Chapter 16
On Global Attractors for Autonomous Damped Wave Equation with Discontinuous Nonlinearity

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Abstract We consider autonomous damped wave equation with discontinuous nonlinearity. The long-term prognosis of the state functions when the conditions on the parameters of the problem do not guarantee uniqueness of solution of the corresponding Cauchy problem are studied. We prove the existence of a global attractor and investigate its structure. It is obtained that trajectory of every weak solution defined on $[0; +\infty)$ tends to a fixed point.

16.1 Introduction

This manuscript is devoted to the research of asymptotical behavior of the autonomous damped wave equation with discontinuous nonlinearity. The investigated problem is considered in a bounded domain Ω with a sufficiently regular boundary $\partial\Omega$. The interaction function $f : \mathbb{R} \to \mathbb{R}$ satisfies the standard growth and sign conditions. Wave equation with a non-smooth nonlinearity f can be interpreted as the mathematical model of the controlled piezoelectric fields or processes. The asymptotic behavior of solutions for such problems were studied by Ball [1, 2], Sell [11], Zgurovsky et al. [17–19] and many others. The case of the continuous function f is well-known [2]. The case of the non-autonomous equation with continuous non-

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linearity was investigated by Kapustyan [6], Melnik [8, 10], Valero [13]. The case when extension of f admits the maximal monotone graph was studied by Zgurovsky and his scholars [6, 7, 15, 16].

Here we provide sufficient conditions for existence of compact in natural phase space global attractor for the nonlinear damped equation with discontinuous non-monotone in general case interaction function.

16.2 Setting of the Problem

Let $\beta > 0$ be a constant, $\Omega \subset \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Consider the problem

$$\begin{cases} u_{tt} + \beta u_t - \triangle u + f(u) = 0, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{16.1}$$

where $u(x, t)$ is unknown state function defined on $\Omega \times \mathbb{R}_+$; $f : \mathbb{R} \to \mathbb{R}$ is an interaction function such that

$$\lim_{|u|\to\infty} \frac{f(u)}{u} > -\lambda_1, \tag{16.2}$$

where λ_1 is the first eigenvalue for $-\triangle$ in $H_0^1(\Omega)$;

$$\exists\ D \geq 0 : \quad |f(u)| \leq D(1 + |u|), \quad \forall u \in \mathbb{R}. \tag{16.3}$$

Further, we use such denotation

$$\overline{f}(s) := \overline{\lim_{t\to s}} f(t), \quad \underline{f}(s) := \underline{\lim_{t\to s}} f(t), \quad G(s) := [\underline{f}(s), \overline{f}(s)], \quad s \in \mathbb{R}.$$

Let us set $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. The space $X = V \times H$ is a phase space of Problem (16.1). For the Hilbert space X as $(\cdot, \cdot)_X$ and $\|\cdot\|_X$ denote the inner product and the norm in X respectively.

Definition 16.1 Let $T > 0$, $\tau < T$. The function $\varphi(\cdot) = (u(\cdot), u_t(\cdot))^T \in L^\infty(\tau, T; X)$ is called *a weak solution* of Problem (16.1) on (τ, T) if for a.e. $(x, t) \in \Omega \times (\tau, T)$, there exists $l = l(x, t) \in L^2(\tau, T; L^2(\Omega))$ $l(x, t) \in G(u(x, t))$, such that $\forall \psi \in H_0^1(\Omega)$, $\forall \eta \in C_0^\infty(\tau, T)$,

$$-\int_\tau^T (u_t, \psi)_H \eta_t dt + \int_\tau^T (\beta(u_t, \psi)_H + (u, \psi)_V + (l, \psi)_H)\eta dt = 0. \tag{16.4}$$

The main goal of the manuscript is to obtain the existence of the global attractor generated by the weak solutions of Problem (16.1) in the phase space X.

16.3 Preliminaries

Lemma 16.1 Zgurovsky et al. [19] *For any* $\varphi_\tau = (u_0, u_1)^T \in X$ *and* $\tau < T$ *there exists a weak solution* $\varphi(\cdot)$ *of Problem* (16.1) on (τ, T) such that $\varphi(\tau) = \varphi_\tau$.

Show that in the general case, when the interaction function f is typically multi-valued, the m-semiflow generated by all solutions of Problem (16.1) have no a compact global attractor.

Example 16.1 Consider the problem

$$\begin{cases} u_{tt} + \beta u_t - \Delta u + [-\varepsilon, \varepsilon] \ni 0, \quad (x,t) \in (0,\pi) \times \mathbb{R}_+, \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = \frac{\varepsilon}{\beta}\varphi_n(x), \quad u_t(x,0) = 0, \quad |\varphi_n'(x)| \le 1. \end{cases} \tag{16.5}$$

There exists a solution $u_n(x,t)$ of Problem (16.5) such that $\{u_n(\cdot, t_n)\}_{n\ge1}$ is not pre-compact set in $H_0^1(0,\pi)$ for some $\{t_n\}_{n\ge1}$, $t_n \to \infty$, and some bounded in $H_0^1(0,\pi)$ sequence $\{\varphi_n\}$.

D'Alembert's formula implies that Problem (16.5) has the solution of the form

$$u_n(x,t) = \frac{\varepsilon}{2\beta}\left(\varphi_n(x+t) - \varphi_n(t-x)\right)$$

for any sufficiently smooth $\varphi_n : \mathbb{R} \to \mathbb{R}$ such that $\varphi_n(x) = -\varphi_n(-x) = -\varphi_n(2\pi - x)$. Indeed, $u_{n_{tt}} - \Delta u_n = 0$ and

$$\beta u_{n_t}(x,t) = \beta\frac{\varepsilon}{2\pi}\left(\varphi_n'(x+t) - \varphi_n'(t-x)\right) \in [-\varepsilon, \varepsilon].$$

Let $\varphi_n(x) = \frac{1}{n}\sin nx$, $x \in (0,\pi)$. Then

$$u_n(x,t) = \frac{1}{n}\frac{\varepsilon}{2\beta}\left(\sin n(x+t) - \sin n(t-x)\right), \quad (x,t) \in (0,\pi) \times \mathbb{R}_+;$$

$$u_{n_x}'(x,t) = \frac{\varepsilon}{2\beta}\left(\cos n(x+t) + \cos n(t-x)\right), \quad (x,t) \in (0,\pi) \times \mathbb{R}_+.$$

Let $\{t_n\}_{n\ge1} \subset \mathbb{R}_+$ be the sequence such that $t_n = \frac{2\pi}{n} + 2\pi n$, $\forall n \ge 1$. Then

$$\|u_n(\cdot, t_n) - u_m(\cdot, t_m)\|^2_{H_0^1(0,\pi)} =$$

$$= \frac{\varepsilon^2}{4\beta^2} \int_0^\pi (\cos n(x + t_n) + \cos n(t_n - x) - \cos m(x + t_m) - \cos m(t_m - x))^2 \, dx =$$

$$= \frac{\varepsilon^2}{\beta^2} \int_0^\pi (\cos nx - \cos mx)^2 dx = \frac{\pi \varepsilon^2}{\beta^2}, \quad \forall n, m \geq 1.$$

Thus $\{u_n(\cdot, t_n)\}_{n \geq 1}$ is not precompact set in $H_0^1(0, \pi)$, $n \to +\infty$.

Further, we assume that

$$f(s) = f_1(s) - f_2(s), \quad s \in \mathbb{R},$$

where $f_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are nondecreasing functions.

We remark that

$$[\underline{f}(s), \overline{f}(s)] \subseteq [\underline{f_1}(s), \overline{f_1}(s)] - [\underline{f_2}(s), \overline{f_2}(s)], \quad s \in \mathbb{R}.$$

Thus we consider more general evolution inclusion

$$\begin{cases} u_{tt} + \beta u_t - \Delta u + [\underline{f_1}(u), \overline{f_1}(u)] - [\underline{f_2}(u), \overline{f_2}(u)] \ni 0, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{16.6}$$

Let us set

$$G_i(s) := \int_0^s f_i(\xi) d\xi, \quad J_i(u) := \int_\Omega G_i(u(x)) dx, \quad J(u) = J_1(u) - J_2(u), \quad u \in H, \ i = 1, 2.$$

The functionals G_i and J_i are locally Lipschitz and regular; Clarke [3, Chap. I]. Thus the next result holds.

Lemma 16.2 Kasyanov et al. [9] *Let $u \in C^1([\tau, T]; H)$. Then for a.e. $t \in (\tau, T)$, the functions $J_i \circ u$ are classically differentiable at the point t. Moreover,*

$$\frac{d}{dt}(J_i \circ u)(t) = (p, u_t(t)) \ \forall p \in \partial J_i(u(t)), \ \ i = 1, 2,$$

and $\frac{d}{dt}(J_i \circ u)(\cdot) \in L_1(\tau, T)$.

Consider $W_\tau^T = C([\tau, T]; X)$. Lebourgue's mean value theorem (see Clarke [3, Chap. 2]) provides the existence of constants $c_1, c_2 > 0$ and $\mu \in (0, \lambda_1)$ such that

$$|J(u)| \leq c_1(1 + \|u\|_H^2), \quad J(u) \geq -\frac{\mu}{2}\|u\|_H^2 - c_2 \quad \forall u \in H. \tag{16.7}$$

The weak solution of the Problem (16.1) with initial data

$$u(\tau) = a, \quad u'(\tau) = b \tag{16.8}$$

on the interval $[\tau, T]$ exists for any $a \in V$, $b \in H$. It follows from Zadoianchuk and Kasyanov [15, Theorem 1.4]. Thus the next lemma holds true (see Kasyanov et al. [9, Lemma 3.2]).

Lemma 16.3 Kasyanov et al. [9, Lemma 3.2] *For any $\tau < T$, $a \in V$, $b \in H$, Cauchy Problem* (16.1), (16.8) *has the weak solution $(u, u_t)^T \in L_\infty(\tau, T; X)$. Moreover, each weak solution $(u, u_t)^T$ of Cauchy Problem* (16.1), (16.8) *on the interval $[\tau, T]$ belongs to the space $C([\tau, T]; X)$ and $u_{tt} \in L_2(\tau, T; V^*)$.*

16.4 Properties of Solutions

For any $\varphi_\tau = (a, b)^T \in X$, denote

$$\mathscr{D}_{\tau,T}(\varphi_\tau) = \left\{ (u(\cdot), u_t(\cdot))^T \,\middle|\, \begin{array}{l} (u, u_t)^T \text{is a weak solution of Problem (16.1) on } [\tau, T], \\ u(\tau) = a, u_t(\tau) = b \end{array} \right\}.$$

From Lemma 16.3 it follows that $\mathscr{D}_{\tau,T}(\varphi_\tau) \subset C([\tau, T]; X) = W_\tau^T$. Let us check that translation and concatenation of weak solutions are weak solutions too.

Lemma 16.4 *If $\tau < T$, $\varphi_\tau \in X$, $\varphi(\cdot) \in \mathscr{D}_{\tau,T}(\varphi_\tau)$, then $\forall s$ $\psi(\cdot) = \varphi(\cdot + s) \in \mathscr{D}_{\tau-s,T-s}(\varphi_\tau)$. If $\tau < t < T$, $\varphi_\tau \in X$, $\varphi(\cdot) \in \mathscr{D}_{\tau,t}(\varphi_\tau)$ and $\psi(\cdot) \in \mathscr{D}_{t,T}(\varphi_\tau)$, then*

$$\theta(s) = \begin{cases} \varphi(s), & s \in [\tau, t], \\ \psi(s), & s \in [t, T] \end{cases} \in \mathscr{D}_{\tau,T}(\varphi_\tau).$$

Proof The proof is trivial (see Kasyanov et al. [9, Lemma 4.1]).

Let $\varphi = (a, b)^T \in X$ and

$$\mathscr{V}(\varphi) = \frac{1}{2}\|\varphi\|_X^2 + J_1(a) - J_2(a). \tag{16.9}$$

Lemma 16.5 *Let $\tau < T$, $\varphi_\tau \in X$, $\varphi(\cdot) = (u(\cdot), u_t(\cdot))^T \in \mathscr{D}_{\tau,T}(\varphi_\tau)$. Then $\mathscr{V} \circ \varphi : [\tau, T] \to \mathbb{R}$ is absolutely continuous and for a.e. $t \in (\tau, T)$, $\frac{d}{dt}\mathscr{V}(\varphi(t)) = -\beta\|u_t(t)\|_H^2$.*

Proof Let $-\infty < \tau < T < +\infty$, $\varphi(\cdot) = (u(\cdot), u_t(\cdot))^T \in W_\tau^T$ be an arbitrary weak solution of Problem (16.1) on (τ, T). Since $\partial J(u(\cdot)) \subset L_2(\tau, T; H)$, from Temam [12] and Zgurovsky et al. [19, Chap. 2] we obtain that the function $t \to \|u_t(t)\|_H^2 + \|u(t)\|_V^2$ is absolutely continuous and for a.e. $t \in (\tau, T)$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left[\|u_t(t)\|_{\mathrm{H}}^2+\|u(t)\|_{\mathrm{V}}^2\right]&=(u_{tt}(t)-\triangle u(t),u_t(t))_{\mathrm{H}}=\\&=-\beta\|u_t(t)\|_{\mathrm{H}}^2-(d_1(t),u_t(t))_{\mathrm{H}}+(d_2(t),u_t(t))_{\mathrm{H}},\end{aligned}\tag{16.10}$$

where $d_i(t)\in\partial J_i(u(t))$ for a.e. $t\in(\tau,T)$ and $d_i(\cdot)\in L_2(\tau,T;H)$, $i=1,2$. As $u(\cdot)\in C^1([\tau,T];H)$ and $J_i:H\to\mathbb{R}$, $i=1,2$ is regular and locally Lipschitz, due to Lemma 16.2 we obtain that for a.e. $t\in(\tau,T)$, $\exists\frac{d}{dt}(J_i\circ u)(t)$, $i=1,2$. Moreover, $\frac{d}{dt}(J_i\circ u)(\cdot)\in L_1(\tau,T)$, $i=1,2$ and for a.e. $t\in(\tau,T)$, $\forall p\in\partial J_i(u(t))$,

$$\frac{d}{dt}(J_i\circ u)(t)=(p,u_t(t))_H,\quad i=1,2.$$

In particular for a.e. $t\in(\tau,T)$, $\frac{d}{dt}(J_i\circ u)(t)=(d_i(t),u_t(t))_H$. Taking into account (16.10) we finally obtain the necessary statement.

This completes the proof.

Lemma 16.6 *Let $T>0$. Then any weak solution of Problem* (16.1) on $[0,T]$ can be extended to a global one defined on $[0,+\infty)$.

Proof The statement of this lemma follows from Lemmas 16.3–16.5, (16.7) and from the next estimates

$$\forall\tau<T,\ \ \forall t\in[\tau,T],\ \ \forall\varphi_\tau\in X,\ \ \forall\varphi(\cdot)=(u(\cdot),u_t(\cdot))^T\in\mathscr{D}_{\tau,T}(\varphi_\tau),$$

$$\begin{aligned}2c_1+\left(1+\tfrac{2c_1}{\lambda_1}\right)\|u(\tau)\|_V^2+\|u_t(\tau)\|_H^2\ &\geq 2\mathscr{V}(\varphi(\tau))\geq 2\mathscr{V}(\varphi(t))\ =\\=\|u(t)\|_V^2+\|u_t(t)\|_H^2+2J(u(t))&\geq\left(1-\tfrac{\mu}{\lambda_1}\right)\|u(t)\|_V^2+\|u_t(t)\|_H^2-2c_2.\end{aligned}$$

The lemma is proved.

For an arbitrary $\varphi_0\in X$ let $\mathscr{D}(\varphi_0)$ be the set of all weak solutions (defined on $[0,+\infty)$) of Problem (16.1) with initial data $\varphi(0)=\varphi_0$. We remark that from the proof of Lemma 16.6 we obtain the next corollary.

Corollary 16.1 *For any $\varphi_0\in X$ and $\varphi\in\mathscr{D}(\varphi_0)$, the next inequality is fulfilled*

$$\|\varphi(t)\|_X^2\leq\frac{\lambda_1+2c_1}{\lambda_1-\mu}\|\varphi(0)\|_X^2+\frac{2(c_1+c_2)\lambda_1}{\lambda_1-\mu}\quad\forall t>0.\tag{16.11}$$

From Corollary 16.1 in a standard way we obtain such statement.

Theorem 16.1 *Let $\tau<T$, $\{\varphi_n(\cdot)\}_{n\geq1}\subset W_\tau^T$ be an arbitrary sequence of weak solutions of Problem* (16.1) on $[\tau,T]$ such that $\varphi_n(\tau)\to\varphi_\tau$ weakly in X, $n\to+\infty$, and let $\{t_n\}_{n\geq1}\subset[\tau,T]$ be a sequence such that $t_n\to t_0$, $n\to+\infty$. Then there exists $\varphi\in\mathscr{D}_{\tau,T}(\varphi_\tau)$ such that up to a subsequence $\varphi_n(t_n)\to\varphi(t_0)$ weakly in X, $n\to+\infty$.

Proof We prove this theorem in several steps.

Step 1. Let $\tau < T$, $\{\varphi_n(\cdot) = (u_n(\cdot), u_n'(\cdot))\}_{n\geq 1} \subset W_\tau^T$ be an arbitrary sequence of weak solutions of Problem (16.1) on $[\tau, T]$ and $\{t_n\}_{n\geq 1} \subset [\tau, T]$ such that

$$\varphi_n(\tau) \to \varphi_\tau \text{ weakly in } X, \quad t_n \to t_0, \quad n \to +\infty. \tag{16.12}$$

In virtue of Corollary 16.1 we have that $\{\varphi_n(\cdot)\}_{n\geq 1}$ is bounded on $W_\tau^T \subset L_\infty(\tau, T; X)$. Therefore up to a subsequence $\{\varphi_{n_k}(\cdot)\}_{k\geq 1} \subset \{\varphi_n(\cdot)\}_{n\geq 1}$ we have

$$\begin{array}{c} u_{n_k} \to u \text{ weakly star in } L_\infty(\tau, T; V),\ k \to +\infty, \\ u_{n_k}' \to u' \text{ weakly star in } L_\infty(\tau, T; H),\ k \to +\infty, \\ u_{n_k}'' \to u'' \text{ weakly star in } L_\infty(\tau, T; V^*),\ k \to +\infty, \\ d_{n_k,i} \to d_i \text{ weakly star in } L_\infty(\tau, T; H),\ i = \overline{1, 2},\ k \to +\infty, \\ u_{n_k} \to u \text{ in } L_2(\tau, T; H),\ k \to +\infty, \\ u_{n_k}(t) \to u(t) \text{ in } H \text{ for a.e. } t \in [\tau, T],\ k \to +\infty, \\ u_{n_k}'(t) \to u'(t) \text{ in } V^* \text{ for a.e. } t \in (\tau, T),\ k \to +\infty, \\ \triangle u_{n_k} \to \triangle u \text{ weakly in } L_2(\tau, T; V^*),\ k \to +\infty, \end{array} \tag{16.13}$$

where $\forall n \geq 1$ $d_{n,i} \in L_2(\tau, T; H)$ and

$$\begin{array}{c} u_n''(t) + \beta u_n'(t) + d_{n,1}(t) - d_{n,2}(t) - \triangle u_n(t) = \bar{0}, \\ d_{n,i}(t) \in \partial J_i(u_n(t)), \quad i = 1, 2, \quad \text{for a.e. } t \in (\tau, T). \end{array} \tag{16.14}$$

Step 2. ∂J_i, $i = 1, 2$ are demiclosed. So, by a standard way we get that $d_i(\cdot) \in \partial J_i(u(\cdot))$, $i = 1, 2$, $\varphi := (u, u') \in \mathscr{D}_{\tau,T}(\varphi_\tau) \subset W_\tau^T$.

Step 3. From (16.13) it follows that for arbitrary fixed $h \in V$ the sequences of real functions $(u_{n_k}(\cdot), h)_H$, $(u_{n_k}'(\cdot), h)_H : [\tau, T] \to \mathbb{R}$ are uniformly bounded and equipotentionally continuous. Taking into account (16.13), (16.11) and density of the embedding $V \subset H$ we obtain that $u_{n_k}'(t_{n_k}) \to u'(t_0)$ weakly in H and $u_{n_k}(t_{n_k}) \to u(t_0)$ weakly in V as $k \to +\infty$.

The theorem is proved.

Theorem 16.2 *Let* $\tau < T$, $\{\varphi_n(\cdot)\}_{n\geq 1} \subset W_\tau^T$ *be an arbitrary sequence of weak solutions of Problem* (16.1) on $[\tau, T]$ such that $\varphi_n(\tau) \to \varphi_\tau$ strongly in X, $n \to +\infty$, then up to a subsequence $\varphi_n(\cdot) \to \varphi(\cdot)$ in $C([\tau, T]; X)$, $n \to +\infty$.

Proof Let $\tau < T$, $\{\varphi_n(\cdot) = (u_n(\cdot), u_n'(\cdot))^T\}_{n\geq 1} \subset W_\tau^T$ be an arbitrary sequence of weak solutions of Problem (16.1) on $[\tau, T]$ and $\{t_n\}_{n\geq 1} \subset [\tau, T]$:

$$\varphi_n(\tau) \to \varphi_\tau \text{ strongly in } X,\ n \to +\infty. \tag{16.15}$$

From Theorem 16.1 we have that there exists $\varphi \in \mathscr{D}_{\tau,T}(\varphi_\tau)$ such that up to the subsequence $\{\varphi_{n_k}(\cdot)\}_{k\geq 1} \subset \{\varphi_n(\cdot)\}_{n\geq 1}$ $\varphi_n(\cdot) \to \varphi(\cdot)$ weakly in X, uniformly on $[\tau, T]$, $k \to +\infty$. Let us prove that

$$\varphi_{n_k} \to \varphi \text{ in } W_\tau^T, \quad k \to +\infty. \tag{16.16}$$

By contradiction, suppose the existence of $L > 0$ and the subsequence $\{\varphi_{k_j}\}_{j\geq 1} \subset \{\varphi_{n_k}\}_{k\geq 1}$ such that $\forall j \geq 1$,

$$\max_{t\in[\tau,T]} \|\varphi_{k_j}(t) - \varphi(t)\|_X = \|\varphi_{k_j}(t_j) - \varphi(t_j)\|_X \geq L.$$

Without loss of generality we suggest that $t_j \to t_0 \in [\tau, T]$, $j \to +\infty$. Therefore by virtue of a continuity of $\varphi : [\tau, T] \to X$ we have

$$\varliminf_{j\to+\infty} \|\varphi_{k_j}(t_j) - \varphi(t_0)\|_X \geq L. \tag{16.17}$$

On the other hand, we prove that

$$\varphi_{k_j}(t_j) \to \varphi(t_0) \text{ in } X, \;\; j \to +\infty. \tag{16.18}$$

First we remark that

$$\varphi_{k_j}(t_j) \to \varphi(t_0) \text{ weakly in } X, \quad j \to +\infty \tag{16.19}$$

(see Theorem 16.1 for details). Secondly let us prove that

$$\varlimsup_{j\to+\infty} \|\varphi_{k_j}(t_j)\|_X \leq \|\varphi(t_0)\|_X. \tag{16.20}$$

Since J is sequentially weakly continuous, $\mathscr{V}$ is sequentially weakly lower semi-continuous on X. Hence we obtain

$$\begin{gathered} \mathscr{V}(\varphi(t_0)) \leq \varliminf_{j\to+\infty} \mathscr{V}(\varphi_{k_j}(t_j)), \\ \int_\tau^{t_0} \|u'(s)\|_H^2 ds \leq \varliminf_{j\to+\infty} \int_\tau^{t_j} \|u'_{k_j}(s)\|_H^2 ds \end{gathered} \tag{16.21}$$

and

$$\mathscr{V}(\varphi(t_0)) + \beta \int_\tau^{t_0} \|u'(s)\|_H^2 ds \leq \varliminf_{j\to+\infty} \left(\mathscr{V}(\varphi_{k_j}(t_j)) + \beta \int_\tau^{t_j} \|u'_{k_j}(s)\|_{\mathrm{H}}^2 ds \right). \tag{16.22}$$

Since by the energy equation both sides of (16.22) equal $\mathscr{V}(\varphi(\tau))$ (see Lemma 16.5), it follows from (16.21) that $\mathscr{V}(\varphi_{k_j}(t_j)) \to \mathscr{V}(\varphi(t_0))$, $j \to +\infty$ and (16.20). Convergence (16.18) directly follows from (16.19), (16.20) and Gajewski et al. [5, Chap. I]. To finish the proof of the theorem we remark that (16.18) contradicts (16.17). Therefore (16.16) holds.

The theorem is proved.

Define the m-semiflow $\mathscr{G}$ as

$$\mathscr{G}(t, \xi_0) = \{\xi(t) \mid \xi(\cdot) \in \mathscr{D}(\xi_0)\}, \quad t \geq 0.$$

Denote the set of all nonempty (nonempty bounded) subsets of X by $P(X)(\beta(X))$. Note that the multivalued map $\mathscr{G} : \mathbb{R}_+ \times X \to P(X)$ is *a strict m-semiflow*, i.e., (see Lemma 16.4)

1. $\mathscr{G}(0, \cdot) = \mathrm{Id}$ (the identity map);
2. $\mathscr{G}(t + s, x) = \mathscr{G}(t, \mathscr{G}(s, x))\ \forall x \in X, t, s \in \mathbb{R}_+$.

Further, $\varphi \in \mathscr{G}$ means that $\varphi \in \mathscr{D}(\xi_0)$ for some $\xi_0 \in X$.

Definition 16.2 $\mathscr{G}$ is called *an asymptotically compact* m-semiflow if for any sequence $\{\varphi_n\}_{n\geq 1} \subset \mathscr{G}$ with $\{\varphi_n(0)\}_{n\geq 1}$ bounded, and for any sequence $\{t_n\}_{n\geq 1}$: $t_n \to +\infty$, $n \to \infty$, the sequence $\{\varphi_n(t_n)\}_{n\geq 1}$ has a convergent subsequence Ball [2, p. 35].

Theorem 16.3 *$\mathscr{G}$ is an asymptotically compact m-semiflow.*

Proof Let $\xi_n \in \mathscr{G}(t_n, v_n)$, $v_n \in B$, $B \in \beta(X)$, $n \geq 1$, $t_n \to +\infty$, $n \to +\infty$. Let us check a precompactness of $\{\xi_n\}_{n\geq 1}$ in X. Without loss of the generality, we extract a convergent in X subsequence from $\{\xi_n\}_{n\geq 1}$. From Corollary 16.1 we obtain that there exists $\{\xi_{n_k}\}_{k\geq 1}$ and $\xi \in X$ such that $\xi_{n_k} \to \xi$ weakly in X, $\|\xi_{n_k}\|_X \to a \geq \|\xi\|_X$, $k \to +\infty$. Show that $a \leq \|\xi\|_X$.

Let us fix an arbitrary $T_0 > 0$. Then for rather big $k \geq 1$, $\mathscr{G}(t_{n_k}, v_{n_k}) \subset \mathscr{G}(T_0, \mathscr{G}(t_{n_k} - T_0, v_{n_k}))$. Hence $\xi_{n_k} \in \mathscr{G}(T_0, \beta_{n_k})$, where $\beta_{n_k} \in \mathscr{G}(t_{n_k} - T_0, v_{n_k})$ and $\sup\limits_{k\geq 1} \|\beta_{n_k}\|_X < +\infty$ (see Corollary 16.1). From Theorem 16.1 for some $\{\xi_{k_j}, \beta_{k_j}\}_{j\geq 1} \subset \{\xi_{n_k}, \beta_{n_k}\}_{k\geq 1}$, $\beta_{T_0} \in X$, we obtain

$$\xi \in \mathscr{G}(T_0, \beta_{T_0}), \quad \beta_{k_j} \to \beta_{T_0} \text{ weakly in } X, \quad j \to +\infty. \tag{16.23}$$

From the definition of $\mathscr{G}$ we set $\forall j \geq 1$, $\xi_{k_j} = (u_j(T_0), u_j'(T_0))^T$, $\beta_{k_j} = (u_j(0), u_j'(0))^T$, $\xi = (u_0(T_0), u_0'(T_0))^T$, $\beta_{T_0} = (u_0(0), u_0'(0))^T$, where $\varphi_j = (u_j, u_j')^T \in C([0, T_0]; X)$, $u_j'' \in L_2(0, T_0; V^*)$, $d_j \in L_\infty(0, T_0; H)$,

$$u_j''(t) + \beta u_j'(t) - \Delta u_j(t) + d_{j,1}(t) - d_{j,2}(t) = \bar{0},$$

$$d_{j,i}(t) \in \partial J_i(u_j(t)), \ \ i = 1, 2 \quad \text{for a.e.} \quad t \in (0, T_0).$$

Let for every $t \in [0, T_0]$,

$$I(\varphi_j(t)) := \frac{1}{2}\|\varphi_j(t)\|_X^2 + J_1(u_j(t)) - J_2(u_j(t)) + \frac{\beta}{2}(u_j'(t), u_j(t))_H.$$

Then in virtue of Lemma 16.2, Gajewski et al. [5, Chap. IV], Temam [12] and Zgurovsky et al. [19]

$$\frac{dI(\varphi_j(t))}{dt} = -\beta I(\varphi_j(t)) + \beta \mathscr{H}(\varphi_j(t)), \text{ for a.e. } t \in (0, T_0),$$

where

$$\mathscr{H}(\varphi_j(t)) = J_1(u_j(t)) - \frac{1}{2}(d_{j,1}(t), u_j(t))_H - J_2(u_j(t)) + \frac{1}{2}(d_{j,2}(t), u_j(t))_H.$$

From (16.11), (16.23) we have $\exists \bar{R} > 0 : \forall j \geq 0, \forall t \in [0, T_0]$,

$$\|u_j'(t)\|_{\mathrm{H}}^2 + \|u_j(t)\|_{\mathrm{V}}^2 \leq \bar{R}^2.$$

Moreover,

$$\begin{gathered} u_j \to u_0 \text{ weakly in } L_2(0, T_0; V), \ \ j \to +\infty, \\ u_j' \to u_0' \text{ weakly in } L_2(0, T_0; H), \ \ j \to +\infty, \\ u_j \to u_0 \text{ in } L_2(0, T_0; H), \ \ j \to +\infty, \\ d_{j,i} \to d_i \text{ weakly in } L_2(0, T_0; H), \ \ i = 1, 2, \ \ j \to +\infty, \\ u_j'' \to u_0'' \text{ weakly in } L_2(0, T_0; V^*), \ \ j \to +\infty, \\ \forall t \in [0, T_0] \ \ u_j(t) \to u_0(t) \text{ in } H, \ \ j \to +\infty. \end{gathered} \tag{16.24}$$

For every $j \geq 0$ and $t \in [0, T_0]$,

$$I(\varphi_j(t)) = I(\varphi_j(0))e^{-\beta t} + \int_0^t \mathscr{H}(\varphi_j(s))e^{-\beta(t-s)}ds.$$

In particular $I(\varphi_j(T_0)) = I(\varphi_j(0))e^{-\beta T_0} + \int_0^{T_0} \mathscr{H}(\varphi_j(s))e^{-\beta(T_0-s)}ds$.

From (16.24) and Lemma 16.2 we have

$$\int_0^{T_0} \mathscr{H}(\varphi_j(s))e^{-\beta(T_0-s)}ds \to \int_0^{T_0} \mathscr{H}(\varphi_0(s))e^{-\beta(T_0-s)}ds, \ \ j \to +\infty.$$

Therefore

$$\begin{aligned} \overline{\lim_{j\to+\infty}} I(\varphi_j(T_0)) &\leq \overline{\lim_{j\to+\infty}} I(\varphi_j(0))e^{-\beta T_0} + \int_0^{T_0} \mathscr{H}(\varphi_0(s))e^{-\beta(T_0-s)}ds = \\ &= I(\varphi_0(T_0)) + \left[\overline{\lim_{j\to+\infty}} I(\varphi_j(0)) - I(\varphi_0(0))\right] e^{-\beta T_0} \leq I(\varphi_0(T_0)) + c_3 e^{-\beta T_0}, \end{aligned}$$

where c_3 does not depend on $T_0 > 0$.

On the other hand, from (16.24) we have

$$\overline{\lim_{j\to+\infty}} I(\varphi_j(T_0)) \geq \frac{1}{2}\lim_{j\to+\infty}\|\varphi_j(T_0)\|_X^2 + J(u_0(T_0)) + \frac{\beta}{2}(u_0'(T_0), u_0(T_0))_H.$$

Therefore we obtain $\frac{1}{2}a^2 \leq \frac{1}{2}\|\xi\|_X^2 + c_3 e^{-\beta T_0}\ \forall T_0 > 0$.

Thus, $a \leq \|\xi\|_X$.

The Theorem is proved.

Let us consider the family $\mathscr{K}_+ = \cup_{u_0\in X}\mathscr{D}(u_0)$ of all weak solutions of Problem (16.1) defined on $[0, +\infty)$. Note that $\mathscr{K}_+$ is *translation invariant one*, i.e., $\forall u(\cdot) \in \mathscr{K}_+$, $\forall h \geq 0$, $u_h(\cdot) \in \mathscr{K}_+$, where $u_h(s) = u(h+s)$, $s \geq 0$. On $\mathscr{K}_+$ we set the *translation semigroup* $\{T(h)\}_{h\geq 0}$, $T(h)u(\cdot) = u_h(\cdot)$, $h \geq 0$, $u \in \mathscr{K}_+$. In view of the translation invariance of $\mathscr{K}_+$ we conclude that $T(h)\mathscr{K}_+ \subset \mathscr{K}_+$ as $h \geq 0$.

On $\mathscr{K}_+$ we consider a topology induced from the Fréchet space $C^{loc}(\mathbb{R}_+; X)$. Note that

$$f_n(\cdot) \to f(\cdot) \text{ in } C^{loc}(\mathbb{R}_+; X) \iff \forall M > 0,\ \Pi_M f_n(\cdot) \to \Pi_M f(\cdot) \text{ in } C([0, M]; X),$$

where Π_M is the restriction operator to the interval $[0, M]$; Vishik and Chepyzhov [14, p. 179]. We denote the restriction operator to $[0, +\infty)$ by Π_+.

Let us consider Problem (16.1) on the entire time axis. Similarly to the space $C^{loc}(\mathbb{R}_+; X)$ the space $C^{loc}(\mathbb{R}; X)$ is endowed with the topology of a local uniform convergence on each interval $[-M, M] \subset \mathbb{R}$ (cf. Vishik and Chepyzhov [14, p. 180]). A function $u \in C^{loc}(\mathbb{R}; X) \cap L_\infty(\mathbb{R}; X)$ is said to be *a complete trajectory* of Problem (16.1) if $\forall h \in \mathbb{R}$, $\Pi_+ u_h(\cdot) \in \mathscr{K}_+$; Vishik and Chepyzhov [14, p. 180].

Let $\mathscr{K}$ be a family of *all complete trajectories* of Problem (16.1). Note that $\forall h \in \mathbb{R}$, $\forall u(\cdot) \in \mathscr{K}$ $u_h(\cdot) \in \mathscr{K}$. We say that the complete trajectory $\varphi \in \mathscr{K}$ is *stationary* if $\varphi(t) = z$ for all $t \in \mathbb{R}$ for some $z \in X$. Following Ball [1, p. 486] we denote by $Z(\mathscr{G})$ the set of all rest points of $\mathscr{G}$. Note that

$$Z(\mathscr{G}) = \{(\bar{0}, u) \mid u \in V,\ -\triangle(u) + \partial J(u) \ni \bar{0}\}.$$

Lemma 16.7 *$Z(\mathscr{G})$ is an bounded set in X.*

The existence of a Lyapunov function for $\mathscr{G}$ follows from Lemma 16.5 (see Ball [1, p. 486]).

Lemma 16.8 *A functional $\mathscr{V} : X \to \mathbb{R}$ defined by (16.9) is a Lyapunov function for $\mathscr{G}$.*

16.5 The Existence of a Global Attractor

At first we consider constructions presented in Ball [1], Mel'nik and Valero [10]. We recall that the set $\mathscr{A}$ is said to be a *global attractor* $\mathscr{G}$ if

(1) $\mathscr{A}$ is negatively semiinvariant (i.e., $\mathscr{A} \subset \mathscr{G}(t, \mathscr{A})\ \forall t \geq 0$);

(2) $\mathscr{A}$ is attracting set, i.e.,

$$\operatorname{dist}(\mathscr{G}(t, B), \mathscr{A}) \to 0, \quad t \to +\infty, \quad \forall B \in \beta(X), \tag{16.25}$$

where $\operatorname{dist}(C, D) = \sup_{c \in C} \inf_{d \in D} \|c - d\|_X$ is the Hausdorff semidistance;

(3) for any closed set $Y \subset H$ satisfying (16.25), we have $\mathscr{A} \subset Y$ (minimality).

The global attractor is said to be *invariant* if $\mathscr{A} = \mathscr{G}(t, \mathscr{A})$, $\forall t \geq 0$.

Note that by definition a global attractor is unique.

We prove the existence of an invariant compact global attractor.

Theorem 16.4 *The m-semiflow $\mathscr{G}$ has an invariant compact in the phase space X global attractor $\mathscr{A}$. For each $\psi \in \mathscr{K}$ the limit sets*

$$\alpha(\psi) = \{z \in \mathrm{X} |\ \psi(t_j) \to z \ \textit{ for some sequence } \ t_j \to -\infty\},$$

$$\omega(\psi) = \{z \in \mathrm{X} |\ \psi(t_j) \to z \ \textit{ for some sequence } \ t_j \to +\infty\}$$

are connected subsets of $Z(\mathscr{G})$ on which $\mathscr{V}$ is constant. If $Z(\mathscr{G})$ is totally disconnected (in particular if $Z(\mathscr{G})$ is countable) the limits

$$z_- = \lim_{t \to -\infty} \psi(t), \quad z_+ = \lim_{t \to +\infty} \psi(t)$$

exist and z_-, z_+ are rest points; furthermore, $\varphi(t)$ tends to a rest point as $t \to +\infty$ for every solution $\varphi \in \mathscr{K}_+$.

Proof The existence of a global attractor for Second Order Evolution Inclusions directly follows from Lemmas 16.3, 16.4, 16.7, 16.8, Theorems 16.1–16.3 and Ball [2, Theorem 2.7].

16.6 Global Attractors for Typically Discontinuous Interaction Functions

Let $\beta > 0$ be a constant, $\Omega \in \mathbb{R}^n$ be a bounded domain with sufficiently smooth boundary $\partial\Omega$. Consider the problem

$$\begin{cases} u_{tt} + \beta u_t - \Delta u \in -f(u) + G(u) + h, \\ \qquad\qquad u|_{\partial\Omega} = 0, \end{cases} \tag{16.26}$$

where $u(x, t)$ is unknown state function defined on $\Omega \times \mathbb{R}_+$, $h \in L^2(\Omega)$, $f : \mathbb{R} \to \mathbb{R}$ is an interaction function such that

$$f \in \mathbf{C}(\mathbb{R}), \quad G = [g_1, g_2], \quad g_i \in \mathbf{C}(\mathbb{R}), \quad i = 1, 2. \tag{16.27}$$

There exist a small constant $C \geq 0$ ($C < \min\{\beta, \lambda_1\}$), and $D_i \geq 0$, $i = 1, 2$ such that

$$\varliminf_{|u|\to\infty} \frac{f(u)}{u} > -\lambda_1, \tag{16.28}$$

where λ_1 is the first eigenvalue for $-\triangle$ in $H_0^1(\Omega)$,

$$|g_i(u)| \leq C|u| + D_1, \quad \forall u \in \mathbb{R},\ i = 1, 2, \tag{16.29}$$

$$|f(u)| \leq D_2(1 + |u|^{\frac{n}{n-2}}), \quad \forall u \in \mathbb{R}. \tag{16.30}$$

Remark 16.1 The case of ε–neighborhood of $f(u)$ satisfies conditions (16.27)–(16.30), i.e., if $\varliminf\limits_{|u|\to\infty} \frac{f(u)}{u} > -\lambda_1$, $G(u) = [-\varepsilon, \varepsilon]$.

Let us set $V = H_0^1(\Omega)$ and $H = L^2(\Omega)$. The space $X = V \times H$ is a phase space of Problem (16.26).

Definition 16.3 Let $T > 0$. The function $\varphi(\cdot) = (u(\cdot), u_t(\cdot))^T \in L^\infty(0, T, X)$ is called *a weak solution* of Problem (16.26) on $(0, T)$ if for a.e. $(x, t) \in \Omega \times (0, T)$, there exists $l = l(x, t) \in L^2(0, T : L^2(\Omega))$, $l(x, t) \in G(u(x, t))$ such that $\forall \psi \in H_0^1(\Omega)$, $\eta \in \mathbf{C}_0^\infty(0, T)$

$$-\int_0^T (u_t, \psi)_H \eta_t\, dt + \int_0^T [(\beta(u_t, \psi)_H +$$

$$+(u, \psi)_V + (f(u), \psi)_H - (l, \psi)_H - (h, \psi)_H)\eta]\, dt = 0.$$

Lemma 16.9 *For all* $\varphi_0 = (u_0, u_1)^T \in X$, $T > 0$, *there exists a weak solution* $\varphi(\cdot)$ *of Problem* (16.26) such that $\varphi(0) = \varphi_0$. Moreover, if $\varphi(\cdot) = (u(\cdot), u_t(\cdot))^T$ is a weak solution of Problem (16.26) with respective $l \in L^2(0, T; L^2(\Omega))$, then $\varphi \in \mathbf{C}([0, T]; X)$, functions

$$t \mapsto \|u_t(t)\|_H^2 + \|u(t)\|_V^2, \quad t \mapsto (F(u(t)), 1)_H$$

are absolutely continuous on $[0, T]$, and for $t, s \in [0, T]$, $s \leq t$,

$$\frac{1}{2}\frac{d}{dt}(\|u_t(t)\|_H^2 + \|u(t)\|_V^2 + (F(u(t)), 1)_H) =$$

$$= -\beta\|u_t(t)\|_H^2 + (l(t), u_t(t))_H + (h, u_t(t))_H, \tag{16.31}$$

$$\|u_t(t)\|_H^2 + \|u(t)\|_V^2 \leq e^{-\delta(t-s)}\left(\|u_t(s)\|_H^2 + \|u(s)\|_V^{\frac{2n-2}{n-2}}\right) + D_3, \tag{16.32}$$

where $F(u) = \int_0^u f(s)ds$, $u \in \mathbb{R}$ and constants $\delta > 0$, $D_3 > 0$ do not depend on φ.

Proof Let us deduce condition (16.32). Consider

$$Y(t) = \frac{1}{2}\|u_t(t)\|_H^2 + \frac{1}{2}\|u(t)\|_V^2 + (F(u(t)), 1)_H + \alpha(u_t(t), u(t))_H, \ t \in [0, T],$$

where $\alpha > 0$. Then for sufficiently small $C > 0$ and $\delta > 0$

$$\begin{aligned}\tfrac{dY(t)}{dt} &= (u_{tt}(t), u_t(t))_H - (\triangle u, u_t(t))_H + (f(u(t)), u_t(t))_H + \\ &+ \alpha(u_{tt}(t), u(t))_H + \alpha\|u_t(t)\|_H^2 = \\ &= (-\beta u_t(t) + l(t) + h, u_t(t))_H + \alpha\|u_t(t)\|_H^2 + \\ &+ \alpha(-\beta u_t(t) + \triangle u - f(u(t)) + l(t) + h, u(t))_H = \\ &= -(\beta - \alpha)\|u_t(t)\|_H^2 + (l(t), u_t(t))_H + (h, u_t(t))_H - \\ &- \alpha\beta(u_t(t), u(t))_H - \alpha\|u(t)\|_V^2 - \\ &- \alpha(f(u(t)), u(t))_H + \alpha(l(t) + h, u(t))_H \le \\ &\le -(\beta - \alpha - \varepsilon)\|u_t(t)\|_H^2 + C\|u(t)\|_H\|u_t(t)\|_H \\ &\le -\alpha\|u(t)\|_V^2 - \alpha(-\lambda_1 + C + \varepsilon)\|u(t)\|_H^2 + \\ &+ \alpha C\|u(t)\|_H^2 + K \le -\delta Y(t) + \tilde{K}.\end{aligned}$$

Therefore the inequalities

$$F(u) \ge \left(-\frac{\lambda_1}{2} + \varepsilon\right)u^2 + L, \quad F(u) \le M\left(1 + |u|^{\frac{2u-2}{u-2}}\right), \quad \forall u \in \mathbb{R}, \tag{16.33}$$

imply (16.32). All the other statements follow from Ball [2], Temam [12]. The existence of a solution follows from the existence of a continuous selector for G.

Remark 16.2 The set of solutions of Problem (16.26) is not covered by all continuous selectors of $G : \mathbb{R} \mapsto 2^{\mathbb{R}}$.

Indeed, let $f \equiv 0$, $G(u) \equiv [-\varepsilon, \varepsilon]$, $h \equiv 0$. Consider solutions of the problem

$$\begin{cases} \triangle u \in [-\varepsilon, \varepsilon], & \text{in } \Omega = (0, \pi), \\ u(0) = u(\pi) = 0, \end{cases}$$

i.e., consider stationary solutions of Problem (16.26). Then the function

$$u(x) = \frac{\varepsilon}{2}\sin x + \frac{\varepsilon}{8}\sin 2x, \quad x \in (0, \pi),$$

is a solution of the given problem but there is no $g \in \mathbf{C}(\mathbb{R})$ such that $g(u) \in [-\varepsilon, \varepsilon]$, $\forall u \in \mathbb{R}$, and $\triangle u(x) = g(u(x))$, $x \in (0, \pi)$. Indeed, assume the converse. Suppose that such function exists. The equation

$$\frac{\varepsilon}{2}\sin x + \frac{\varepsilon}{8}\sin 2x = \frac{\varepsilon}{2}$$

has two solutions

$$x = \frac{\pi}{2} \text{ and } x = x^* \neq \frac{\pi}{2} \in (0, \pi).$$

If $x = \frac{\pi}{2}$, then

$$g(\frac{\varepsilon}{2}) = u''(\frac{\pi}{2}) = -\frac{\varepsilon}{2}.$$

If $x = x^*$, then

$$g(\frac{\varepsilon}{2}) = -\frac{\varepsilon}{2}\sin x^* - \frac{\varepsilon}{2}\sin 2x^* = -\frac{\varepsilon}{2} - \frac{3\varepsilon}{8}\sin 2x^* \neq -\frac{\varepsilon}{2}.$$

This contradiction concludes the example.

Remark 16.3 If $G(u) \equiv g(u)$ is a single-valued function, then the existence of a global attractor was proved in Ball [2].

Select the class of solutions for which there exists a global attractor. For this purpose we use the notion of "energy" equation Ball [2], which describes the conservation laws of energy.

Let $\varphi \in C([0, +\infty); X)$ is a solution of Problem (16.26). Denote

$$I(\varphi) = \frac{1}{2}\|u_t(t)\|_H^2 + \frac{1}{2}\|u(t)\|_V^2 + (F(u(t)), 1)_H + \frac{\beta}{2}(u_t(t), u(t))_H,$$

$$g_\lambda(u) = \lambda g_1(u) + (1 - \lambda g_2(u)), \quad G_\lambda(u) = \int_0^u g_\lambda(s)ds, \quad \lambda \in [0, 1],$$

$$H(\varphi) = \beta(F(u), 1)_H - \frac{\beta}{2}(f(u), u)_H + \frac{\beta}{2}(h, u)_H + (h, u_t)_H.$$

Definition 16.4 A weak solution φ of Problem (16.26) with the corresponding function l is called an *energy solution* if there exists $\lambda \in [0, 1]$ $(\lambda = \lambda(\varphi))$ such that $\forall t \geq 0$,

$$\frac{d}{dt}I(\varphi(t)) + \beta I(\varphi(t)) - \frac{d}{dt}(G_\lambda(u(t)), 1)_H = \frac{\beta}{2}(l(t), u(t))_H + H(\varphi(t)). \quad (16.34)$$

Remark 16.4 Any solution satisfies the equation

$$\frac{d}{dt}I(\varphi(t)) + \beta I(\varphi(t)) - (l(t), u_t(t))_H = \frac{\beta}{2}(l(t), u(t))_H + H(\varphi(t)).$$

Any "selector" solution satisfies the equation

$$\frac{d}{dt}I(\varphi(t)) + \beta I(\varphi(t)) - (g(u(t)), u_t(t))_H = \frac{\beta}{2}(g(u(t)), u(t))_H + H(\varphi(t)).$$

Remark 16.5 Any stationary solution $u(t)$ obviously satisfies (16.34). So, the set of all "selector" solutions (solutions of Problem (16.26) with $l(x,t) = g(u(x,t))$, $g \in G$) does not include the set of energy solutions. Moreover, the set of all energy solutions is wider than the set of all solutions of (16.26) with $l(x,t) = g_\lambda(u(x,t))$.

Let us set

$$\mathcal{G}(t, \varphi_0) = \{\varphi(t) \mid \varphi(\cdot) \quad \text{is an energy solution of}(16.26), \varphi(0) = \varphi_0\} \quad (16.35)$$

Theorem 16.5 *The m-semiflow $\mathcal{G}$ has an invariant compact in the phase space X global attractor.*

Proof $\mathcal{G}$ is the m-semiflow (but not strict; it will be strict if in the definition 16.4 $[0, +\infty)$ is divided into intervals with different λ). Note that $\mathcal{G}$ is dissipative; $\mathcal{G}$ has a closed graph (it is necessary to pass to the limit in (16.34)); $\mathcal{G}$ is asymptotically semicompact m-semiflow. Indeed, similarly to Ball [2] we obtain the equation

$$\begin{aligned} &I(\varphi_j(t_j)) - (G_{\lambda_j}(u_j(t_j)), 1)_H = \\ &= \big(I(\varphi_j(t_j - M)) - (G_{\lambda_j}(\varphi_j(t_j - M)), 1)_H\big)\, e^{-\beta M} + \int_0^M e^{\beta(t-M)} \cdot \\ &\cdot \Big(H(\varphi_j(t)) + \tfrac{\beta}{2}(l_j(t), u_j(t))_H - \beta(G_{\lambda_j}(\varphi_j(t)), 1)_H\Big)\, dt. \end{aligned} \quad (16.36)$$

Since up to a subsequence $\lambda_j \to \lambda$, $\varphi_j(t_j) \to \chi$ weakly in $H_0^1(\Omega)$, we obtain

$$\big(G_{\lambda_j}(\varphi_j(t_j)), 1\big)_H \to (G_\lambda(\chi), 1)_H$$

and similarly Ball [2] we have

$$I(\varphi_j(t_j)) \to I(\chi).$$

Remark 16.6 It is possible to build another multivalued semiflow generated by selector solutions, i.e.,

$$\mathcal{G}(t, \varphi_0) = \left\{ \varphi(t) \;\middle|\; \begin{array}{l} \varphi(\cdot) \quad \text{is a solution of (16.26)}, \\ \varphi(0) = \varphi_0, \\ \exists g \in \mathrm{G} : \;\; \varphi(\cdot) \text{ is a solution of the resp. equation with } g \end{array} \right\}.$$

However in this case, for the sequence $\{\varphi_j\}_{j=1}^\infty$, we have $\{g_j\}_{j=1}^\infty$, $g_j(u) \in G(u)$, $\forall u \in \mathbb{R}$. In order to $g_j(u) \to g(u)$ $\forall u \in \mathbb{R}, g \in G$, it is necessary to strengthen the conditions for G. But in this case, the question about solvability of Problem (16.26) arises.

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