Chapter 11
Multivalued Dynamics of Solutions for Autonomous Operator Differential Equations in Strongest Topologies

Mikhail Z. Zgurovsky and Pavlo O. Kasyanov

Abstract We consider nonlinear autonomous operator differential equations with pseudomonotone interaction functions satisfying (S)-property. The dynamics of all weak solutions on the positive time semi-axis is studied. We prove the existence of a trajectory and a global attractor in a strongest topologies and study their structure. As a possible application, we consider the class of high-order nonlinear parabolic equations.

11.1 Introduction: Statement of the Problem

In this chapter, we study the limiting behavior as time $t \to +\infty$ of the solutions of first-order general nonlinear evolution equations of the form

$$u'(t) + A(u(t)) = \bar{0}, \tag{11.1}$$

It is assumed that the nonlinear operator $A : V \to V^*$, acts in a Banach space V, which is reflexive and separable and, for some Hilbert space H, the embeddings $V \Subset H \equiv H \subset V^*$ are valid. Suppose that the nonlinear operator A is *pseudomonotone* and satisfies dissipation conditions of the form

$$\langle A(u), u\rangle_V \geq \alpha \|u\|_V^p - \beta \quad \forall u \in V, \tag{11.2}$$

where $p \geq 2$, and $\alpha, \beta > 0$, and also power growth conditions of the form

M. Z. Zgurovsky · P. O. Kasyanov (✉)
Institute for Applied System Analysis, National Technical University of Ukraine
“Kyiv Polytechnic Institute”, Peremogy ave., 37, build, 35, Kyiv 03056, Ukraine
e-mail: kasyanov@i.ua

M. Z. Zgurovsky
e-mail: zgurovsm@hotmail.com

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$$\|A(u)\|_{V^*} \leq c(1 + \|u\|_V^{p-1}) \quad \forall u \in V, \tag{11.3}$$

for some $c > 0$. Here $\langle \cdot, \cdot \rangle_V : V^* \times V \to \mathbb{R}$ is the pairing in $V^* \times V$ coinciding on $H \times V$ with the inner product $(\cdot, \cdot)$ in the Hilbert space H.

By a *weak solution* of the operator differential equation (11.1) on a closed interval $[\tau, T]$ we mean an element u of the space $L_p(\tau, T; V)$ such that

$$\forall \xi \in C_0^\infty([\tau, T]; V) \quad -\int_\tau^T (\xi'(t), u(t))dt + \int_\tau^T \langle A(u(t)), \xi(t) \rangle_V dt = 0. \tag{11.4}$$

Many evolution partial differential equations in a domain Ω whose leading part is a pth power nonlinear monotone differential operator and which may contain lower (now nonmonotone) summands with subordinate nonlinearity growth can be reduced to the form (11.1). In this case, the space V is a Sobolev space of the corresponding order, while the space H is $H = L_2(\Omega)$. Such equations are very often used to describe complicated evolution processes in various models in physics and mechanics. For equations of the form (11.1), there is a well-developed technique for constructing global (i.e., for all $t \geq 0$) weak solutions $u(t)$, $t \geq 0$, from the space $L_p^{loc}(\mathbb{R}_+; V)$ such that $u'(\cdot) \in L_q^{loc}(\mathbb{R}_+; V^*)$ (here $1/p + 1/q = 1$). It is well known that such weak solutions $u(t)$ are continuous functions with values in H, i.e., $u(\cdot) \in C(\mathbb{R}_+; H)$.

The problem is to study the asymptotic behavior as $t \to +\infty$ of the families of weak solutions $\{u(t)\}$ of Eq. (11.1) in the norm of H under the assumption that the initial data $\{u(0)\}$ constitute a bounded set in H.

Note that, under certain additional conditions on the nonlinear operator $A(u)$ ensuring, for Eq. (11.1), the unique solvability of the Cauchy problem $u|_{t=0} = u_0$ for any $u_0 \in H$, the study of the class of weak solutions under consideration involves the highly fruitful theory of dynamical semigroups and their global attractors in infinite-dimensional phase spaces. This theory has been successfully developed over a period of more than 30 years; its foundations were created by Ladyzhenskaya, Babin, Vishik, Hale, Temam and other well-known mathematicians [4, 11, 14–16, 21].

The problem becomes significantly more complicated if the corresponding Cauchy problem is not uniquely solvable or the proof of the relevant theorem is not known. Such a situation often occurs in complicated mathematical models. In this case, the "classical" method based on unique semigroups and global attractors cannot be applied directly. However, two approaches to the study of the dynamics of the corresponding weak solutions are well known. The first method is based on the theory of multivalued semigroups; it was developed in ground-breaking papers of Babin and Vishik (see, for example, [3]). The second approach uses the method of trajectory attractors; it was proposed in the papers [5, 7] of Chepyzhov and Vishik as well as in the independent work [20] of Sell.

The new result contained in the present chapter consists in the application of these two approaches to the study of the asymptotic behavior of the weak solutions of equations of the form (11.1) with general nonlinear pseudomonotone operator $A(u)$ satisfying (S)-property without any conditions guaranteeing the unique solvability of the Cauchy problem; cf. [1, 2, 4, 9, 14, 16, 19, 21, 23] . In this chapter, we prove a theorem on the existence of a global attractor $\mathscr{A}$ in the space H for the multivalued semigroup corresponding to Eq. (11.1) as well as a theorem on the existence of a trajectory attractor $\mathscr{P}$ in the space $C^{loc}(\mathbb{R}_+; H) \cap L_p^{loc}(\mathbb{R}_+; V)$ for the corresponding translation semigroup in the space of all weak trajectories (weak solutions on the half-line) of Eq. (11.1). We also describe the structure of the global and trajectory attractors as well as establish a simple relation between these attractors.

11.2 Additional Properties of Solutions

For fixed $\tau < T$ let us set

$$X_{\tau,T} = L_p(\tau, T; V), \quad X^*_{\tau,T} = L_q(\tau, T; V^*), \quad W_{\tau,T} = \{u \in X_{\tau,T} \mid u' \in X^*_{\tau,T}\},$$

where u' is a derivative of an element $u \in X_{\tau,T}$ in the sense of the space of distributions $\mathscr{D}([\tau, T]; V^*)$ (see, for example, [10, Definition IV.1.10, p. 168]). We note that

$$A(u)(t) = A(u(t)), \quad \text{for any } u \in X_{\tau,T} \text{ and a.e. } t \in (\tau, T).$$

The space $W_{\tau,T}$ is a reflexive Banach space with the graph norm of a derivative (see, for, example [24, Proposition 4.2.1, p. 291]):

$$\|u\|_{W_{\tau,T}} = \|u\|_{X_{\tau,T}} + \|u'\|_{X^*_{\tau,T}}, \quad u \in W_{\tau,T}. \tag{11.5}$$

Properties of A and (V, H, V^*) provide the existence of a weak solution of Cauchy problem (11.1) with initial data

$$u(\tau) = u_\tau \tag{11.6}$$

on the interval $[\tau, T]$ for an arbitrary $y_\tau \in H$. Therefore, the next result takes place:

Lemma 11.1 Kasyanov [12] *For any* $\tau < T$, $y_\tau \in H$ *Cauchy problem* (11.1), (11.6) *has a weak solution on the interval* $[\tau, T]$*. Moreover, each weak solution* $u \in X_{\tau,T}$ *of Cauchy problem* (11.1), (11.6) *on the interval* $[\tau, T]$ *belongs to* $W_{\tau,T} \subset C([\tau, T]; H)$.

For fixed $\tau < T$ we denote

$$\mathscr{D}_{\tau,T}(u_\tau) = \{u(\cdot) \mid u \text{ is a weak solution of (11.1) on } [\tau, T],\ u(\tau) = u_\tau\}, \quad u_\tau \in H.$$

From Lemma 11.1 it follows that $\mathscr{D}_{\tau,T}(u_\tau) \neq \emptyset$ and $\mathscr{D}_{\tau,T}(u_\tau) \subset W_{\tau,T}$ $\forall \tau < T$, $u_\tau \in H$.

We note that the translation and concatenation of weak solutions is a weak solution too.

Lemma 11.2 Kasyanov [12] *If* $\tau < T$, $u_\tau \in H$, $u(\cdot) \in \mathscr{D}_{\tau,T}(u_\tau)$, *then* $v(\cdot) = u(\cdot + s) \in \mathscr{D}_{\tau-s,T-s}(u_\tau)$ $\forall s$. *If* $\tau < t < T$, $u_\tau \in H$, $u(\cdot) \in \mathscr{D}_{\tau,t}(u_\tau)$ *and* $v(\cdot) \in \mathscr{D}_{t,T}(u(t))$, *then*

$$z(s) = \begin{cases} u(s), & s \in [\tau, t], \\ v(s), & s \in [t, T] \end{cases}$$

belongs to $\mathscr{D}_{\tau,T}(u_\tau)$.

As a rule, the proof of the existence of compact global and trajectory attractors for equations of type (11.1) is based on the properties of the set of weak solutions of problem (11.1) related to the absorption of the generated m-semiflow of solutions and its asymptotic compactness (see, for example, [18, 22] and the references therein). The following lemma on a priori estimates of solutions and Theorem 11.1 on the dependence of solutions on initial data will play a key role in the study of the dynamics of the solutions of problem (11.1) as $t \to +\infty$.

Lemma 11.3 Kasyanov [12] *There exist* $c_4, c_5, c_6, c_7 > 0$ *such that for any finite interval of time* $[\tau, T]$ *every weak solution* u *of problem* (11.1) *on* $[\tau, T]$ *satisfies estimates:* $\forall t \geq s$, $t, s \in [\tau, T]$

$$\|u(t)\|_H^2 + c_4 \int_s^t \|u(\xi)\|_V^p d\xi \leq \|u(s)\|_H^2 + c_5(t-s), \tag{11.7}$$

$$\|u(t)\|_H^2 \leq \|u(s)\|_H^2 e^{-c_6(t-s)} + c_7. \tag{11.8}$$

We recall that $A : V \to V^*$ satisfies (S)-*property*, if from $u_n \to u$ weakly in V and $\langle A(u_n), u_n - u \rangle_V \to 0$, as $n \to \infty$, it follows that $u_n \to u$ strongly in V, as $n \to +\infty$.

Further we assume that A satisfies (S)-property.

Theorem 11.1 *Let* $\tau < T$, $\{u_n\}_{n\geq 1}$ *be an arbitrary sequence of weak solutions of* (11.1) *on* $[\tau, T]$ *such that* $u_n(\tau) \to \eta$ *weakly in* H. *Then there exist* $\{u_{n_k}\}_{k\geq 1} \subset \{u_n\}_{n\geq 1}$ *and* $u(\cdot) \in \mathscr{D}_{\tau,T}(\eta)$ *such that*

$$\forall \varepsilon \in (0, T-\tau) \quad \max_{t \in [\tau+\varepsilon, T]} \|u_{n_k}(t) - u(t)\|_H + \int_{\tau+\varepsilon}^T \|u_{n_k}(t) - u(t)\|_V^p dt \to 0, \quad k \to +\infty. \tag{11.9}$$

Before the proof of Theorem 11.1 let us provide some auxiliary statements.

Lemma 11.4 *Let $\tau < T$, $y_n \to y$ weakly in $W_{\tau,T}$, and*

$$\overline{\lim_{n\to+\infty}} \langle A(y_n), y_n - y\rangle_{X_{\tau,T}} \le 0. \tag{11.10}$$

Then

$$\lim_{n\to+\infty} \int_\tau^T |\langle A(y_n(t)), y_n(t) - y(t)\rangle_V|\, dt = 0. \tag{11.11}$$

Proof There exists a set of measure zero, $\Sigma_1 \subset (\tau, T)$ such that for $t \notin \Sigma_1$, we have that

$$y_n(t) \in V \text{ for all } n \ge 1.$$

Similarly to [13, p. 7] we verify the following claim.

Claim Let $y_n \to y$ weakly in $W_{\tau,T}$ and let $t \notin \Sigma_1$. Then

$$\underline{\lim}_{n\to+\infty} \langle A(y_n(t)), y_n(t) - y(t)\rangle_V \ge 0.$$

Proof of the claim Fix $t \notin \Sigma_1$ and suppose to the contrary that

$$\underline{\lim}_{n\to+\infty} \langle A(y_n(t)), y_n(t) - y(t)\rangle_V < 0. \tag{11.12}$$

Then up to a subsequence $\{y_{n_k}\}_{k\ge1} \subset \{y_n\}_{n\ge1}$ we have

$$\lim_{k\to+\infty} \langle A(y_{n_k}(t)), y_{n_k}(t) - y(t)\rangle_V = \underline{\lim}_{n\to+\infty} \langle A(y_n(t)), y_n(t) - y(t)\rangle_V < 0. \tag{11.13}$$

Therefore, for all rather large k, growth and dissipation conditions imply

$$\alpha\|y_{n_k}(t)\|_V^p - \beta \le \|A(y_{n_k}(t))\|_{V^*}\|y(t)\|_V \le c(1 + \|y_{n_k}(t)\|_V^{p-1})\|y(t)\|_V.$$

which implies that the sequences $\{\|y_{n_k}(t)\|_V\}_{k\ge1}$ and consequently $\{\|A(y_{n_k}(t))\|_{V^*}\}_{k\ge1}$ are bounded sequences. In virtue of the continuous embedding $W_{\tau,T} \subset C([\tau, T]; H)$ we obtain that $y_{n_k}(t) \to y(t)$ weakly in H. Due to boundedness of $\{y_{n_k}(t)\}_{k\ge1}$ in V we finally have

$$\forall t \in [\tau, T]\setminus\Sigma_1 \;\; y_{n_k}(t) \to y(t) \text{ weakly in } V,\;\; k \to +\infty. \tag{11.14}$$

The pseudomonotony of A, (11.12), (11.13) and (11.14) imply that

$$\underline{\lim}_{n\to+\infty}\langle A(y_n(t)), y_n(t)-y(t)\rangle_V \geq \langle A(y(t)), y(t)-y(t)\rangle_V$$
$$= 0 > \underline{\lim}_{n\to+\infty}\langle A(y_n(t)), y_n(t)-y(t)\rangle_V.$$

We obtain a contradiction.

The claim is proved.

Now let us continue the proof of Lemma 11.4. The claim provides that for a.e. $t \in [\tau, T]$, in fact for any $t \notin \Sigma_1$, we have

$$\underline{\lim}_{n\to+\infty}\langle A(y_n(t)), y_n(t)-y(t)\rangle_V \geq 0. \tag{11.15}$$

Dissipation and growth conditions imply that, if $\omega \in X_{\tau,T}$, then

$$\langle A(y_n(t)), y_n(t)-\omega(t)\rangle_V \geq \alpha\|y_n(t)\|_V^p - \beta - c(1+\|y_n(t)\|_V^{p-1})\|\omega(t)\|_V \text{ for a.e. } t\in[\tau,T]\backslash\Sigma_1.$$

Using $p-1=\frac{p}{q}$, the right side of the above inequality equals to

$$\alpha\|y_n(t)\|_V^p - \beta - c\|y_n(t)\|_V^{\frac{p}{q}}\|\omega(t)\|_V - c\|\omega(t)\|_V.$$

Now using Young's inequality, we can obtain a constant $c(c,\alpha)$ depending on c, α such that

$$c\|y_n(t)\|_V^{\frac{p}{q}}\|\omega(t)\|_V \leq \frac{\alpha}{2}\|y_n(t)\|_V^p + \|\omega(t)\|_V^p\cdot c(c,\alpha).$$

Letting $\bar{c} = \max\{\beta+\frac{c}{q}; c(c,\alpha)+\frac{c}{p}\}$ it follows that

$$\langle A(y_n(t)), y_n(t)-\omega(t)\rangle_V \geq -\bar{c}(1+\|\omega(t)\|_V^p) \text{ for a.e. } t\in[\tau,T]. \tag{11.16}$$

Letting $\omega = y$, we can use Fatou's lemma and we obtain

$$\underline{\lim}_{n\to+\infty}\int_0^T [\langle A(y_n(t)), y_n(t)-y(t)\rangle_V + \bar{c}(1+\|y(t)\|_V^p)]dt \geq$$

$$\geq \int_0^T \underline{\lim}_{n\to+\infty}[\langle A(y_n(t)), y_n(t)-y(t)\rangle_V + \bar{c}(1+\|y(t)\|_V^p)]dt \geq \bar{c}\int_0^T(1+\|y(t)\|_V^p)dt.$$

Therefore,

$$0 \geq \overline{\lim}_{n\to+\infty}\langle A(y_n), y_n-y\rangle_{X_{\tau,T}} \geq \underline{\lim}_{n\to+\infty}\int_\tau^T \langle A(y_n(t)), y_n(t)-y(t)\rangle_V dt =$$

$$= \varliminf_{n\to+\infty} \langle A(y_n), y_n - y\rangle_{X_{\tau,T}} \geq \int_\tau^T \varliminf_{n\to+\infty} \langle A(y_n(t)), y_n(t) - y(t)\rangle_V dt = 0,$$

showing that

$$\lim_{n\to+\infty} \langle A(y_n), y_n - y\rangle_{X_{\tau,T}} = 0. \tag{11.17}$$

From (11.16),

$$\forall n \geq 1 \ \forall t \notin \Sigma_1 \ 0 \leq \langle A(y_n(t)), y_n(t) - y(t)\rangle_V^- \leq \bar{c}(1 + \|y(t)\|_V^p),$$

where $a^- = \max\{0, -a\}$, for $a \in \mathbb{R}$. Due to (11.15) we know that for a.e. t, $\langle A(y_n(t)), y_n(t) - y(t)\rangle_V \geq -\varepsilon$ for all rather large n. Therefore, for such n, $\langle A(y_n(t)), y_n(t) - y(t)\rangle_V^- \leq \varepsilon$, if $\langle A(y_n(t)), y_n(t) - y(t)\rangle_V < 0$ and $\langle A(y_n(t)), y_n(t) - y(t)\rangle_V^- = 0$, if $\langle A(y_n(t)), y_n(t) - y(t)\rangle_V \geq 0$. Therefore, $\lim_{n\to+\infty} \langle A(y_n(t)), y_n(t) - y(t)\rangle_V^- = 0$ and we can apply the dominated convergence theorem and from (11.15) we conclude that

$$\lim_{n\to+\infty} \int_\tau^T \langle A(y_n(t)), y_n(t) - y(t)\rangle_V^- = \int_\tau^T \lim_{n\to+\infty} \langle A(y_n(t)), y_n(t) - y(t)\rangle_V^- dt = 0.$$

Now by (11.17) and the above equation we have

$$\lim_{n\to+\infty} \int_\tau^T \langle A(y_n(t)), y_n(t) - y(t)\rangle_V^+ dt =$$

$$= \lim_{n\to+\infty} \int_0^T \left[\langle A(y_n(t)), y_n(t) - y(t)\rangle_V + \langle A(y_n(t)), y_n(t) - y(t)\rangle_V^-\right] dt =$$

$$= \lim_{n\to+\infty} \langle A(y_n), y_n - y\rangle_{X_{\tau,T}} = 0.$$

Therefore,

$$\lim_{n\to+\infty} \int_\tau^T |\langle A(y_n(t)), y_n(t) - y(t)\rangle_V| dt = 0.$$

The lemma is proved.

Lemma 11.5 *Let $\tau < T$, $y_n \to y$ weakly in $W_{\tau,T}$, and* (11.10) *holds. Then there exists a subsequence $\{y_{n_k}\}_{k\geq 1} \subset \{y_n\}_{n\geq 1}$ such that for a.e. $t \in (\tau, T)$ we have that $y_{n_k}(t) \to y(t)$ weakly in V, and $\langle A(y_{n_k}(t)), y_{n_k}(t) - y(t)\rangle_V \to 0$, $k \to +\infty$.*

Proof Let $y_n \to y$ weakly in $W_{\tau,T}$ and

$$\overline{\lim_{n\to+\infty}} \langle A(y_n), y_n - y\rangle_{X_{\tau,T}} \leq 0.$$

In virtue of Lemma 11.4 we obtain

$$\lim_{n\to+\infty} \int_\tau^T |\langle A(y_n(t)), y_n(t) - y(t)\rangle_V| dt = 0. \tag{11.18}$$

Due to the continuous embedding $W_{\tau,T} \subset C([\tau, T]; H)$ we have

$$\forall t \in [\tau, T]\ y_n(t) \to y(t) \text{ weakly in } H,\ \ n \to +\infty. \tag{11.19}$$

From (11.18) it follows that there exists a subsequence $\{y_{n_k}\}_{k\geq 1} \subset \{y_n\}_{n\geq 1}$ such that

$$\text{for a.e. } t \in [\tau, T]\ \langle A(y_{n_k}(t)), y_{n_k}(t) - y(t)\rangle_V \to 0,\ \ k \to +\infty.$$

Let $\Sigma_1 \subset [\tau, T]$ be a set of measure zero such that for $t \notin \Sigma_1$ $y_{n_k}(t)$, $y(t)$ are well-defined $\forall k \geq 1$, and

$$\langle A(y_{n_k}(t)), y_{n_k}(t) - y(t)\rangle_V \to 0,\ \ k \to +\infty.$$

In virtue of growth and dissipation conditions we obtain

$$\forall t \notin \Sigma_1\ \forall k \geq 1\ \ \overline{\lim_{k\to+\infty}} \left(\alpha \|y_{n_k}(t)\|_V^p - \beta - c(1 + \|y_{n_k}(t)\|_V^{p-1})\|y(t)\|_V\right) \leq 0.$$

Thus $\forall t \notin \Sigma_1$

$$\overline{\lim_{k\to+\infty}} \|y_{n_k}(t)\|_V^p \leq c(c, \alpha, \beta, p)(1 + \|y(t)\|_V^p).$$

Therefore, due to (11.19) we obtain that for a.e. $t \in (\tau, T)$ $y_{n_k}(t) \to y(t)$ weakly in V, $k \to +\infty$.

The lemma is proved.

Proof (Proof of Theorem 11.1) Let $\tau < T$, $\{u_n\}_{n\geq 1}$ be an arbitrary sequence of weak solutions of (11.1) on $[\tau, T]$ such that $u_n(\tau) \to \eta$ weakly in H. Theorem 1 from [12] implies the existence of a subsequence $\{u_{n_k}\}_{k\geq 1} \subset \{u_n\}_{n\geq 1}$ and $u(\cdot) \in \mathscr{D}_{\tau,T}(\eta)$ such that

$$\forall \varepsilon \in (0, T - \tau) \quad \max_{t \in [\tau+\varepsilon, T]} \|u_{n_k}(t) - u(t)\|_H \to 0, \quad k \to +\infty. \tag{11.20}$$

Let us prove that

$$\forall \varepsilon \in (0, T - \tau) \quad \int_{\tau+\varepsilon}^{T} \|u_{n_k}(t) - u(t)\|_V^p dt \to 0, \quad k \to +\infty. \tag{11.21}$$

On the contrary, without loss of generality we assume that for some $\varepsilon \in (0, T - \tau)$ and $\delta > 0$ it is fulfilled

$$\int_{\tau+\varepsilon}^{T} \|u_{n_k}(t) - u(t)\|_V^p dt \geq \delta, \quad \forall k \geq 1. \tag{11.22}$$

In virtue of (11.7), without loss of generality we claim that

$$u_{n_k} \to u \text{ weakly in } W_{\tau+\varepsilon, T}, \quad k \to +\infty. \tag{11.23}$$

Moreover, due to (11.20), we have

$$\varlimsup_{k \to \infty} \int_{\tau+\varepsilon}^{T} \langle A(u_{n_k}(t)), u_{n_k}(t) - u(t) \rangle_V dt \leq 0. \tag{11.24}$$

Thus, Lemma 11.5 and (S)-property for A imply that up to a subsequence which we denote again as $\{u_{n_k}\}_{k \geq 1}$ for a.e. $t \in (\tau + \varepsilon, T)$ we have that $u_{n_k}(t) \to u(t)$ strongly in V, $k \to +\infty$. Moreover, Lemma 11.4 provides that

$$\lim_{k \to +\infty} \int_{\tau+\varepsilon}^{T} \left| \langle A(u_{n_k}(t)), u_{n_k}(t) - u(t) \rangle_V \right| dt = 0.$$

Dissipation and growth conditions follow the existence a constant $C > 0$ such that

$$\|u_{n_k}(t) - u(t)\|_V^p \leq C(1 + \|u(t)\|_V^p + \left| \langle A(u_{n_k}(t)), u_{n_k}(t) - u(t) \rangle_V \right|)$$

for a.e. $t \in (\tau + \varepsilon, T)$ and any $k \geq 1$. Therefore,

$$\lim_{k \to +\infty} \int_{\tau+\varepsilon}^{T} \|u_{n_k}(t) - u(t)\|_V^p dt = 0.$$

We obtain a contradiction.

The theorem is proved.

11.3 Attractors in Strongest Topologies

First we consider constructions presented in [18]. Denote the set of all nonempty (nonempty bounded) subsets of H by $P(H)$ ($\mathscr{B}(H)$). We recall that the multivalued map $G : \mathbb{R} \times H \to P(H)$ is said to be a *m-semiflow* if:

(a) $G(0, \cdot) = \mathrm{Id}$ (the identity map),
(b) $G(t + s, x) \subset G(t, G(s, x)) \; \forall x \in H, t, s \in \mathbb{R}_+$;

m-semiflow *is a strict* one if $G(t + s, x) = G(t, G(s, x)) \; \forall x \in H, t, s \in \mathbb{R}_+$.

From Lemmas 11.2 and 11.3 it follows that any weak solution can be extended to a global one defined on $[0, +\infty)$. For an arbitrary $y_0 \in H$ let $\mathscr{D}(y_0)$ be the set of all weak solutions (defined on $[0, +\infty)$) of problem (11.1) with initial data $y(0) = y_0$.

We define the m-semiflow G as $G(t, y_0) = \{y(t) \mid y(\cdot) \in \mathscr{D}(y_0)\}$.

Lemma 11.6 Kasyanov [12] *G is the strict m-semiflow.*

We recall that the set $\mathscr{A}$ is said to be *a global attractor G*, if:

(1) $\mathscr{A}$ is negatively semiinvariant (i.e. $\mathscr{A} \subset G(t, \mathscr{A}) \; \forall t \geq 0$);
(2) $\mathscr{A}$ is attracting, that is,

$$\mathrm{dist}(\mathrm{G}(\mathrm{t}, \mathrm{B}), \mathscr{A}) \to 0, \quad \mathrm{t} \to +\infty \quad \forall \mathrm{B} \in \mathscr{B}(\mathrm{H}), \tag{11.25}$$

where $\mathrm{dist}(\mathrm{C}, \mathrm{D}) = \sup_{c \in C} \inf_{d \in D} \|\mathrm{c} - \mathrm{d}\|_{\mathrm{H}}$ is the Hausdorff semidistance;
(3) For any closed set $Y \subset H$ satisfying (11.25), we have $\mathscr{A} \subset Y$ (minimality).

The global attractor is said to be *invariant* if $\mathscr{A} = G(t, \mathscr{A}) \; \forall t \geq 0$.

Theorem 11.2 Kasyanov [12] *The m-semiflow G has the invariant compact in the phase space H global attractor $\mathscr{A}$.*

Let us consider the family $\mathscr{K}_+ = \cup_{y_0 \in H} \mathscr{D}(y_0)$ of all weak solutions of inclusion (11.1) defined on the semi-infinite interval $[0, +\infty)$. Note that $\mathscr{K}_+$ is *translation invariant* one, i.e. $\forall u(\cdot) \in \mathscr{K}_+$, $\forall h \geq 0 \; u_h(\cdot) \in \mathscr{K}_+$, where $u_h(s) = u(h + s)$, $s \geq 0$. We set the *translation semigroup* $\{T(h)\}_{h \geq 0}$, $T(h)u(\cdot) = u_h(\cdot)$, $h \geq 0$, $u \in \mathscr{K}_+$ on $\mathscr{K}_+$.

We shall construct the attractor of the translation semigroup $\{T(h)\}_{h \geq 0}$ acting on $\mathscr{K}_+$. On $\mathscr{K}_+$ we consider a topology induced from the Fréchet space $C^{loc}(\mathbb{R}_+; H) \cap L_p^{loc}(\mathbb{R}_+; V)$. Note that Π_M is the restriction operator to the interval $[0, M]$. We denote the restriction operator to the semi-infinite interval $[0, +\infty)$ by Π_+.

We recall that the a $\mathscr{P} \subset C^{loc}(\mathbb{R}_+; H) \cap L_\infty(\mathbb{R}_+; H)$ is said to be *attracting* for the trajectory space $\mathscr{K}_+$ of inclusion (11.1) in the topology of $C^{loc}(\mathbb{R}_+; H)$ if for any bounded in $L_\infty(\mathbb{R}_+; H)$ set $\mathscr{B} \subset \mathscr{K}_+$ and any number $M \geq 0$ the following relation holds:

$$\mathrm{dist}_{\mathrm{C([0,M];H)}}(\Pi_{\mathrm{M}}\mathrm{T(t)}\mathscr{B}, \Pi_{\mathrm{M}}\mathscr{P}) \to 0, \quad \mathrm{t} \to +\infty. \tag{11.26}$$

A set $\mathscr{U} \subset \mathscr{K}_+$ is said to be *trajectory attractor* in the trajectory space $\mathscr{K}_+$ with respect to the topology of $C^{loc}(\mathbb{R}_+; H) \cap L_p^{loc}(\mathbb{R}_+; V)$ if

(i) $\mathscr{U}$ is a compact set in $C^{loc}(\mathbb{R}_+; H) \cap L_p^{loc}(\mathbb{R}_+; V)$ and bounded in $L_\infty(\mathbb{R}_+; H)$;
(ii) $\mathscr{U}$ is strictly invariant with respect to $\{T(h)\}_{h\geq 0}$, i.e. $T(h)\mathscr{U} = \mathscr{U}$ $\forall h \geq 0$;
(iii) $\mathscr{U}$ is an attracting set in the trajectory space $\mathscr{K}_+$ in the topology $C^{loc}(\mathbb{R}_+; H) \cap L_p^{loc}(\mathbb{R}_+; V)$.

Let us consider inclusions (11.1) on the entire time axis. Similarly to the space $C^{loc}(\mathbb{R}_+; H)$ the space $C^{loc}(\mathbb{R}; H)$ is equipped with the topology of local uniform convergence on each interval $[-M, M] \subset \mathbb{R}$ (see, for example, [22, p. 198]). A function $u \in C^{loc}(\mathbb{R}; H) \cap L_\infty(\mathbb{R}; H)$ is called a *complete trajectory* of inclusion (11.1) if $\forall h \in \mathbb{R}$ $\Pi_+ u_h(\cdot) \in \mathscr{K}_+$ [22, p. 198]. Let $\mathscr{K}$ be a family all complete trajectories of inclusion (11.1). Note that

$$\forall h \in \mathbb{R}, \ \forall u(\cdot) \in \mathscr{K} \quad u_h(\cdot) \in \mathscr{K}. \tag{11.27}$$

Lemma 11.7 *The set $\mathscr{K}$ is nonempty, compact in $C^{loc}(\mathbb{R}; H) \cap L_p^{loc}(\mathbb{R}_+; V)$ and bounded in $L_\infty(\mathbb{R}; H)$. Moreover,*

$$\forall y(\cdot) \in \mathscr{K}, \ \forall t \in \mathbb{R} \quad y(t) \in \mathscr{A}, \tag{11.28}$$

where $\mathscr{A}$ is the global attractor from Theorem 11.2.

Proof The statement of lemma follows from [12] and Theorem 11.1.

Lemma 11.8 Kasyanov [12] *Let $\mathscr{A}$ be a global attractor from Theorem 11.2. Then*

$$\forall y_0 \in \mathscr{A} \quad \exists y(\cdot) \in \mathscr{K} : \quad y(0) = y_0. \tag{11.29}$$

Theorem 11.3 *Let $\mathscr{A}$ be a global attractor from Theorem 11.2. Then there exists the trajectory attractor $\mathscr{P} \subset \mathscr{K}_+$ in the space $\mathscr{K}_+$. At that the next formula holds:*

$$\mathscr{P} = \Pi_+\mathscr{K} = \Pi_+\{y \in \mathscr{K} \mid y(t) \in \mathscr{A} \ \forall t \in \mathbb{R}\}, \tag{11.30}$$

Proof The statement of theorem follows from Theorem 11.1 and [12].

11.4 Application

Consider an example of the class of nonlinear boundary value problems for which we can investigate the dynamics of solutions as $t \to +\infty$. Note that in discussion we do not claim generality.

Let $n \geq 2$, $m \geq 1$, $p \geq 2$, $1 < q \leq 2$, $\frac{1}{p} + \frac{1}{q} = 1$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with rather smooth boundary $\Gamma = \partial\Omega$. We denote a number of differentiations by x of order $\leq m - 1$ (correspondingly of order $= m$) by N_1 (correspondingly by N_2). Let $A_\alpha(x, \eta; \xi)$ be a family of real functions ($|\alpha| \leq m$), defined in $\Omega \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and satisfying the next properties:

(C_1) for a.e. $x \in \Omega$ the function $(\eta, \xi) \to A_\alpha(x, \eta, \xi)$ is continuous one in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$;

(C_2) $\forall(\eta, \xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ the function $x \to A_\alpha(x, \eta, \xi)$ is measurable one in Ω;

(C_3) exist such $c_1 \geq 0$ and $k_1 \in L_q(\Omega)$, that for a.e. $x \in \Omega$, $\forall(\eta, \xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

$$|A_\alpha(x, \eta, \xi)| \leq c_1[|\eta|^{p-1} + |\xi|^{p-1} + k_1(x)];$$

(C_4) exist such $c_2 > 0$ and $k_2 \in L_1(\Omega)$, that for a.e. $x \in \Omega$, $\forall(\eta, \xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$

$$\sum_{|\alpha|=m} A_\alpha(x, \eta, \xi)\xi_\alpha \geq c_2|\xi|^p - k_2(x);$$

(C_5) there exists increasing function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ such that for a.e. $x \in \Omega$, $\forall\eta \in \mathbb{R}^{N_1}$, $\forall\xi, \xi^* \in \mathbb{R}^{N_2}$, $\xi \neq \xi^*$ the inequality

$$\sum_{|\alpha|=m} (A_\alpha(x, \eta, \xi) - A_\alpha(x, \eta, \xi^*))(\xi_\alpha - \xi^*_\alpha) \geq (\varphi(|\xi_\alpha|) - \varphi(|\xi^*_\alpha|))(|\xi_\alpha| - |\xi^*_\alpha|)$$

takes place.

Consider such denotations: $D^k u = \{D^\beta u, |\beta| = k\}$, $\delta u = \{u, Du, ..., D^{m-1}u\}$ (see [17, p. 194]).

For an arbitrary fixed interior force $f \in L_2(\Omega)$ we investigate the dynamics of all weak (generalized) solutions defined on $[0, +\infty)$ of such problem:

$$\frac{\partial y(x,t)}{\partial t} + \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha \left(A_\alpha(x, \delta y(x,t), D^m y(x,t))\right) = f(x) \text{ on } \Omega \times (0, +\infty), \tag{11.31}$$

$$D^\alpha y(x,t) = 0 \text{ on } \Gamma \times (0, +\infty), \quad |\alpha| \leq m - 1 \tag{11.32}$$

as $t \to +\infty$.

Consider such denotations (see for detail [17, p. 195]): $H = L_2(\Omega)$, $V = W_0^{m,p}(\Omega)$ is a real Sobolev space,

$$a(u, \omega) = \sum_{|\alpha| \le m} \int_{\Omega} A_\alpha(x, \delta u(x), D^m u(x)) D^\alpha \omega(x) dx, \quad u, \omega \in V.$$

Note that the operator $A : V \to V^*$, defined by the formula $\langle A(u), \omega \rangle_V = a(u, \omega)$ $\forall u, \omega \in V$, satisfies all mentioned assumptions. Hence, we can pass from problem (11.31)–(11.32) to corresponding problem in "generalized" setting (11.1). Note that

$$A(u) = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha \left(A_\alpha(x, \delta u, D^m u) \right) \quad \forall u \in C_0^\infty(\Omega).$$

Therefore, all statements from previous sections are fulfilled for weak (generalized) solutions of problem (11.31)–(11.32).

Remark 11.1 As applications we can also consider new classes of problems with degenerations, problems on a manifold, problems with delay, stochastic partial differential equations etc. [6, 8, 17, 20] with differential operators of pseudomonotone type as corresponding choice of the phase space.

11.5 Conclusions

For the class of autonomous differential operator equations with pseudomonotone nonlinear dependence between the defining parameters of the problem, we have studied the dynamics as $t \to +\infty$ in strongest topologies of all global weak solutions defined on $[0, +\infty)$. We have proved the existence of a global compact attractor and a compact trajectory attractor, studied their structure. The results obtained allow one to study the dynamics of solutions for new classes of evolution equations related to nonlinear mathematical models of geophysical and socioeconomic processes and for fields with interaction functions of pseudomonotone type satisfying the (S)-property, the condition of "power growth", and the standard sign condition.

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