{2n}\int_r^1(s^{2n+1}-s^{1-2n})\tilde{u}_{2n-1}(s)ds\Big),$$

$$b_n'(r)=\frac{1}{2}(r^{2n-1}+r^{-2n-1})\int_0^r s^{2n+1}\tilde{u}_{2n-1}(s)ds+ \qquad (21.21)$$

$$+\frac{1}{2}r^{2n-1}\int_r^1(s^{2n+1}-s^{1-2n})\tilde{u}_{2n-1}(s)ds,$$

(summands which do not contain integrals are mutually canceled)

$$b_n''(r)=\frac{1}{2}\Big((2n-1)r^{2n-2}+(-2n-2)r^{-2n-2}\Big)\int_0^r s^{2n+1}\tilde{u}_{2n-1}(s)ds+$$

$$+\frac{1}{2}(2n-1)r^{2n-2}\int_r^1(s^{2n+1}-s^{1-2n})\tilde{u}_{2n-1}(s)ds+\tilde{u}_{2n-1}(r). \qquad (21.22)$$

Since $\int_0^r s^{2n+1}ds=\frac{r^{2n+1}}{2n+2}$,

$$\int_r^1 (s^{2n+1} - s^{1-2n})ds = -\frac{n}{1-n^2} - \frac{r^{2n+1}}{2n+2} + \frac{r^{2-2n}}{2-2n},$$

then $\exists C_1 > 0$ such that

$$|b_n'(r)| \le \frac{C}{n} \cdot \frac{\alpha^{2n-1}}{n}(|p_{2n-1}| + |p_{2n}|),$$

so, the series $\sum_{n=2}^{\infty} \frac{\partial}{\partial r} B_n(r,\theta)$, $\sum_{n=2}^{\infty} \frac{\partial}{\partial r} C_n(r,\theta)$, $\sum_{n=2}^{\infty} \frac{\partial^2}{\partial r \partial\theta} B_n(r,\theta)$, $\sum_{n=2}^{\infty} \frac{\partial^2}{\partial r\partial\theta} C_n(r,\theta)$ converge uniformly on $[a,b] \times [c,d]$.

From the same estimates $|b_n''(r)| \le C_2 \cdot \frac{\alpha^{2n-1}}{n}(|p_{2n-1}| + |p_{2n}|)$ and, thereby the series $\sum_{n=2}^{\infty} \frac{\partial^2}{\partial r^2} B_n(r,\theta)$, $\sum_{n=2}^{\infty} \frac{\partial^2}{\partial r^2} C_n(r,\theta)$ converge uniformly on $[a,b] \times [c,d]$.

For the function $d_n(r)$ we have $\forall r \in [a,b]$:

$$\begin{aligned}
d_n(r) =& \frac{1}{8n^2} r^{-2n} \cdot \int_0^r s^{2n+1}\tilde{u}_{2n-1}(s)ds - \frac{1}{8n^2} \cdot r^{2n} \int_0^r s^{2n+1}\tilde{u}_{2n-1}(s)ds + \\
&+ \frac{1}{4n} r^{2n} \int_0^r s^{2n+1} \ln s\tilde{u}_{2n-1}(s)ds + \frac{1}{4n} r^{2n} \ln r \int_0^r s^{2n+1}\tilde{u}_{2n-1}(s)ds - \\
&- \frac{r^{-2n}}{4n} \int_0^r s^{2n+1} \ln s\tilde{u}_{2n-1}(s)ds + \frac{r^{-2n}\ln r}{4n} \int_0^r s^{2n+1}\tilde{u}_{2n-1}(s)ds + \\
&+ \frac{1}{8n^2} r^{2n} \int_r^1 s^{-2n+1}\tilde{u}_{2n-1}(s)ds - \frac{1}{8n^2} r^{2n} \int_r^1 s^{2n+1}\tilde{u}_{2n-1}(s)ds + \\
&+ \frac{1}{4n} r^{2n} \int_r^1 s^{2n+1} \ln s\tilde{u}_{2n-1}(s)ds + \frac{1}{4n} r^{2n} \ln s \int_r^1 s^{2n+1}\tilde{u}_{2n-1}(s)ds - \\
&- \frac{1}{4n} r^{2n} \ln s \int_r^1 s^{-2n+1}\tilde{u}_{2n-1}(s)ds + \frac{1}{4n} r^{2n} \int_r^1 s^{-2n+1} \ln s\tilde{u}_{2n-1}(s)ds,
\end{aligned}$$

$$\begin{aligned}
d_n'(r) =& -\frac{1}{4}\frac{1}{n} r^{-2n-1} \int_0^r s^{2n+1}\tilde{u}_{2n-1}(s)ds - \frac{1}{4n} r^{2n-1} \int_0^r s^{2n+1}\tilde{u}_{2n-1}(s)ds + \\
&+ \frac{1}{2} r^{2n-1} \int_0^r s^{2n+1} \ln s\tilde{u}_{2n-1}(s)ds + \frac{1}{4n}\left(2nr^{2n-1}\ln r + r^{2n-1}\right) \int_0^r s^{2n+1}\tilde{u}_{2n-1}(s)ds + \\
&+ \frac{1}{2} r^{-2n-1} \int_0^r s^{2n+1} \ln s\tilde{u}_{2n-1}(s)ds + \frac{1}{4n}\left(-2nr^{-2n-1}\ln r + r^{-2n-1}\right) \int_0^r s^{2n+1}\tilde{u}_{2n-1}(s)ds +
\end{aligned}$$

$$+\frac{1}{4n}r^{2n-1}\int_r^1 s^{-2n+1}\tilde{u}_{2n-1}(s)ds-\frac{1}{4n}r^{2n-1}\int_r^1 s^{2n+1}\tilde{u}_{2n-1}(s)ds+$$

$$+\frac{1}{2}r^{2n-1}\int_r^1 s^{2n+1}\ln s\tilde{u}_{2n-1}(s)ds+\frac{1}{4n}\left(2nr^{2n-1}\ln r+r^{2n-1}\right)\int_r^1 s^{2n+1}\tilde{u}_{2n-1}(s)ds-$$

$$-\frac{1}{4n}\left(2nr^{2n-1}\ln r+r^{2n-1}\right)\int_r^1 s^{-2n+1}\tilde{u}_{2n-1}(s)ds+\frac{1}{2}r^{2n-1}\int_r^1 s^{-2n+1}\ln s\tilde{u}_{2n-1}(s)ds.$$

Since

$$\int_0^r s^{2n+1}\ln sds=\frac{1}{2n+1}r^{2n+1}\ln r-\frac{r^{2n+1}}{(2n+1)^2},$$

then $\exists C_2>0$ such that

$$|d_n'(r)|\leq\frac{C_2}{n}\cdot\frac{\alpha^{2n-1}}{n}(|p_{2n-1}|+|p_{2n}|),$$

so, the series $\sum\limits_{n=2}^{\infty}\frac{\partial}{\partial r}D_n(r,\theta)$, $\sum\limits_{n=2}^{\infty}\frac{\partial^2}{\partial r\partial\theta}D_n(r,\theta)$ converge uniformly on $[a,b]\times[c,d]$.

It is easy to see that $\exists C_3>0$ such that

$$|d_n''(r)|\leq C_3\cdot\frac{\alpha^{2n-1}}{n}(|p_{2n-1}|+|p_{2n}|).$$

Hence, the series $\sum\limits_{n=2}^{\infty}\frac{\partial^2}{\partial r^2}D_n(r,\theta)$ converges uniformly on $[a,b]\times[c,d]$.

Thereby, $\tilde{y}\in C(\bar{Q})\bigcap C^2(Q)$ and Theorem is proved.

Remark 21.1 If $u(r,\theta)\in C(\bar{Q})$ and for some constant $C>0$ $\forall n\geq 1$ $|u_n(r)|\leq\frac{C}{n^2}$, then the control u is admissible in the problem (21.1)–(21.2), that is the corresponding function $y(r,\theta)$ from (21.5) defines classical solution of (21.1).

21.4 Conclusions

In this paper we proved a solvability of the optimal control problem on the classical solutions of elliptic boundary value problem in a circular sector with equality of flows on radiuses and equality of the solution on the one from radiuses to zero in distributed control class for quadratic cost functional.

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