Chapter 14
Inertial Manifolds and Spectral Gap Properties for Wave Equations with Weak and Strong Dissipation

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Abstract Sufficient conditions for the existence of an inertial manifold for the equation $u_{tt} - 2\gamma_s \Delta u_t + 2\gamma_w u_t - \Delta u = f(u)$, $\gamma_s > 0$, $\gamma_w \geq 0$ are found. The nonlinear function f is supposed to satisfy Lipschitz property. The proof is based on construction of a new inner product in the phase space in which the conditions of a general theorem on the existence of inertial manifolds for an abstract differential equation in a Hilbert space are satisfied.

14.1 Introduction

In the theory of nonlinear evolution partial differential equations, great attention is paid to long-time behavior of dynamic systems. Some way of such description relates with notion of an inertial manifold (see [5, 6, 9]).

Let us consider an initial-value problem for an abstract differential equation in a Hilbert space,

$$\frac{d}{dt}y + \mathbf{A}y = F(y), \qquad y \in \mathscr{H}, \tag{14.1}$$

$$y\big|_{t=0} = y_0 \in \mathscr{H}. \tag{14.2}$$

Here $\mathbf{A}$ is a linear operator and F is a nonlinear operator. Suppose problem (14.1), (14.2) has a unique solution y for any $y_0 \in \mathscr{H}$. Hence, this problem generates a continuous semigroup $\{S(t) \mid t \geq 0\}$, acting in the space $\mathscr{H}$ by the formula $S(t)y_0 = y(t) \in \mathscr{H}$.

Definition 14.1 A Lipschitz finite dimensional manifold $\mathscr{M} \subset \mathscr{H}$ is an *inertial manifold* for the semigroup $S(t)$ if it is invariant (i.e., $S(t)\mathscr{M} = \mathscr{M}$, $\forall t \geq 0$) and it satisfies the following asymptotic completeness property:

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$$\forall y_0 \in \mathscr{H} \exists \tilde{y}_0 \in \mathscr{M} \text{ such that } \|S(t)y_0 - S(t)\tilde{y}_0\|_{\mathscr{H}} \le q(\|y_0\|_{\mathscr{H}})e^{-ct}, t \ge 0,$$

where the positive constant c and the monotonic function q are independent of y_0.

Inertial manifolds enable one to reduce the study of the behavior of an infinite-dimensional dynamical system to the investigation of this problem for some finite-dimensional dynamical system generated by original system on an inertial manifold.

For the abstract equation of the form (14.1), there are known sufficient conditions under which there is an inertial manifold in the Hilbert space $\mathscr{H}$ (see [3]). Let us present these conditions. Let $\mathbf{A}$ be a linear closed (possibly unbounded) operator with dense domain $\mathscr{D}(\mathbf{A})$ in $\mathscr{H}$ and let the spectrum $\sigma(\mathbf{A})$ of $\mathbf{A}$ be disjoint from the strip $\{m < \Re\zeta < M\}$, where $M \ge 0$, $M > m$. Denote by P the orthogonal projection to the invariant subspace of $\mathbf{A}$ corresponding to the part of the spectrum $\sigma \cap \{\Re\zeta \le m\}$ and write $Q = \mathrm{Id} - P$. Assume that the space $P(\mathscr{H})$ is finite-dimensional.

Theorem 14.1 *Let the space $\mathscr{H}$ be equiped with an inner product in such a way that the space $P(\mathscr{H})$ and $Q(\mathscr{H})$ are orthogonal and the following relations hold:*

$$\begin{aligned} (\mathbf{A}y, y) &\le m|y|^2 \quad \forall y \in P(\mathscr{H}), \\ (\mathbf{A}y, y) &\ge M|y|^2 \quad \forall y \in Q(\mathscr{H}) \cap \mathscr{D}(\mathbf{A}). \end{aligned} \tag{14.3}$$

Moreover, let $F(y)$ be a nonlinear function such that $F(0) = 0$ and let F satisfy the Lipschitz condition with the constant L, where

$$2L < M - m. \tag{14.4}$$

In this case, there is an inertial manifold $\mathscr{M}$ in the Hilbert space $\mathscr{H}$, and this manifold is the graph of a Lipschitz continuous function $\Phi\colon P(H) \to Q(H)$.

In the present chapter, an initial-boundary value problem for a wave equation with weak and strong dissipation is considered. The nonlinear term depends on the unknown function u, these term is assumed to be Lipschitzian,

$$u_{tt} - 2\gamma_s \Delta u_t + 2\gamma_w u_t - \Delta u = f(u).$$

For this equation, we obtain a condition on the Lipschitz constant of the function f which ensures the existence of an inertial manifold. The result is stated in Theorems 14.2 and 14.3. The proof is based on construction of a new inner product in the phase space in which the conditions of Theorem 14.1 are satisfied.

14.2 Statement of the Problem and Spectrum of the Linear Operator

In a bounded domain Ω, we consider the inertial-boundary value problem for a wave equation with dissipation,

$$u_{tt} - 2\gamma_s \Delta u_t + 2\gamma_w u_t - \Delta u = f(u), \qquad u|_{\partial\Omega} = 0, \tag{14.5}$$

$$u|_{t=0} = u_0(x) \in H_0^1(\Omega), \qquad u_t|_{t=0} = p_0 \in L_2(\Omega). \tag{14.6}$$

Here γ_w and γ_s are positive coefficients of the dissipation, and the nonlinear function f is continuously differentiable and satisfy the global Lipschitz condition,

$$|f(v_1) - f(v_2)| \le l|v_1 - v_2| \qquad \forall v_1, v_2 \in \mathbb{R}, \tag{14.7}$$

Moreover, let $f(0) = g(0) = 0$.

Under these assumptions, problem (14.5), (14.6) has a unique weak solution $u \in C\big([0,T]; H_0^1(\Omega)\big)$, $\partial_t u \in C\big([0,T]; L_2(\Omega)\big)$ for any $T > 0$ (see [7, 8, 10]). Hence, this problem generates a continuous semigroup $\{S(t)\}$, $t \ge 0$, acting in the phase space $\mathscr{H} = H_0^1(\Omega) \times L_2(\Omega)$ by the formula

$$S(t)(u_0(x), p_0(x)) = y(t) \equiv (u(t,x), p(t,x)) \in H,$$

where $u(t,x)$ is a solution of the problem (14.5), (14.6), $p(t,x) = \partial_t u(t,x)$ stands for the derivative of this solution w.r.t. t, and $y = (u, p) \in \mathscr{H}$.

Let us represent the initial-boundary value problem in the form of an ordinary differential equation to find the unknown vector function $y = (u, p) \in \mathscr{H}$,

$$\frac{d}{dt} y(t) + \mathbf{A} y = F(y), \quad \mathbf{A} y = \begin{pmatrix} 0 & -1 \\ -\Delta & 2\gamma_w - 2\gamma_s \Delta \end{pmatrix} y, \quad F(y) = \begin{pmatrix} 0 \\ f(u) \end{pmatrix}.$$

Let $e_k(x)$ and λ_k be the eigenfunctions and the eigenvalues of the operator $-\Delta$ in the domain Ω with the Dirichlet conditions on the boundary,

$$-\Delta e_k(x) = \lambda_k e_k(x), \quad e_k(x)\big|_{\partial\Omega} = 0, \quad e_k(x) \not\equiv 0,$$
$$0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \cdots \to +\infty.$$

Denote by $(\cdot,\cdot)_{\mathscr{H}}$ and $\|\cdot\|$ the standard inner product and the corresponding norm in the space $\mathscr{H}$, namely,

$$(y, \tilde{y})_{\mathscr{H}} = (\nabla u, \nabla \tilde{u}) + (p, \tilde{p}) = \sum_{k=1}^{\infty} (\lambda_k u_k \tilde{u}_k + p_k \tilde{p}_k),$$

where $u_k = (u, e_k)$, $p_k = (p, e_k)$, and $(\cdot,\cdot)$ stands for the inner product in $L_2(\Omega)$.

The two-dimensional subspace $\mathscr{H}_k$ with basis $(e_k, 0)$, $(0, e_k)$ is invariant under the operator $\mathbf{A}$. The restriction of the operator $\mathbf{A}$ to the subspace $\mathscr{H}_k$ has the matrix $A_k = \begin{pmatrix} 0 & -1 \\ \lambda_k & 2(\gamma_w + \gamma_s \lambda_k) \end{pmatrix}$. The eigenvalues of A_k are equal to

$$\mu_k = \gamma_k - \sqrt{\gamma_k^2 - \lambda_k} \quad \text{and} \quad \nu_k = \gamma_k + \sqrt{\gamma_k^2 - \lambda_k}$$

where we denote $\gamma_k = \gamma_w + \gamma_s \lambda_k$. In Figs. 14.1 and 14.2, we show the qualitative displacement of these eigenvalues on the complex plane in two cases, namely, $4\gamma_w \gamma_s < 1$ and $4\gamma_w \gamma_s \geq 1$. In the first case, the operator A has both real and nonreal eigenvalues and, in the other case, all eigenvalues are real.

If the orthogonal projection P satisfies the assumptions of the Theorem 14.1, then the image $P(\mathscr{H})$ (which is finite-dimensional) must correspond to finitely many eigenvalues of $\mathbf{A}$ belonging to the domain $\{\mathrm{Re}\zeta \leq m\}$. However, $\mu_k \to 1/(2\gamma_s)$ and

Fig. 14.1 $4\gamma_w \gamma_s < 1$

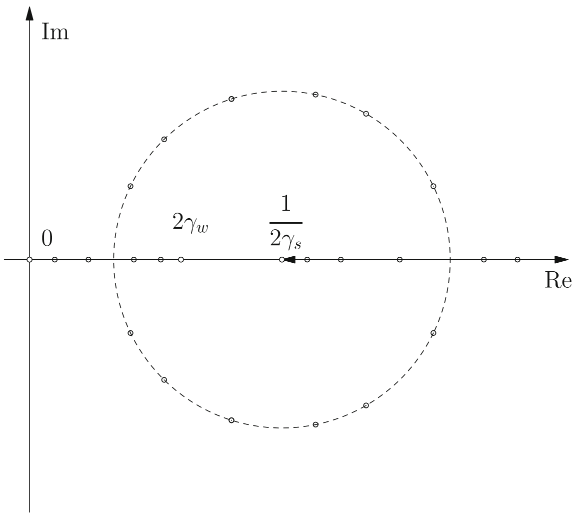
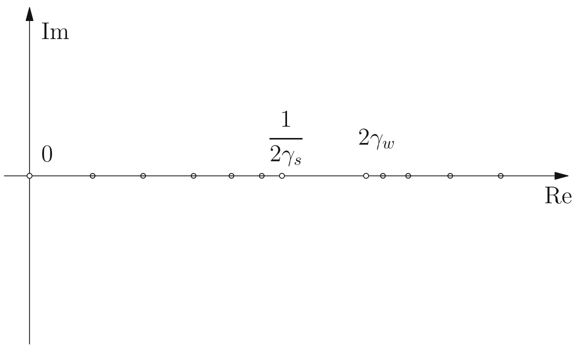


Fig. 14.2 $4\gamma_w \gamma_s > 1$



$\nu_k \to +\infty$ as $\lambda_k \to +\infty$, and thus the quantity m must be less than $1/(2\gamma_s)$. In the case $4\gamma_w\gamma_s < 1$, to the values μ_k and ν_k lying to the left of the accumulation point $1/(2\gamma_s)$ there correspond values $\lambda_k < \frac{1-2\gamma_w\gamma_s}{2\gamma_s^2}$. If $4\gamma_w\gamma_s \geq 1$, then $\mu_k < 1/(2\gamma_s)$ for any k.

14.3 Sufficient Conditions for the Existence of Inertial Manifolds

In this section, we present conditions for the existence of a gap both in the real part (Theorem 14.2) and in the nonreal part (Theorem 14.3) of the spectrum of **A**.

First let us consider a gap in the real part of the spectrum. Thus, for $4\gamma_w\gamma_s < 1$, the additional condition $m < \frac{1-\sqrt{1-4\gamma_w\gamma_s}}{2\gamma_s}$ is imposed, which corresponds to the inequality $\lambda_k < \frac{1-2\gamma_w\gamma_s-\sqrt{1-4\gamma_w\gamma_s}}{2\gamma_s^2}$.

Remark 14.1 If Eq. (14.5) has not strongly dissipative term (i.e., $\gamma_s = 0$), then the circle to which a part of eigenvalues of the operator **A** belongs (see Fig. 14.1) is transformed to the vertical line $\{\Re\zeta = \gamma_w\}$ (see Fig. 14.3), and the condition on m becomes $m < \gamma_w$.

Write

$$\gamma_\star = \begin{cases} \gamma_1, & \text{if } 1 \leq 2\gamma_s\gamma_1; \\ 1/(2\gamma_s), & \text{if } 2\gamma_s\gamma_1 \leq 1 \leq 2\gamma_s\gamma_{N+1}; \\ \gamma_{N+1}, & \text{if } 2\gamma_s\gamma_{N+1} \leq 1; \end{cases} \quad \lambda_\star = \frac{\gamma_\star - \gamma_w}{\gamma_s}.$$

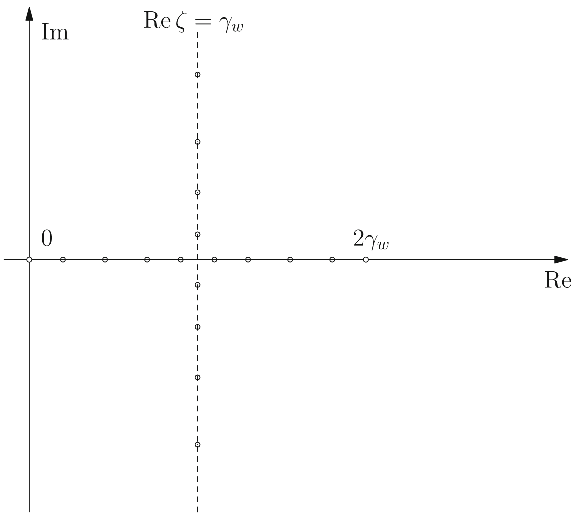


Fig. 14.3 Weak dissipation, $\gamma_s = 0$

Theorem 14.2 *Let f satisfy condition* (14.7). *Moreover, suppose that there is an N such that the following inequality holds:*

$$2\frac{l}{\sqrt{\gamma_\star^2-\lambda_\star}} < \mu_{N+1}-\mu_N = \gamma_{N+1}-\sqrt{\gamma_{N+1}^2-\lambda_{N+1}}-\gamma_N+\sqrt{\gamma_N^2-\lambda_N}, \quad (14.8)$$

and, if $4\gamma_w\gamma_s < 1$, then the following inequality also holds:

$$\lambda_{N+1} < \frac{1-2\gamma_w\gamma_s-\sqrt{1-4\gamma_w\gamma_s}}{2\gamma_s^2}.$$

In this case, there is an N-dimensional inertial manifold for problem (14.5), (14.6) *in the space $\mathscr{H}$.*

Remark 14.2 If $\gamma_s = 0$, then condition (14.8) coincides with the similar condition obtained in [4].

Remark 14.3 If there is no weak dissipation, then all real point of the spectrum of the operator $\mathbf{A}$ are located to the right of the number $1/(2\gamma_s)$ (see Fig. 14.4), and Theorem 14.2 cannot be applied to this situation.

Now we consider case of spectral gap in nonreal part of spectrum. Hence we assume that $4\gamma_w\gamma_s < 1$.

Let values m and M be chosen in such a way that

$$\frac{1-\sqrt{1-4\gamma_w\gamma_s}}{2\gamma_s} \le m < M \le \frac{1}{2\gamma_s}, \quad (14.9)$$

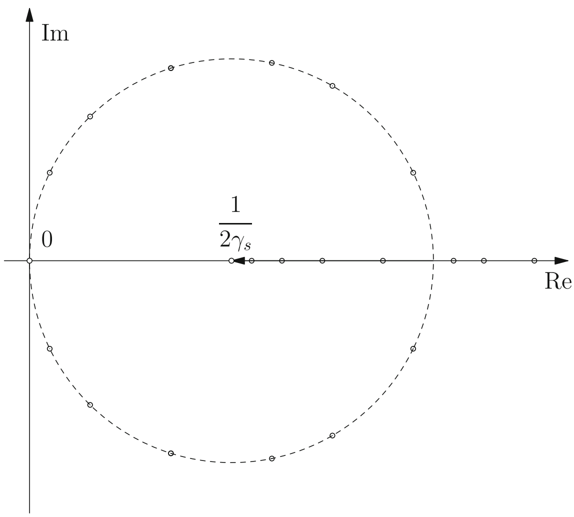


Fig. 14.4 Strong dissipation, $\gamma_w = 0$

and the spectrum $\sigma(\mathbf{A})$ of $\mathbf{A}$ be disjoint from the strip $\{m < \Re\zeta < M\}$, but the set $\sigma(\mathbf{A}) \cap \{\Re\zeta \le m\}$ is not empty.

Let numbers k_1, k_2 are such that values ν_{k_1} and ν_{k_2+1} belong to the domain $\{\Re\zeta \ge M\}$, and numbers ν_{k_1+1} and ν_{k_2} belong to the domain $\{\Re\zeta \le m\}$ (see Fig. 14.5). Thus for $\nu_1 \notin \mathbb{R}$ or $\nu_1 \in \mathbb{R}$, $\nu_1 \le m$ we have $k_1 = 0$; for the converse case we get $\Re\nu_{k_1+1} \le m < M \le \Re\nu_{k_1}$.

If there are not numbers ν_k to the left of the strip, then we have $M \le \Re\nu_{k_2+1}$ and $M \le \Re\nu_{k_1} = \nu_{k_1} = \nu_{k_2}$. Otherwise number k_2 is such that $\Re\nu_{k_2} \le m < M \le \Re\nu_{k_2+1}$.

Denote numbers $\varkappa_I$, $\varkappa_{II}$, $\varkappa_{III}$ and $\varkappa_{IV}$. First if $k_1 = 0$ then formally write $\varkappa_I = +\infty$. In the other case write $\varkappa_I = \sqrt{\gamma_{k_1}^2 - \lambda_{k_1}}$. Secondly if $k_2 = k_1$ then formally write $\varkappa_{II} = \varkappa_{III} = +\infty$. Otherwise denote $\varkappa_{II} = s_{k_1+1}$, $\varkappa_{III} = s_{k_2}$, where

$$s_k = \sqrt{m^2 - 2m\gamma_k + \lambda_k} + m - \gamma_k. \tag{14.10}$$

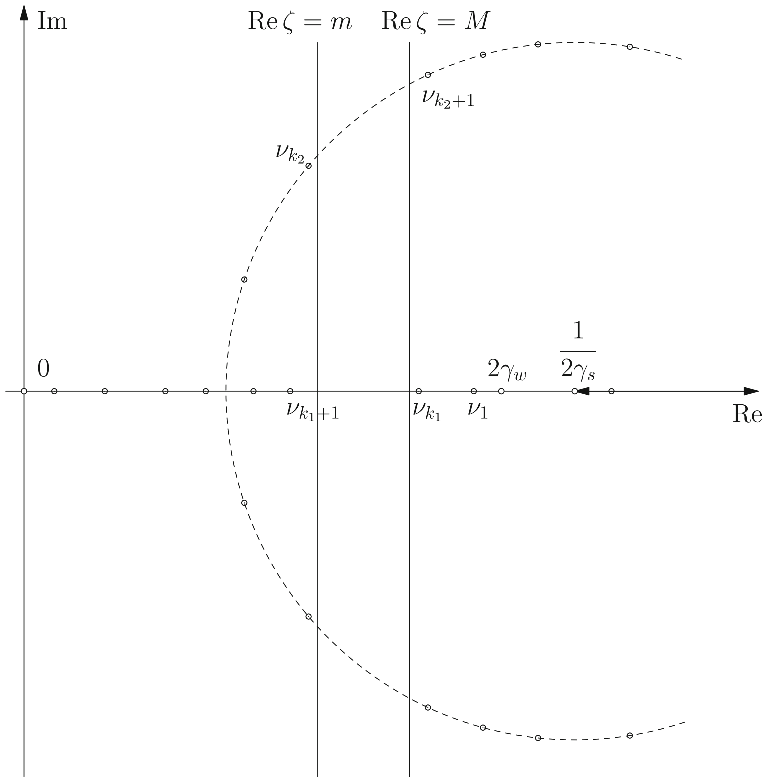


Fig. 14.5 A spectral gap in nonreal part of the spectrum

Finally write $\lambda_M = (M - \gamma_w)/\gamma_s$ and $\varkappa_{IV} = \sqrt{\lambda_M - M^2}$.

Theorem 14.3 *Let nonlinear function f satisfies condition (14.7). Moreover, suppose that the following inequality holds:*

$$2l < (M - m)\min\{\varkappa_I, \varkappa_{II}, \varkappa_{III}, \varkappa_{IV}\}. \tag{14.11}$$

Then there is a $(2k_2 - k_1)$-dimensional inertial manifold for problem (14.5), (14.6) in the space $\mathscr{H}$.

Remark 14.4 It follows from condition (14.11) that there are enough large gaps in the spectrum of operator $-\Delta$ in domain Ω. Actually, we have

$$\varkappa_{IV} = \sqrt{\lambda_M - M^2} = \sqrt{\frac{M - \gamma_w - \gamma_s M^2}{\gamma_s}} =$$
$$= \sqrt{\frac{4\gamma_s M - 4\gamma_w\gamma_s - 4\gamma_s^2 M^2}{4\gamma_s^2}} = \sqrt{\frac{1 - 4\gamma_w\gamma_s - (2\gamma_s M - 1)^2}{4\gamma_s^2}} < \frac{\sqrt{1 - 4\gamma_w\gamma_s}}{2\gamma_s}.$$

Moreover, the inequalities $\gamma_{k_2} \leq m$ and $M \leq \gamma_{k_2+1}$ hold by definition of the number k_2. Indeed if $\nu_{k_2} \in \mathbb{R}$, then we have $\nu_{k_2} < \frac{1}{2\gamma_s}$, $\gamma_{k_2} < \frac{1-\sqrt{1-4\gamma_w\gamma_s}}{2\gamma_s} \leq m$ (see (14.9)); otherwise we have $\gamma_{k_2} = \Re\nu_{k_2} \leq m$. Similarly if $\nu_{k_2+1} \in \mathbb{R}$, then we have $\nu_{k_2} > \frac{1}{2\gamma_s}$, $\gamma_{k_2} > \frac{1+\sqrt{1-4\gamma_w\gamma_s}}{2\gamma_s} > M$; otherwise we get $\gamma_{k_2+1} = \Re\nu_{k_2+1} \geq M$.

Thus, by (14.11) it follows the inequality,

$$2l < (\gamma_{k_2+1} - \gamma_{k_2})\frac{\sqrt{1 - 4\gamma_w\gamma_s}}{2\gamma_s} = (\lambda_{k_2+1} - \lambda_{k_2})\frac{\sqrt{1 - 4\gamma_w\gamma_s}}{2}.$$

This means that there are spectral gaps on the order of l:

$$\lambda_{k_2+1} - \lambda_{k_2} > 4l\sqrt{1 - 4\gamma_w\gamma_s}.$$

The proofs of Theorems 14.2 and 14.3 are based on the construction of a new norm in the phase space $\mathscr{H}$, in which the assumptions of Theorem 14.1 are satisfied. Note the schemes of the new inner product construction are essentially different for gaps in the real part and in the nonreal part of the spectrum. Then this two cases are considered separately. In the present chapter we prove Theorem 14.3. The proof of Theorem 14.2 presented in [1].

Remark 14.5 The case of the gap in the nonreal part of the spectrum was partially studied in [2], where a strongly dissipative wave equation (i.e., $\gamma_w = 0$) was considered.

14.4 Proof of Theorem 14.3

Let us decompose the entire phase space $\mathscr{H}$ in direct sum of spaces that are pairwise orthogonal, $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \ldots \oplus \mathscr{H}_{k_2} \oplus \mathscr{H}_\infty$, where every subspace $\mathscr{H}_k$, $k = 1, \ldots, k_2$, is two-dimensional and corresponds to the eigenvector e_k with respect to u and p, and $\mathscr{H}_\infty = (\mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \ldots \oplus \mathscr{H}_{k_2})^\perp$ is the subspace of codimension $2k_2$ which corresponds to the eigenvectors $e_{k_2+1}, e_{k_2+2}, \ldots$ of the Laplace operator. Note that the spaces $\mathscr{H}_k$, $k = 1, \ldots, k_2$, and $\mathscr{H}_\infty$ are invariant with respect to the action of the linear operator $\mathbf{A}$.

The new inner product $[\cdot, \cdot]$ introduced below preserves the condition that the spaces $\mathscr{H}_k, k = 1, \ldots, k_2, \infty$, are pairwise orthogonal and modifies the inner product in each of these subspaces. Thus, if $y = (u, p) \in \mathscr{H}$ and the orthogonal projections of y to $\mathscr{H}_k$ are denoted by $y_k = (u_k e_k, p_k e_k) \in \mathscr{H}_k, k = 1, \ldots, k_2, \infty$, then the new norm in $\mathscr{H}$ is defined by the formula

$$|||y|||^2 = \sum_{k=1}^{k_2} |||y_k|||_k^2 + |||y_\infty|||_\infty^2.$$

14.4.1 New Norm in the Spaces $\mathscr{H}_k$, $k = 1, \ldots, k_1$

By definition the number k_1, for $k = 1, \ldots, k_1$ the eigenvalues μ_k and ν_k are real and lie to the different sides of the strip $\{m < \Re\zeta < M\}$. We introduce the new inner product in such a way that the eigenvectors ξ_k and η_k, which correspond to the eigenvalues μ_k and ν_k, are orthogonal with respect to this inner product.

Define a new inner product $[\cdot, \cdot]_k$ of vectors $y = (u, p)$, $\tilde{y} = (\tilde{u}, \tilde{p})$, $y, \tilde{y} \in \mathscr{H}_k$ by the rule

$$[y, \tilde{y}]_k = (2\gamma_k^2 - \lambda_k)(u, \tilde{u}) + \gamma_k(u, \tilde{p}) + \gamma_k(p, \tilde{u}) + (p, \tilde{p}).$$

The following assertions hold.

Lemma 14.1 *The eigenvectors ξ_k and η_k corresponding to the eigenvalues μ_k and ν_k, are orthogonal with respect to the new inner product.*

Proof The eigenvectors of the matrix A_k in the space $\mathscr{H}_k$ are the vectors $\xi_k = (1, -\mu_k)$ and $\eta_k = (1, -\nu_k)$. It follows from $\mu_k + \nu_k = 2\gamma_k$ and $\mu_k \nu_k = \lambda_k$ that

$$[\xi_k, \eta_k]_k = 2\gamma_k^2 - \lambda_k - \gamma_k(\mu_k + \nu_k) + \mu_k \nu_k = 0.$$

Since $\gamma_k^2 > \lambda_k$ for $k \le k_1$, it follows that the new inner product defines the norm

$$|||y|||_k^2 = [y, y]_k = (\gamma_k^2 - \lambda_k)\|u\|^2 + \|\gamma_k u + p\|^2.$$

Let us prove that

Lemma 14.2 *The minimum of the function* $\kappa_1(\gamma) = \gamma^2 - \lambda(\gamma)$, *where* $\lambda(\gamma) = \dfrac{\gamma - \gamma_w}{\gamma_s}$, *on the interval* $\gamma \in [\gamma_1, \gamma_{k_1}]$ *is achieved at the point* $\gamma = \gamma_{k_1}$.

Proof Let us show that the derivative of $\kappa_1(\gamma)$ is negative on the interval $\gamma \in [\gamma_1, \gamma_{k_1}]$. Indeed, by definition of the number k_1 we get $\gamma < \gamma_{k_1} < 1/(2\gamma_s)$. Hence for $\gamma < \gamma_{k_1}$ we have

$$\gamma_s {\kappa_1}'_\gamma = 2\gamma\gamma_s - 1 < 0.$$

Thus, the function $\kappa_1(\gamma)$ decreases on the interval $\gamma \in [\gamma_1, \gamma_{k_1}]$, and its minimum is attained at $\gamma = \gamma_{k_1}$.

Since Lemma 14.2 the following estimate of the norm of the vector $y = y_1 + \cdots + y_{k_1}$, $y_k = (u_k, p_k) \in \mathscr{H}_k$, holds

$$|||y|||^2 = \sum_{k=1}^{k_1} |||y_k|||_k^2 \geq \sum_{k=1}^{k_1} (\gamma_k^2 - \lambda_k)\|u_k\|^2 \geq \min_{1 \leq k \leq k_1} \left\{\gamma_k^2 - \lambda_k\right\} \cdot \sum_{k=1}^{k_1} \|u_k\|^2 =$$
$$= (\gamma_{k_1}^2 - \lambda_{k_1})\|u\|^2 = \varkappa_I^2 \|u\|^2. \tag{14.12}$$

14.4.2 New Norm in the Spaces $\mathscr{H}_k$, $k = k_1 + 1, \ldots, k_2$

By definition the numbers k_1, k_2 for $k = k_1 + 1, \ldots, k_2$ the eigenvalues μ_k and ν_k belong to the domain $\{\Re\zeta < m\}$. In this section, we introduce the new inner product $[\cdot, \cdot]_k$ in the spaces $\mathscr{H}_k$, $k = k_1 + 1, \ldots, k_2$, in such a way that $[\mathbf{A}y, y]_k \leq m[y, y]_k$ for any vector $y \in \mathscr{H}_k$.

Define the new inner product $[\cdot, \cdot]_k$ of the vectors $y = (u, p)$, $\tilde{y} = (\tilde{u}, \tilde{p})$, $y, \tilde{y} \in \mathscr{H}_k$ by the rule

$$[y, \tilde{y}]_k = b_k(u, \tilde{u}) + \gamma_k(u, \tilde{p}) + \gamma_k(p, \tilde{u}) + (p, \tilde{p}),$$

where $b_k = \gamma_k^2 + s_k^2$ and the numbers s_k are defined in (14.10).

Define the auxiliary function

$$s(\gamma) = \sqrt{m^2 - 2\gamma m + \lambda(\gamma)} + m - \gamma,$$

where $\lambda(\gamma) = (\gamma - \gamma_w)/\gamma_s$. Then $s(\gamma_k) = s_k$. For $\gamma \in [\gamma_{k_1+1}, \gamma_{k_2}]$ the value $s(\gamma)$ is real. Actually, by the choice of k_1, k_2 we have $m \geq \Re\nu = \Re\left(\gamma + \sqrt{\gamma^2 - \lambda(\gamma)}\right)$ for $\gamma \in [\gamma_{k_1+1}, \gamma_{k_2}]$. Hence $m \geq \gamma$, $m^2 - 2\gamma m + \lambda(\gamma) \geq 0$.

Since the numbers s_k are real, we see that the inner product defines the norm

$$|||y|||_k^2 = [y, y]_k = s_k^2\|u\|^2 + \|\gamma_k u + p\|^2.$$

The following assertions hold.

Lemma 14.3 *For any vector* $y = (u, p) \in \mathscr{H}_k$, $[\mathbf{A}y, y]_k \le m[y, y]$.

Proof Since $\gamma_k = \gamma_w + \gamma_s\lambda_k$, we see that $\mathbf{A}y = (-p, \lambda_k u + 2\gamma_k p)$ and

$$\begin{aligned}[\mathbf{A}y, y]_k &= -b_k(p, u) - \gamma_k(p, p) + \gamma_k(\lambda_k u + 2\gamma_k p, u) + (\lambda_k u + 2\gamma_k p, p) = \\ &= \gamma_k\lambda_k\|u\|^2 + (2\gamma_k^2 - b_k + \lambda_k)(u, p) + \gamma_k\|p\|^2.\end{aligned}$$

Then

$$\begin{aligned}[\mathbf{A}y, y]_k - m[y, y]_k = (\gamma_k\lambda_k - mb_k)\|u\|^2 + \\ + (2\gamma_k^2 - b_k + \lambda_k - 2m\gamma_k)(u, p) + (\gamma_k - m)\|p\|^2.\end{aligned}$$

Simple monomorphisms can show that the determinant of the last quadratic form is equal to

$$\begin{aligned}D &= (2\gamma_k^2 - b_k + \lambda_k - 2m\gamma_k)^2 - 4(\gamma_k\lambda_k - mb_k)(\gamma_k - m) = \\ &= (b_k - \lambda_k - 2(m - \gamma_k)^2)^2 - 4(\gamma_k - m)^2(m^2 - 2\gamma_k m + \lambda_k).\end{aligned}$$

The reader will easily prove that

$$b_k - 2m^2 + 2\gamma_k(2m - \gamma_k) - \lambda_k = 2(m - \gamma_k)\sqrt{m^2 - 2\gamma_k m + \lambda_k}.$$

Thus $D = 0$. Moreover, since $\gamma_k - m \le 0$ then the quadratic form $[\mathbf{A}y, y]_k - m[y, y]_k$ is confluent and nonpositive. This completes the proof of the lemma.

Let us show that $\min_{k_1+1\le k\le k_2}\{s_k\} = \min\{s_{k_1+1}, s_{k_2}\}$.

Lemma 14.4 *The minimum of the function* $s(\gamma)$ *on the closed interval* $I = [\gamma_{k_1+1}, \gamma_{k_2}]$ *is attained at the ends of the closed interval.*

Proof The derivative of $s(\gamma)$ is given by

$$s'_\gamma = \frac{-2\gamma_s m + 1}{2\gamma_s\sqrt{m^2 - 2\gamma m + \lambda(\gamma)}} - 1.$$

Since $2\gamma_s m < 1$ then s'_γ has the same sign as the following expression

$$\begin{aligned}(1 - 2\gamma_s m)^2 - 4\gamma_s^2(m^2 - 2\gamma m + \lambda(\gamma)) = 1 - 4\gamma_s m + 4\gamma_s^2 m^2 - \\ -4\gamma_s^2(m^2 - 2\gamma m) - 4\gamma_s(\gamma - \gamma_w) = 1 - 4\gamma_s m + 4\gamma_s\gamma_w + 4\gamma_s(2\gamma_s m - 1)\gamma.\end{aligned}$$

The last expression is linear with respect to γ and the leading coefficient is negative. Hence, s'_γ may have only one root on the interval I and this root corresponds to the maximum of $s(\gamma)$. We get that the minimum of s is attained at the ends of the closed interval.

By Lemma 14.4 the minimum of s_k for $k_1 + 1 \le k \le k_2$ is achieved either at $k = k_1 + 1$ or at $k = k_2$. This implies the following estimate of the norm of vector $y = y_{k_1+1} + \ldots + y_{k_2}$, $y_k \in \mathscr{H}_k$,

$$\|\!|y|\!\|^2 = \sum_{k=k_1+1}^{k_2} \|\!|y_k|\!\|_k^2 \ge \sum_{k=k_1+1}^{k_2} s_k^2 \|u_k\| \ge \min_{k_1+1 \le k \le k_2} \left\{ s_k^2 \right\} \sum_{k=k_1+1}^{k_2} \|u_k\|^2 \ge$$
$$\ge \min\{s_{k_1+1}^2, s_{k_2}^2\} \|u\|^2 = \min\{\varkappa_{II}^2, \varkappa_{III}^2\} \|u\|^2. \quad (14.13)$$

14.4.3 New Norm in the Space $\mathscr{H}_\infty$

The space $\mathscr{H}_\infty$ is infinitely-dimensional. We introduce the new inner product $[\cdot, \cdot]_\infty$, which is equivalent to the standard one, in such a way that for any vector $y \in \mathscr{H}_\infty$, $[\mathbf{A}y, y]_\infty \ge M[y, y]_\infty$.

Define the inner product of vectors $y = (u, p) \in \mathscr{H}_\infty$, $\tilde{y} = (\tilde{u}, \tilde{p}) \in \mathscr{H}_\infty$, by the rule

$$[y, \tilde{y}]_\infty = (1 - 2M\gamma_s)(\nabla u, \nabla \tilde{u}) + 2M\gamma_s \lambda_M (u, \tilde{u}) + M(u, \tilde{p}) + M(p, \tilde{u}) + (p, \tilde{p}),$$

where $\lambda_M = \frac{M - \gamma_w}{\gamma_s}$. By (14.9) we have $\lambda_M > M^2$. Moreover, for any vector $y = (u, p) \in \mathscr{H}_\infty$,

$$\|\nabla u\|^2 \ge \lambda_{k_2+1} \|u\|^2 = \frac{\gamma_{k_2+1} - \gamma_w}{\gamma_s} \|u\|^2 \ge \lambda_M \|u\|^2. \quad (14.14)$$

Corresponding norm is defined by the formula

$$\|\!|y|\!\|_\infty^2 = (1 - 2M\gamma_s)\|\nabla u\|^2 + M(2\gamma_s \lambda_M - M)\|u\|^2 + \|Mu + p\|^2.$$

Lemma 14.5 *The norms $\|\!|y|\!\|_\infty$ and $\|y\|_H$ are equivalent on the space H_∞.*

Proof Since $2\gamma_s M < 1$ and

$$\|\!|y|\!\|_\infty^2 \le (1 - 2M\gamma_s)\|\nabla u\|^2 + M(2\gamma_s \lambda_M - M)\|u\|^2 + (M\|u\| + \|p\|)^2,$$

it follows that the quantity $\|\!|y|\!\|_\infty^2$ is bounded above by a quantity depending on $\|\nabla u\|^2$ and $\|p\|^2$.

Let us find a lower bound for $\|\!|y|\!\|_\infty^2$. For some $\varepsilon > 0$, we have $\lambda_M(1 - \varepsilon) > M^2$. With regard to (14.14), we have

$$\|y\|_\infty^2 = (1-2M\gamma_s)\|\nabla u\|^2 + (2M\gamma_s\lambda_M - M^2)\|u\|^2 + \|Mu+p\|^2 \geq$$
$$\geq \varepsilon\|\nabla u\|^2 + (1-2M\gamma_s-\varepsilon)\|\nabla u\|^2 + \lambda_M(2M\gamma_s - 1 + \varepsilon)\|u\|^2 +$$
$$+ \|Mu+p\|^2 \geq \varepsilon\|\nabla u\|^2 + \|Mu+p\|^2 \geq \frac{\varepsilon}{2}\|\nabla + u\|^2 + \frac{\varepsilon\lambda_M}{2}\|u\|^2 +$$
$$+ (M\|u\| - \|p\|)^2.$$

The expression on the right-hand side is a positive-defined quadratic form in $\|\nabla u\|$, $\|u\|$ and $\|p\|$, which can be estimated below by multiple of $\|\nabla u\|^2 + \|p\|^2$.

Lemma 14.6 *For any vector* $y = (u, p) \in \mathscr{H}_\infty$,

$$\|y\|_\infty \geq \sqrt{\lambda_M - M^2}\|u\| = \varkappa_{IV}\|u\|. \tag{14.15}$$

Proof By (14.14) we have

$$\|y\|_\infty^2 \geq ((1-2M\gamma_s)\lambda_M + 2M\gamma_s\lambda_M)\|u\|^2 - 2M\|u\|\|p\| + \|p\|^2 =$$
$$= (\lambda_M - M^2)\|u\|^2 + (M\|u\| - \|p\|)^2 \geq (\lambda_M - M^2)\|u\|^2.$$

Lemma 14.7 *For any vector* $y = (u, p) \in \mathscr{H}_\infty \cap \mathscr{D}(\mathbf{A})$, $[\mathbf{A}y, y]_\infty \geq M[y, y]_\infty$.

Proof With regard to $M = \gamma_w + \gamma_s\lambda_M$, we have

$$[y, y]_\infty = (1-2M\gamma_s)\|\nabla u\|^2 + 2M(M-\gamma_w)\|u\|^2 + 2M(u, p) + \|p\|^2;$$
$$\mathbf{A}y = (-p, -\Delta u + 2\gamma_w p - 2\gamma_s\Delta p);$$

$$[\mathbf{A}y, y]_\infty = -(1-2M\gamma_s)(\nabla p, \nabla u) + 2M(M-\gamma_w)(-p, u) + M(-p, p) +$$
$$+ (-\Delta u + 2\gamma_w p - 2\gamma_s\Delta p, Mu + p) = M\|\nabla u\|^2 + 4M\gamma_s(\nabla p, \nabla u) +$$
$$+ 2\gamma_s\|\nabla p\|^2 + 2M(2\gamma_w - M)(u, p) + (2\gamma_w - M)\|p\|^2.$$

It follows that

$$[\mathbf{A}y, y]_\infty - M[y, y]_\infty = 2M^2\gamma_s\|\nabla u\|^2 + 4M\gamma_s(\nabla p, \nabla u) + 2\gamma_s\|\nabla p\|^2 -$$
$$- 2M^2(M-\gamma_w)\|u\|^2 + 2M(2\gamma_w - 2M)(u, p) +$$
$$+ (2\gamma_w - 2M)\|p\|^2 =$$
$$= 2\gamma_s\|M\nabla u + \nabla p\|^2 - 2\gamma_s\lambda_M\|Mu + p\|^2.$$

The last expression is nonnegative by (14.14).

14.4.4 End of the Proof of Theorem 14.3

Denote $\mathscr{H}^\eta = \langle \eta_1 e_1, \dots, \eta_{k_1} e_{k_1} \rangle$, $\mathscr{H}^\xi = \langle \xi_1 e_1, \dots, \xi_{k_1} e_{k_1} \rangle$, $\mathscr{H}^I = \mathscr{H}^\eta \oplus \mathscr{H}_{k_1+1} \oplus \dots \mathscr{H}_{k_2}$, $\mathscr{H}^{II} = \mathscr{H}^\xi \oplus \mathscr{H}_\infty$. The spaces $\mathscr{H}^I$ and $\mathscr{H}^{II}$ are orthogonal to each other with respect to the new inner product.

Since $\mathbf{A}(\xi_k e_k) = \mu_k(\xi_k e_k)$, $\mathbf{A}(\eta_k e_k) = \nu_k(\eta_k e_k)$ for $k = 1, \dots, k_1$, it follows that

$$[\mathbf{A}y, y] \le \max_{1 \le k \le k_1} \mu_k \cdot [y, y] = \mu_{k_1}[y, y] \qquad \forall y \in \mathscr{H}^\xi, \tag{14.16}$$

$$[\mathbf{A}y, y] \ge \min_{1 \le k \le k_1} \nu_k \cdot [y, y] = \nu_{k_1}[y, y] \qquad \forall y \in \mathscr{H}^\eta. \tag{14.17}$$

It follows from condition (14.16), Lemma 14.3, and the inequality $m > \mu_{k_1}$ that

$$[\mathbf{A}y, y] \le m[y, y] \qquad \forall y \in \mathscr{H}^I. \tag{14.18}$$

Also, condition (14.17), Lemma 14.7, and the inequality $M < \nu_{k_1}$ imply that

$$[\mathbf{A}y, y] \ge M[y, y] \qquad \forall y \in \mathscr{H}^{II} \cap \mathscr{D}(\mathbf{A}). \tag{14.19}$$

Since the vector $F(y)$ has zero u-component, it follows that

$$|||F(y_1) - F(y_2)||| = \|F(y_1) - F(y_2)\|_{\mathscr{H}} = \|f(u_1) - f(u_2)\| \le l\|u_1 - u_2\|. \tag{14.20}$$

By estimates (14.12), (14.13), (14.15) of the vector $y = y_1 - y_2 = y_1 + \dots + y_{k_2} + y_\infty$, $y_k \in \mathscr{H}_k$, $y_\infty \in \mathscr{H}_\infty$, we obtain

$$|||y|||^2 = \sum_{k=1}^{k_1} |||y_k|||_k^2 + \sum_{k=k_1+1}^{k_2} |||y_k|||_k^2 + |||y_\infty|||_\infty^2 \ge \min\{\varkappa_I^2, \varkappa_{II}^2, \varkappa_{III}^2, \varkappa_{IV}^2\}\|u\|^2. \tag{14.21}$$

It follows from inequalities (14.20) and (14.21) that

$$|||F(y_1) - F(y_2)||| \le l\|u_1 - u_2\| \le \frac{l\,|||y_1 - y_2|||}{\min\{\varkappa_I, \varkappa_{II}, \varkappa_{III}, \varkappa_{IV}\}}.$$

Thus the global Lipschitz constant L for the function $F(y)$ is equal to

$$L = \frac{l}{\min\{\varkappa_I, \varkappa_{II}, \varkappa_{III}, \varkappa_{IV}\}}.$$

Let us define the orthogonal projection to the $(2k_2 - k_1)$-dimensional space $\mathscr{H}^I = P(\mathscr{H})$ and denote it by P and define the orthogonal projection $Q = \mathrm{Id} - P$ to $\mathscr{H}^{II} \oplus \mathscr{H}_\infty = Q(\mathscr{H})$. Then the inequalities (14.18) and (14.19) acquire the form (14.3), and the spectral gap condition (14.4) is equivalent to condition (14.11).

Thus, all conditions of Theorem 14.1 are satisfied, and thus the space $\mathscr{H}$ contains an integral manifold which dimension is equal to that of the subspace $\mathscr{H}^I$, i.e., to $2k_2 - k_1$. This completes the proof of the theorem.

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