Chapter 2
On Hyperbolic Zeta Function of Lattices

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Dedicated to the 95th Birth Anniversary
of Nikolai Mikhailovich Korobov
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Abstract This chapter provides an overview of the theory of hyperbolic zeta function of lattices. A functional equation for the hyperbolic zeta function of Cartesian lattice is obtained. Information about the history of the theory of the hyperbolic zeta function of lattices is provided. The relations with the hyperbolic zeta function of nets and Korobov optimal coefficients are considered.

2.1 Introduction

The introduction contains necessary definitions, results and historical facts about the appearance of the concepts of the hyperbolic zeta functions of nets and lattices, and gives its general theoretical review. The article is partly based on the monographs

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[8, 15], but it addresses the given problems from a more unified point of view. The article also utilizes the data from Chap. 6 of the monograph [30].

2.1.1 Lattices

First, we will recall some definitions.

Definition 2.1 Let $\boldsymbol{\lambda_1}, \dots, \lambda_{\mathbf{m}}$, $m \leq s$ be linearly independent system of vectors from $\mathbb{R}^s$. The set Λ of all vectors $a_1\lambda_{\mathbf{1}} + \dots + a_m\lambda_{\mathbf{m}}$, where a_i, $1 \leq i \leq m$ independently run through all integers, is called an m-dimensional lattice in $\mathbb{R}^s$, and the vectors $\lambda_{\mathbf{1}}, \dots, \lambda_{\mathbf{m}}$ are considered its basis.

If $m = s$, then a lattice is considered complete, otherwise it is incomplete. In this chapter we assume all lattices to be complete. Obviously, $\mathbb{Z}^s$ is a lattice. It is also called the fundamental lattice.

A lattice Λ is called an integer lattice in $\mathbb{R}^s$, if Λ is a sublattice of the fundamental lattice $\mathbb{Z}^s$, i.e.

$$\Lambda = \{m_1\lambda_{\mathbf{1}} + \dots + m_s\lambda_{\mathbf{s}} | m_1, \dots, m_s \in \mathbb{Z}\}$$

and $\lambda_{\mathbf{1}}, \dots, \lambda_{\mathbf{s}}$ is a linearly independent system of integer vectors.

Definition 2.2 For a lattice Λ there is a dual lattice Λ^*, which is the set

$$\Lambda^* = \{\mathbf{y} \mid \forall \mathbf{x} \in \Lambda \; (\mathbf{y}, \mathbf{x}) \in \mathbb{Z}\}. \tag{2.1}$$

Obviously, a dual lattice Λ^* for a lattice Λ is set by the dual basis $\lambda^*_{\mathbf{1}}, \dots, \lambda^*_{\mathbf{s}}$, determined by the equations

$$\left(\lambda^*_{\mathbf{i}}, \lambda_{\mathbf{j}}\right) = \delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases} \tag{2.2}$$

It's easy to see that the fundamental lattice $\mathbb{Z}^s$ coincides with its dual lattice and is also a sublattice of a dual lattice of any integer lattice. Moreover, if $\Lambda_1 \subset \Lambda \subset \mathbb{Z}^s$, then $\mathbb{Z}^s \subset \Lambda^* \subset \Lambda_1^*$; thus, for any $C \neq 0$ we see that $(C\Lambda)^* = \Lambda^*/C$. The equality $\det\Lambda^* = (\det\Lambda)^{-1}$ is true for any lattice.

The set of all s-dimensional complete lattices from $\mathbb{R}^s$ will be denoted as PR_s. The set of $\Lambda + \mathbf{x}$, where $\Lambda \in PR_s$ and $\mathbf{x} \in \mathbb{R}^s$ is called a shifted lattice. The set of all shifted lattices $\Lambda + \mathbf{x}$ from $\mathbb{R}^s$ will be denoted as CPR_s.

Concepts of lattices, shifted lattices and lattice projections on coordinate subspaces let us to discuss various issues of number theory in the uniform language.

E.g., if $(a_j, N) = 1 (1 \leq j \leq s)$, then the set $\Lambda = \Lambda(a_1, \dots, a_s; N)$ of solutions of the linearly homogeneous comparison is the lattice Λ with $\det\Lambda = N$

$$a_1 \cdot x_1 + \cdots + a_s \cdot x_s \equiv 0 \pmod{N}.$$

If F is a totally real algebraic extension of degree s of the field of rational numbers $\mathbb{Q}$ and $\mathbb{Z}_F$ is a ring of algebraic integers of the field F, then the set $\Lambda(F)$, which has been derived in the following way from $\mathbb{Z}_F$, is an s-dimensional lattice

$$\Lambda(F) = \{(\Theta^{(1)}, \ldots, \Theta^{(s)}) \mid \Theta^{(1)} \in \mathbb{Z}_F\}, \tag{2.3}$$

where $(\Theta^{(1)}, \ldots, \Theta^{(s)})$ is a system of algebraic conjugates, and if d is the discriminant of the field F, then $\det \Lambda(F) = \sqrt{d}$.

These two examples, namely the lattice $\Lambda(a_1, \ldots, a_s; N)$ of solutions of a linear equation and the algebraic lattice $\Lambda(F)$, are the focus of this chapter.

A lot of problems of geometry of numbers are defined in terms of shifted lattices $\Lambda + \mathbf{x}$, norms $N(\mathbf{x}) = |x_1 \cdot \ldots \cdot x_s|$, lattice norm minimum and shifted lattice norm minimum.

For an arbitrary lattice $\Lambda \in PR_s$, a norm minimum is the value

$$N(\Lambda) = \inf_{\mathbf{x} \in \Lambda \setminus \{\mathbf{0}\}} N(\mathbf{x}).$$

For an arbitrary shifted lattice $\Lambda + \mathbf{b} \in CPR_s$, a norm minimum is the value

$$N(\Lambda + \mathbf{b}) = \inf_{\mathbf{x} \in (\Lambda + \mathbf{b}) \setminus \{\mathbf{0}\}} N(\mathbf{x}).$$

Littlewood hypothesis has the following formula in these terms:

for $s > 1$ *and any non-zero real numbers* $\alpha_1, \ldots, \alpha_s$ *for the lattice*

$$\Lambda(\alpha_1, \ldots, \alpha_s) = \{(q, q \cdot \alpha_1 + p_1, \ldots, q \cdot \alpha_s + p_s) \mid q, p_1, \ldots, p_s \in \mathbb{Z}\}$$

$$N(\Lambda(\alpha_1, \ldots, \alpha_s)) = 0.$$

Oppenheim hypothesis, from which follows the Littlewood hypothesis, states in lattice terms that

for $s > 2$ *any* s*-dimensional lattice* Λ $N(\Lambda) > 0$ *is similar to an algebraic lattice.*

These two hypotheses are closely related to the Korobov's method of optimal coefficients.

A norm minimum is closely connected with a truncated norm minimum, or a hyperbolic lattice parameter. This is the value ([14, 17])

$$q(\Lambda) = \min_{\mathbf{x} \in \Lambda \setminus \{\mathbf{0}\}} q(\mathbf{x}),$$

which has simple geometrical meaning:

the hyperbolic cross $K_s(T)$ *does not contain nonzero points of the lattice* Λ *with* $T < q(\Lambda)$.

A hyperbolic cross is the area

$$K_s(T) = \{\mathbf{x} \mid q(\mathbf{x}) \leq T\},$$

where $q(\mathbf{x}) = \overline{x}_1 \cdot \ldots \cdot \overline{x}_s$ is a truncated norm of $\mathbf{x}$, and for a real x we will define $\overline{x} = \max(1, |x|)$ ([31], 1963).

Since $\max(1, N(\mathbf{x})) \leq q(\mathbf{x})$, it follows that $\max(1, N(\Lambda)) \leq q(\Lambda)$ for any lattice Λ, and the Minkowski's theorem on convex bodies states that

$$q(\Lambda) \leq \max(\det \Lambda,\ 1).$$

The issue of calculation of the hyperbolic parameter of the lattice of solutions of a linear equation has been addressed in the article [21].

2.1.2 Exponential Sums of Lattices

We will use $G_s = [0; 1)^s$ to denote a s-dimensional half-open cube. A net is an arbitrary nonempty finite set M in G_s. A net with weights is an ordered pair (M, ρ), where ρ is an arbitrary numerical function on M. For the sake of convenicence, we will indentify a net M with an ordered pair $(M, 1)$, that is, with a net with unit weights: $\rho \equiv 1$.

Definition 2.3 A product of two nets with weights (M_1, ρ_1) and (M_2, ρ_2) in G_s is a net with weights (M, ρ):

$$M = \{\, \{\mathbf{x} + \mathbf{y}\} \mid \mathbf{x} \in M_1,\ \mathbf{y} \in M_2\,\}, \quad \rho(\mathbf{z}) = \sum_{\substack{\{\mathbf{x}+\mathbf{y}\} = \mathbf{z},\\ \mathbf{x} \in M_1,\ \mathbf{y} \in M_2}} \rho_1(\mathbf{x})\rho_2(\mathbf{y}),$$

where $\{\mathbf{z}\} = (\{z_1\}, \ldots, \{z_s\})$.

The product of nets with weights (M_1, ρ_1) and (M_2, ρ_2) is denoted by

$$(M_1, \rho_1) \cdot (M_2, \rho_2).$$

Moreover, if $(M, \rho) = (M_1, \rho_1) \cdot (M_2, \rho_2)$, then we will write $M = M_1 \cdot M_2$ assuming that a net M is the product of nets M_1 and M_2 (see [23]).

Definition 2.4 An exponential sum of a net with weights (M, ρ) for an arbitrary integer vector $\mathbf{m}$ is

$$S(\mathbf{m}, (M, \rho)) = \sum_{\mathbf{x} \in M} \rho(\mathbf{x}) e^{2\pi i (\mathbf{m}, \mathbf{x})}, \tag{2.4}$$

and a normed exponential sum of a net with weights is

$$S^*(\mathbf{m}, (M, \rho)) = \frac{1}{|M|} S(\mathbf{m}, (M, \rho)).$$

Let $\rho(M) = \sum_{j=1}^{N} |\rho_j|$, then the following trivial estimate is true for all normed exponential sums of a net with weights:

$$|S^*(\mathbf{m}, (M, \rho))| \le \frac{1}{|M|} \rho(M).$$

It is easy to see, that for any nets with weights (M_1, ρ_1) and (M_2, ρ_2) the following equality is true:

$$S(\mathbf{m}, (M_1, \rho_1) \cdot (M_2, \rho_2)) = S(\mathbf{m}, (M_1, \rho_1)) \cdot S(\mathbf{m}, (M_2, \rho_2)). \tag{2.5}$$

Definition 2.5 If the following equality is true:

$$(M, 1) = (M_1, 1) \cdot (M_2, 1),$$

then nets M_1 and M_2 are called coprime nets.

Thus, if M_1 and M_2 are coprime nets then the equation $\mathbf{z} = \{\mathbf{x} + \mathbf{y}\}$ does not have more than one solution for $\mathbf{x} \in M_1$ and $\mathbf{y} \in M_2$. That is why the following equality is only true for coprime nets: $|M_1 \cdot M_2| = |M_1| \cdot |M_2|$.

When $\rho \equiv 1$ we obtain a definition of an exponential sum of a net.

Definition 2.6 An exponential sum of a net M for an arbitrary integer vector $\mathbf{m}$ is the value

$$S(\mathbf{m}, M) = \sum_{\mathbf{x} \in M} e^{2\pi i (\mathbf{m}, \mathbf{x})},$$

and a normed exponential sum of a net is

$$S^*(\mathbf{m}, M) = \frac{1}{|M|} S(\mathbf{m}, M).$$

It is easy to see, that for any coprime nets M_1 and M_2 the following equality is true:

$$S(\mathbf{m}, M_1 \cdot M_2) = S(\mathbf{m}, M_1) \cdot S(\mathbf{m}, M_2). \tag{2.6}$$

Let us take for an arbitrary integer lattice Λ, an integer vector $\mathbf{m}$ and an arbitrary vector $\mathbf{x}$ from a dual lattice Λ^* the following values:

$$\delta_\Lambda(\mathbf{m}) = \begin{cases} 1, & \text{if } \mathbf{m} \in \Lambda, \\ 0, & \text{if } \mathbf{m} \in \mathbb{Z}^s \setminus \Lambda, \end{cases} \qquad \delta^*_\Lambda(\mathbf{x}) = \begin{cases} 1, & \text{if } \mathbf{x} \in \mathbb{Z}^s, \\ 0, & \text{if } \mathbf{x} \in \Lambda^* \setminus \mathbb{Z}^s. \end{cases}$$

The $\delta_\Lambda(\mathbf{m})$ is the multidimensional generalisation of the famous Korobov's number-theoretical symbol

$$\delta_N(m) = \begin{cases} 1, & \text{if} \quad m \equiv 0 \pmod N, \\ 0, & \text{if} \quad m \not\equiv 0 \pmod N. \end{cases}$$

Definition 2.7 A generalised parallelepipedal net $M(\Lambda)$ is the set $M(\Lambda) = \Lambda^* \cap G_s$.

For an integer lattice Λ its generalised parallelepipedal net $M(\Lambda)$ is a complete system of residues of a dual lattice Λ^* modulo the fundamental sublattice $\mathbb{Z}^s$. Thus, we have the equality $|M(\Lambda)| = \det \Lambda$.

Definition 2.8 A complete linear multiple exponential sum of an integer lattice Λ is

$$s(\mathbf{m}, \Lambda) = \sum_{\mathbf{x} \in M(\Lambda)} e^{2\pi i(\mathbf{m},\mathbf{x})} = \sum_{\mathbf{x} \in \Lambda^*/\mathbb{Z}^s} e^{2\pi i(\mathbf{m},\mathbf{x})},$$

where $\mathbf{m}$ is an arbitrary integer vector.

It is clear, that for a generalised parallelepipedal net $M(\Lambda)$ the following equality is true: $S(\mathbf{m}, M(\Lambda)) = s(\mathbf{m}, \Lambda)$.

Definition 2.9 A complete linear multiple exponential sum of a dual lattice Λ^* of an integer lattice Λ is

$$s^*(\mathbf{x}, \Lambda) = \sum_{\mathbf{m} \in \mathbb{Z}^s/\Lambda} e^{2\pi i(\mathbf{m},\mathbf{x})} = \sum_{j=1}^{N} e^{2\pi i(\mathbf{m}_j,\mathbf{x})},$$

where $\mathbf{x}$ is an arbitrary vector of the dual lattice Λ^* and $\mathbf{m}_1, \dots, \mathbf{m}_N$ is a complete system of residues of the lattice $\mathbb{Z}^s$ modulo the sublattice Λ.

The following dual statements are true:

Theorem 2.1 *For $s(\mathbf{m}, \Lambda)$ the following equality is true:*

$$s(\mathbf{m}, \Lambda) = \delta_\Lambda(\mathbf{m}) \cdot \det \Lambda.$$

Theorem 2.2 *For any integer lattice Λ with* $\det \Lambda = N$ *and for an arbitrary* $\mathbf{x} \in \Lambda^*$ *the following equality is true:*

$$s^*(\mathbf{x}, \Lambda) = \delta^*_\Lambda(\mathbf{x}) \cdot \det \Lambda.$$

2.1.3 Multidimensional Quadrature Formulas and Hyperbolic Zeta Function of a Net

First works by Korobov were published in 1957–1959 [33–35], where the methods of number theory were applied to the problems of numerical integration of multiple integrals. After the class of periodical functions E_s^α had been defined, it has become possible to use methods of harmonic analysis and the theory of exponential sums (an important branch of analytic number theory) to estimate errors of approximate integration. The history of the creation of this method is presented in the chapter [32].

Banach space E_s^α consists of functions $f(\mathbf{x})$, where each of s variables $x_1, \ldots, x_s$ has a period of one, for which their Fourier series

$$f(\mathbf{x}) = \sum_{\mathbf{m}\in\mathbb{Z}^s} C(\mathbf{m})e^{2\pi i(m_1x_1+\cdots+m_sx_s)} \tag{2.7}$$

comply with the conditions

$$\sup_{\mathbf{m}\in\mathbb{Z}^s} |C(\mathbf{m})|(\overline{m}_1 \ldots \overline{m}_s)^\alpha = \|f(\mathbf{x})\|_{E_s^\alpha} < \infty. \tag{2.8}$$

Clearly, such Fourier series are absolutely convergent, since

$$\|f(\mathbf{x})\|_{l_1} = \sum_{\mathbf{m}\in\mathbb{Z}^s} |C(\mathbf{m})| \leq \|f(\mathbf{x})\|_{E_s^\alpha}(1 + 2\zeta(\alpha))^s,$$

and thus for any ($\alpha > 1$) they are continuous functions. Here and hereafter, as usual, $\zeta(\alpha)$ is the Riemann zeta function.

A truncated norm surface with parameter $t \geq 1$ is the set $N_s(t) = \{\mathbf{x} \mid q(\mathbf{x}) = t, \mathbf{x} \neq \mathbf{0}\}$, which is the boundary of the hyperbolic cross $K_s(t)$.

For a natural t on a truncated norm surface there is $\tau_s^*(t)$ of integer nonzero points, where[1]

$$\tau_s^*(t) = {\sum_{\mathbf{m}\in N(t)}}' 1 \tag{2.9}$$

is the number of presentations of the natural number t as $t = \overline{m_1} \ldots \overline{m_s}$.

Using new definitions, we can rewrite the expression for the norm $||f(\mathbf{x})||_{E_s^\alpha}$. The following equality is true:

$$||f(\mathbf{x})||_{E_s^\alpha} = \max\left(|C(\mathbf{0})|, \sup_{t\in\mathbb{N}}\left(t^\alpha \max_{\mathbf{m}\in N(t)} |C(\mathbf{m})|\right)\right).$$

It is easy to see, that an arbitrary periodic function $f(\mathbf{x})$ from $E_s^\alpha(C)$ is bounded in absolute value by $C\,(1 + 2\zeta(\alpha))^s$, and this estimate is achieved by the function

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[1] Here and hereafter $\sum'$ denotes summation over systems: $(m_1, \ldots, m_s) \neq (0, \ldots, 0)$.

$$f(\mathbf{x}) = \sum_{\mathbf{m}=-\infty}^{\infty} \frac{C}{(\overline{m}_1 \cdot \ldots \cdot \overline{m}_s)^\alpha} e^{2\pi i(\mathbf{m},\mathbf{x})}$$

in the point $\mathbf{x} = \mathbf{0}$.

Obviously, $E_s^\alpha(C) \subset E_s^\beta(C)$ for $\alpha \geq \beta$. For any periodic function

$$f(\mathbf{x}) \in E_s^\alpha(C) \subset E_s^\beta(C)$$

the following inequality is true

$$||f(\mathbf{x})||_{E_s^\alpha} \geq ||f(\mathbf{x})||_{E_s^\beta}.$$

The equality is true only for finite exponential polynomials

$$f(\mathbf{x}) = C(\mathbf{0}) + \sum_{\mathbf{m}\in N(1)} C(\mathbf{m})\, e^{2\pi i(\mathbf{m},\mathbf{x})}.$$

Let us take *the quadrature formula with weights*

$$\int_0^1 \ldots \int_0^1 f(x_1, \ldots, x_s) dx_1 \ldots dx_s = \frac{1}{N} \sum_{k=1}^{N} \rho_k f[\xi_1(k), \ldots, \xi_s(k)] - R_N[f]. \quad (2.10)$$

Here, $R_N[f]$ is the error resulting from the replacement of the integral

$$\int_0^1 \ldots \int_0^1 f(x_1, \ldots, x_s) dx_1 \ldots dx_s$$

with the weighted average value of the function $f(x_1, \ldots, x_s)$, calculated in points

$$M_k = (\xi_1(k), \ldots, \xi_s(k)) \qquad (k = 1, \ldots, N).$$

The set M of points M_k is *a net* M, and the points themselves are *the nodes of the quadrature formula*. The values $\rho_k = \rho(M_k)$ are the weights of the quadrature formula. In this chapter we assume all weights to be real-valued.

Definition 2.10 Zeta function of a net M with weights ρ and parameter $p \geq 1$ is the function $\zeta(\alpha, p|M, \rho)$ defined in the right half-plane $\alpha = \sigma + it$ $(\sigma > 1)$ by the Dirichlet series

$$\zeta(\alpha, p|M, \rho) = {\sum_{m_1,\ldots,m_s=-\infty}^{\infty}}' \frac{|S^*(\mathbf{m}, (M, \rho))|^p}{(\overline{m_1} \ldots \overline{m_s})^\alpha} = \sum_{n=1}^{\infty} \frac{S^*(p, M, \rho, n)}{n^\alpha}, \quad (2.11)$$

where

$$S^*(p, M, \rho, n) = \sum_{\mathbf{m}\in N(n)} |S^*(\mathbf{m}, (M, \rho))|^p. \tag{2.12}$$

The definition provides us with the following inequality:

$$\zeta(p\alpha, p|M, \rho) \le \zeta^p(\alpha, 1|M, \rho). \tag{2.13}$$

When all the weights are 1, we get the zeta function of a net M with parameter p and denote it as $\zeta(\alpha, p|M)$.

The formula (2.11) provides that the zeta function $\zeta(\alpha, p|M, \rho)$ of a net M with weights ρ and parameter $p \ge 1$ is a Dirichlet series, which converges in the right half-plane $\alpha = \sigma + i \cdot t$ $(\sigma > 1)$.

The following two Korobov's generalised theorems on errors of quadrature formulas are true:

Theorem 2.3 *Let the Fourier series of a function $f(\mathbf{x})$ absolutely converge, with $C(\mathbf{m})$ being its Fourier coefficients and $S(\mathbf{m}, (M, \rho))$ be an exponential sum of a lattice with weights, then the following equation is true:*

$$R_N[f] = C(\mathbf{0})\left(\frac{1}{N}S(\mathbf{0}, (M, \rho)) - 1\right) + \frac{1}{N}\sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}' C(\mathbf{m})S(\mathbf{m}, (M, \rho)) =$$
$$= C(\mathbf{0})\left(S^*(\mathbf{0}, (M, \rho)) - 1\right) + \sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}' C(\mathbf{m})S^*(\mathbf{m}, (M, \rho)) \tag{2.14}$$

and with $N \to \infty$ the error $R_N[f]$ will tend to zero only if the weighted nodes of the quadrature formula are evenly distributed in a s-dimensional unit cube.

Theorem 2.4 *If $f(x_1, \ldots, x_s) \in E_s^\alpha(C)$, then the following estimate is true for the error of the quadrature formula:*

$$|R_N[f]| \le C\left|\frac{1}{N}S(\mathbf{0}, (M, \rho)) - 1\right| + \frac{C}{N}\sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}' \frac{|S(\mathbf{m}, (M, \rho))|}{(\overline{m_1}\ldots\overline{m_s})^\alpha} =$$
$$= C\left|S^*(\mathbf{0}, (M, \rho)) - 1\right| + C \cdot \zeta(\alpha, 1|M, \rho), \tag{2.15}$$

where the sum $S(\mathbf{m}, (M, \rho))$ is defined by the equality (2.4). On the class $E_s^\alpha(C)$ this estimate cannot be improved.

The Theorem 2.4 can also be formulated as:

For the norm $\|R_N[f]\|_{E_s^\alpha}$ of the linear functional of the error of approximate integration with quadrature formula (2.10) the following equality is true:

$$\|R_N[f]\|_{E_s^\alpha} = \left|\frac{1}{N}S(\mathbf{0}, (M, \rho)) - 1\right| + \frac{1}{N}\sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}' \frac{|S(\mathbf{m}, (M, \rho))|}{(\overline{m_1}\ldots\overline{m_s})^\alpha} =$$
$$= \left|S^*(\mathbf{0}, (M, \rho)) - 1\right| + \zeta(\alpha, 1|M, \rho). \quad (2.16)$$

The method of optimal coefficients has proven to be the most productive for construction for the s-dimensional cube $G_s = [0; 1)^s$ of multidimensional quadrature formulas with parallelepipedal nets of the from:

$$\iint\limits_{Gs} f(\mathbf{x})d\mathbf{x} = \frac{1}{N}\sum_{k=1}^{N} f\left(\left\{\frac{a_1 k}{N}\right\}, \ldots, \left\{\frac{a_s k}{N}\right\}\right) - R_N(f),$$

where $R_N(f)$ is the error of the quadrature formula, and integers a_j $(a_j, N) = 1$ $(j = 1, .., s)$ are the optimal coefficients, chosen in a special way.

The first algorithms for calculation of optimal coefficients were created by Korobov in 1959. He is also the author of the most efficient and high-performance algorithms we use nowadays (see [38]). These algorithms are based on the lemma on hyperbolic parameter of the lattice of solutions of a linear equation by Gelfand (see [13, 28, 37, 38]). Based on the Korobov's suggestion, Dobrovol'skii and Klepikova have made tables of optimal coefficients for dimensions $s \le 30$ and modulo $N = 2^k \quad 3 \le k \le 22$ [11], which is far beyond the scope of the famous tables by Saltykov. The chapter by Bocharova, Van'kova and Dobrovol'skii [2] describes the modification of the Korobov's algorithm, which allows to find not only one optimal net modulo $N = 2^k$, but the whole class of such lattices. One more class of high-performance algorithms for optimal coefficients calculation has been found in the article [3]. Problems of finding optimal coefficients for combined lattices have been addressed in the articles [22, 39].

A series of important articles on applying divisor theory to the optimal coefficients search for parallelepipedal nets have been produced by Voronin and Timergaliyev (see [41–44]). In fact, these articles describe algorithms for the search of integer lattices with high-value hyperbolic lattice parameter.

In the study of the error of approximate integration for quadrature formulas with parallelepipedal nets on the class of periodical functions E_s^α Korobov in his article [34] for the first time mentions a special case of the hyperbolic zeta function of a lattice $\Lambda = \Lambda(a_1, \ldots, a_s; N)$ for real $\alpha > 1$:

$$\zeta_H(\Lambda|\alpha) = \sum_{m_1,\ldots,m_s=-\infty}^{+\infty}{}' \frac{\delta_N(a_1 \cdot m_1 + \cdots + a_s \cdot m_s)}{(\overline{m}_1 \cdot \ldots \cdot \overline{m}_s)^\alpha}, \quad (2.17)$$

where the Korobov's symbol $\delta_N(m)$ is defined by the following equalities:

$$\delta_N(m) = \begin{cases} 1 & \text{if} \quad m \equiv 0 \pmod N, \\ 0 & \text{if} \quad m \not\equiv 0 \pmod N, \end{cases}$$

and $(a_j, N) = 1 \quad (j = 1, 2, \dots, s)$.

The hyperbolic zeta function of a lattice $\Lambda = \Lambda(a_1, \dots, a_s; N)$ is important, because for the parallelepipedal net $M(\mathbf{a}, N)$, defined by the formula

$$M(\mathbf{a}, N) = \left\{ M_k = \left(\left\{ \frac{a_1 k}{N} \right\}, \dots, \left\{ \frac{a_s k}{N} \right\} \right) \middle| k = 0, \dots, N-1 \right\},$$

there is an equality $\zeta_H(\Lambda|\alpha) = \zeta(\alpha, 1|M(\mathbf{a}, N))$, i.e. the norm of the linear functional of the error of approximate integration with quadrature formulas with parallelpipedal nets equals the hyperbolic zeta function of the corresponding integer lattice of solutions of a linear equation.

The hyperbolic zeta function of the form (2.17) appears in a lot of articles addressing the estimate of errors of multidimensional quadrature formulas with parallelepipedal nets on the class E_s^α. Specifically, Bakhvalov [1] proved the estimate

$$\zeta_H(\Lambda|\alpha) \ll \frac{(\ln q(\Lambda) + 1)^{s-1}}{q(\Lambda)^\alpha}. \tag{2.18}$$

Korobov ([35], 1959) proved, that for such lattices

$$\zeta_H(\Lambda|\alpha) \gg \frac{\ln^{s-1} \det \Lambda}{(\det \Lambda)^\alpha} \tag{2.19}$$

for any integers $a_1, \dots, a_s$, which are coprime with N.

There are algorithms for finding $a_1, \dots, a_s$ such that

$$\zeta_H(\Lambda|\alpha) \ll \frac{\ln^{s\alpha} \det \Lambda}{(\det \Lambda)^\alpha} \quad \text{(Korobov 1960)},$$

$$\zeta_H(\Lambda|\alpha) \ll \frac{\ln^{(s-1)\alpha} \det \Lambda}{(\det \Lambda)^\alpha} \quad \text{(Bakhvalov and Korobov)}. \tag{2.20}$$

In its general form the hyperbolic zeta function of lattices appears in works by Frolov [26, 27]. Frolov's thesis [26] states, that for any $\alpha > 1$ and an arbitrary s-dimensional lattice Λ the series

$$\sum_{\mathbf{x} \in \Lambda}{}' (\bar{x}_1 \cdot \ldots \cdot \bar{x}_s)^{-\alpha}$$

absolutely converges.

Having studied an algebraic lattice of the form (2.3), Frolov proved, that for $t > 1$ and the lattice $\Lambda(t, F) = t\Lambda(F)$ with $\det \Lambda(t, F) = t^s \det \Lambda(F)$ the following estimate is true:

$$\zeta_H(\Lambda(t, F)|\alpha) \ll \frac{\ln^{s-1} \det \Lambda(t, F)}{(\det \Lambda(t, F))^\alpha}. \tag{2.21}$$

The Frolov's method is further developed in works by Bykovskii [4, 5] and by Dobrovol'skii [14, 16]. Construction from the chapter [14] shows, that the methods of Korobov and Frolov are two opposite poles of the theory of quadrature formulas with generalised parallelepipedal nets and special weight-function. At the same time, the problem of calculation of errors of approximate integration by these formulas can be turned into a number-theoretic problem of estimating the hyperbolic zeta function of the corresponding lattice once and for all. There's no need to estimate the norm of linear functional of errors of approximate integration for each new type of generalised parallelepipedal nets all over again.

The problems of integration over modified nets have been addressed in chapters [9, 10].

2.1.4 Hyperbolic Zeta Function of Lattices

The term "hyperbolic zeta function of lattice" has been introduces by Dobrovol'skii in 1984 in his works [14, 16], where systematic study of the function $\zeta_H(\Lambda|\alpha)$ has been started.

Specifically, lower estimates for the hyperbolic zeta function of an arbitrary s-dimensional lattice have been obtained:

$$\begin{cases} \zeta_H(\Lambda|\alpha) \geq C_1(\alpha, s)(\det \Lambda)^{-1}, & \text{if } 0 < \det \Lambda \leq 1, \\ \zeta_H(\Lambda|\alpha) \geq C_2(\alpha, s)(\det \Lambda)^{-\alpha} \ln^{s-1} \det \Lambda, & \text{if } \det \Lambda > 1, \end{cases} \tag{2.22}$$

where $C_1(\alpha, s), C_2(\alpha, s) > 0$ are constants depending only on α and s.

An upper estimate for the hyperbolic zeta function of an s-dimensional lattice has been proven:

$$\begin{cases} \zeta_H(\Lambda|\alpha) \leq C_3(\alpha, s)C_1(\Lambda)^s, & \text{if } q(\Lambda) = 1, \\ \zeta_H(\Lambda|\alpha) \leq C_4(\alpha, s)q^{-\alpha}(\Lambda)(\ln q(\Lambda) + 1)^{s-1}, & \text{if } q(\Lambda) > 1. \end{cases} \tag{2.23}$$

This result is a generalisation of the Bakhvalov's theorem, i.e. the inequality (2.18). The estimate (2.23) provides us with the following conclusions. Specifically, it unconditionally provides us with the result, obtained by Frolov (2.21), as the hyperbolic parameter $q(\Lambda(t, F)) = t^s$ for $t > 1$.

Dobrovol'skii has also proven the following theorem: *for any integer lattice Λ and a natural n we have the following presentation:*

$$\zeta_H(\Lambda|2n) = -1 + (\det \Lambda)^{-1} \sum_{\mathbf{x}\in M(\Lambda)} \prod_{j=1}^{s} \left(1 - \frac{(-1)^n (2\pi)^{2n}}{(2n)!} B_{2n}(x_j)\right), \tag{2.24}$$

where $B_{2n}(x)$ is a Bernoulli polynomial of the order $2n$ and $M(\Lambda)$ is the generalised parallelepipedal net of the lattice Λ, which consists of the points of the dual lattice Λ^*, lying in the s-dimensional half-open unit cube $G_s = [0; 1)^s$;

$$\zeta_H(\Lambda \mid 2n+1) = -1 + \frac{1}{\det \Lambda} \sum_{\mathbf{x} \in M(\Lambda)} \prod_{j=1}^{s} \left(1 - (-1)^n \frac{(2\pi)^{2n+1}}{(2n+1)!} \times \right.$$
$$\left. \times \int_0^1 \frac{B_{2n+1}(\{y+x_j\}) + B_{2n+1}(\{y-x_j\})}{2} \mathrm{ctg}(\pi y)\, dy \right).$$

This theorem points out an analogy between the hyperbolic zeta function of a lattice and the Riemann zeta function, for which

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1}\pi^{2n}}{(2n)!} B_{2n},$$

$$\zeta(2n+1) = (-1)^{n+1} \frac{2^{2n}\pi^{2n+1}}{(2n+1)!} \int_0^1 B_{2n+1}(y)\mathrm{ctg}(\pi y)\, dy.$$

Also, the following equality is true:

$$\zeta(\alpha) = \frac{1}{2}\zeta_H(\mathbb{Z}|\alpha) \quad \alpha = \sigma + it \quad \sigma > 1.$$

The presentation (2.24) unconditionally states that for any integer lattice Λ and an even $\alpha = 2n$ the value of $\zeta_H(\Lambda|2n)$ is a transcendental number.

The formula (2.24) allows to utilize $O(ns \det \Lambda)$ of operations to calculate $\zeta_H(\Lambda|2n)$. In their joint article, Dobrovol'skii, Esayan, Pihtilkov, Rodionova and Ustyan [20] have obtained the formula, which allows to calculate $\zeta_H(\Lambda(a; N)|2)$ using $O(\ln N)$ operations.

For the hyperbolic zeta function of the lattice $\Lambda(t, F)$ Dobrovol'skii, Van'kova and Kozlova in their joint article [12] have obtained the asymptotic formula

$$\zeta_H(\Lambda(t,F)|\alpha) = \frac{2(\det \Lambda(F))^{\alpha}}{R(s-1)!} \left(\sum_{(w)} \frac{1}{|N(w)|^{\alpha}} \right) \frac{\ln^{s-1} \det \Lambda(t,F)}{(\det \Lambda(t,F))^{\alpha}} + $$
$$+ O\left(\frac{\ln^{s-2} \det \Lambda(t,F)}{(\det \Lambda(t,F))^{\alpha}} \right), \tag{2.25}$$

where R is the regulator of a field F, and in the sum $\sum_{(w)} \frac{1}{|N(w)|^{\alpha}}$ the summation is over all the main ideals of the ring $\mathbb{Z}_F$.

At the first stage of research (1984–1990), the function $\zeta_H(\Lambda|\alpha)$ had been studied only for real $\alpha > 1$. But the joint articles by Dobrovol'skii, Rebrova and Roshchenya in 1995 ([17, 19]) introduced a new stage of research of the hyperbolic zeta function

$\zeta_H(\Lambda|\alpha)$ of a lattice Λ from different aspects: firstly, as a function of a complex argument α, and secondly, as a function on a metric space of lattices.

Thus, we have the following most general definition of the hyperbolic zeta function of a lattice Λ for a complex α.

Definition 2.11 The hyperbolic zeta function of a lattice Λ is the function $\zeta_H(\Lambda|\alpha)$, $\alpha = \sigma + it$ defined for $\sigma > 1$ by the absolutely convergent series

$$\zeta_H(\Lambda|\alpha) = \sum_{\mathbf{x}\in\Lambda}{}^{'} (\overline{x}_1 \cdot \ldots \cdot \overline{x}_s)^{-\alpha}. \tag{2.26}$$

By Abel's theorem ([6], p. 106) the hyperbolic zeta function of lattices can be represented in the following integral form:

$$\zeta_H(\Lambda|\alpha) = \alpha \int_1^\infty \frac{D(t|\Lambda)dt}{t^{\alpha+1}},$$

where $D(T|\Lambda)$ is the number of nonzero points of the lattice Λ in the hyperbolic cross $K_s(T)$.

First, we note that the hyperbolic zeta function of lattices is a Dirichlet series. Let us give some definitions.

The norm spectrum of a lattice Λ is the set of norm values in the nonzero points of the lattice Λ:

$$N_{sp}(\Lambda) = \{\lambda \mid \lambda = N(\mathbf{x}),\ \mathbf{x} \in \Lambda\backslash\{\mathbf{0}\}\}.$$

Correspondingly, the truncated norm spectrum of a lattice Λ is the set of truncated norm values in the nonzero points of the lattice:

$$Q_{sp}(\Lambda) = \{\lambda \mid \lambda = q(\mathbf{x}),\ \mathbf{x} \in \Lambda\backslash\{\mathbf{0}\}\}.$$

The truncated norm spectrum is a discrete numerical set, i.e.

$$Q_{sp}(\Lambda) = \{\lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots\} \qquad \lim_{k\to\infty} \lambda_k = \infty.$$

Obviously,

$$N(\Lambda) = \inf_{\lambda\in N_{sp}(\Lambda)} \lambda, \quad q(\Lambda) = \min_{\lambda\in Q_{sp}(\Lambda)} \lambda = \lambda_1.$$

The order of a point of the spectrum is the number of lattice points with the given norm value. If the number of such lattice points is infinite, then we assume that the point of the spectrum has an infinite order. The order of a point λ of the norm spectrum is denoted by $n(\lambda)$, and the order of a point λ of the truncated norm spectrum is denoted by $q(\lambda)$ correspondingly.

The concept of the order of a point of the spectrum provides a better understanding of the definition of the hyperbolic zeta function of a lattice. In it instead of the norm of a point $\mathbf{x}$ appears the truncated norm.

Let us give an example of a lattice Λ, for which the series

$$\sum_{\mathbf{x}\in\Lambda}{}' |x_1 \cdot \ldots \cdot x_s|^{-\alpha}$$

diverges for any $\alpha > 1$.

Actually, let $\Lambda = t\Lambda(F)$ be an algebraic lattice, then

$$\sum_{\mathbf{x}\in\Lambda}{}' |x_1 \cdot \ldots \cdot x_s|^{-\alpha} = \sum_{w\in\mathbb{Z}_F}{}' |t^s \cdot N(w)|^{-\alpha}, \quad (2.27)$$

where $N(w)$ is the norm of an algebraic integer from the ring $\mathbb{Z}_F$. By Dirichlet's unit theorem the series on the right side of the equality (2.27) diverges for any $\alpha > 1$, as the ring $\mathbb{Z}_F$ of algebraic integers of a totally real algebraic number field F of the power s has an infinite number of units ε and for them $|N(\varepsilon)| = 1$. Thus, in this case each point of the spectrum has an infinite order, which leads to the series' divergence for any α.

This example shows that the usage of the truncated norm of the vector $q(\mathbf{x}) = \overline{x}_1 \cdot \ldots \cdot \overline{x}_s$ instead of the norm $N(\mathbf{x}) = |x_1 \cdot \ldots \cdot x_s|$ in the definition of $\zeta_H(\Lambda \mid \alpha)$ has substantial meaning, as it provides absolute convergence of the series of the hyperbolic zeta function of any lattice Λ.

The discrete nature of the truncated norm spectrum provides that the hyperbolic zeta function of an arbitrary lattice Λ can be presented as a Dirichlet series:

$$\zeta_H(\Lambda|\alpha) = \sum_{\mathbf{x}\in\Lambda}{}' (\overline{x}_1 \cdot \ldots \cdot \overline{x}_s)^{-\alpha} = \sum_{\mathbf{x}\in\Lambda}{}' q(\mathbf{x})^{-\alpha} = \sum_{k=1}^{\infty} q(\lambda_k)\lambda_k^{-\alpha} =$$
$$= \sum_{\lambda\in Q_{sp}(\Lambda)} q(\lambda)\lambda^{-\alpha}. \quad (2.28)$$

As $D(T|\Lambda) = 0$ for $T < q(\Lambda)$, then

$$\zeta_H(\Lambda|\alpha) = \alpha \int_{q(\Lambda)}^{\infty} \frac{D(t|\Lambda)dt}{t^{\alpha+1}}.$$

The equality (2.28) provides, that for any complex $\alpha = \sigma + it$ in the right half-plane ($\sigma > 1$) there is a regular function of a complex variable, defined by the series (2.26) and the following inequality is true:

$$|\zeta_H(\Lambda|\alpha)| \le \zeta_H(\Lambda|\sigma).$$

A reasonable question arises, whether the hyperbolic zeta function $\zeta_H(\Lambda|\alpha)$ of an arbitrary lattice Λ can be extended to the whole complex plane. In their works, Dobrovol'skii, Rebrova and Roshchenya ([17, 19]) addressed these issues for PZ_s, i.e. the set of all integer lattices, PQ_s, i.e. the set of all rational lattices, PD_s i.e. the set of all lattices with diagonal matrices. It has been proven, that

for any integer lattice $\Lambda \in PZ_s$ the hyperbolic zeta function $\zeta_H(\Lambda|\alpha)$ is a regular function on all α-plane, excluding the point $\alpha = 1$, where it has a pole of order s.

For any lattice $\Lambda \in PQ_s$ the hyperbolic zeta function $\zeta_H(\Lambda|\alpha)$ is also a regular analytic function on all the α-plane, excluding the point $\alpha = 1$, where it has a pole of order s.

The behavior of the hyperbolic zeta function of lattices on the lattice space has been studied. In particular, it was found that

if a sequence of lattices $\{\Lambda_n\}$ converges to the lattice Λ, then the sequence of the hyperbolic zeta functions of lattices $\zeta_H(\Lambda_n|\alpha)$ converges uniformly to the hyperbolic zeta function of the lattice $\zeta_H(\Lambda|\alpha)$ in any half-plane $\sigma \geq \sigma_0 > 1$.

Another result of this kind can be formulated as follows:

for any point α on the α-plane, except of the point $\alpha = 1$, there is neighborhood $|\alpha - \beta| < \delta$ such that for any lattice $\Lambda = \Lambda(d_1, \dots, d_s) \in PD_s$

$$\lim_{M \to \Lambda, M \in PD_s} \zeta_H(M|\beta) = \zeta_H(\Lambda|\beta),$$

and this convergence is uniform in the neighborhood of the point α.

The derivation of these results is principally based on the asymptotic formula for the number of points of an arbitrary lattice in the hyperbolic cross as a function of the parameter of the hyperbolic cross. The formula has been obtained by Dobrovol'skii and Roshchenya ([18]):

$$D(T \mid \Lambda) = \frac{2^s T \ln^{s-1} T}{(s-1)! \det \Lambda} + \Theta C(\Lambda) \frac{2^s T \ln^{s-2} T}{\det \Lambda},$$

where $C(\Lambda)$ is an effective constant, calculated through the lattice basis, and $|\Theta| \leq 1$.

Gelfond has already pointed out an important relationship between the value of the hyperbolic parameter $q(\Lambda)$ of a lattice $\Lambda(a_1, \dots, a_{s-1}, 1; N)$ and the valule

$$Q = \min_{k=1,\dots,N-1} \overline{k} \cdot \overline{k}_1 \cdot \dots \cdot \overline{k}_{s-1},$$

where integers $k, k_1, \dots, k_{s-1}$ comply with the system of equations

$$\begin{cases} k_1 \equiv a_1 \cdot k \\ k_2 \equiv a_2 \cdot k \\ \dots\dots\dots \\ k_{s-1} \equiv a_{s-1} \cdot k \end{cases} \pmod N$$

with the lattice of solutions $\Lambda^{(p)}(a_1, \ldots, a_{s-1}, 1; N)$. This result is known as the Gelfond's lemma. It turned out that this relationship manifests itself during the analytic continuation into the left half-plane too.

Theorem 2.5 *In the left half-plane* $\alpha = \sigma + it$ $(\sigma < 0)$ *the following equalities are true:*

$$\zeta_H(\Lambda(a_1, \ldots, a_{s-1}, 1; N) \mid \alpha) =$$

$$= \sum_{t=1}^{s} M_\alpha^t N^{-\alpha t} \sum_{\mathbf{j}_t \in J_{t,s}} N^{t-1} \zeta(\Lambda^{(p)}(a_{j_1}, \ldots, a_{j_t}; N) \mid 1-\alpha),$$

$$\zeta_H(\Lambda^{(p)}(a_1, \ldots, a_{s-1}, 1; N) \mid \alpha) = -1 + \left(1 + \frac{M_\alpha}{N^\alpha} \zeta(\mathbb{Z} \mid 1-\alpha)\right)^s -$$

$$- \frac{M_\alpha^s}{N^{\alpha s}} \zeta^s(\mathbb{Z} \mid 1-\alpha) + \zeta(\Lambda(a_1, \ldots, a_{s-1}, 1; N) \mid 1-\alpha) \frac{M_\alpha^s N}{N^{\alpha s}},$$

where

$$M(\alpha) = \frac{2\Gamma(1-\alpha)}{(2\pi)^{1-\alpha}} \sin\frac{\pi\alpha}{2}.$$

This theorem provides the following result for the values of the hyperbolic zeta function of these lattices in negative odd points:

Theorem 2.6 *For* $\alpha = 1 - 2n,\ n \in \mathbb{N}$ *the following equalities are true:*

$$\zeta_H(\Lambda(a_1, \ldots, a_{s-1}, 1; N) \mid \alpha) =$$

$$= \sum_{t=1}^{s} \frac{(-1)^t N^{2nt-t}}{n^t} \sum_{\mathbf{j}_t \in J_{t,s}} \sum_{k_1, \ldots, k_{t-1}=0}^{N-1} \prod_{\nu=1}^{t-1} B_{2n}\left(\left\{\frac{k_\nu a_{j_\nu}}{N}\right\}\right) \times$$

$$\times B_{2n}\left(\left\{\frac{-(a_{j_1}k_1 + \cdots + a_{j_{s-1}}k_{s-1})}{N}\right\}\right),$$

$$\zeta_H(\Lambda^{(p)}(a_1, \ldots, a_{s-1}, 1; N) \mid \alpha) = -1 + \left(1 + \frac{N^{2n-1}B_{2n}}{n}\right)^s -$$

$$- \left(\frac{N^{2n-1}B_{2n}}{n}\right)^s + \left(\frac{1}{n}\right)^s \sum_{k=0}^{N-1} \prod_{j=1}^{s} B_{2n}\left(\left\{\frac{a_j k}{N}\right\}\right),$$

and negative even points are trivial zeroes.

2.1.5 Generalised Hyperbolic Zeta Function of Lattices

Based on the analogy between the hyperbolic zeta function of lattices and the Riemann zeta function, Rebrova in the article [40] studied the generalisation of the hyperbolic zeta function of lattices as an s-dimensional analogue of the Hurwitz zeta function. In her research she tried to answer the questions, naturally arising from such an approach: to what extent can the results regarding the hyperbolic zeta function of a lattice be transferred onto a general case? Can we obtain an analytic continuation of the generalised hyperbolic zeta function of a lattice to the whole complex plane? What is the behaviour of the generalised hyperbolic zeta function of a lattice as a function on the metric lattice space?

Definition 2.12 The generalised hyperbolic zeta function of a lattice Λ is the function $\zeta_H(\Lambda+\mathbf{b}|\alpha)$, defined in the right half-plane $\alpha=\sigma+it \quad (\sigma>1)$ by the absolutely convergent series

$$\zeta_H(\Lambda+\mathbf{b}\mid\alpha)=\sum_{\mathbf{x}\in\Lambda}{}'(\overline{x_1+b_1}\cdot\ldots\cdot\overline{x_s+b_s})^{-\alpha}=\sum_{\mathbf{x}\in(\Lambda+\mathbf{b})\setminus\{\mathbf{0}\}}q(\mathbf{x})^{-\alpha}, \tag{2.29}$$

where $\sum'$ means, that the point $\mathbf{x}=-\mathbf{b}$ is excluded from the summation.

From this point of view, we have to examine the place of shifted lattices and explore the possibility to define metrics on them.

Chapter 2 of the monograph [15] (see also [8]) addresses CPR_s i.e. the set of all shifted lattices $\Lambda(\mathbf{x})=\Lambda+\mathbf{x}$, where $\Lambda\in PR_s$ is an arbitrary s-dimensional real lattice, and $\mathbf{x}\in R^s$ is an arbitrary vector. A metric is defined on this set.

For the construction of an analytic continuation of the generalised hyperbolic zeta function, a fairly broad class of lattices is allocated—Cartesian lattices. We need the following definitions.

Definition 2.13 A simple Cartesian lattice is a shifted lattice $\Lambda+\mathbf{x}$ of the form

$$\Lambda+\mathbf{x}=(t_1\mathbb{Z}+x_1)\times(t_2\mathbb{Z}+x_2)\times\cdots\times(t_s\mathbb{Z}+x_s),$$

where $t_j\neq 0 \quad (j=1,\ldots,s)$.

In other words, if the lattice $\Lambda+\mathbf{x}$ is a simple Cartesian lattice then it is the result of the stretching of the fundamental lattice along the axes with coefficients $t_1,\ldots,t_s$ followed by a shift by the vector $\mathbf{x}$.

Definition 2.14 A Cartesian lattice is a shifted lattice, which can be presented as a union of a finite number of simple Cartesian lattices.

Definition 2.15 A Cartesian lattice is a shifted lattice with a shifted sublattice which is a simple Cartesian lattice.

Theorem 2.7 *Definitions 2.14 and 2.15 are equivalent.*

Theorem 2.8 *Any shift of a rational lattice is a Cartesian lattice.*

Two lattices Λ and Γ are considered similar, if

$$\Gamma = D(d_1, \dots, d_s) \cdot \Lambda, \quad \Lambda = D\left(\frac{1}{d_1}, \dots, \frac{1}{d_s}\right) \cdot \Gamma,$$

where

$$D(d_1, \dots, d_s) = \begin{pmatrix} d_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_s \end{pmatrix}$$

is an arbitrary diagonal matrix, $d_1 \cdot \ldots \cdot d_s \neq 0$.

The set of all nonsingular real diagonal matrices of an order s will be denoted as

$$D_s(\mathbb{R}) = \{D(d_1, \dots, d_s) \mid d_1 \cdot \ldots \cdot d_s \neq 0\}.$$

Regarding the operation of matrix multiplication $D_s(\mathbb{R})$ is a multiplicative abelian group.

The set of all unimodular real diagonal matrices $DU_s(\mathbb{R})$ is a subgroup of the group $D_s(\mathbb{R})$. Moreover,

$$D_s(\mathbb{R}) \cong DU_s(\mathbb{R}) \times \mathbb{R}^+,$$

where isomorphism φ between $D_s(\mathbb{R})$ and the direct product $DU_s(\mathbb{R}) \times \mathbb{R}^+$ is given by the rule

$$\varphi(D(d_1, \dots, d_s)) =$$

$$= \left(D\left(\frac{d_1}{\sqrt[s]{|d_1 \cdot \ldots \cdot d_s|}}, \dots, \frac{d_s}{\sqrt[s]{|d_1 \cdot \ldots \cdot d_s|}}\right), \sqrt[s]{|d_1 \cdot \ldots \cdot d_s|}\right).$$

Theorem 2.9 *An arbitrary Cartesian lattice is similar to a shifted integer lattice.*

Definition 2.16 An integer lattice Λ is simple, if its projections on any axis coincide with $\mathbb{Z}$.

Theorem 2.10 *Any integer lattice Λ is similar to a simple lattice uniquely determined by the lattice Λ.*

Theorem 2.11 *For any Cartesian lattice Λ there is only one presentation:*

$$\Lambda = D(t_1, \dots, t_s)\Lambda_0, \quad t_1, \dots, t_s > 0,$$

where Λ_0 is a simple lattice.

Let $M^*(\Lambda)$ be the set of points of the lattice Λ located in the s-dimensional half-open cube $[0; \det \Lambda)^s$. Thus, for any integer lattice Λ the set $M^*(\Lambda)$ is the complete system of residues of the lattice Λ modulo the sublattice $\det \Lambda \times \mathbb{Z}^s$.

Theorem 2.12 *Let*

$$\mathbf{x}(k_1, \dots, k_{s-1}) = \left(k_1, \dots, k_{s-1}, N\left\{\frac{-(a_1k_1 + \dots + a_{s-1}k_{s-1})}{N}\right\}\right),$$

then for the lattice $\Lambda = \Lambda(a_1, \dots, a_{s-1}, 1; N)$

$$M^*(\Lambda) = \{\mathbf{x}(k_1, \dots, k_{s-1}) \mid 0 \le k_\nu \le N-1 \ (\nu = 1, \dots, s-1)\} \tag{2.30}$$

and the following partition is true:

$$\Lambda(a_1, \dots, a_{s-1}, 1; N) = \bigcup_{\mathbf{x}\in M^*(\Lambda)} (N\mathbb{Z}^s + \mathbf{x}) =$$

$$= \bigcup_{k_1,\dots,k_{s-1}=0}^{N-1} (N\mathbb{Z}^s + \mathbf{x}(k_1, \dots, k_{s-1})). \tag{2.31}$$

Corollary 2.1 *The following partition is true:*

$$\Lambda(a_1, \dots, a_{s-1}, 1; N) =$$

$$= \bigcup_{k_1,\dots,k_{s-1}=0}^{N-1} \left(\prod_{j=1}^{s-1}(N\mathbb{Z} + k_j)\right) \times (N\mathbb{Z} - a_1k_1 - \dots - a_{s-1}k_{s-1}).$$

For the lattice $\Lambda(a_1, \dots, a_{s-1}, 1; N)$ we will examine its combined lattice $\Lambda^{(p)}$ $(a_1, \dots, a_{s-1}; N)$ of solutions of the system of linear equations

$$\begin{cases} m_1 \equiv a_1 \cdot m_s \\ m_2 \equiv a_2 \cdot m_s \\ \dots\dots\dots \\ m_{s-1} \equiv a_{s-1} \cdot m_s \end{cases} \pmod{N}. \tag{2.32}$$

For $(a_j, N) = 1$ $(j = 1, \dots, s-1)$ the lattice $\Lambda^{(p)}(a_1, \dots, a_{s-1}, 1; N)$ is also simple.

Corollary 2.2 *The following partition is true:*

$$\Lambda^{(p)}(a_1, \dots, a_{s-1}; N) = \bigcup_{k=0}^{N-1} \left(\prod_{j=1}^{s-1}(N\mathbb{Z} + a_jk)\right) \times (N\mathbb{Z} + k).$$

For an arbitrary shifted lattice $\Lambda + \mathbf{b} \in CPR_s$ a truncated norm minimum, or a hyperbolic parameter, is the value

$$q(\Lambda + \mathbf{b}) = \min_{\mathbf{x} \in (\Lambda + \mathbf{b}) \setminus \{\mathbf{0}\}} q(\mathbf{x}).$$

As $\max(1, N(\mathbf{x})) \leq q(\mathbf{x})$, then $\max(1, N(\Lambda + \mathbf{b})) \leq q(\Lambda + \mathbf{b})$, for any lattice Λ.

The norm spectrum of the shifted lattice $\Lambda + \mathbf{b}$ is the set of norm values in the nonzero points of the shifted lattice $\Lambda + \mathbf{b}$:

$$N_{sp}(\Lambda + \mathbf{b}) = \{\lambda \mid \lambda = N(\mathbf{x}),\ \mathbf{x} \in (\Lambda + \mathbf{b}) \setminus \{\mathbf{0}\}\}.$$

Correspondingly, the truncated norm spectrum of the shifted lattice $\Lambda + \mathbf{b}$ is the set of truncated norm values in the nonzero points of the shifted lattice:

$$Q_{sp}(\Lambda + \mathbf{b}) = \{\lambda \mid \lambda = q(\mathbf{x}),\ \mathbf{x} \in (\Lambda + \mathbf{b}) \setminus \{\mathbf{0}\}\}.$$

Obviously,

$$N(\Lambda + \mathbf{b}) = \inf_{\lambda \in N_{sp}(\Lambda + \mathbf{b})} \lambda,$$

$$q(\Lambda + \mathbf{b}) = \min_{\lambda \in Q_{sp}(\Lambda + \mathbf{b})} \lambda.$$

An order of a point of the spectrum is the number of points of the shifted lattice with the given norm value. If the number of such points of the shifted lattice is infinite, then we assume the point of the spectrum to have an infinite order. The order of a point λ of the spectrum is denoted by $n(\lambda)$, and the order of a point λ of the truncated norm spectrum is denoted by $q(\lambda)$.

The following analogue of the Lemma 1 from the article [17] is true.

Lemma 2.1 *For any lattice $\Lambda + \mathbf{b}$ and any point λ of the truncated norm spectrum $Q_{sp}(\Lambda + \mathbf{b})$ the order of the point λ is finite and $Q_{sp}(\Lambda + \mathbf{b})$—discrete.*

The Lemma 2.1 provides, that

$$Q_{sp}(\Lambda + \mathbf{b}) = \{\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots\}$$

and

$$q(\Lambda + \mathbf{b}) = \lambda_1, \quad \lim_{n \to \infty} \lambda_n = \infty.$$

That provides, that the hyperbolic zeta function of an arbitrary shifted lattice $\Lambda + \mathbf{b}$ can be presented as a Dirichlet series:

$$\zeta_H(\Lambda + \mathbf{b} \mid \alpha) = \sum_{\mathbf{x} \in (\Lambda + \mathbf{b}) \setminus \{\mathbf{0}\}} q(\mathbf{x})^{-\alpha} = \sum_{k=1}^{\infty} q(\lambda_k) \lambda_k^{-\alpha} = \sum_{\lambda \in Q_{sp}(\Lambda + \mathbf{b})} q(\lambda) \lambda^{-\alpha}.$$

Theorem 2.13 *For any $\alpha = \sigma + it$ in the right half-plane $\sigma > 1$ the Dirichlet series for $\zeta_H(\Lambda + \mathbf{b} \mid \alpha)$ is absolutely convergent; and in the half-plane $\sigma \geq \sigma_0 > 1$ it is uniformly convergent.*

As for $\alpha = \sigma + it$ and $\sigma \geq \sigma_0 > 0$

$$\sum_{k=1}^{\infty} \left| \frac{q(\lambda_k)}{\lambda_k^{\alpha}} \right| \leq \sum_{k=1}^{\infty} \frac{q(\lambda_k)}{\lambda_k^{\sigma_0}} = \zeta_H(\Lambda + \mathbf{b} \mid \sigma_0),$$

then the Theorem 2.13 provides, that for any complex $\alpha = \sigma + it$ in the right half-plane ($\sigma > 1$) there is a regular function of a complex variable, defined by the series (2.29) and the following inequality is true:

$$|\zeta_H(\Lambda + \mathbf{b} \mid \alpha)| \leq \zeta_H(\Lambda + \mathbf{b} \mid \sigma).$$

Theorem 2.14 *The generalised hyperbolic zeta function of the unidimensional fundamental lattice is an analytic function on the whole α-plane, excluding the point $\alpha = 1$, where it has a pole of order 1 with the residue equal to 2.*

Theorem 2.15 *For an arbitrary shifted unidimensional lattice $\Lambda + b = d\mathbb{Z} + b$ the generalised hyperbolic zeta function $\zeta_H(d \cdot \mathbb{Z} + b \mid \alpha)$ is analytic on the whole α-plane, excluding the point $\alpha = 1$, where it has a pole of order 1 with the residue equal to $\dfrac{2}{\det \Lambda}$.*

Theorem 2.16 *The generalised hyperbolic zeta function $\zeta_H(\Lambda \mid \alpha)$ of any simple Cartesian lattice $\Lambda = \prod_{j=1}^{s}(d_j\mathbb{Z} + a_j)$ is analytic on the whole α-plane, excluding the point $\alpha = 1$, where it has a pole of order s.*

Theorem 2.17 *For any Cartesian lattice Λ the generalised hyperbolic zeta function $\zeta_H(\Lambda + \mathbf{b} \mid \alpha)$ is analytic on the whole α-plane, excluding the point $\alpha = 1$, where it has a pole of order s.*

After that the problem of behavior of the generalised hyperbolic zeta function on the orbit of Cartesian lattices is addressed. Again, we start the examination with the unidimensional case.

Theorem 2.18 *For any point α on the α-plane, excluding the point $\alpha = 1$, there is neighborhood $|\alpha - \beta| < \delta$ such that for any shifted lattice $\Lambda + b \in CPR_1$*

$$\lim_{\Gamma + g \to \Lambda + b} \zeta_H(\Gamma + g \mid \beta) = \zeta_H(\Lambda + b \mid \beta),$$

and this convergence is uniform in the neighborhood of the point α.

Theorem 2.19 *For any point α on the α-plane, excluding the point $\alpha = 1$, there is neighborhood $|\alpha - \beta| < \delta$ such that for any Cartesian lattice $\Lambda + \mathbf{b} \in CPR_s$*

$$\lim_{D(q_1,\ldots,q_s)\cdot\Lambda+\mathbf{g}\to\Lambda+\mathbf{b}} \zeta_H(D(q_1,\ldots,q_s)\cdot\Lambda+\mathbf{g}\mid\beta)=\zeta_H(\Lambda+\mathbf{b}\mid\beta),$$

and this convergence is uniform in the neighborhood of the point α.

2.2 Functional Equation for Hyperbolic Zeta Function of Integer Lattices

The articles [24, 25] utilized a new approach to obtain the functional equation for the hyperbolic zeta function. Earlier, to prove the existence of an analytic continuation of the hyperbolic zeta function of an arbitrary Cartesian lattice only the method of expansion of the integer lattice Λ on sublattice $\det\Lambda\cdot\mathbb{Z}^s$ was used followed by the Hurwitz functional equation. Now exponential sums of a lattice were used, which allowed to apply the known features of Dirichlet series with periodic coefficients. Moreover, the concept of the zeta function helps to simplify the arguments and formulas.

As usual, we will use $N(\mathbf{x})=|x_1\ldots x_s|$ to denote the multiplicative norm of the vector $\mathbf{x}$. It has non-zero values only in points of general position, i.e. points without zero coordinates. Let us present new definitions using the multiplicative norm.

Definition 2.17 The zeta function of a lattice Λ is the function $\zeta(\Lambda|\alpha)$, $\alpha=\sigma+it$, defined for $\sigma>1$ by the series

$$\zeta(\Lambda|\alpha)=\sum_{\mathbf{x}\in\Lambda,\,N(\mathbf{x})\neq0}|x_1\ldots x_s|^{-\alpha}. \tag{2.33}$$

Generally speaking, there is no zeta function for certain lattices Λ, as the corresponding series can diverge for any value of $\alpha=\sigma+it$ but for an arbitrary Cartesian lattice Λ it is obviously exist for $\sigma>1$.

Also, the hyperbolic zeta function is not homogeneous (as a function of a lattice), while the zeta function of a lattice is homogeneous:

$$\zeta(T\Lambda|\alpha)=T^{-s\alpha}\zeta(\Lambda|\alpha). \tag{2.34}$$

The concept of the zeta function of a lattice is the special case with $\mathbf{b}=\mathbf{0}$ of the concept of the generalised zeta function of a lattice.

Definition 2.18 A generalised zeta function of a lattice Λ is the function $\zeta\,(\Lambda+\mathbf{b}|\,\alpha)$, $\alpha=\sigma+it$, defined for $\sigma>1$ by the series

$$\zeta\,(\Lambda+\mathbf{b}|\,\alpha)=\sum_{\mathbf{x}\in\Lambda+\mathbf{b},\,N(\mathbf{x})\neq0}|x_1\ldots x_s|^{-\alpha}. \tag{2.35}$$

It is easy to see, that the hyperbolic zeta function of a lattice Λ is directly defined by the sum of the zeta function of a lattice Λ and the zeta functions of corresponding integer lattices of smaller dimensions, which are obtained by discarding of zero coordinates.

Let

$$J_{t,s} = \{\mathbf{j}_t = (j_1, \ldots, j_s) \mid 1 \leq j_1 < \cdots < j_t \leq s,\ 1 \leq j_{t+1} < \cdots < j_s \leq s,$$

$$\{j_1, \ldots, j_s\} = \{1, 2, \ldots, s\}\}.$$

In other words, the set $J_{t,s}$ consists of integer vectors $\mathbf{j}_t$, coordinates of which form a permutation of numbers from 1 to s, while coordinates from 1 to t and from $t+1$ to s form increasing sequences.

If we denote the coordinate subspace as $\Pi(\mathbf{j}_t)$

$$\Pi(\mathbf{j}_t) = \{\mathbf{x} \mid x_{j_\nu} = 0\ (\nu = t+1, \ldots, s)\},$$

and denote the projection of intersection of $(\Lambda + \mathbf{a}) \bigcap \Pi(\mathbf{j}_t)$ on $\mathbb{R}^t$ as $(\Lambda + \mathbf{a})_{\mathbf{j}_t}$, then for any shifted lattice the following equality is true:

$$\zeta_H(\Lambda + \mathbf{a} \mid \alpha) = \sum_{t=1}^{s} \sum_{\mathbf{j}_t \in J_{t,s}} \zeta((\Lambda + \mathbf{a})_{\mathbf{j}_t} \mid \alpha).$$

2.2.1 Periodized in the Parameter b Hurwitz Zeta Function

Hereafter we will use the periodized in the parameter b Hurwitz zeta function

$$\zeta^*(\alpha; b) = \sum_{0 < n+b} (n+b)^{-\alpha} = \begin{cases} \sum\limits_{n=1}^{\infty} n^{-\alpha}, & \{b\} = 0, \\ \sum\limits_{n=0}^{\infty} (n + \{b\})^{-\alpha}, & \{b\} > 0 \end{cases}, \quad (\sigma > 1).$$

It's easy to write out various explicit formulas for analytic continuation on the whole complex plane except the point $\alpha = 1$ of the periodized Hurwitz zeta function. In this point for any real value of b the periodized Hurwitz zeta function has a pole of order 1 with residue equal to 1.

The following formulas cover the whole complex plane and define the explicit analytic continuation of $\zeta^*(\alpha; b)$.

$$\zeta^*(\alpha; b) = \\ = \begin{cases} \sum\limits_{0<n+b} (n+b)^{-\alpha}, & \sigma > 1, \\ \frac{1}{2} + \frac{1}{\alpha-1} - \alpha(\alpha+1) \int\limits_1^{\infty} \frac{\{x\}^2 - \{x\} dx}{2x^{\alpha+2}}, \qquad \{b\} = 0, & \sigma > -1, \\ \frac{1}{2\{b\}^{\alpha}} + \frac{1}{(\alpha-1)\{b\}^{\alpha-1}} - \alpha(\alpha+1) \int\limits_1^{\infty} \frac{\{x\}^2 - \{x\}\, dx}{2\,(x+\{b\})^{\alpha+2}}, & \{b\} \neq 0,\ \sigma > -1, \\ 2(2\pi)^{\alpha-1}\Gamma(1-\alpha)\left(\sin\frac{\pi\alpha}{2}\sum\limits_{n=1}^{\infty}\frac{\cos 2\pi nb}{n^{1-\alpha}} + \cos\frac{\pi\alpha}{2}\sum\limits_{n=1}^{\infty}\frac{\sin 2\pi nb}{n^{1-\alpha}}\right), & \sigma < 0. \end{cases} \tag{2.36}$$

2.2.2 Dirichlet Series with Periodic Coefficients

Let us examine the special case of Dirichlet series with periodic coefficients of the form

$$l\left(\alpha, \frac{b}{n}\right) = \sum_{m=1}^{\infty} \frac{e^{2\pi i \frac{bm}{n}}}{m^{\alpha}} \quad (\sigma > 1) \tag{2.37}$$

and prove for them the special case of the general theorem (see [7]) on analytic continuation of Dirichlet series with periodic coefficients on the whole complex plane.

Lemma 2.2 *For $\sigma > 1$ the following equality is true:*

$$l\left(\alpha, \frac{b}{n}\right) = \begin{cases} \zeta(\alpha) & \text{if} \quad \delta_n(b) = 1, \\ \frac{1}{n^{\alpha}} \sum\limits_{j=1}^{n} e^{2\pi i \frac{bj}{n}} \zeta^*\left(\alpha, \frac{j}{n}\right) & \text{if} \quad \delta_n(b) = 0. \end{cases} \tag{2.38}$$

Lemma 2.3 *For $\sigma > 0$ and $\delta_n(b) = 0$ the following equality is true:*

$$\int\limits_1^{\infty} \frac{e^{2\pi i \frac{b[t]}{n}}}{t^{\alpha+1}} dt = (\alpha+1) \int\limits_1^{\infty} \frac{\frac{e^{2\pi i \frac{b[t]}{n}} - e^{2\pi i \frac{b}{n}}}{e^{2\pi i \frac{b}{n}} - 1} + e^{2\pi i \frac{b[t]}{n}} \{t\}}{t^{\alpha+2}} dt. \tag{2.39}$$

Theorem 2.20 *For a natural n, an integer b with $\delta_n(b) = 0$ and analytic continuation of the function $l\left(\alpha, \frac{b}{n}\right)$ on the whole complex plane the following presentations are true:*

$$l\left(\alpha, \frac{b}{n}\right) =$$
$$= \begin{cases} \sum\limits_{m=1}^{\infty} \dfrac{e^{2\pi i \frac{bm}{n}}}{m^{\alpha}}, & \sigma > 1, \\ \dfrac{\alpha e^{2\pi i \frac{b}{n}}}{e^{2\pi i \frac{b}{n}} - 1} \displaystyle\int\limits_1^{\infty} \dfrac{e^{2\pi i \frac{b[t]}{n}}}{t^{\alpha+1}}\, dt - \dfrac{e^{2\pi i \frac{b}{n}}}{e^{2\pi i \frac{b}{n}} - 1}, & \sigma > 0, \\ \dfrac{\alpha(\alpha+1) e^{2\pi i \frac{b}{n}}}{e^{2\pi i \frac{b}{n}} - 1} \displaystyle\int\limits_1^{\infty} \dfrac{\frac{e^{2\pi i \frac{b[t]}{n}} - e^{2\pi i \frac{b}{n}}}{e^{2\pi i \frac{b}{n}} - 1} + e^{2\pi i \frac{b[t]}{n}} \{t\}}{t^{\alpha+2}}\, dt - \dfrac{e^{2\pi i \frac{b}{n}}}{e^{2\pi i \frac{b}{n}} - 1}, & \sigma > -1, \\ (2\pi)^{\alpha-1} \Gamma(1-\alpha) \left(\sum\limits_{m=1}^{\infty} \dfrac{e^{\frac{\pi i(\alpha-1)}{2}}}{\left(m - \left\{\frac{b}{n}\right\}\right)^{1-\alpha}} + \sum\limits_{m=0}^{\infty} \dfrac{e^{\frac{-\pi i(\alpha-1)}{2}}}{\left(m + \left\{\frac{b}{n}\right\}\right)^{1-\alpha}} \right), & \sigma < 0. \end{cases} \tag{2.40}$$

This result can be applied to another type of Dirichlet series with periodic coefficients. Let

$$l^{*}\left(\alpha, \frac{b}{n}\right) = \sum_{m=-\infty}^{\infty} \frac{e^{2\pi i \frac{bm}{n}}}{\overline{m}^{\alpha}} \quad (\Re\alpha > 1). \tag{2.41}$$

The Dirichlet series of the latest form can directly define the hyperbolic zeta function of integer lattices for $\sigma > 1$, if we use exponential sums of lattices, and namely, for any integer lattice Λ:

$$\begin{aligned} \zeta_H(\Lambda|\alpha) + 1 &= \sum_{\mathbf{x} \in \Lambda}{}' (\overline{x}_1 \cdot \ldots \cdot \overline{x}_s)^{-\alpha} + 1 = \sum_{\mathbf{m} \in \mathbb{Z}^s} \frac{\delta_{\Lambda}(\mathbf{m})}{(\overline{m}_1 \cdot \ldots \cdot \overline{m}_s)^{\alpha}} = \\ &= \frac{1}{\det \Lambda} \sum_{\mathbf{x} \in M(\Lambda)} \sum_{\mathbf{m} \in \mathbb{Z}^s} \frac{e^{2\pi i (\mathbf{m}, \mathbf{x})}}{(\overline{m}_1 \cdot \ldots \cdot \overline{m}_s)^{\alpha}} = \\ &= \frac{1}{\det \Lambda} \sum_{\mathbf{x} \in M(\Lambda)} \prod_{j=1}^{s} \sum_{m_j=-\infty}^{\infty} \frac{e^{2\pi i m_j x_j}}{\overline{m}_j^{\alpha}} \\ &= \frac{1}{\det \Lambda} \sum_{\mathbf{x} \in M(\Lambda)} \prod_{j=1}^{s} l^{*}\left(\alpha, \frac{b_j(\mathbf{x})}{\det \Lambda}\right), \end{aligned} \tag{2.42}$$

where $b_j(\mathbf{x}) = x_j \det \Lambda$ is an integer ($j = 1, \ldots, s$) for any point $\mathbf{x} = (x_1, \ldots, x_s) \in M(\Lambda)$.

Theorem 2.21 *For a natural n, an integer b with $\delta_n(b) = 0$ and analytic continuation of the function $l^{*}\left(\alpha, \frac{b}{n}\right)$ on the whole complex plane the following presentations are true:*

$$l^*\left(\alpha, \frac{b}{n}\right) =$$

$$= \begin{cases} \sum\limits_{m=-\infty}^{\infty} \dfrac{e^{2\pi i \frac{bm}{n}}}{\overline{m}^{\alpha}}, & \sigma > 1, \\ \dfrac{\alpha}{e^{2\pi i \frac{b}{n}} - 1} \int\limits_{1}^{\infty} \dfrac{e^{2\pi i \frac{b([t]+1)}{n}} - e^{-2\pi i \frac{b[t]}{n}}}{t^{\alpha+1}} dt, & \sigma > 0, \\ \dfrac{\alpha(\alpha+1)}{e^{2\pi i \frac{b}{n}} - 1} \int\limits_{1}^{\infty} \dfrac{g(t, b, n)}{t^{\alpha+2}} dt, & \sigma > -1, \\ 1 + 2(2\pi)^{\alpha-1} \Gamma(1-\alpha) \cos \dfrac{\pi(\alpha-1)}{2} \cdot n^{1-\alpha} \sum\limits_{m=-\infty}^{\infty} \dfrac{1}{\left(\overline{nm+b}\right)^{1-\alpha}} & \sigma < 0, \end{cases} \tag{2.43}$$

where

$$g(t, b, n) = \frac{e^{2\pi i \frac{b}{n}} \left(e^{2\pi i \frac{b[t]}{n}} - e^{2\pi i \frac{b}{n}} + e^{-2\pi i \frac{b[t]}{n}} - e^{-2\pi i \frac{b}{n}}\right)}{e^{2\pi i \frac{b}{n}} - 1} +$$
$$+ \left(e^{2\pi i \frac{b([t]+1)}{n}} - e^{-2\pi i \frac{b[t]}{n}}\right) \{t\}.$$

Note 2.1 The latest equality won't change if rewritten as follows

$$l^*\left(\alpha, \frac{b}{n}\right) = 1 + 2(2\pi)^{\alpha-1} \Gamma(1-\alpha) \cos \frac{\pi(\alpha-1)}{2} \cdot n^{1-\alpha} \sum_{\substack{m=-\infty, \\ nm+b \neq 0}}^{\infty} \frac{1}{\left(\overline{nm+b}\right)^{1-\alpha}},$$

which remains true with $\delta_n(b) = 1$:

$$l^*\left(\alpha, \frac{0}{n}\right) = 1 + 2\zeta(\alpha) = 1 + 2(2\pi)^{\alpha-1} \Gamma(1-\alpha) \cos \frac{\pi(\alpha-1)}{2} \sum_{m=1}^{\infty} \frac{1}{m^{1-\alpha}} =$$
$$= 1 + 2(2\pi)^{\alpha-1} \Gamma(1-\alpha) \cos \frac{\pi(\alpha-1)}{2} \cdot n^{1-\alpha} \sum_{\substack{m=-\infty, \\ nm \neq 0}}^{\infty} \frac{1}{(\overline{nm})^{1-\alpha}}.$$

2.2.3 Functional Equation for Hyperbolic Zeta Zunction of Integer Lattices

Let us obtain the explicit form of the $\zeta_H(\Lambda \mid \alpha)$ in the left half-plane for an arbitrary integer lattice Λ. For this, we will need a combined lattice $\Lambda^{(p)}$, which is defined by the following relationship:

$$\Lambda^{(p)} = \det \Lambda \cdot \Lambda^*. \qquad (2.44)$$

For any integer lattice Λ its combined latice $\Lambda^{(p)}$ is also integer. As these lattices are special cases of Cartesian lattices, then, as we know, there are analytic continuations

$$\zeta_H(\Lambda \mid \alpha) \quad \text{and} \quad \zeta_H(\Lambda^{(p)} \mid \alpha)$$

on the whole complex α–plane, excluding the point $\alpha = 1$, where they have a pole of order s.

For the sake of convenience, we will use the following notations:

$$N = \det \Lambda, \quad M^{(p)}(\Lambda) = \det \Lambda \cdot M(\Lambda), \quad M^*(\Lambda) = \Lambda \cap [0; \det \Lambda)^s. \qquad (2.45)$$

It is clear, that the following expansions are true:

$$\Lambda = \bigcup_{\mathbf{x} \in M^*(\Lambda)} \left(\mathbf{x} + N\mathbb{Z}^s\right), \quad \Lambda^{(p)} = \bigcup_{\mathbf{x} \in M^{(p)}(\Lambda)} \left(\mathbf{x} + N\mathbb{Z}^s\right). \qquad (2.46)$$

Let $\mathbf{j}_t \in J_{t,s}$. We will denote the coordinate subspace as $\Pi(\mathbf{j}_t)$

$$\Pi(\mathbf{j}_t) = \{\mathbf{x} \mid x_{j_\nu} = 0 \ (\nu = t+1, \dots, s)\}.$$

If we assume, that $\mathbf{j}_t^* = (j_{t+1}, \dots, j_s, j_1, \dots, j_t)$, then $\mathbf{j}_t^* \in J_{s-t,s}$ and

$$\mathbb{R}^s = \Pi(\mathbf{j}_t) \bigoplus \Pi(\mathbf{j}_t^*)$$

is decomposition into the direct sum of coordinate subspaces. If we denote projections of a shifted lattice on coordinate subspaces $\Pi(\mathbf{j}_t)$ and $\Pi(\mathbf{j}_t^*)$ according to decomposition of the space in the direct sum of these coordinate subspaces as $(\Lambda + \mathbf{a})^{(1)}_{\mathbf{j}_t}$ and $(\Lambda + \mathbf{a})^{(2)}_{\mathbf{j}_t}$; and denote its intersections with coordinate subspaces as $(\Lambda + \mathbf{a})_{\mathbf{j}_t} = (\Lambda + \mathbf{a}) \bigcap \Pi(\mathbf{j}_t)$ and $(\Lambda + \mathbf{a})_{\mathbf{j}_t^*} = (\Lambda + \mathbf{a}) \bigcap \Pi(\mathbf{j}_t^*)$, then, generally speaking, $(\Lambda+\mathbf{a})^{(1)}_{\mathbf{j}_t} \neq (\Lambda+\mathbf{a})_{\mathbf{j}_t}$ and $(\Lambda+\mathbf{a})^{(2)}_{\mathbf{j}_t} \neq (\Lambda+\mathbf{a})_{\mathbf{j}_t^*}$. The equality is possible, if and only if $\Lambda+\mathbf{a} = (\Lambda_1+\mathbf{a}_1)\times(\Lambda_2+\mathbf{a}_2)$, $\Lambda_1+\mathbf{a}_1 = (\Lambda+\mathbf{a})_{\mathbf{j}_t}$ $\Lambda_2+\mathbf{a}_2 = (\Lambda+\mathbf{a})_{\mathbf{j}_t^*}$.

We need to recall that

$$M(\alpha) = \frac{2\Gamma(1-\alpha)}{(2\pi)^{1-\alpha}} \sin \frac{\pi\alpha}{2}$$

and that for an arbitrary integer lattice Λ its zeta function $\zeta(\Lambda \mid \alpha)$ in the right half-plane is defined by the equality

$$\zeta(\Lambda \mid \alpha) = \sum_{\mathbf{x}\in\Lambda,\, N(\mathbf{x})\neq 0} |x_1 \dots x_s|^{-\alpha}.$$

Theorem 2.22 *For the zeta function of an arbitrary integer lattice Λ in the left half-plane $\sigma < 0$ the following functional equation is true:*

$$\zeta(\Lambda \mid \alpha) = \frac{1}{N}\left(M(\alpha)N^{1-\alpha}\right)^s \zeta\left(\Lambda^{(p)}\middle| 1-\alpha\right). \tag{2.47}$$

If we address dual lattices, then this theorem can be rewritten in the following way:

Theorem 2.23 *For the zeta function of an arbitrary integer lattice Λ in the left half-plane $\sigma < 0$ the following functional equation is true:*

$$\zeta(\Lambda \mid \alpha) = \frac{M(\alpha)^s}{N}\zeta\left(\Lambda^*\middle| 1-\alpha\right). \tag{2.48}$$

Proof As we can see, $\Lambda^{(p)} = N \cdot \Lambda^*$, therefore

$$\left(N^{1-\alpha}\right)^s \zeta\left(\Lambda^{(p)}\middle| 1-\alpha\right) = \left(N^{1-\alpha}\right)^s \sum_{\mathbf{x}\in\Lambda^{(p)},\, N(\mathbf{x})\neq 0} |x_1 \dots x_s|^{\alpha-1} =$$

$$= \sum_{\mathbf{x}\in\Lambda^{(p)},\, N(\mathbf{x})\neq 0} \left|\frac{x_1}{N}\cdots\frac{x_s}{N}\right|^{\alpha-1} = \sum_{\mathbf{y}\in\Lambda^*,\, N(\mathbf{y})\neq 0} |y_1 \dots y_s|^{\alpha-1} = \zeta\left(\Lambda^*\middle| 1-\alpha\right),$$

which proves the statement of the theorem.

According to the aforementioned definitions, $(\Lambda)_{\mathbf{j}_t} = \Lambda \bigcap \Pi(\mathbf{j}_t)$ is the intersection of the lattice and the coordinate subspace. Let us denote a t-dimensional lattice derived from the lattice $(\Lambda)_{\mathbf{j}_t}$ by discarding $s-t$ zero coordinates from each point as $\Lambda_{\mathbf{j}_t}$ and denote its determinant as $N_{\mathbf{j}_t}$. Thus, $\Lambda_{\mathbf{j}_t}^{(p)}$ is the "combined" t-dimensional lattice, $N_{\mathbf{j}_t} = \det \Lambda_{\mathbf{j}_t}$ and $N_{\mathbf{j}_t} | N$.

Theorem 2.24 *For the zeta function of an arbitrary integer lattice Λ in the left half-plane $\sigma < 0$ the following functional equation is true:*

$$\zeta_H(\Lambda \mid \alpha) = \sum_{t=1}^{s} M(\alpha)^t \sum_{\mathbf{j}_t\in J_{t,s}} N_{\mathbf{j}_t}^{t(1-\alpha)-1} \zeta\left(\Lambda_{\mathbf{j}_t}^{(p)}\middle| 1-\alpha\right). \tag{2.49}$$

If we use the Theorem 2.23 and denote the t-dimensional dual lattice as $\Lambda^*_{\mathbf{j}_t}$, then we will obtain a new form of the functional equation for the hyperbolic zeta function of an integer lattice.

Theorem 2.25 *For the hyperbolic zeta function of an arbitrary integer lattice Λ in the left half-plane $\sigma < 0$ the following functional equation is true:*

$$\zeta_H(\Lambda \mid \alpha) = \sum_{t=1}^{s} \sum_{\mathbf{j}_t \in J_{t,s}} \frac{M(\alpha)^t}{N_{\mathbf{j}_t}} \zeta\left(\Lambda^*_{\mathbf{j}_t} \middle| 1-\alpha\right). \tag{2.50}$$

Proof The definitions of the hyperbolic zeta function of a lattice and the zeta function of a lattice provide, that

$$\zeta_H(\Lambda \mid \alpha) = \sum_{t=1}^{s} \sum_{\mathbf{j}_t \in J_{t,s}} \zeta\left(\Lambda_{\mathbf{j}_t} \middle| \alpha\right). \tag{2.51}$$

Applying to each term of the right side the Theorem 2.23 we obtain the required result.

2.3 Functional Equation for Hyperbolic Zeta Function of Cartesian Lattices

First of all, we need the main result on the form of an arbitrary Cartesian lattice (see Theorem 2.11). According to this theorem, a Cartesian lattice Λ can be unambiguously presented as

$$\Lambda = D(d_1, \dots, d_s) \cdot \Lambda_0, \quad d_1, \dots, d_s > 0,$$

where Λ_0 is a simple lattice, and $D(d_1, \dots, d_s)$ is a diagonal matrix.

Similarly to the aforementioned definitions, $(\Lambda_0)_{\mathbf{j}_t} = \Lambda_0 \bigcap \Pi(\mathbf{j}_t)$ is the intersection of the lattice and the coordinate space. Let us denote the t-dimensional lattice derived from the lattice $(\Lambda_0)_{\mathbf{j}_t}$ by discarding $s-t$ zero coordinates from each point as $\Lambda_{0,\mathbf{j}_t}$. Thus, $\Lambda^{(p)}_{0,\mathbf{j}_t}$ is the "combined" t-dimensional lattice.

First, let us examine the simpler case, where all the elements $d_j \geq 1$ $(j = 1, \dots, s)$.

Theorem 2.26 *For the hyperbolic zeta function of a Cartesian lattice Λ of the form $\Lambda = D(d_1, \dots, d_s) \cdot \Lambda_0$, where Λ_0 is a simple lattice and all its elements $d_j \geq 1$ $(j = 1, \dots, s)$, in the left half-plane $\sigma < 0$ the following functional equation is true:*

$$\zeta_H(\Lambda \mid \alpha) = \sum_{t=1}^{s} M(\alpha)^t \sum_{\mathbf{j}_t \in J_{t,s}} \prod_{\nu=1}^{t} (d_{j_\nu})^{-\alpha} N_{0,\mathbf{j}_t}^{t(1-\alpha)-1} \zeta\left(\Lambda^{(p)}_{0,\mathbf{j}_t} \middle| 1-\alpha\right), \tag{2.52}$$

where $N_{0,\mathbf{j}_t} = \det \Lambda_{0,\mathbf{j}_t}$.

Proof The definitions of the hyperbolic zeta function of a lattice and the zeta function of a lattice provide that

$$\zeta_H(\Lambda \mid \alpha) = \sum_{t=1}^{s} \sum_{\mathbf{j}_t \in J_{t,s}} \prod_{\nu=1}^{t} (d_{j_\nu})^{-\alpha} \zeta\left(\Lambda_{0,\mathbf{j}_t} \middle| \alpha\right). \tag{2.53}$$

Applying to each term of the right side the Theorem 2.22 we obtain the required result.

Now we will obtain a functional equation using a dual lattice.

Theorem 2.27 *For the hyperbolic zeta function of a Cartesian lattice* Λ *of the form* $\Lambda = D(d_1, \dots, d_s) \cdot \Lambda_0$, *where* Λ_0 *is a simple lattice and all elements* $d_j \geq 1$ $(j = 1, \dots, s)$, *in the left half-plane* $\sigma < 0$ *the following functional equation is true:*

$$\zeta_H(\Lambda \mid \alpha) = \sum_{t=1}^{s} \sum_{\mathbf{j}_t \in J_{t,s}} \frac{M(\alpha)^t}{\det \Lambda_{\mathbf{j}_t}} \zeta\left(\Lambda^*_{\mathbf{j}_t} \middle| 1 - \alpha\right). \tag{2.54}$$

Proof First of all, we need to state, that $\Lambda^* = (D(d_1, \dots, d_s) \cdot \Lambda_0)^* = D\left(\frac{1}{d_1}, \dots, \frac{1}{d_s}\right) \cdot \Lambda_0^*$ and $\det(D(d_1, \dots, d_s) \cdot \Lambda_0) = d_1 \cdots d_s \cdot \det \Lambda_0$.

If we address the projections of $\Lambda_{\mathbf{j}_t}$, then we will obtain that

$$\begin{aligned}
\Lambda_{\mathbf{j}_t} &= D(d_{j_1}, \dots, d_{j_t}) \cdot \Lambda_{0,\mathbf{j}_t}, \\
\Lambda^*_{\mathbf{j}_t} = \left(D(d_{j_1}, \dots, d_{j_t}) \cdot \Lambda_{0,\mathbf{j}_t}\right)^* &= D\left(\frac{1}{d_{j_1} N_{0,\mathbf{j}_t}}, \dots, \frac{1}{d_{j_t} N_{0,\mathbf{j}_t}}\right) \cdot \Lambda^{(p)}_{0,\mathbf{j}_t} = \\
&= D\left(\frac{1}{d_{j_1}}, \dots, \frac{1}{d_{j_t}}\right) \cdot \Lambda^*_{0,\mathbf{j}_t}, \\
\Lambda^*_{0,\mathbf{j}_t} &= D(d_{j_1}, \dots, d_{j_t}) \Lambda^*_{\mathbf{j}_t}, \\
\det\left(D(d_{j_1}, \dots, d_{j_t}) \cdot \Lambda_{0,\mathbf{j}_t}\right) &= d_{j_1} \cdot \ldots \cdot d_{j_t} \cdot \det \Lambda_{0,\mathbf{j}_t} = d_{j_1} \cdot \ldots \cdot d_{j_t} \cdot N_{0,\mathbf{j}_t}, \\
\zeta\left(\Lambda^*_{0,\mathbf{j}_t} \middle| 1 - \alpha\right) &= \left(d_{j_1} \cdot \ldots \cdot d_{j_t}\right)^{\alpha - 1} \zeta\left(\Lambda^*_{\mathbf{j}_t} \middle| 1 - \alpha\right).
\end{aligned}$$

The definitions of the hyperbolic zeta function of a lattice and the zeta function of a lattice provide that

$$\zeta_H(\Lambda \mid \alpha) = \sum_{t=1}^{s} \sum_{\mathbf{j}_t \in J_{t,s}} \prod_{\nu=1}^{t} (d_{j_\nu})^{-\alpha} \zeta\left(\Lambda_{0,\mathbf{j}_t} \middle| \alpha\right). \tag{2.55}$$

Applying to each term of the right side the Theorem 2.23, we obtain that

$$\zeta_H(\Lambda \mid \alpha) = \sum_{t=1}^{s} \sum_{\mathbf{j}_t \in J_{t,s}} \prod_{\nu=1}^{t} (d_{j_\nu})^{-\alpha} \frac{M(\alpha)^t}{N_{0,\mathbf{j}_t}} \zeta\left(\Lambda^*_{0,\mathbf{j}_t} \middle| 1-\alpha \right) =$$
$$= \sum_{t=1}^{s} \sum_{\mathbf{j}_t \in J_{t,s}} \prod_{\nu=1}^{t} (d_{j_\nu})^{-\alpha} \frac{M(\alpha)^t}{N_{0,\mathbf{j}_t}} \left(d_{j_1} \cdots d_{j_t}\right)^{\alpha-1} \zeta\left(\Lambda^*_{\mathbf{j}_t} \middle| 1-\alpha \right) =$$
$$= \sum_{t=1}^{s} \sum_{\mathbf{j}_t \in J_{t,s}} \frac{M(\alpha)^t}{\det \Lambda_{\mathbf{j}_t}} \zeta\left(\Lambda^*_{\mathbf{j}_t} \middle| 1-\alpha \right), \tag{2.56}$$

which proves the statement of the theorem.

Now, let us examine a general case, where the set $D_1 = \{j|0 < d_j < 1\} \neq \emptyset$. For this, we need to examine one more type of Dirichlet series with periodic coefficients. Let

$$l^{**}\left(\alpha, d, \frac{b}{n}\right) = \sum_{m=-\infty}^{\infty} \frac{e^{2\pi i \frac{bm}{n}}}{\overline{dm}^{\alpha}} \quad (\Re\alpha > 1, \quad d > 0). \tag{2.57}$$

The Dirichlet series of the latest form can directly define the hyperbolic zeta function of Cartesian lattices for $\sigma > 1$, if we use exponential sums of lattices, and namely, for any Cartesian lattice $\Lambda = D(d_1, \ldots, d_s) \cdot \Lambda_0$, where Λ_0 is a simple lattice, and $D(d_1, \ldots, d_s)$ is a diagonal matrix:

$$\zeta_H(\Lambda|\alpha) + 1 = \sum_{\mathbf{x} \in \Lambda}{}' (\overline{x}_1 \cdots \overline{x}_s)^{-\alpha} + 1 =$$
$$= \sum_{\mathbf{m} \in \mathbb{Z}^s} \frac{\delta_{\Lambda_0}(\mathbf{m})}{(\overline{d_1 m_1} \cdots \overline{d_s m_s})^{\alpha}} =$$
$$= \frac{1}{\det \Lambda_0} \sum_{\mathbf{x} \in M(\Lambda_0)} \sum_{\mathbf{m} \in \mathbb{Z}^s} \frac{e^{2\pi i (\mathbf{m},\mathbf{x})}}{(\overline{d_1 m_1} \cdots \overline{d_s m_s})^{\alpha}} =$$
$$= \frac{1}{\det \Lambda_0} \sum_{\mathbf{x} \in M(\Lambda_0)} \prod_{j=1}^{s} \sum_{m_j=-\infty}^{\infty} \frac{e^{2\pi i m_j x_j}}{\overline{d_j m_j}^{\alpha}} =$$
$$= \frac{1}{\det \Lambda_0} \sum_{\mathbf{x} \in M(\Lambda_0)} \prod_{j=1}^{s} l^{**}\left(\alpha, d_j, \frac{b_j(\mathbf{x})}{\det \Lambda_0}\right), \tag{2.58}$$

where $b_j(\mathbf{x}) = x_j \det \Lambda_0$ is an integer ($j = 1, \ldots, s$) for any point $\mathbf{x} = (x_1, \ldots, x_s) \in M(\Lambda_0)$.

As it was stated above the hyperbolic zeta function of a lattice is not homogeneous, while the zeta function is. Our previous arguments provide, that the homogeneous zeta function of a lattice is crucial for the analytic continuation. In the general case, the hyperbolic zeta function of a lattice can not be presented as a sum of homogeneous components (as it can be done with integer lattices), but in the case of Cartesian lattices we can define $\mathbf{j}_t$-components.

As it has been done above, for a Cartesian lattice Λ we will use $\Lambda_{\mathbf{j}_t}$ to denote the projection of the intersection $\Lambda \bigcap \Pi(\mathbf{j}_t)$ on $\mathbb{R}^t$.

Definition 2.19 The $\mathbf{j}_t$-component of the hyperbolic zeta function of the lattice Λ is the function $\zeta_{H,\mathbf{j}_t}(\Lambda|\alpha)$, $\alpha = \sigma + it$, defined for $\sigma > 1$ by the series

$$\zeta_{H,\mathbf{j}_t}(\Lambda|\alpha) = \sum_{\mathbf{x} \in \Lambda_{H,\mathbf{j}_t},\, N(\mathbf{x}) \neq 0} |x_1 \cdots x_t|^{-\alpha}. \tag{2.59}$$

It is easy to see, that for the $\mathbf{j}_t$-component of the hyperbolic zeta function of a lattice Λ the analogue of the formula (2.58) is true.

$$\zeta_{H,\mathbf{j}_t}(\Lambda|\alpha) = \frac{1}{\det \Lambda_{0,\mathbf{j}_t}} \sum_{\mathbf{x} \in M(\Lambda_{0,\mathbf{j}_t})} \prod_{\nu=1}^{t} \left(l^{**}\left(\alpha, d_{j_\nu}, \frac{b_j(\mathbf{x})}{\det \Lambda_{0,\mathbf{j}_t}}\right) - 1 \right). \tag{2.60}$$

Moreover, we can see the decomposition into components:

$$\zeta_H(\Lambda|\alpha) = \sum_{t=1}^{s} \sum_{\mathbf{j}_t \in J(t,s)} \zeta_{H,\mathbf{j}_t}(\Lambda|\alpha). \tag{2.61}$$

Definition 2.20 Let the $\mathbf{j}_s$-component of the hyperbolic zeta function of a lattice Λ be called the main component and denoted as $\zeta_{H,s}(\Lambda|\alpha)$.

It is clear, that the following equality is true:

$$\zeta_{H,\mathbf{j}_t}(\Lambda|\alpha) = \zeta_{H,\,t}(\Lambda_{\mathbf{j}_t}|\alpha). \tag{2.62}$$

Theorem 2.28 *For a natural n, an integer b with $\delta_n(b) = 0$, a positive d and the analytic continuation of the function $l^{**}\left(\alpha, d, \frac{b}{n}\right)$ on the whole complex plane the following presentations are true:*

$$l^{**}\left(\alpha, d, \frac{b}{n}\right) = 1 + \frac{1}{d^\alpha}\left(l^{*}\left(\alpha, \frac{b}{n}\right) - 1\right) + f\left(\alpha, d, \frac{b}{n}\right), \tag{2.63}$$

where

$$f\left(\alpha, d, \frac{b}{n}\right) = \sum_{1 \leq |m| \leq \left[\frac{1}{d}\right]} e^{2\pi i \frac{bm}{n}} \left(1 - \frac{1}{|dm|^\alpha}\right)$$

and $f\left(\alpha, d, \frac{b}{n}\right) = 0$ *with* $d \geq 1$.

Proof For $\sigma > 1$ from the definition follows that

$$l^{**}\left(\alpha, d, \frac{b}{n}\right) = 1 + \sum_{1\le|m|\le\left[\frac{1}{d}\right]} e^{2\pi i\frac{bm}{n}} + \sum_{|m|>\left[\frac{1}{d}\right]} \frac{e^{2\pi i\frac{bm}{n}}}{|dm|^{\alpha}} =$$

$$= 1 + \sum_{1\le|m|\le\left[\frac{1}{d}\right]} e^{2\pi i\frac{bm}{n}}\left(1 - \frac{1}{|dm|^{\alpha}}\right) + \sum_{|m|\ge 1} \frac{e^{2\pi i\frac{bm}{n}}}{|dm|^{\alpha}} =$$

$$= 1 + \frac{1}{d^{\alpha}}\left(l^{*}\left(\alpha, \frac{b}{n}\right) - 1\right) + f\left(\alpha, d, \frac{b}{n}\right).$$

As there are analytic functions in the right side of the equality, which are defined on the whole complex α-plane, excluding the point $\alpha = 1$, where is a pole of order 1, then the theorem is proven.

Let us introduce some additional definitions. For $1 \le r \le |D_1|$ and $1 \le t \le s-r$ let us define the set of integer vectors

$$J_{t,r,s}(D_1) = \{\mathbf{j}_{t,r} = (j_1, \ldots, j_s) \mid 1 \le j_1 < \cdots < j_t \le s, \quad 1 \le j_{t+r+1} < \cdots < j_s \le s,$$

$$1 \le j_{t+1} < \cdots < j_{t+r} \le s, \quad \{j_1, \ldots, j_s\} = \{1, 2, \ldots, s\},$$

$$j_{t+\nu} \in D_1 \text{ if } 1 \le \nu \le r\}.$$

In other words, the set $J_{t,r,s}(D_1)$ consists of integer vectors $\mathbf{j}_{t,r}$, coordinates of which form the permutation of numbers from 1 to s, wile coordinates from 1 to t, and from $t+1$ to $t+r$, and from $t+r+1$ to s form increasing sequences. Moreover, all coordinates from $t+1$ to $t+r$ belong to the set D_1. Obviously, $J_{t,r,s}|D_1| = C^t_{s-r}C^r_{|D_1|}$.

Theorem 2.29 *For the main component of the hyperbolic zeta function of an arbitrary Cartesian lattice Λ of the form $\Lambda = D(d_1, \ldots, d_s) \cdot \Lambda_0$, where Λ_0 is a simple lattice and all its elements $d_j > 0$ $(j = 1, \ldots, s)$, in the left half-plane $\sigma < 0$ the following functional equation is true:*

$$\zeta_{H,s}(\Lambda \mid \alpha) = \frac{M(\alpha)^s}{\det \Lambda}\zeta\left(\Lambda^*\middle| 1-\alpha\right) + \frac{1}{\det \Lambda_0}\sum_{\mathbf{x}\in M(\Lambda_0)}\sum_{r=1}^{|D_1|} M(\alpha)^{s-r} N_0^{s-r-\alpha(s-r)}$$

$$\cdot \sum_{\mathbf{j}_{s-r,r}\in J_{s-r,r,s}(D_1)} \prod_{\nu=1}^{s-r}(d_{j_\nu})^{-\alpha} \prod_{\nu=s-r+1}^{s} f\left(\alpha, d_{j_\nu}, \frac{b_{j_\nu}(\mathbf{x})}{\det \Lambda_0}\right)\zeta\left(N_0\mathbb{Z}^{s-r} + \mathbf{b}_{s-r}(\mathbf{x})\middle| 1-\alpha\right), \tag{2.64}$$

where $N_0 = \det \Lambda_0$.

Proof According to the equality (2.60) and the Theorem 2.28 for the main component of the hyperbolic zeta function of an arbitrary Cartesian lattice $\Lambda = D(d_1, \ldots, d_s) \cdot \Lambda_0$ on the whole complex α-plane, excluding the point $\alpha = 1$, which has a pole of order s, the following equality is true:

$$\zeta_{H,s}(\Lambda|\alpha) = \frac{1}{\det \Lambda} \sum_{\mathbf{x} \in M(\Lambda_0)} \prod_{j=1}^{s} \left(l^{**}\left(\alpha, d_j, \frac{b_j(\mathbf{x})}{\det \Lambda_0}\right) - 1 \right). \qquad (2.65)$$

For $\sigma < 0$, let us apply the Theorems 2.28 and 2.21, and therefore obtain that

$$\zeta_{H,s}(\Lambda|\alpha) = \frac{1}{\det \Lambda_0} \sum_{\mathbf{x} \in M(\Lambda_0)} \prod_{j=1}^{s} \left(\frac{1}{d_j^{\alpha}} \left(l^{*}\left(\alpha, \frac{b_j(\mathbf{x})}{\det \Lambda_0}\right) - 1 \right) + f\left(\alpha, d_j, \frac{b_j(\mathbf{x})}{\det \Lambda_0}\right) \right) =$$

$$= \frac{1}{\det \Lambda_0} \sum_{\mathbf{x} \in M(\Lambda_0)} \prod_{j=1}^{s} \left(\frac{M(\alpha)}{d_j^{\alpha}} N_0^{1-\alpha} \sum_{\substack{m=-\infty, \\ N_0 \cdot m + b_j(\mathbf{x}) \neq 0}}^{\infty} \frac{1}{\left|N_0 \cdot m + b_j(\mathbf{x})\right|^{1-\alpha}} + f\left(\alpha, d_j, \frac{b_j(\mathbf{x})}{\det \Lambda_0}\right) \right). \qquad (2.66)$$

To expand the product in the right side of the formula (2.66) let us use the following equality:

$$\prod_{j=1}^{s} \left(\frac{M(\alpha)}{d_j^{\alpha}} N_0^{1-\alpha} \sum_{\substack{m=-\infty, \\ N_0 \cdot m + b_j(\mathbf{x}) \neq 0}}^{\infty} \frac{1}{\left|N_0 \cdot m + b_j(\mathbf{x})\right|^{1-\alpha}} + f\left(\alpha, d_j, \frac{b_j(\mathbf{x})}{\det \Lambda_0}\right) \right) =$$

$$= \prod_{j \in D_1} \left(\frac{M(\alpha)}{d_j^{\alpha}} N_0^{1-\alpha} \sum_{\substack{m=-\infty, \\ N_0 \cdot m + b_j(\mathbf{x}) \neq 0}}^{\infty} \frac{1}{\left|N_0 \cdot m + b_j(\mathbf{x})\right|^{1-\alpha}} + f\left(\alpha, d_j, \frac{b_j(\mathbf{x})}{\det \Lambda_0}\right) \right) \times$$

$$\times \prod_{j \notin D_1} \left(\frac{M(\alpha)}{d_j^{\alpha}} N_0^{1-\alpha} \sum_{\substack{m=-\infty, \\ N_0 \cdot m + b_j(\mathbf{x}) \neq 0}}^{\infty} \frac{1}{\left|N_0 \cdot m + b_j(\mathbf{x})\right|^{1-\alpha}} \right) =$$

$$= \frac{M(\alpha)^s}{N_0^{(\alpha-1)s}} \prod_{j=1}^{s} (d_j)^{-\alpha} \sum_{\substack{m_j=-\infty (1 \le j \le s), \\ N_0 \cdot m_j + b_j(\mathbf{x}) \neq 0}}^{\infty} \frac{1}{|(N_0 \cdot m_1 + b_1(\mathbf{x})) \cdots (N_0 \cdot m_s + b_s(\mathbf{x}))|^{1-\alpha}} +$$

$$+ \sum_{r=1}^{|D_1|} \left(M(\alpha)^{s-r} N_0^{s-r-\alpha(s-r)} \sum_{\mathbf{j}_{s-r,r} \in J_{s-r,r,s}(D_1)} \prod_{\nu=1}^{s-r} (d_{j_\nu})^{-\alpha} \prod_{\nu=s-r+1}^{s} f\left(\alpha, d_{j_\nu}, \frac{b_{j_\nu}(\mathbf{x})}{\det \Lambda_0}\right) \times \right.$$

$$\times \sum_{\substack{m_{j_\nu}=-\infty (1\le\nu\le s-r),\\ N_0\cdot m_{j_\nu}+b_{j_\nu}(\mathbf{x})\neq 0}}^{\infty} \frac{1}{\left|(N_0\cdot m_{j_1}+b_{j_1}(\mathbf{x}))\cdots(N_0\cdot m_{j_{s-r}}+b_{j_{s-r}}(\mathbf{x}))\right|^{1-\alpha}}\Bigg). \tag{2.67}$$

From (2.66) and (2.67), assuming that $\mathbf{b}_t(\mathbf{x}) = (b_{j_1}(\mathbf{x}), \dots, b_{j_t}(\mathbf{x}))$, we will obtain that

$$\begin{aligned}
&\zeta_{H,s}(\Lambda|\alpha) \\
&= \frac{1}{\det \Lambda_0} \sum_{\mathbf{x}\in M(\Lambda_0)} \Bigg(M(\alpha)^s N_0^{s-\alpha s} \prod_{j=1}^{s} (d_j)^{-\alpha} \times \\
&\quad \times \sum_{\substack{m_j=-\infty (1\le j\le s),\\ N_0\cdot m_j+b_j(\mathbf{x})\neq 0}}^{\infty} \frac{1}{|(N_0\cdot m_1+b_1(\mathbf{x}))\cdots(N_0\cdot m_s+b_s(\mathbf{x}))|^{1-\alpha}} + \\
&\quad + \sum_{r=1}^{|D_1|} \Bigg(M(\alpha)^{s-r} N_0^{s-r-\alpha(s-r)} \sum_{\mathbf{j}_{s-r,r}\in J_{s-r,r,s}(D_1)} \prod_{\nu=1}^{s-r} (d_{j_\nu})^{-\alpha} \prod_{\nu=s-r+1}^{s} f\left(\alpha, d_{j_\nu}, \frac{b_{j_\nu}(\mathbf{x})}{\det \Lambda_0}\right) \times \\
&\quad \times \sum_{\substack{m_{j_\nu}=-\infty (1\le\nu\le s-r),\\ N_0\cdot m_{j_\nu}+b_{j_\nu}(\mathbf{x})\neq 0}}^{\infty} \frac{1}{\left|(N_0\cdot m_{j_1}+b_{j_1}(\mathbf{x}))\cdots(N_0\cdot m_{j_{s-r}}+b_{j_{s-r}}(\mathbf{x}))\right|^{1-\alpha}}\Bigg)\Bigg) = \\
&= \frac{1}{\det \Lambda_0} \sum_{\mathbf{x}\in M(\Lambda_0)} \Bigg(M(\alpha)^s N_0^{s-\alpha s} \prod_{j=1}^{s} (d_j)^{-\alpha} \zeta\left(N_0\mathbb{Z}^s + \mathbf{b}_s(\mathbf{x})\middle| 1-\alpha\right) + \\
&\quad + \sum_{r=1}^{|D_1|} \Bigg(M(\alpha)^{s-r} N_0^{s-r-\alpha(s-r)} \sum_{\mathbf{j}_{s-r,r}\in J_{s-r,r,s}(D_1)} \prod_{\nu=1}^{s-r} (d_{j_\nu})^{-\alpha} \prod_{\nu=s-r+1}^{s} f\left(\alpha, d_{j_\nu}, \frac{b_{j_\nu}(\mathbf{x})}{\det \Lambda_0}\right) \times \\
&\quad \times \zeta\left(N_0\mathbb{Z}^{s-r} + \mathbf{b}_{s-r}(\mathbf{x})\middle| 1-\alpha\right)\Bigg)\Bigg).
\end{aligned} \tag{2.68}$$

As

$$\begin{aligned}
&\frac{1}{\det \Lambda_0} \sum_{\mathbf{x}\in M(\Lambda_0)} M(\alpha)^s N_0^{s-\alpha s} \prod_{j=1}^{s} (d_j)^{-\alpha} \zeta\left(N_0\mathbb{Z}^s + \mathbf{b}_s(\mathbf{x})\middle| 1-\alpha\right) = \\
&\quad = \frac{1}{\det \Lambda_0} M(\alpha)^s N_0^{s-\alpha s} \prod_{j=1}^{s} (d_j)^{-\alpha} \zeta\left(\Lambda_0^{(p)}\middle| 1-\alpha\right) = \frac{M(\alpha)^s}{\det \Lambda} \zeta\left(\Lambda^*\middle| 1-\alpha\right),
\end{aligned} \tag{2.69}$$

then the statement of the theorem is completely proven.

Theorem 2.30 *For the hyperbolic zeta function of an arbitrary Cartesian lattice Λ of the form $\Lambda = D(d_1, \dots, d_s) \cdot \Lambda_0$, where Λ_0 is a simple lattice and all elements $d_j > 0$ $(j = 1, \dots, s)$, in the left half-plane $\sigma < 0$ the following functional equation is true:*

$$\zeta_H(\Lambda \mid \alpha) = \sum_{t=1}^{s} \sum_{\mathbf{j}_t \in J(t,s)} \frac{M(\alpha)^t}{\det \Lambda_{\mathbf{j}_t}} \zeta\left(\Lambda^*_{\mathbf{j}_t} \middle| 1-\alpha\right) +$$
$$+ \sum_{t=1}^{s} \sum_{\mathbf{j}_t \in J(t,s)} \frac{1}{\det \Lambda_{0,\mathbf{j}_t}} \sum_{\mathbf{x} \in M(\Lambda_{0,\mathbf{j}_t})} \sum_{r=1}^{|D_{1,\mathbf{j}_t}|} M(\alpha)^{t-r} N_{0,\mathbf{j}_t}^{t-r-\alpha(t-r)}$$
$$\times \sum_{\mathbf{j}_{t-r,r} \in J_{t-r,r,t}(D_{1,\mathbf{j}_t})} \prod_{\nu=1}^{t-r} (d_{j_\nu})^{-\alpha} \prod_{\nu=t-r+1}^{t} f\left(\alpha, d_{j_\nu}, \frac{b_{j_\nu}(\mathbf{x})}{\det \Lambda_{0,\mathbf{j}_t}}\right)$$
$$\zeta\left(N_{0,\mathbf{j}_t}\mathbb{Z}^{t-r} + \mathbf{b}_{t-r}(\mathbf{x}) \middle| 1-\alpha\right), \tag{2.70}$$

where $N_{0,\mathbf{j}_t} = \det \Lambda_{0,\mathbf{j}_t}$.

Proof The theorem statement follows from the decomposition into components formula (see (2.61)) and the application of the Theorem 2.29 to each component according to the formula (2.62).

2.4 On Some Unsolved Problems of the Theory of Hyperbolic Zeta Function of Lattices

The article [9] hints at some possible directions of further development of Korobov number-theoretical method in approximate analysis. We are going to examine the problems regarding the theory of the hyperbolic zeta function of lattices in more detail.

The problem of right order The class of algebraic lattices is known for making it possible to achieve the correct order of decreasing hyperbolic zeta function of lattices when increasing the determinant of lattices (see the formulas (2.19) and (2.21)). Moreover, the asymptotic formula (2.25) is true for these lattices. The continuity of the hyperbolic function on the lattice space provides that the correct order of decreasing hyperbolic zeta function of lattices can be achieved on the class of rational lattices. It is enough to take rational lattices from very small neighborhoods of algebraic lattices. A natural question arise: can the correct order of decreasing be achieved in the class of integer lattices, or not? If it can be achieved, we need to provide an algorithm for construction of such optimal parallelepipedal nets, which would have the right order of the error of approximate integration on the classes E_s^α. Otherwise, we will obtain a kind of the theorem, which is analogous to the Liouville-Thue-Siegel-Roth theorem for algebraic lattices, as the impossibility of the right order means that algebraic lattices can not be correctly approximated by integer ones.

The problem of existence of analytic continuation As stated above, any Cartesian lattice has an analytic continuation of the hyperbolic zeta function of an arbitrary Cartesian lattice. Moreover, there's been obtained the functional equation for an arbitrary Cartesian lattice, which explicitly defines this analytic continuation. Naturally, there are questions, whether an analytic continuation of the hyperbolic zeta function exists in the following cases:

for a lattice of joint approximations $\Lambda(\theta_1, \ldots, \theta_s)$, defined by the equality

$$\Lambda(\theta_1, \ldots, \theta_s) = \{(q, q\theta_1 - p_1, \ldots, q\theta_s - p_s) \mid q, p_1, \ldots, p_s \in \mathbb{Z}\},$$

where $\theta_1, \ldots, \theta_s$ are arbitrary irrational numbers.

for an algebraic lattice $\Lambda(t, F) = t\Lambda(F)$, where the lattice $\Lambda(F)$ is defined by the equality (2.3).

for an arbitrary lattice Λ. If the hyperbolic zeta function of an arbitrary lattice can not be continued onto the whole complex plane (and we have strong doubts about that), then we will have to describe a new class, containing all lattices, for which their hyperbolic zeta functions can be analytically continued onto the whole complex plane, excluding the point $\alpha = 1$, which has a pole of order s.

The problem of the critical strip behaviour This problem has been underlined by Korobov. He suggested the hypothesis, according to which the analytic continuation of the hyperbolic zeta function of a lattice into the critical strip from the right half-plane and the analytic continuation of the hyperbolic zeta function of a dual lattice or combined lattices into the critical strip from the left half-plane will allow us to get the constants in the corresponding transfer theorems.

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