Chapter 13
Topological Properties of Strong Solutions for the 3D Navier-Stokes Equations

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Abstract In this chapter we give a criterion for the existence of global strong solutions for the 3D Navier-Stokes system for any regular initial data.

13.1 Introduction

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded open set with sufficiently smooth boundary $\partial\Omega$ and $0 < T < +\infty$. We consider the incompressible Navier-Stokes equations

$$\begin{cases} y_t + (y \cdot \nabla)y = \nu \triangle y - \nabla p + f \text{ in } Q = \Omega \times (0, T), \\ \operatorname{div} y = 0 \text{ in } Q, \\ y = 0 \text{ on } \partial\Omega \times (0, T), \qquad y(x, 0) = y_0(x) \text{ in } \Omega, \end{cases} \tag{13.1}$$

where $\nu > 0$ is a constant. We define the usual function spaces

$$\mathscr{V} = \{u \in (C_0^\infty(\Omega))^3 : \operatorname{div} u = 0\},$$

$$H = \text{closure of } \mathscr{V} \text{ in } (L^2(\Omega))^3, \quad V = \{u \in (H_0^1(\Omega))^3 : \operatorname{div} u = 0\}.$$

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We denote by V^* the dual space of V. The spaces H and V are separable Hilbert spaces and $V \subset H \subset V^*$ with dense and compact embedding when H is identified with its dual H^*. Let $(\cdot,\cdot)$, $\|\cdot\|_H$ and $((\cdot,\cdot))$, $\|\cdot\|_V$ be the inner product and the norm in H and V, respectively, and let $\langle\cdot,\cdot\rangle$ be the pairing between V and V^*. For $u, v, w \in V$, the equality

$$b(u,v,w) = \int\limits_{\Omega} \sum_{i,j=1}^{3} u_i \frac{\partial v_j}{\partial x_i} w_j dx$$

defines a trilinear continuous form on V with $b(u,v,v) = 0$ when $u \in V$ and $v \in (H_0^1(\Omega))^3$. For $u, v \in V$, let $B(u,v)$ be the element of V^* defined by $\langle B(u,v), w\rangle = b(u,v,w)$ for all $w \in V$.

We say that the function y is a *weak solution* of Problem (13.1) on $[0,T]$, if $y \in L^\infty(0,T;H) \cap L^2(0,T;V)$, $\frac{dy}{dt} \in L^1(0,T;V^*)$, if

$$\frac{d}{dt}(y,v) + \nu((y,v)) + b(y,y,v) = \langle f, v\rangle \quad \text{for all } v \in V, \tag{13.2}$$

in the sense of distributions on $(0,T)$, and if y satisfies the energy inequality

$$V(y)(t) \le V(y)(s) \quad \text{for all } t \in [s,T], \tag{13.3}$$

for a.e. $s \in (0,T)$ and for $s = 0$, where

$$V(y)(t) := \frac{1}{2}\|y(t)\|_H^2 + \nu \int\limits_0^t \|y(\tau)\|_V^2 d\tau - \int\limits_0^t \langle f(\tau), y(\tau)\rangle d\tau. \tag{13.4}$$

This class of solutions is called Leray–Hopf or physical one. If $f \in L^2(0,T;V^*)$, and if y satisfies (13.2), then $y \in C([0,T];H_w)$, $\frac{dy}{dt} \in L^{\frac{4}{3}}(0,T;V^*)$, where H_w denotes the space H endowed with the weak topology. In particular, the initial condition $y(0) = y_0$ makes sense for any $y_0 \in H$.

Let $A : V \to V^*$ be the linear operator associated to the bilinear form $((u,v)) = \langle Au, v\rangle$. Then A is an isomorphism from $D(A)$ onto H with $D(A) = (H^2(\Omega))^3 \cap V$. We recall that the embedding $D(A) \subset V$ is dense and continuous. Moreover, we assume $\|Au\|_H$ as the norm on $D(A)$, which is equivalent to the one induced by $(H^2(\Omega))^3$. The Problem (13.1) can be rewritten as

$$\begin{cases} \frac{dy}{dt} + \nu Ay + B(y,y) = f \text{ in } V^*, \\ y(0) = y_0, \end{cases} \tag{13.5}$$

where the first equation we understand in the sense of distributions on $(0,T)$. Now we write

$\mathscr{D}(y_0, f) = \{\, y :\ y$ is a weak solution of Problem (13.1) on $[0, T]\}$.

It is well known (cf. [1]) that if $f \in L^2(0, T; V^*)$, and if $y_0 \in H$, then $\mathscr{D}(y_0, f)$ is not empty.

A weak solution y of Problem (13.1) on $[0, T]$ is called a *strong* one, if it additionally belongs to Serrin's class $L^8(0, T; (L^4(\Omega))^3)$. We note that any strong solution y of Problem (13.1) on $[0, T]$ belongs to $C([0, T]; V) \cap L^2(0, T; D(A))$ and $\frac{dy}{dt} \in L^2(0, T; H)$ (cf. [2, Theorem 1.8.1, p. 296] and references therein).

For any $f \in L^\infty(0, T; H)$ and $y_0 \in V$ it is well known the only local existence of strong solutions for the 3D Navier-Stokes equations (cf. [1–4] and references therein). Here we provide a criterion for existence of strong solutions for Problem (13.1) on $[0, T]$ for any initial data $y_0 \in V$ and $0 < T < +\infty$. Presented results were announced in [5].

13.2 Topological Properties of Strong Solutions

The main result of this note has the following form.

Theorem 13.1 *Let $f \in L^2(0, T; H)$ and $y_0 \in V$. Then either for any $\lambda \in [0, 1]$ there is an $y_\lambda \in C([0, T]; V) \cap L^2(0, T; D(A))$ such that $y_\lambda \in \mathscr{D}(\lambda y_0, \lambda f)$, or the set*

$$\{y \in C([0, T]; V) \cap L^2(0, T; D(A)) \ :\ y \in \mathscr{D}(\lambda y_0, \lambda f), \lambda \in (0, 1)\} \tag{13.6}$$

is unbounded in $L^8(0, T; (L^4(\Omega))^3)$.

In the proof of Theorem 13.1 we use an auxiliary statement connected with continuity property of strong solutions on parameters of Problem (13.1) in Serrin's class $L^8(0, T; (L^4(\Omega))^3)$.

Theorem 13.2 *Let $f \in L^2(0, T; H)$ and $y_0 \in V$. If y is a strong solution for Problem (13.1) on $[0, T]$, then there exist $L, \delta > 0$ such that for any $z_0 \in V$ and $g \in L^2(0, T; H)$, satisfying the inequality*

$$\|z_0 - y_0\|_V^2 + \|g - f\|_{L^2(0,T;H)}^2 < \delta, \tag{13.7}$$

the set $\mathscr{D}(z_0, g)$ is one-point set $\{z\}$ which belongs to $C([0, T]; V) \cap L^2(0, T; D(A))$, and

$$\|z - y\|_{C([0,T];V)}^2 + \frac{\nu}{4}\|z - y\|_{D(A)}^2 \le L\left(\|z_0 - y_0\|_V^2 + \|g - f\|_{L^2(0,T;H)}^2\right). \tag{13.8}$$

Remark 13.1 We note that from Theorem 13.2 with $z_0 \in V$ and $g \in L^2(0, T; H)$ with $\|z_0\|_V^2 + \|g\|_{L^2(0,T;H)}^2$ sufficiently small, Problem (13.1) has only one global strong solution.

Remark 13.2 Theorem 13.2 provides that, if for any $\lambda \in [0,1]$ there is an $y_\lambda \in L^8(0,T;(L^4(\Omega))^3)$ such that $y_\lambda \in \mathscr{D}(\lambda y_0, \lambda f)$, then the set

$$\{y \in C([0,T];V) \cap L^2(0,T;D(A)) \,:\, y \in \mathscr{D}(\lambda y_0, \lambda f), \lambda \in (0,1)\}$$

is bounded in $L^8(0,T;(L^4(\Omega))^3)$.

If Ω is a C^∞-domain and if $f \in C_0^\infty(\overline{(0,T)\times\Omega})^3$, then any strong solution y of Problem (13.1) on $[0,T]$ belongs to $C^\infty((0,T]\times\Omega)^3$ and $p \in C^\infty((0,T]\times\Omega)$ (cf. [2, Theorem 1.8.2, p. 300] and references therein). This fact directly provides the next corollary of Theorems 13.1 and 13.2.

Corollary 13.1 *Let Ω be a C^∞-domain, $f \in C_0^\infty(\overline{(0,T)\times\Omega})^3$. Then either for any $y_0 \in V$ there is a strong solution of Problem* (13.1) *on* $[0,T]$*, or the set*

$$\{y \in C^\infty((0,T]\times\Omega)^3 \,:\, y \in \mathscr{D}(\lambda y_0, \lambda f), \lambda \in (0,1)\}$$

is unbounded in $L^8(0,T;(L^4(\Omega))^3)$ for some $y_0 \in C_0^\infty(\Omega)^3$.

13.3 Proof of Theorem 13.2

Let $f \in L^2(0,T;H)$, $y_0 \in V$, and $y \in C([0,T];V) \cap L^2(0,T;D(A))$ be a strong solution of Problem (13.1) on $[0,T]$. Due to [6], [1, Chap. 3] the set $\mathscr{D}(y_0,f) = \{y\}$. Let us now fix $z_0 \in V$ and $g \in L^2(0,T;H)$ satisfying (13.7) with

$$\delta = \min\left\{1; \frac{\nu}{4}\right\} e^{-2TC}, \; C = \max\left\{\frac{27c^4}{2\nu^3}; \frac{7^7c^8}{2^9\nu^7}\right\}\left(\|y\|^4_{C([0,T];V)} + 1\right)^2, \quad (13.9)$$

$c > 0$ is a constant from the inequalities (cf. [2, 1])

$$|b(u,v,w)| \le c\|u\|_V \|v\|_V^{\frac{1}{2}} \|v\|_{D(A)}^{\frac{1}{2}} \|w\|_H \quad \forall u \in V,\; v \in D(A),\; w \in H; \qquad (13.10)$$

$$|b(u,v,w)| \le c\|u\|_{D(A)}^{\frac{3}{4}} \|u\|_V^{\frac{1}{4}} \|v\|_V \|w\|_H \quad \forall u \in D(A),\; v \in V,\; w \in H. \qquad (13.11)$$

The auxiliary Problem

$$\begin{cases} \frac{d\eta}{dt} + \nu A\eta + B(\eta,\eta) + B(y,\eta) + B(\eta,y) = g - f \text{ in } V^*, \\ \eta(0) = z_0 - y_0, \end{cases} \qquad (13.12)$$

has a strong solution $\eta \in C([0,T];V) \cap L^2(0,T;D(A))$ with $\frac{d\eta}{dt} \in L^2(0,T;H)$, i.e.

$$\frac{d}{dt}(\eta, v) + \nu((\eta, v)) + b(\eta, \eta, v) + b(y, \eta, v) + b(\eta, y, v) = \langle g - f, v\rangle \quad \text{for all } v \in V,$$

in the sense of distributions on $(0, T)$. In fact, let $\{w_j\}_{j\geq 1} \subset D(A)$ be a special basis (cf. [7, p. 56]), i.e. $Aw_j = \lambda_j w_j$, $j = 1, 2, ..., 0 < \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_j \to +\infty$, $j \to +\infty$. We consider Galerkin approximations $\eta_m : [0, T] \to \text{span}\{w_j\}_{j=1}^m$ for solutions of Problem (13.12) satisfying

$$\frac{d}{dt}(\eta_m, w_j) + \nu((\eta_m, w_j)) + b(\eta_m, \eta_m, w_j) + b(y, \eta_m, w_j) + b(\eta_m, y, w_j) = \langle g - f, w_j\rangle,$$

with $(\eta_m(0), w_j) = (z_0 - y_0, w_j)$, $j = \overline{1, m}$. Due to (13.10), (13.11) and Young's inequality we get

$$2\langle g - f, A\eta_m\rangle \leq 2\|g - f\|_H \|\eta_m\|_{D(A)} \leq \frac{\nu}{4}\|\eta_m\|^2_{D(A)} + \frac{4}{\nu}\|f - g\|^2_H;$$

$$-2b(\eta_m, \eta_m, A\eta_m) \leq 2c\|\eta_m\|_V^{\frac{3}{2}}\|\eta_m\|_{D(A)}^{\frac{3}{2}} \leq \frac{\nu}{2}\|\eta_m\|^2_{D(A)} + \frac{27c^4}{2\nu^3}\|\eta_m\|^6_V;$$

$$-2b(y, \eta_m, A\eta_m) \leq 2c\|y\|_V\|\eta_m\|_V^{\frac{1}{2}}\|\eta_m\|_{D(A)}^{\frac{3}{2}} \leq \frac{\nu}{2}\|\eta_m\|^2_{D(A)} + \frac{27c^4}{2\nu^3}\|y\|^4_{C([0,T];V)}\|\eta_m\|^2_V;$$

$$-2b(\eta_m, y, A\eta_m) \leq 2c\|\eta_m\|_{D(A)}^{\frac{7}{4}}\|\eta_m\|_V^{\frac{1}{4}}\|y\|_V \leq \frac{\nu}{2}\|\eta_m\|^2_{D(A)} + \frac{7^7c^8}{2^9\nu^7}\|y\|^8_{C([0,T];V)}\|\eta_m\|^2_V.$$

Thus,

$$\frac{d}{dt}\|\eta_m\|^2_V + \frac{\nu}{4}\|\eta_m\|^2_{D(A)} \leq C(\|\eta_m\|^2_V + \|\eta_m\|^6_V) + \frac{4}{\nu}\|g - f\|^2_H,$$

where $C > 0$ is a constant from (13.9). Hence, the absolutely continuous function $\varphi = \min\{\|\eta_m\|^2_V, 1\}$ satisfies the inequality $\frac{d}{dt}\varphi \leq 2C\varphi + \frac{4}{\nu}\|g - f\|^2_H$, and therefore $\varphi \leq L(\|z_0 - y_0\|^2_V + \|g - f\|^2_{L^2(0,T;H)}) < 1$ on $[0, T]$, where $L = \delta^{-1}$. Thus, $\{\eta_n\}_{n\geq 1}$ is bounded in $L^\infty(0, T; V) \cap L^2(0, T; D(A))$ and $\{\frac{d}{dt}\eta_n\}_{n\geq 1}$ is bounded in $L^2(0, T; H)$. In a standard way we get that the limit function η of η_n, $n \to +\infty$, is a strong solution of Problem (13.12) on $[0, T]$. Due to [6], [1, Chap. 3] the set $\mathscr{D}(z_0, g)$ is one-point $z = y + \eta \in L^8(0, T; (L^4(\Omega))^3)$. So, z is strong solution of Problem (13.1) on $[0, T]$ satisfying (13.8).

The theorem is proved.

13.4 Proof of Theorem 13.1

We provide the proof of Theorem 13.1. Let $f \in L^2(0, T; H)$ and $y_0 \in V$. We consider the 3D controlled Navier-Stokes system (cf. [8, 9])

$$\begin{cases} \frac{dy}{dt} + \nu Ay + B(z, y) = f, \\ y(0) = y_0, \end{cases} \tag{13.13}$$

where $z \in L^8(0, T; (L^4(\Omega))^3)$.

By using standard Galerkin approximations (see [1]) it is easy to show that for any $z \in L^8(0, T; (L^4(\Omega))^3)$ there exists an unique weak solution $y \in L^\infty(0, T; H) \cap L^2(0, T; V)$ of Problem (13.13) on $[0, T]$, that is,

$$\frac{d}{dt}(y, v) + \nu((y, v)) + b(z, y, v) = \langle f, v \rangle, \text{ for all } v \in V, \tag{13.14}$$

in the sense of distributions on $(0, T)$. Moreover, by the inequality

$$|b(u, v, Av)| \le c_1 \|u\|_{(L^4(\Omega))^3} \|v\|_V^{\frac{1}{4}} \|v\|_{D(A)}^{\frac{7}{4}} \le \frac{\nu}{2}\|v\|^2_{D(A)} + c_2\|u\|^8_{(L^4(\Omega))^3}\|v\|^2_V, \tag{13.15}$$

for all $u \in (L^4(\Omega))^3$ and $v \in D(A)$, where $c_1, c_2 > 0$ are some constants that do not depend on u, v (cf. [1]), we find that $y \in C([0, T]; V) \cap L^2(0, T; D(A))$ and $B(z, y) \in L^2(0, T; H)$, so $\frac{dy}{dt} \in L^2(0, T; H)$ as well. We add that, for any $z \in L^8(0, T; (L^4(\Omega))^3)$ and corresponding weak solution $y \in C([0, T]; V) \cap L^2(0, T; D(A))$ of (13.13) on $[0, T]$, by using Gronwall inequality, we obtain

$$\begin{aligned} \|y(t)\|_V^2 &\le \|y_0\|_V^2 e^{2c_2 \int_0^t \|z(t)\|^8_{(L^4(\Omega))^3} dt}, \qquad \forall t \in [0, T]; \\ \nu \int_0^T \|y(t)\|^2_{D(A)} dt &\le \|y_0\|_V^2 \left[1 + 2c_2 e^{2c_2 \int_0^T \|z(t)\|^8_{(L^4(\Omega))^3} dt} \|z\|^8_{L^8(0,T;(L^4(\Omega))^3)} \right]. \end{aligned} \tag{13.16}$$

Let us consider the operator $F : L^8(0, T; (L^4(\Omega))^3) \to L^8(0, T; (L^4(\Omega))^3)$, where $F(z) \in C([0, T]; V) \cap L^2(0, T; D(A))$ is the unique weak solution of (13.13) on $[0, T]$ corresponded to $z \in L^8(0, T; (L^4(\Omega))^3)$.

Let us check that F is a compact transformation of Banach space $L^8(0, T; (L^4(\Omega))^3)$ into itself (cf. [10]). In fact, if $\{z_n\}_{n\ge 1}$ is a bounded sequence in $L^8(0, T; (L^4(\Omega))^3)$, then, due to (13.15) and (13.16), the respective weak solutions y_n, $n = 1, 2, ...,$ of Problem (13.13) on $[0, T]$ are uniformly bounded in $C([0, T]; V) \cap L^2(0, T; D(A))$ and their time derivatives $\frac{dy_n}{dt}$, $n = 1, 2, ...,$ are uniformly bounded in $L^2(0, T; H)$. So, $\{F(z_n)\}_{n\ge 1}$ is a precompact set in $L^8(0, T; (L^4(\Omega))^3)$. In a standard way we deduce that $F : L^8(0, T; (L^4(\Omega))^3) \to L^8(0, T; (L^4(\Omega))^3)$ is continuous mapping.

Since F is a compact transformation of $L^8(0, T; (L^4(\Omega))^3)$ into itself, Schaefer's Theorem (cf. [10, p. 133] and references therein) and Theorem 13.2 provide the statement of Theorem 13.1. We note that Theorem 13.2 implies that the set

$\{z \in L^8(0, T; (L^4(\Omega))^3) : z = \lambda F(z), \lambda \in (0, 1)\}$ is bounded in $L^8(0, T; (L^4(\Omega))^3)$ iff the set defined in (13.6) is bounded in $L^8(0, T; (L^4(\Omega))^3)$.

The theorem is proved.

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