Chapter 12
Structure of Uniform Global Attractor for General Non-Autonomous Reaction-Diffusion System

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Abstract In this paper we study structural properties of the uniform global attractor for non-autonomous reaction-diffusion system in which uniqueness of Cauchy problem is not guarantied. In the case of translation compact time-depended coefficients we prove that the uniform global attractor consists of bounded complete trajectories of corresponding multi-valued processes. Under additional sign conditions on non-linear term we also prove (and essentially use previous result) that the uniform global attractor is, in fact, bounded set in $L^{\infty}(\Omega) \cap H_0^1(\Omega)$.

12.1 Introduction

In this paper we study the structural properties of the uniform global attractor of non-autonomous reaction-diffusion system in which the nonlinear term satisfy suitable growth and dissipative conditions on the phase variable, suitable translation compact conditions on time variable, but there is no condition ensuring uniqueness of Cauchy problem. In autonomous case such system generates in the general case

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a multi-valued semiflow having a global compact attractor (see [1–5]). Also, it is known [1, 2, 6], that the attractor is the union of all bounded complete trajectories of the semiflow. Here we prove the same result for non-autonomous system. More precisely, we prove that the family of multi-valued processes, generated by weak solutions of reaction-diffusion system, has uniform global attractor which is union of bounded complete trajectories of corresponding processes. Using this result, we can prove that under additional restrictions on nonlinear term obtained uniform global attractor is bounded set in the space $L^\infty(\Omega) \cap H_0^1(\Omega)$.

12.2 Setting of the Problem

In a bounded domain $\Omega \subset \mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$ we consider the following non-autonomous parabolic problem (named RD-system) [7–17]

$$\begin{cases} u_t = a\Delta u - f(t,u) + h(t,x), \quad x \in \Omega,\ t > \tau, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{12.1}$$

where $\tau \in \mathbb{R}$ is initial moment of time, $u = u(t,x) = (u^1(t,x), ..., u^N(t,x))$ is unknown vector-function, $f = (f^1, ..., f^N)$, $h = (h^1, ..., h^N)$ are given functions, a is real $N \times N$ matrix with positive symmetric part $\frac{1}{2}(a + a^*) \geq \beta I$, $\beta > 0$,

$$h \in L^2_{loc}(\mathbb{R}; (L^2(\Omega))^N), \quad f \in C(\mathbb{R} \times \mathbb{R}^N; \mathbb{R}^N), \tag{12.2}$$

$\exists\, C_1, C_2 > 0, \quad \gamma_i > 0,\ p_i \geq 2,\ i = \overline{1,N}$ such that $\forall\, t \in \mathbb{R}\, \forall v \in \mathbb{R}^N$

$$\sum_{i=1}^N |f^i(t,v)|^{\frac{p_i}{p_i-1}} \leq C_1(1 + \sum_{i=1}^N |v^i|^{p_i}), \tag{12.3}$$

$$\sum_{i=1}^N f^i(t,v)v^i \geq \sum_{i=1}^N \gamma_i |v^i|^{p_i} - C_2. \tag{12.4}$$

In further arguments we will use standard functional spaces

$$H = (L^2(\Omega))^N \text{ with the norm } |v|^2 = \int_\Omega \sum_{i=1}^N |v^i(x)|^2 dx,$$

$$V = (H_0^1(\Omega))^N \text{ with the norm } \|v\|^2 = \int_\Omega \sum_{i=1}^N |\nabla v^i(x)|^2 dx.$$

Let us denote $V' = (H^{-1}(\Omega))^N$, $q_i = \frac{p_i}{p_i-1}$, $p = (p_1, ..., p_N)$, $q = (q_1, ..., q_N)$, $L^p(\Omega) = L^{p_1}(\Omega) \times \cdots \times L^{p_N}(\Omega)$.

Definition 12.1 The function $u = u(t,x) \in L^2_{loc}(\tau, +\infty; V) \bigcap L^p_{loc}(\tau, +\infty; L^p(\Omega))$ is called a (weak) solution of the problem (12.1) on $(\tau, +\infty)$ if for all $T > \tau$, $v \in V \cap L^p(\Omega)$

$$\frac{d}{dt}\int_\Omega u(t,x)v(x)dx + \int_\Omega \Big(a\nabla u(t,x)\nabla v(x) + f(t,u(t,x))v(x) - h(t,x)v(x)\Big)dx = 0 \tag{12.5}$$

in the sense of scalar distributions on (τ, T).

From (12.3) and Sobolev embedding theorem we see that every solution of (12.1) satisfies inclusion $u_t \in L^q_{loc}(\tau, +\infty; H^{-r}(\Omega))$, where $r = (r_1, ..., r_N)$, $r_k = max\{1, n(\frac{1}{2} - \frac{1}{p_k})\}$. The following theorem is well-known result about global resolvability of (12.1) for initial conditions from the phase space H.

Theorem 12.1 [18, Theorem 2] *or* [8, p.284]. *Under conditions* (12.3), (12.4) *for every* $\tau \in \mathbb{R}$, $u_\tau \in H$ *there exists at least one weak solution of* (12.1) *on* $(\tau, +\infty)$ *with* $u(\tau) = u_\tau$ *(and it may be non unique) and any weak solution of* (12.1) *belongs to* $C([\tau, +\infty); H)$. *Moreover, the function* $t \mapsto |u(t)|^2$ *is absolutely continuous and for a.a.* $t \geq \tau$ *the following energy equality holds*

$$\frac{1}{2}\frac{d}{dt}|u(t)|^2 + (a\nabla u(t), \nabla u(t)) + (f(t, u(t)), u(t)) = (h(t), u(t)). \tag{12.6}$$

Under additional not-restrictive conditions on function f and h it is known that solution of (12.1) generate non-autonomous dynamical system (two-parametric family of m-processes), which has uniform global attractor. The aim of this paper is to give description of the attractor in terms of bounded complete trajectories and show some regularity property of this set.

12.3 Multi-Valued Processes and Uniform Attractors

Let (X, ρ) be a complete metric space. The Hausdorff semidistance from A to B is given by

$$dist(A, B) = \sup_{x\in A}\inf_{y\in B} \rho(x, y),$$

By $\overline{A}$ and $O_\varepsilon(A) = \{x \in X \mid \inf_{y\in A} \rho(x, y) < \varepsilon\}$we denote closure and ε-neighborhood of the set A. Denote by $P(X)$ $(\beta(X), C(X), K(X))$ the set of all non-empty (not-empty bounded, not-empty closed, not-empty compact) subsets of X,

$$\mathbb{R}_d = \{(t, \tau) \in \mathbb{R}^2 | t \geq \tau\}.$$

Let Σ be some complete metric space $\{T(h): \Sigma \mapsto \Sigma\}_{h\geq 0}$ be a continuous semigroup, acting on Σ. Note, that in most applications $T(h)$ is shift semigroup.

Definition 12.2 Two-parameter family of multi-valued mappings $\{U_\sigma : \mathbb{R}_d \times X \mapsto P(X)\}_{\sigma\in\Sigma}$ is said to be the family of m-processes (family of MP), if $\forall\, \sigma \in \Sigma$, $\tau \in \mathbb{R}$:
(1) $U_\sigma(\tau, \tau, x) = x \; \forall x \in X$,
(2) $U_\sigma(t, \tau, x) \subseteq U_\sigma(t, s, U_\sigma(s, \tau, x)), \forall t \geq s \geq \tau \;\; \forall x \in X$,
(3) $U_\sigma(t+h, \tau+h, x) \subseteq U_{T(h)\sigma}(t, \tau, x) \; \forall t \geq \tau \;\; \forall h \geq 0, \; \forall\, x \in X$,
where for $A \subset X, B \subset \Sigma$ $U_B(t, s, A) = \bigcup_{\sigma\in B} \bigcup_{x\in A} U_\sigma(t, s, x)$, in particular

$$U_\Sigma(t, \tau, x) = \bigcup_{\sigma\in\Sigma} U_\sigma(t, \tau, x).$$

Family of MP $\{U_\sigma | \sigma \in \Sigma\}$ is called strict, if in conditions (2), (3) equality take place.

Definition 12.3 A set $A \subset X$ is called uniformly attracting for the family of MP $\{U_\sigma | \sigma \in \Sigma\}$, if for arbitrary $\tau \in \mathbb{R}, B \in \beta(X)$

$$dist(U_\Sigma(t, \tau, B), A) \to 0, \;\; t \to +\infty, \tag{12.7}$$

that is $\forall\varepsilon > 0$, $\tau \in \mathbb{R}$ and $B \in \beta(X)$ there exists $T = T(\tau, \varepsilon, B)$ such that

$$U_\Sigma(t, \tau, B) \subset O_\varepsilon(A) \forall\, t \geq T.$$

For fixed $B \subset X$ and $(s, \tau) \in \mathbb{R}_d$ let us define the following sets

$$\gamma^\tau_{s,\sigma}(B) = \bigcup_{t\geq s} U_\sigma(t, \tau, B), \gamma^\tau_{s,\Sigma}(B) = \bigcup_{t\geq s} U_\Sigma(t, \tau, B),$$

$$\omega_\Sigma(\tau, B) = \bigcap_{s\geq\tau} cl_X(\gamma^\tau_{s,\Sigma}(B)).$$

It is clear that $\omega_\Sigma(\tau, B) = \bigcap_{s\geq p} cl_X(\gamma^\tau_{s,\Sigma}(B)) \;\; \forall p \geq \tau$.

Definition 12.4 The family of MP $\{U_\sigma | \sigma \in \Sigma\}$ is called uniformly asymptotically compact, if for arbitrary $\tau \in \mathbb{R}$ and $B \in \beta(X)$ there exists $A(\tau, B) \in K(X)$ such that

$$U_\Sigma(t, \tau, B) \to A(\tau, B), \;\; t \to +\infty \;\; \text{in}\, X.$$

It is known [19] that if $\forall\tau \in \mathbb{R}, \;\forall B \in \beta(X) \;\exists T = T(\tau, B) \;\gamma^\tau_{T,\Sigma}(B) \in \beta(X)$, then the condition of uniformly asymptotically compactness is equivalent to the following one:

$$\forall\tau \in \mathbb{R} \;\forall B \in \beta(X) \;\forall t_n \nearrow \infty$$

$$\text{every sequence} \;\; \xi_n \in U_\Sigma(t_n, \tau, B) \; \text{is precompact.}$$

Definition 12.5 A set $\Theta_\Sigma \subset X$ is called uniform global attractor of the family of MP $\{U_\sigma | \sigma \in \Sigma\}$, if :

(1) Θ_Σ is uniformly attracting set;
(2) for every uniformly attracting set Y we have $\Theta_\Sigma \subset cl_X Y$.

Uniform global attractor $\Theta_\Sigma \subset X$ is called invariant (semiinvariant), if $\forall\ (t, \tau) \in \mathbb{R}_d$

$$\Theta_\Sigma = U_\Sigma(t, \tau, \Theta_\Sigma) \ \ (\Theta_\Sigma \subset U_\Sigma(t, \tau, \Theta_\Sigma)).$$

If Θ_Σ is compact, invariant uniform global attractor, then it is called stable if $\forall\ \varepsilon > 0 \exists\ \delta > 0 \forall\ (t, \tau) \in \mathbb{R}_d$

$$U_\Sigma(t, \tau, O_\delta(\Theta_\Sigma)) \subset O_\varepsilon(\Theta_\Sigma).$$

The following sufficient conditions we can obtain with slight modifications from [19].

Theorem 12.2 (*I*) *Let us assume that the family of MP* $\{U_\sigma | \sigma \in \Sigma\}$ *satisfies the following conditions:*

(1) $\exists\ B_0 \in \beta(X)\ \forall\ B \in \beta(X)\ \forall \tau \in \mathbb{R}\ \exists\ T = T(\tau, B)$

$$\forall\ t \geq T\ U_\Sigma(t, \tau, B) \subset B_0;$$

(2) $\{U_\sigma | \sigma \in \Sigma\}$ is uniformly asymptotically compact.
Then $\{U_\sigma\}_{\sigma\in\Sigma}$ has compact uniform global attractor

$$\Theta_\Sigma = \bigcup_{\tau\in\mathbb{R}} \bigcup_{B\in\beta(X)} \omega_\Sigma(\tau, B) = \omega_\Sigma(0, B_0) = \omega_\Sigma(\tau, B_0) \ \ \forall \tau \in \mathbb{R}. \tag{12.8}$$

(II) If $\{U_\sigma\}_{\sigma\in\Sigma}$ satisfy (1), (2), Σ is compact and $\forall t \geq \tau$ the mapping

$$(x, \sigma) \mapsto U_\sigma(t, \tau, x) \tag{12.9}$$

has closed graph , then Θ_Σ is semiinvariant.
If, moreover, $\forall h \geq 0\ T(h)\Sigma = \Sigma$ and the family MP $\{U_\sigma | \sigma \in \Sigma\}$ is strict, then Θ_Σ is invariant.

(III) If $\{U_\sigma\}_{\sigma\in\Sigma}$ satisfy (1), (2), Σ is connected and compact, $\forall t \geq \tau$ the mapping (12.9) is upper semicontinuous and has closed and connected values, B_0 is connected set, then Θ_Σ is connected set.

(IV) If $\{U_\sigma | \sigma \in \Sigma\}$ is strict, $T(h)\Sigma = \Sigma$ for any $h \geq 0$, there exists a compact, invariant uniform global attractor Θ_Σ and the following condition hold:

$$\text{if } y_n \in U_\Sigma(t_n, \tau, x_n),\ t_n \to t_0,\ x_n \to x_0,$$

$$\text{then up to subsequence } y_n \to \ y_0 \in U_\Sigma(t_0, \tau, x_0), \tag{12.10}$$

then Θ_Σ is stable.

Proof (I) From conditions (1), (2) due to [19] we have that $\forall \tau \in \mathbb{R}$ $\forall B \in \beta(X)$ $\omega_\Sigma(\tau, B) \neq \emptyset$, is compact , $\omega_\Sigma(\tau, B) \subset B_0$ and the set

$$\Theta_\Sigma = \bigcup_{\tau\in\mathbb{R}} \bigcup_{B\in\beta(X)} \omega_\Sigma(\tau, B)$$

is uniform global attractor. Let us prove that $\omega_\Sigma(\tau, B) \subset \omega_\Sigma(\tau_0, B_0)$ $\forall \tau, \tau_0 \in \mathbb{R}$.

$$U_\sigma(t, \tau, B) \subset U_\sigma(t, \frac{t}{2}, U_\sigma(\frac{t}{2}, \tau, B)) \subset U_{T(\frac{t}{2}-\tau_0)\sigma}(\frac{t}{2} + \tau_0, \tau_0, U_\sigma(\frac{t}{2}, \tau, B)) \subset$$

$$\subset U_\Sigma(\frac{t}{2} + \tau_0, \tau_0, B_0), \text{ if } \frac{t}{2} \geq T(\tau, B) + |\tau_0| + |\tau| := T.$$

So, for $t \geq 2T$

$$U_\Sigma(t, \tau, B) \subset U_\Sigma(\frac{t}{2} + \tau_0, \tau_0, B_0).$$

Then for $s \geq 2T$

$$\bigcup_{t\geq s} U_\Sigma(t, \tau, B) \subset \bigcup_{t\geq s} U_\Sigma(\frac{t}{2} + \tau_0, \tau_0, B_0) = \bigcup_{p\geq \frac{s}{2}+\tau_0} U_\Sigma(p, \tau_0, B_0),$$

$$\bigcap_{s\geq 2T} \overline{\bigcup_{t\geq s} U_\Sigma(t, \tau, B)} = \omega_\Sigma(\tau, B) \subset \bigcap_{s\geq 2T} \overline{\bigcup_{p\geq \frac{s}{2}+\tau_0} U_\Sigma(p, \tau_0, B_0)}$$

$$= \bigcap_{s'\geq T+\tau_0} \overline{\bigcup_{p\geq s'} U_\Sigma(p, \tau_0, B_0)} = \omega_\Sigma(\tau_0, B_0).$$

So we deduce equality (12.8).

(II) Due to (12.8) $\forall \xi \in \Theta_\Sigma = \omega_\Sigma(\tau, B_0)$ $\exists t_n \nearrow +\infty$, $\exists \sigma_n \in \Sigma$ $\exists \xi_n \in U_{\Sigma_n}(t_n, \tau, B_0)$ such that $\xi = \lim_{n\to\infty} \xi_n$. Then

$$\xi_n \in U_{\sigma_n}(t_n - t - \tau + t + \tau, \tau, B_0) \subset$$

$$\subset U_{\sigma_n}(t_n - t - \tau + t + \tau, t_n - t + \tau, U_{\sigma_n}(t_n - t + \tau, \tau, B_0))$$
$$\subset U_{T(t_n-t)\sigma_n}(t, \tau, \eta_n),$$

where $\eta_n \in U_{\sigma_n}(t_n - t + \tau, \tau, B_0)$, $t \geq \tau$ and for sufficiently large $n \geq 1$.

From uniform asymptotically compactness we have that on some subsequence $\eta_n \to \eta \in \omega_\Sigma(\tau, B_0) = \Theta_\Sigma$,

$$T(t_n - t)\sigma_n \to \sigma \in \Sigma.$$

Then from (12.9) we deduce :

$$\xi \in U_\Sigma(t, \tau, \Theta_\Sigma),$$

and therefore $\Theta_\Sigma \subset U_\Sigma(t, \tau, \Theta_\Sigma)$.
Other statements of the theorem are proved analogously to [19]. Theorem is proved.

Corollary 12.1 *If for the family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$ we have :*

(1) $\forall h \geq 0\; T(h)\Sigma = \Sigma$;
(2) $\forall (t, \tau) \in \mathbb{R}_d\; \forall h \geq 0\; \forall \sigma \in \Sigma\; \forall x \in X$

$$U_\sigma(t + h, \tau + h, x) = U_{T(h)\sigma}(t, \tau, x),$$

then all conditions of previous theorem can be verified only for $\tau = 0$.

Proof Under conditions (1), (2) $\forall t \geq \tau$ if $\tau \geq 0$ then

$$U_\sigma(t, \tau, x) = U_{T(\tau)\sigma}(t - \tau, 0, x),$$

and if $\tau \leq 0$ then $\exists \sigma' \in \Sigma : \sigma = T(-\tau)\sigma'$, so

$$U_\sigma(t, \tau, x) = U_{T(-\tau)\sigma'}(t, \tau, x) = U_{\sigma'}(t - \tau, 0, x).$$

In the single-valued case it is known [8], that the uniform global attractor consists of bounded complete trajectories of processes $\{U_\sigma\}_{\sigma\in\Sigma}$.

Definition 12.6 The mapping $\varphi : [\tau, +\infty) \mapsto X$ is called trajectory of MP U_σ, if $\forall t \geq s \geq \tau$

$$\varphi(t) \in U_\sigma(t, s, \varphi(s)). \tag{12.11}$$

If for $\varphi : \mathbb{R} \mapsto X$ the equality (12.11) takes place $\forall t \geq s$, then φ is called complete trajectory.

Now we assume that for arbitrary $\sigma \in \Sigma$ and $\tau \in \mathbb{R}$ we have the set K_σ^τ of mappings $\varphi : [\tau, +\infty) \mapsto X$ such that :

(a) $\forall x \in X\; \exists \varphi(\cdot) \in K_\sigma^\tau$ such, that $\varphi(\tau) = x$;
(b) $\forall \varphi(\cdot) \in K_\sigma^\tau\;\; \forall s \geq \tau\;\; \varphi(\cdot)|_{[s,+\infty)} \in K_\sigma^s$;
(c) $\forall h \geq 0\; \forall \varphi(\cdot) \in K_\sigma^{\tau+h}\;\; \varphi(\cdot + h) \in K_{T(h)\sigma}^\tau$.

Let us put

$$U_\sigma(t,\tau,x)=\{\varphi(t)|\varphi(\cdot)\in K_\sigma^\tau,\ \varphi(\tau)=x\}. \tag{12.12}$$

Lemma 12.1 *Formula (12.12) defines the family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$, and $\forall\varphi(\cdot)\in K_\sigma^\tau$*

$$\forall t\geq s\geq\tau\quad \varphi(t)\in U_\sigma(t,s,\varphi(s)). \tag{12.13}$$

Proof Let us check conditions of the Definition 12.2.

(1) $U_\sigma(\tau,\tau,x)=\varphi(\tau)=x$;
(2) $\forall\xi\in U_\sigma(t,\tau,x)\quad \xi=\varphi(t)$, where $\varphi\in K_\sigma^\tau,\varphi(\tau)=x$. Then for $s\in[\tau,t]$ $\varphi(s)\in U_\sigma(s,\tau,x)$ and from $\varphi|_{[s,+\infty)}\in K_\sigma^s$ we have $\varphi(t)\in U_\sigma(t,s,\varphi(s))$. So

$$\xi\in U_\sigma(t,s,U_\sigma(s,\tau,x)).$$

(3) $\forall\xi\in U_\sigma(t+h,\tau+h,x)$ $\xi=\varphi(t+h)$, where $\varphi\in K_\sigma^{\tau+h},\varphi(\tau+h)=x$. Then $\psi(\cdot)=\varphi(\cdot+h)\in K_{T(h)\sigma}^\tau$, $\psi(\tau)=x$, so $\xi=\psi(t)\in U_{T(h)\sigma}(t,\tau,x)$. Lemma is proved.

It is easy to show that under conditions (a)–(c), if $\forall s\geq\tau\ \forall\psi\in K_\sigma^\tau,\forall\varphi\in K_\sigma^s$ such that $\psi(s)=\varphi(s)$, we have

$$\theta(p)=\begin{cases}\psi(p),\ p\in[\tau,s]\\ \varphi(p),\ p>s,\end{cases}\in K_\sigma^\tau, \tag{12.14}$$

then in the condition (2) of Definition 12.2 equality takes place.

If $\forall h\geq 0\ \forall\varphi\in K_{T(h)\sigma}^\tau\varphi(\cdot-h)\in K_\sigma^{\tau+h}$, then in the condition (3) of Definition 12.2 equality takes place.

From (12.13) we immediately obtain that if for mapping $\varphi(\cdot):\mathbb{R}\mapsto X$ for arbitrary $\tau\in\mathbb{R}$ we have $\varphi(\cdot)|_{[\tau,+\infty)}\in K_\sigma^\tau$, then $\varphi(\cdot)$ is complete trajectory of U_σ.

The next result is generalization on non-autonomous case results from [20, 21].

Lemma 12.2 *Let the family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$ be constructed by the formula (12.12), $\forall\varphi(\cdot)\in K_\sigma^\tau$ is continuous on $[\tau,+\infty)$, the condition (12.14) takes place and the following one: if $\varphi_n(\cdot)\in K_\sigma^\tau,\ \ \varphi_n(\tau)=x$, then $\exists\varphi(\cdot)\in K_\sigma^\tau,\ \ \varphi(\tau)=x$ such that on some subsequence*

$$\varphi_n(t)\to\varphi(t)\ \ \forall t\geq\tau.$$

Then every continuous on $[\tau,+\infty)$ trajectory of MP U_σ belongs to K_σ^τ.

Proof Let $\psi:[\tau,+\infty)\mapsto X$ be continuous trajectory. Let us construct sequence $\{\varphi_n(\cdot)\}_{n=1}^\infty\subset K_\sigma^\tau$ such that

$$\varphi_n(\tau+j2^{-n})=\psi(\tau+j2^{-n}),\quad j=0,1,...,n2^n.$$

For $\varphi_1(\cdot)$ we have

$$\psi(\tau+\frac{1}{2}) \in U_\sigma(\tau+\frac{1}{2},\tau,\psi(\tau)),$$

$$\psi(\tau+1) \in U_\sigma(\tau+1,\tau+\frac{1}{2},\psi(\tau+\frac{1}{2})).$$

So there exists $\widetilde{\varphi}(\cdot) \in K_\sigma^\tau$, there exists $\widetilde{\widetilde{\varphi}}(\cdot) \in K_\sigma^{\tau+\frac{1}{2}}$ such that

$$\psi(\tau+\frac{1}{2}) = \widetilde{\varphi}(\tau+\frac{1}{2}),\ \ \widetilde{\varphi}(\tau) = \psi(\tau),$$

$$\psi(\tau+1) = \widetilde{\widetilde{\varphi}}(\tau+1),\ \ \widetilde{\widetilde{\varphi}}(\tau+\frac{1}{2}) = \psi(\tau+\frac{1}{2}).$$

Therefore due to (12.14) for function

$$\varphi_1(p) = \begin{cases} \widetilde{\varphi}(p), & \tau \le p \le \tau+\frac{1}{2}, \\ \widetilde{\widetilde{\varphi}}(p), & p > \tau+\frac{1}{2} \end{cases} \quad \text{we have:}$$

$$\varphi_1(\cdot) \in K_\sigma^\tau,\ \ \varphi_1(\tau) = \psi(\tau),\ \ \varphi_1(\tau+\frac{1}{2}) = \psi(\tau+\frac{1}{2}),\ \ \varphi_1(\tau+1) = \psi(\tau+1).$$

Further, using (12.14), we obtain required property for every $n \ge 1$. As $\varphi_n(\tau) = \psi(\tau)$, so $\exists \varphi(\cdot) \in K_\sigma^\tau$, $\varphi(\tau) = \psi(\tau)$ such that on subsequence $\forall t \ge \tau\ \varphi_n(t) \to \varphi(t)$. As $\forall t = \tau + j2^{-n}\ \ \varphi(t) = \psi(t)$, so from continuity $\varphi(t) = \psi(t)\ \ \forall t \ge \tau$. Lemma is proved.

The following theorem declare structure of uniform global attractor in terms of bounded complete trajectories of corresponding m-processes. It should be noted that this result is known for single-valued case [8] and in multi-valued case for very special class of strict processes, generated by strict compact semiprocesses, which act in Banach spaces [22].

Theorem 12.3 *Let Σ is compact, $T(h)\Sigma = \Sigma\ \forall\, h \ge 0$, the family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$ satisfies (12.12), in condition (3) of Definition 12.2 equality takes place, the mapping $(x,\sigma) \mapsto U_\sigma(t,0,x)$ has closed graph. Let us assume that there exists Θ_Σ—compact uniform global attractor of the family $\{U_\sigma\}_{\sigma\in\Sigma}$, and one of two conditions hold: either the family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$ is strict, or*

$$\textit{for every } \sigma_n \to \sigma_0,\ \ x_n \to x_0 \textit{ if } \varphi_n(\cdot) \in K_{\sigma_n}^0,\ \varphi_n(0) = x_n,$$

$$\textit{so } \exists\, \varphi(\cdot) \in K_{\sigma_0}^0,\ \varphi(0) = x_0 \textit{ such that on subsequnce } \forall t \ge 0\ \varphi_n(t) \to \varphi(t). \quad (12.15)$$

Then the following structural formula holds

$$\Theta_\Sigma = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(0), \tag{12.16}$$

where $\mathcal{K}_\sigma$ is the set of all bounded complete trajectories of MP U_σ.

Proof First let us consider situation when the family of MP $\{U_\sigma\}_{\sigma\in\Sigma}$ is strict. In this case one can consider multivalued semigroup (m-semiflow) on the extended phase space $X \times \Sigma$ by the rule

$$G(t, \{x, \sigma\}) = \{U_\sigma(t, 0, x), T(t)\sigma\}. \tag{12.17}$$

Then G is strict, has closed graph and compact attracting set $\Theta_\Sigma \times \Sigma$. So G has compact invariant global attractor

$$\mathscr{A} = \bigcap_{s\geq 0} \overline{\bigcup_{t\geq s} G(t, \Theta_\Sigma \times \Sigma)} = \{\gamma(0)|\gamma \text{ is bounded complete trajectories of } G\}.$$

Here under complete trajectory of m-semiflow G we mean the mapping $\mathbb{R} \ni t \mapsto \gamma(t)$ such that

$$\forall\, t \in \mathbb{R}\, \forall\, s \geq 0\ \gamma(t+s) \in G(s, \gamma(t)).$$

Let us consider two projectors Π_1 and Π_2, $\Pi_1(u, \sigma) = u$, $\Pi_2(u, \sigma) = \sigma$. As $T(t)\Sigma = \Sigma$, so $\Pi_2\mathscr{A} = \Sigma$. Let us prove that $\Pi_1\mathscr{A} = \Theta_\Sigma$.

As $\forall\, B \in \beta(X)\ G(t, B \times \Sigma) \to \mathscr{A},\ \ t \to +\infty$, so

$$U_\Sigma(t, \tau, B) \to \Pi_1\mathscr{A},$$

so $\Theta_\Sigma \subset \Pi_1\mathscr{A}$. Let us prove that $\Pi_1\mathscr{A} = \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(0)$. For this purpose we take $(u_0, \sigma_0) \in \mathscr{A}$. Then there exists $\gamma(\cdot) = \{u(\cdot), \sigma(\cdot)\}$, which is bounded complete trajectory of G and such that $\gamma(0) = (u_0, \sigma_0)$. Then $\forall t \geq \tau$

$$u(t) \in U_{\sigma(\tau)}(t-\tau, 0, u(\tau)),\ \ \sigma(t) = T(t-\tau)\sigma(\tau).$$

If $\tau \geq 0$, then $\sigma(\tau) = T(\tau)\sigma_0$, that is

$$u(t) \in U_{T(\tau)\sigma_0}(t-\tau, 0, u(\tau)) = U_{\sigma_0}(t, \tau, u(\tau)).$$

If $\tau < 0$, then $\sigma_0 = T(-\tau)\sigma(\tau)$, so

$$u(t) \in U_{\sigma(\tau)}(t-\tau, \tau-\tau, u(\tau)) = U_{T(-\tau)\sigma(\tau)}(t, \tau, u(\tau)) = U_{\sigma_0}(t, \tau, u(\tau)).$$

Therefore $u_0 = u(0) \in \mathcal{K}_{\sigma_0}(0) \subset \bigcup_{\sigma\in\Sigma} \mathcal{K}_\sigma(0)$.

Now let $u_0 = u(0) \in K_{\sigma_0}(0), \quad u(t) \in U_{\sigma_0}(t, \tau, u(\tau)) \ \ \forall t \geq \tau$. As $T(t)\Sigma = \Sigma$, so there exists $\sigma(s)$, $s \in \mathbb{R}$, such that $\sigma(t) = T(t-\tau)\sigma(\tau), \quad \forall t \geq \tau, \quad \sigma(0) = \sigma_0$. Then for $s \geq 0$ we have

$$\begin{aligned} G(t, \{u(s), \sigma(s)\}) &= (U_{\sigma(s)}(t, 0, u(s)), T(t)\sigma(s)) \\ &= (U_{T(s)\sigma_0}(t, 0, u(s)), \sigma(t+s)) = (U_{\sigma_0}(t+s, s, u(s)), \sigma(t+s)), \\ &\quad \{u(t+s), \sigma(t+s)\} \in (U_{\sigma_0}(t+s, s, u(s)), \sigma(t+s)). \end{aligned}$$

If $s < 0$, then $\sigma_0 = T(-s)\sigma(s)$, and

$$u(t+s) \in U_{\sigma_0}(t+s, s, u(s)) = U_{T(-s)\sigma(s)}(t+s, s, u(s)) = U_{\sigma(s)}(t, 0, u(s)).$$

Then $u_0 \in \Pi_1 \mathscr{A}$ and $\Pi_1 \mathscr{A} = \bigcup_{\sigma \in \Sigma} \mathscr{K}_\sigma(0)$.

Since for arbitrary attracting set P and for arbitrary bounded complete trajectory $\Gamma = \{u(s)\}_{s \in \mathbb{R}}$ of the process U_σ we have

$$u(0) \in U_\sigma(0, -n, u(-n)) = U_{T(n)\sigma(-n)}(0, -n, u(-n))$$

$$\subset U_\Sigma(n, 0, \Gamma) \to P, \ \ n \to +\infty,$$

so $u(0) \in P$, and we obtain (12.16).

Now let us consider another case, when family of m-processes is not strict, but the condition (12.15) holds. Let us show that $\mathscr{K}_\sigma(0) \subset \Theta_\Sigma$. If $z \in \mathscr{K}_\sigma(0)$, then there exists bounded complete trajectory $\varphi(\cdot)$ of m-process U_σ, such that $\varphi(0) = z$. Let us denote $\Gamma = \bigcup\limits_{t \in \mathbb{R}} \varphi(t) \in \beta(X)$. Then for $z = \varphi(0)$ we have

$$\varphi(0) \in U_\sigma(0, -n, \varphi(-n)) = U_{T(n)\sigma_n}(0, -n, \varphi(-n)) \subset U_\Sigma(n, 0, \Gamma).$$

Since $\forall \varepsilon > 0 \ \ \exists n_0 \ \ \forall n \geq n_0 \ \ U_\Sigma(n, 0, \Gamma) \subset O_\varepsilon(\Theta_\Sigma)$, then $z \in \Theta_\Sigma$ and we obtain required embedding.

Now let $z \in \Theta_\Sigma = \omega_\Sigma(0, B_0)$. Then $z = \lim\limits_{n \to +\infty} \xi_n, \quad \xi_n \in U_\Sigma(t_n, 0, B_0)$. Therefore on some subsequence

$$z = \lim_{n \to +\infty} \varphi_n(t_n), \quad \varphi_n(\cdot) \in K^0_{\sigma_n}, \quad \varphi_n(0) \in B_0, \quad \sigma_n \to \sigma.$$

For $\forall n \geq 1$ let us consider

$$\psi_n(\cdot) := \varphi_n(\cdot + t_n) \in K^{-t_n}_{T(t_n)\sigma_n},$$

that is $\psi_n(\cdot) \in K^{-t_n}_{\tilde{\sigma}_n}$, where $\tilde{\sigma}_n = T(t_n)\sigma_n$. Then $\psi_n(\cdot) \in K^0_{\tilde{\sigma}_n}, \quad \tilde{\sigma}_n \to \tilde{\sigma}, \quad \psi_n(0) = \varphi_n(t_n) \to z$, so there exists $\psi^{(0)}(\cdot) \in K^0_{\tilde{\sigma}}, \quad \psi^{(0)}(0) = z$, such that

$$\forall t \geq 0 \;\; \psi_n(t) = \varphi_n(t + t_n) \to \psi^{(0)}(t).$$

For $\tau = -1 \;\forall n \geq n_1 \;\; -t_n < -1$, therefore $\psi_n(\cdot) \in K^{-1}_{\tilde{\sigma}_n}$ and on some subsequence

$$\psi_n(-1) = \varphi_n(t_n - 1) \to z_1.$$

Herewith there exists $\psi^{(-1)}(\cdot) \in K^{-1}_{\tilde{\sigma}}$ such that on subsequence

$$\psi_n(t) = \varphi_n(t + t_n) \to \psi^{(-1)}(t) \;\; \forall t \geq -1,$$

and $\forall t \geq 0 \;\; \psi^{(0)}(t) = \psi^{(-1)}(t)$. By standard diagonal procedure we construct sequence of functions

$$\psi^{(-k)}(\cdot) \in K^{-k}_{\tilde{\sigma}}, \;\; k \geq 0,$$

with $\psi^{(-k+1)}(t) = \psi^{(-k)}(t) \;\; \forall t \geq -k + 1$. Let us put

$$\psi(t) := \psi^{(-k)}(t), \; if \; t \geq -k.$$

Then the function $\psi(\cdot)$ is correctly defined, $\psi : \mathbb{R} \mapsto X$.

Moreover $\forall \tau < 0 \;\; \exists k$ such that $[\tau, +\infty) \subset [-k, +\infty)$, on $[-k, +\infty) \;\; \psi(\cdot) \equiv \psi^{(-k)}$, so $\psi(\cdot) \in K^{-k}_{\tilde{\sigma}}$, and from this

$$\psi(\cdot) \in K^{\tau}_{\tilde{\sigma}}, \;\; \psi(0) = \psi^{(0)}(0) = z.$$

Since on subsequence

$$\forall t \in \mathbb{R} \;\; \psi(t) = \lim_{n \to +\infty} \varphi_n(t + t_n) \in \omega_{\Sigma}(0, B_0) \in \beta(X),$$

then $z = \psi(0) \in \mathscr{K}_{\tilde{\sigma}}$ and theorem is proved.

12.4 Uniform Global Attractor for RD-System

Definition 12.7 Let Θ be some topological space of functions from $\mathbb{R}$ to topological space E. The function $\xi \in \Theta$ is called translation compact in Θ, if the set

$$H(\xi) = cl_{\Theta}\{\xi(\cdot + s) \mid s \in \mathbb{R}\}$$

is compact in Θ.

To construct family of m-processes for the problem (12.1) we suppose that time-depended functions f and h are translation compact in natural spaces [8]. More precisely, we will assume that

$$h \text{ is translation compact in } L^{2,w}_{loc}(\mathbb{R}; H), \tag{12.18}$$

where $L^{2,w}_{loc}(\mathbb{R}; H)$ is the space $L^2_{loc}(\mathbb{R}; H)$ with the local weak convergence topology, and

$$f \text{ is translation compact in } C(\mathbb{R}; C(\mathbb{R}^N, \mathbb{R}^N)), \tag{12.19}$$

where $C(\mathbb{R}; C(\mathbb{R}^N, \mathbb{R}^N))$ equipped with local uniform convergence topology.

It is known that condition (12.18) is equivalent to

$$|h|^2_+ := \sup_{t\in\mathbb{R}} \int_t^{t+1} |h(s)|^2 ds < \infty \tag{12.20}$$

It is also known that condition (12.19) is equivalent to

$$\begin{aligned} &\forall R > 0\, f \text{ is bounded and uniformly continuous on} \\ &Q(R) = \{(t, v) \in \mathbb{R} \times \mathbb{R}^N \mid |v|_{\mathbb{R}^N} \le R\}. \end{aligned} \tag{12.21}$$

If conditions (12.18),(12.19) take place, then the symbol space

$$\Sigma = cl_{C(\mathbb{R}; C(\mathbb{R}^N, \mathbb{R}^N)) \times L^{2,w}_{loc}(\mathbb{R}; H)} \{(f(\cdot + s), h(\cdot + s)) \mid s \in \mathbb{R}\} \tag{12.22}$$

is compact, and $\forall\, s \ge 0\, T(s)\Sigma = \Sigma$, where $T(s)$ is translation semigroup, which is continuous on Σ.

For every $\sigma = (f_\sigma, h_\sigma) \in \Sigma$ we consider the problem

$$\begin{cases} u_t = a\Delta u - f_\sigma(t, u) + h_\sigma(t, x), & x \in \Omega,\ t > \tau, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{12.23}$$

It is proved in [19] that $\forall\, \sigma \in \Sigma\, f_\sigma$ satisfies (12.3), (12.4) with the same constants $C_1, C_2, \gamma_i, |h_\sigma|_+ \le |h|_+$. So we can apply Theorem 2 and obtain that $\forall\, \tau \in \mathbb{R}, u_\tau \in H$ the problem (12.23) has at least one solution on $(\tau, +\infty)$, each solution of (12.23) belongs to $C([\tau, +\infty); H)$ and satisfies energy equality (12.6). For every $\sigma \in \Sigma, \tau \in \mathbb{R}$ we define

$$K^\tau_\sigma = \{u(\cdot) \mid u(\cdot) \text{ is solution of } (12.23) \text{ on } (\tau, +\infty)\} \tag{12.24}$$

and according to (12.12) we put $\forall\, \sigma \in \Sigma, \forall\, t \ge \tau, \forall\, u_\tau \in H$

$$U_\sigma(t, \tau, u_\tau) = \{u(t) | u(\cdot) \in K^\tau_\sigma,\ u(\tau) = u_\tau\}. \tag{12.25}$$

From [19] and Theorem 13 we obtain the following result

Theorem 12.4 *Under conditions* (12.3)*,* (12.4)*,* (12.18)*,* (12.19) *formula* (12.25) *defines a strict family of MP* $\{U_\sigma\}_{\sigma\in\Sigma}$ *which has compact, invariant, stable and con-*

nected uniform global attractor Θ_Σ*, which consists of bounded complete trajectories, that is*

$$\Theta_\Sigma = \bigcup_{\sigma\in\Sigma} \mathscr{K}_\sigma(0), \tag{12.26}$$

where $\mathscr{K}_\sigma$ *is the set of all bounded complete trajectories of MP* U_σ*.*

Now we want to use formula (12.26) for proving that the uniform global attractor of RD-system is bounded set in the space $(L^\infty(\Omega))^N \cap V$.

First let us consider the following conditions:

$\exists\, M_i > 0,\ i = \overline{1, N}$ *such that for all* $v = (v^1, ..., v^N) \in \mathbb{R}^N$ *for a.a.* $x \in \Omega\ \forall\, t \in \mathbb{R}$

$$\sum_{i=1}^{N} (f^i(t, v) - h^i(t, x))(v^i - M_i)^+ \geq 0 \tag{12.27}$$

$$\sum_{i=1}^{N} (f^i(t, v) - h^i(t, x))(v^i + M_i)^- \leq 0 \tag{12.28}$$

where $\varphi^+ = max\{0, \varphi\}, \varphi^- = max\{0, -\varphi\}, \varphi = \varphi^+ - \varphi^-$.

Let us consider some example, which allow to verify conditions (12.27), (12.28).

Lemma 12.3 *If* $N = 1$ *(scalar equation), then from* (12.3), (12.4) *and* $h \in L^\infty(\mathbb{R} \times \Omega)$ *we have* (12.27), (12.28).

Proof From (12.3) and $h \in L^\infty(\Omega)$ for a.a. $x \in \Omega$ and $u \in \mathbb{R}$,

$$\tilde{\gamma}|u|^p - \tilde{C}_2 \leq g(t, x, u)u \leq \tilde{C}_1|u|^p + \tilde{C}_1,$$

where $g(t, x, u) = f(t, u) - h(t, x)$, $\tilde{\gamma}$ does not depend on t, u, x.

If $u \leq M$, then $g(t, x, u)(u - M)^+ = 0$.

If $u > M$, then

$$\begin{aligned} g(t, x, u)(u - M)^+ &= g(t, x, u)u\frac{(u - M)^+}{u} = g(t, x, u)u(1 - \frac{M}{u}) \\ &\geq (\tilde{\gamma}u^p - \tilde{C}_2)(1 - \frac{M}{u}) \geq (\tilde{\gamma}M^p - \tilde{C}_2)(1 - \frac{M}{u}) \end{aligned}$$

and if we choose $M = \left(\frac{\tilde{C}_2}{\tilde{\gamma}}\right)^{\frac{1}{p}}$, then $g(t, x, u)(u - M)^+ \geq 0$ a.e.

Lemma 12.4 *If for arbitrary* $N \geq 1$ $h \equiv 0, f(t, u) = (f^1(t, u), ...f^N(t, u))$*, where* $f^i(t, u) = (\sum_{i=1}^{N} |u^i|^2 - R^2)u^i, R > 0$ *is positive constant, then conditions* (12.27), (12.28) *hold for* $M_i = R$.

Proof If $\sum\limits_{i=1}^{N} |u^i|^2 < R^2$, so $\forall\, i = \overline{1, N} |u^i| < R$ and

$$\sum_{i=1}^{N} f^i(t, u)(u^i - R)^+ = 0,$$

$$\sum_{i=1}^{N} f^i(t, u)(u^i + R)^- = 0.$$

If $\sum\limits_{i=1}^{N} |u^i|^2 \geq R^2$, then

$$\sum_{i=1}^{N} f^i(t, u)(u^i - R)^+ = (\sum_{i=1}^{N} |u^i|^2 - R^2) \sum_{i=1}^{N} u^i(u^i - R)^+ \geq 0,$$

$$\sum_{i=1}^{N} f^i(t, u)(u^i + R)^- = (\sum_{i=1}^{N} |u^i|^2 - R^2) \sum_{i=1}^{N} u^i(u^i + R)^- \leq 0.$$

Theorem 12.5 *If conditions* (12.3), (12.4), (12.18), (12.19), (12.27), (12.28) *hold and matrix a is diagonal, then the uniform global attractor Θ_Σ is bounded set in the space $(L^\infty(\Omega))^N \cap V$.*

Proof First let us prove that $\forall \sigma \in \Sigma$ functions f_σ, h_σ satisfy (12.27), (12.28). Indeed, there exists sequence $t_n \nearrow \infty$ such that $\forall\ T > 0,\ R > 0,\ \eta \in L^2((-T, T) \times \Omega)$

$$\sup_{|t| \leq T} \sup_{|v| \leq R} \sum_{i=1}^{N} |f^i(t + t_n, v) - f^i_\sigma(t, v)|^2 \to 0,\ n \to \infty,$$

$$\sum_{i=1}^{N} \int_{-T}^{T} \int_{\Omega} (h^i(t + t_n, x) - h^i_\sigma(t, x))\eta(t, x)dxdt \to 0,\ n \to \infty.$$

From (12.27)

$$\sum_{i=1}^{N} (f^i(t + t_n, v) - h^i(t + t_n, x))(v^i - M_i)^+ \geq 0. \tag{12.29}$$

Therefore for fixed v and for arbitrary $\varepsilon > 0$ there exists $N \geq 1$ such that $\forall\, n \geq N$

$$\sum_{i=1}^{N} h^i(t+t_n,x)(v^i-M_i)^+ \le \sum_{i=1}^{N} f^i(t+t_n,v)(v^i-M_i)^+ < \sum_{i=1}^{N} f^i_\sigma(t,v)(v^i-M_i)^+ + \varepsilon.$$

Because

$$\sum_{i=1}^{N} h^i(t\,+\,t_n,x)(v^i-M_i)^+ \to \sum_{i=1}^{N} h^i_\sigma(t,x)(v^i-M_i)^+ \text{ weakly in } L^2((-T,T)\times\Omega),$$

from Mazur's Theorem we deduce that

$$\sum_{i=1}^{N} h^i_\sigma(t,x)(v^i-M_i)^+ \le \sum_{i=1}^{N} f^i_\sigma(t,v)(v^i-M_i)^+ + \varepsilon \text{ for a.a. } x\in\Omega.$$

From arbitrary choice of ε we can obtain required result.

It is easy to obtain that for arbitrary weak solution of (12.1) and for every $\eta \in C_0^\infty(\tau,T)$

$$\int_\tau^T (u_t,u^+)\eta dt = -\frac{1}{2}\int_\tau^T |u^+|^2\eta_t dt. \tag{12.30}$$

Then putting $g_\sigma = f_\sigma - h_\sigma$ and for numbers $M_1, ..., M_N$ from condition (12.27) we have

$$\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N}|(u^i-M_i)^+|^2+\beta\sum_{i=1}^{N}\|(u^i-M_i)^+\|^2+\int_\Omega\sum_{i=1}^{N} g^i_\sigma(t,x,u)(u^i-M_i)^+dx=0.$$

Then from (12.27)

$$\frac{d}{dt}\sum_{i=1}^{N}|(u^i-M_i)^+|^2+2\beta\sum_{i=1}^{N}|(u^i-M_i)^+|^2\le 0$$

and for all $t>\tau$

$$\sum_{i=1}^{N}|(u^i-M_i)^+(t)|^2\le\sum_{i=1}^{N}|(u^i-M_i)^+(\tau)|^2e^{-2\lambda_1\beta(t-\tau)}. \tag{12.31}$$

If $u(\cdot)\in\mathscr{K}_\sigma$ then from (12.31) taking $\tau\to-\infty$ we obtain $u^i(x,t)\le M_i,\ i=\overline{1,N},\ \forall t\in\mathbb{R}$, for a.a. $x\in\Omega$.

In the same way we will have $u^i(x,t)\ge M_i$ (using $(u^i+M_i)^-$).

Then

$$ess\sup_{x\in\Omega}|z^i(x)|\le M_i\ \ \forall z=(z^1,...,z^N)\in\Theta_\Sigma.$$

So we obtain that Θ_Σ is bounded set in the space $(L^\infty(\Omega))^N$. From the equality $\Theta_\Sigma = U_\Sigma(t,\tau,\Theta_\Sigma) \forall\, t \geq \tau$ we deduce that $\forall\, \sigma \in \Sigma\ U_\sigma(t,\tau,\Theta_\Sigma) \subset \Theta_\Sigma$. Now let us consider arbitrary complete trajectory $u(\cdot) \in \mathcal{K}_\sigma$. Due to definition of weak solution for a.a . $t \in \mathbb{R}\ u(t) \in V$. We take such $\tau \in \mathbb{R}$ that $u(\tau) \in V$ and consider the following Cauchy problem

$$\begin{cases} v_t = a\Delta v - f_\sigma(t,u) + h_\sigma(t,x), \quad x \in \Omega,\ t > \tau, \\ v|_{\partial\Omega} = 0, \\ v|_{t=\tau} = u(\tau). \end{cases} \tag{12.32}$$

Because $\forall\ t \geq \tau\ \ u(t) \in \Theta_\Sigma$, which is bounded in $(L^\infty(\Omega))^N$, we have that $f_\sigma(t,u(t,x)) \in (L^\infty(\Omega))^N$. Thus for linear problem (12.32) from well-known results one can deduce that $\forall\, T > \tau\ v \in C([\tau,T];V)$. So from uniqueness of the solution of Cauchy problem (12.32) $v \equiv u$ on $[\tau,+\infty)$ and, therefore, $\forall\, t \geq \tau\ u(t) \in V$. It means that $\forall\, t \in \mathbb{R}\ u(t) \in V$ and from the formula (12.26) $\Theta_\Sigma \subset V$.

From the energy equality, applying to function u, and boundness of Θ_Σ in the space H we deduce, that $\exists\, C > 0$, which does not depend on σ, such that $\forall\, t \in \mathbb{R}$

$$\int_t^{t+1} \|u(s)\|^2 ds \leq C(1 + \int_t^{t+1} |h_\sigma(s)|^2 ds).$$

From translation compactness of h we have

$$\int_t^{t+1} \|u(s)\|^2 ds \leq C(1 + |h|_+^2).$$

So for arbitrary $t \in \mathbb{R}$ we find $\tau \in [t, t+1]$ such that $\|u(\tau)\|^2 \leq C(1 + |h|_+^2)$. Then for the problem (12.32) we obtain inequality

$$\forall\, t \geq \tau\ \ \|v(t)\|^2 \leq e^{-\delta(t-\tau)}\|u(\tau)\|^2 + D,$$

where positive constants $\delta,\ D$ do not depend on σ. Thus

$$\forall\, t \in \mathbb{R}\ \ \|u(t)\|^2 \leq C(1 + |h|_+^2) + D$$

and theorem is proved.

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