Chapter 1
Algebra and Geometry Through Hamiltonian Systems

Anatoly T. Fomenko and Andrei Konyaev

Abstract Hamiltonian systems are considered to be the prime tool of classical and quantum mechanics. The proper investigation of such systems usually requires deep results from algebra and geometry. Here we present several results which in some sense go the opposite way: the knowledge about the integrable system enables us to obtain results on geometric and algebraic structures which naturally appear in such problems. All the results were obtained by employees of the Chair of Differential Geometry and Applications in Moscow State University in 2011–2012.

1.1 Introduction

Hamiltonian systems are common in classical and quantum mehanics. Usually the investigation of such system's properties requires deep results from different fields of algebra, geometry, topology etc. Here we present several results which in some sence go the opposite ways. It means that the study of the objects, which naturally appear in such problems gets some extra perspective from the study of the dynamics of the system itself.

The chapter consists of five parts, each related to some topic in study of Hamiltonian systems. All the results presented were obtained by the employees of the Chair of Differential Geometry and Applications in Moscow State Univercity during the period of 2011–2012.

The first part is dedicated to the study of symmetry groups of atoms, which are the main building block of Fomenko-Ziechang invariants (FZ invariants). These

A. T. Fomenko (✉) · A. Konyaev
Faculty of Mechanics and Mathematics, Lomonosov Moscow State University,
GSP-1, Leninskie Gory, Moscow, Russian Federation 119991
e-mail: fomenko@mech.math.msu.su

A. Konyaev
e-mail: maodzund@bk.ru

M. Z. Zgurovsky and V. A. Sadovnichiy (eds.), *Continuous and Distributed Systems*,
Solid Mechanics and Its Applications 211, DOI: 10.1007/978-3-319-03146-0_1,

invariants provide the classification for the integrable systems with two degrees of freedom. The main result of this section is due to Fomenko and Kudryavtseva: every finite group can be realized as a symmetry group for some atom. Moreover, there are some restriction on atom's topological complexity in terms of the genus of the atom.

The second part is dedicated to so called integer lattice on bifurcation diagrams. The similar objects naturally appear in quantum mechanics, but study of their classical counterparts is a relatively new topic. The results of this section deal with so called Fomenko hypothesis. It states that the properties of the lattice are closely related to the FZ invariants and, as a result, to the property of the system. Kantonistova studies the system called the Spherical Pendulum. The computation of the ZF invariats for this system is relatively simple which allows the direct verification of the hypothesis which, in this case, as Kantonistova showed, is valid.

The third section deals with integrable billiards, that is the motion of the material point inside two-dimensional domain in this case bounded by segments of the quadric from the same family (this section also contains some extra information about FZ inveriants). The computaion and thorough discription of invariants is done by Fokicheva for a large family of such systems. She showed, that the properties of integrable system are closely related to the shape of the domain.

The fourth section is dedicated to the generalization of classical Bertrand's problem, that is the discription of such potentials that the movement of a particle (moving point) in the corresponding field on a surface of bounded revolution has only periodic trajectories. This problem was formulated in nineteenth century and is yet still interesting. The main result of this section is due to Fedoseev et al.: in two theorems a discriprion of the class of so called Bertrand's manifolds (the manifolds that admit the necessary potentials) is presented.

The final part is the only one that has nothing to do with the integrable systems with FZ invariants. It deals with the classification of linear Poisson structures with generic leaves of dimension two. These structures are of an interest as Hamiltonian systems associated with them have the following property: after the restriction on the leaf every such system is integrable. The other aspect is that two-dimentional leaves are the leaves of the smallest possible non-zero dimension while linear brackets are the simplest possible non-trivial structures. Konyaev proved a theorem that provide a full description od such brackets.

1.2 Atoms and Their Symmetries

The notion of atom was introduced by Fomenko in [17, 19]. Atoms encode typical bifurcations of Liouville tori in non-degenerate integrable Hamiltonian systems. Now many notorious integrable systems with two degrees of freedom and the equivalence classes have been described in terms of two-dimensional atoms and molecules (the set of atoms with additional structure). Moreover many bifurcations of integrable systems in higher dimensions can be represented as semidirect product of two-dimensional atoms [34]. Because the notion of semidirect product uses the

symmetry group of atom, the study of such groups becomes essential. Two-dimensional saddle atoms (2-atoms) are a special class of atoms which can be described in many equivalent ways: f-graphs [35], maps also known as abstract polytopes.

Map (abstract polytope) is an equivalence class of particular cellular decomposition of closed two-dimensional surface up to cellular homeomorphisms. We call two decomposition preserving automorphisms of the surface equivalent iff they differ by the homeomorphism, which sends every cell to itself, preserving any orientation on it. It turns out that every finite group can be realized as a symmetry group of some 2-atom, orientable map or chord diagram. All three approaches are equivalent.

In [5] Fomenko and Bolsinov formulated a question: is it true that every finite group can be realized as symmetry group of bifurcation diagram of some integrable Hamiltonian system or in other terms as symmetry group of some 2-atom? For maximally symmetric atoms (the definition is given below) the description of their symmetries was done by Oshemkov and Brailov in [9]. In [10] Brailov and Kudryavtseva discovered an unexpected link between several of the infinite series of maximally symmetric atoms and stable topological non-conjugacy of integrable Hamiltonian systems. Thorough investigation of the group symmetries of the atoms was done by Fomenko et al. [30, 31].

Siran and Skoviera in [39] notice, that there are a number of results about various classes of combinatorial structures saying that every finite group is the automorphism group of some member of the class. Examples are provided by graphs [22], cubic graphs [23], Steiner triple systems [32], "pictures" [1], and others. Results of this type indicate that a given class is, to some extent, rich. On the other hand, there are some very natural classes that do not have this property, for instance, trees [14]. Similar questions have been asked in connection with graph embeddings on surfaces. As was proven in [11], every finite group is the automorphism group of some map on an orientable surface. However, it is by no means obvious that the same holds for non-orientable maps. The main result of the chapter [39] gives an affirmative answer to this question.

In [29] Fomenko and Kudryavtseva prove that every finite group can be realized as a symmetry group of some orientable two-dimensional atom. The method used in the work differs from the ones described above. In particular the algorithm of the construction of the atom by the given group is presented together with formulas for upper estimates of the atom's genus. The proof utilizes the notion of the covering of the atoms.

Let us recollect some of the basic notions of the atoms theory. Let M be a connected closed two-dimensional surface (orientable or non-orientabe) and $f : M \to R$ is a Morse function with exactly three singular values: maximum, minimum and saddle. We call this kind of function proper Morse function. For such a function its level surface for saddle value can be considered a connected graph K with only degree 4 vertices. The complement to K consists of two-dimensional cells homeomorphic to standard two-dimensional discs. Therefore we have degree 4 cellular decomposition. In particular this means, that such decomposition allows chess coloring, e.g. coloring of the discs in black and white, such that every edge borders discs of different colors.

Proper Morse functions f and f on the surfaces M and M respectively are called leaf-wise equivalent in the neighborhoods P and P of critical levels $\{f = c\}$ and $\{f = c\}$ iff there exist small $\varepsilon > 0$ and $\epsilon > 0$ together with diffeomorphism $D : P = \{|f - c| < e\} \to P' = \{|f' - c'| < e'\}$ such that the connected components of level surfaces of f map with D into the connected components of the level surfaces of f. If D preserves the direction of the growth of the functions f and f then we call them leaf-wise equipped equivalent.

Atom (P, K) is a class of leaf-wise equipped equivalence of Morse function f in the neighborhood $P = \{|f| < \varepsilon\}$ of its saddle critical level $K = \{f = 0\}$. We also call an atom any representative of such class, that is the pair of the surface P with the graph K embedded into it. The complement to the K consists of rings, which we denote positive and negative according to the sign of f. To define the symmetry group of an atom we start with the group of all homeomorphisms of (P, K) onto itself, which preserve the structure of an atom and the direction of the growth of th function. Then we take the quotient by the subgroup which sends every edge of the K into itself with preservation of any given orientation on it. This group is oviously descrete and we call it the symmetry group of an atom. In case of orientable surface P we call a symmetry proper if it preserves the orientation on P. Saddle critical values of the f are called atom vertecies and their number is atom's complexity.

As the boundary of P is disjoint union of circles the surface M with the proper Morse function f described in the beginning of this section is obtained from P by gluing the boundary of some two-dimentional disc to every such circle (different discs are taken for different circles). The proper Morse function on M is a continuation of f such that in every glued disc there's exactly one maximum or minimum in its center. The genus of an atom is the genus of M. Orientable atom is called maximally symmetric [30] if its symmetry group acts transitively on its edges.

There's a natural bijection between atoms and maps (abstract polytopes) [30]. Recollect that [30, 33] starting with a given map we can construct 2-atom (same as proper Morse function on M). That's why all the results about atoms can be if needed reformulated equivalently in terms of maps and their symmetry groups.

These are the main results.

Theorem 1.1 (Fomenko and Kudryavtseva [29]) *Every finite group G is a symmetry group of some orientable two-dimensional atom*

It should be noted that the atom $X(G)$ is constructed explicitly. Moreover there are lots of atoms with symmetry group G, but the properties of the $X(G)$ constructed in the the proof of the theorem can be described in great detail, e.g. there is an upper estimation on the genus of M.

Let k, d be a pair of non-negative integers with condition $d \geq 5$ for any $k \geq 0$ and $d \geq 6$ for $k = 0$. For every such a pair it's possible to construct an orientable atom $T(k, d)$ of genus k with boundary consisting of exactly d circles and trivial symmetry group. Let n be a number of $T(k, d)$ vertices. Define d_- as the number of non-positive boundary circles and d_+ as number of positive boundary circles of $T(k, d)$. Obviously $d = d_+ + d_-$.

Theorem 1.2 (Fomenko and Kudryavtseva [29]) *Let k be a number (not necessary least possible) of generators of a given group G. Among all the atoms with symmetry group isomorphic to G there exists orientable atom $X(G)$ of genus $g = (k-1)|G|+1$ where $|G|$ is the order of the group G. At the same time atom $X(G)$ is G-regular covering of $T(k, d)$ and has $n|G|$ vertices, $d_-|G|$ negative boundary circles and $d_+|G|$ positive boundary circles. Every symmetry of sucn atom preserves its orientation.*

We should notice that the symmetry group G of an atom $X(G)$ acts freely on its vertices and boundary circles, that is every symmetry is either trivial or doesn't have any invariant vertices and boundary circles. More accurate estimates in some special cases given in [29].

1.3 Integer Lattices of Action Variables for "Spherical Pendulum" System

The problem of constructing integer lattices of action variables for integrable Hamiltonian systems with two degrees of freedom is relatively new. It appeared due to Fomenko's hypothesis, which states that the structure of integer lattice of action variables (see the definition below) in "typical case" is completely determined by Fomenko-Ziechang invariants (FZ invariants). On the other hand such integer lattice allows one to calculate at least some of the FZ-invariants of such systems, particularly the marks on the ribs of "molecule" (see [4] and also Sect. 1.4). It should be noted that in this case we need not only the lattice but the level lines of action variables.

This hypothesis was proved in case of the Spherical Pendulum by Kantonistova [26]. She obtained analytical description of the action variables, momentum map and bifurcation diagram. It turns out, that in this case it's possible to describe the algorithm for calculating the monodromy matrix of the isolated singular value and, therefore, the marks of the molecule.

Definition 1.1 Let (M^{2n}, ω, H) be an integrable Hamiltonian system with n degrees of freedom, and $F_1, \ldots, F_n$ are its first integrals ($F_1 = H$). The map $\Phi = (F_1, \ldots, F_n) : M^{2n} \to R^n$ is called *momentum map*.

The approach of Kantonistova is as following. From the Liouville theorem (see [4, Sect. 1.5]) it's well-known that regular connected compact level surface T_{ξ_0} of integrals $F_1, \ldots, F_n$ on Hamiltonian integrable system on M^{2n} described in definition is an $n-$dimensional torus T^n. Moreover there exists a set of coordinates $(I_1, \ldots, I_n, \varphi_1, \ldots, \varphi_n)$ in the neighbourhood $U(T_{\xi_0})$ of the T_{ξ_0} such that I_i depend only on the first integrals, and expressed by the formulas

$$I_i(\xi) = \frac{1}{2\pi} \oint\limits_{\gamma_i(\xi)} \alpha, \tag{1.1}$$

where $\gamma_1(\xi), \ldots, \gamma_n(\xi)$ are 1–cycles on the Liouville torus T_ξ, which form the basis of the homology group $H_1(T_\xi)$ continiously depending on $\xi \in U(\xi_0)$, and α being any 1–form in $U(\xi)$, such that $d\alpha = \omega$. This coordinates are called action-angle variables, I_i being the angle variables.

Definition 1.2 Fix 1–form α on the connected open domain P^{2n} in M^{2n}, such that $d\alpha = \omega$. Let all nonsingular Liouville fibers laying in the domain P^{2n} be compact and connected. Than a set of points in $\Phi(P^{2n})\backslash\Sigma \subset R^n$, formed by the intersections of n level-surfaces $\{\xi \in \Phi(P^{2n})\backslash\Sigma \mid I_1(\xi) = c_1, \ldots, I_n(\xi) = c_n,\ c_i \in Z\}$ of functions I_i, defined above, is called *the integer lattice* $\mathscr{R}$ *o*f action variables (or simply *the lattice*).

In the case of spherical pendulum such 1–form α exists on the cotangent bundle $M^4 = T^*S^2$ and we can assume that $P^4 = M^4$.

Definition 1.3 The Spherical Pendulum is the system which describes the movement of the particle with mass m, confined to the surface of the sphere with radius R in a uniform gravitational field of strength g.

The phase space of the system is

$$T^*S^2 \cong \{(\mathbf{x}, \mathbf{p}) \in R^3 \times R^3 \mid x^2 + y^2 + z^2 = 1,\ xp_x + yp_y + zp_z = 0\}. \quad (1.2)$$

For further calculations it is convenient to introduce the following coordinates on the phase space: φ, $p_\varphi = M_z$, θ, p_θ.

The system has two independent integrals $F_1 = E$ (energy antegral) and $F_2 = p_\varphi = M_z$ (cyclic integral). Hence, the phase space is fibered into two-dimensional surfaces, which are according to Liouville theorem the 2−tori (in regular connected case).

Consider the functions $W(E, M_z, z)$ and $M_z(E)$, where:

$$W(E, M_z, z) = 2(E - z)(1 - z^2) - M_z^2, \quad (1.3)$$

where

$$M_z(E) := \frac{29}{(3 - E^2 + E\sqrt{E^2 + 3})}\sqrt{E + \sqrt{E^2 + 3}}. \quad (1.4)$$

Theorem 1.3 (Kantonistova [26]) *The image of the momentum map for given integrals is the variety* $\{(E, M_z) |\ E \geq -1,\ |M_z| \leq M_z(E)\}$.

The only interesting pairs (E, M_z), are the ones laying in the image of the momentum map.

Theorem 1.4 (Kantonistova [26]) *For all pairs* $(E, M_z) \in \Phi(M^4)$ $I_1(E, M_z) = M_z$ *is an action variable. For all pairs* $\{(E, M_z) \in \Phi(M^4) \mid M_z > 0\}$ *(similar for all pairs* $\{(E, M_z) \in \Phi(M^4) \mid M_z < 0\}$*) the action variable* $I_2(E, M_z)$ *is defined by the formula*

$$I_2(E, M_z) = \frac{1}{\pi} \int_{z_1}^{z_2} \frac{\sqrt{2(E-z)(1-z^2) - M_z^2}}{1-z^2} dz, \tag{1.5}$$

where z_1, z_2 are the roots of the equation $W(E, M_z, z) = 2(E-z)(1-z^2) - M_z^2 = 0$ such that $-1 < z_1 < z_2 < 1$ and $W(E, M_z, z) > 0$ for all $z \in (z_1, z_2)$.

It turns out that there exists exactly two singular values of rank 0, namely $(E, M_z) = (\pm 1, 0)$. In the preimage of every singular value there is exactly one singular point of the momentum map on M^4, and its rank is 0. Moreover, the bifurcation diagram of momentum map of the system is composed of two sets: piecewise smooth curve defined by the equation $|M_z| = M_z(E)$, where

$$M_z(E) := \frac{29}{(3 - E^2 + E\sqrt{E^2+3})} \sqrt{E + \sqrt{E^2+3}}, \quad E \geq -1 \tag{1.6}$$

and isolated singular point with the coordinates $(E, M_z) = (1, 0)$. The singular point of rank 0, corresponding to isolated singular value $(E, M_z) = (1, 0)$, is nonsingular and has a focus–focus type. The singular point of rank 0, corresponding to isolated singular value $(E, M_z) = (-1, 0)$, is nonsingular and has a center–center type.

To calculate the lattice for this system the computer program on C++ (with the help of Wolfram Mathematica 7.0 package) was written by Kantonistova. This program numerically solves the system of equations with respect to variables E and M_z:

$$\begin{cases} I_1(E, M_z) = A \\ I_2(E, M_z) = B \end{cases} \tag{1.7}$$

for all possible pairs $(A, B) \in Z \times Z$.

The result, i.e. the pair of numbers (E, M_z), is drawn on the plane $R^2(E, M_z)$. The set of all pairs which are the solutions of this system forms the required integer lattice of action variables.

The order of coordinate axes on $R^2(E, M_z)$ containing the image of momentum map determines the orientation (i.e. defines the positive direction of circuit) on $R^2(E, M_z)$ and on the image of momentum map. Fix the numbering of functions I_1, I_2 in such way that the orientation given by them in every regular point of the image of momentum map coincide with the orientation induced by the numbering of coordinate axes. Then fix the order of level-lines of action variables I_1, I_2 by the method described above. There appears uniquely defined direction of the circuit around the isolated singular value of the momentum map. Start "basis" (e_1, e_2) of the lattice $\mathscr{R}$, which is not passing through the singular point. Go around the singular value in a closed loop in the positive direction. As the result a new "basis" (e_1', e_2') is derived from (e_1, e_2).

Let $\widetilde{M}$ be the transition matrix between the bases (e_1, e_2) and (e_1', e_2'), where $(e_1', e_2') = \widetilde{M}(e_1, e_2)$. $\widetilde{M}$ is related to monodromy matrix. Recall the definition of monodromy matrix.

Definition 1.4 Let (γ_1, γ_2) be the basis cycles on the torus T_ξ (where ξ is the regular value of momentum map), (γ_1', γ_2') be the result of deformation of the cycles around the singularity. Matrix M such that $(\gamma_1', \gamma_2') = M\dot{(}\gamma_1, \gamma_2)$ is called *the monodromy matrix of singular value of the system with respect to the basis* (γ_1, γ_2) *on the torus.*

Recall that there exists the canonical morphism between the set of cycles on torus T_ξ (where the value ξ is regular) and the set of the integer covectors of the lattice in $\mathscr{R}$. For every such cycle on the torus there exists some function (action variable) on the base of Liouville fibration, which diferential is the desired covector.

Let $(\varepsilon^1, \varepsilon^2)$ and $(\varepsilon^{1'}, \varepsilon^{2'})$ be the covector bases corresponding to vector bases (e_1, e_2) and (e_1', e_2'). Using the morphism above we can reformulate the definition of monodromy matrix.

Definition 1.5 Matrix M such that $(\varepsilon_1', \varepsilon_2') = M\dot{(}\varepsilon_1, \varepsilon_2)$ is called the monodromy matrix of singular value of the system with respect to the covector basis $(\varepsilon_1, \varepsilon_2)$ of the lattice $\mathscr{R}$.

There exits a relation between matrices M and $\widetilde{M}$: $M = \widetilde{M}^{t^{-1}}$. Moreover, for the "Spherical Pendulum" system monodromy matrix corresponding to isolated singular value $(E, M_z) = (1, 0)$ belongs to the conjugation class of matrix $M = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ in the group $\mathrm{SL}(2, Z)$. The main result of is the following.

Theorem 1.5 (Kantonistova [26]) *For the "Spherical Pendulum" system for the energy levels* $-1 < E < 1$ *there is a mark* $r = 0$*, and for the energy levels* $E > 1$ *there is a mark* $r = \frac{1}{2}$.

If $E_0 > 1$ than the level surface $Q^3_{E_0} = \{E = E_0\}$ is diffeomorfic to RP^3, and if $-1 < E_0 < 1$ than the $Q^3_{E_0}$ is diffeomorfic to S^3.

1.4 Billiards in Confocal Quadrics

Let Ω be the domain in R^2, bounded by some quadrics from the family:

$$\frac{x^2}{a - \Lambda} + \frac{y^2}{b - \Lambda} = 1, \quad a > 0,\ b > 0, \tag{1.8}$$

where $\Lambda \in R$ is a numerical parameter. Assume that angles of the boundary of the Ω differ from $\frac{3\pi}{2}$. Consider the dynamical system describing the motion of the material point inside this domain with the reflection rule on the boundary $P = \partial\Omega$ (including the vertices) being the equality of angles before and after reflection.

The phase space of this system is the manifold

$$M^4 := \{(x, v) \mid x \in \Omega, v \in T_x R^2, |v| > 0\}/\sim, \tag{1.9}$$

with the following equivalence relation:

$$(x_1, v_1) \sim (x_2, v_2) \quad \Leftrightarrow \quad x_1 = x_2 \in P, \quad |v_1| = |v_2| \quad \text{and} \quad v_1 - v_2 \perp T_{x_1} P. \tag{1.10}$$

Where $T_x P$ stands for a tangent line in the point x to the smooth part of boundary $P = \partial\Omega$ and $|v|$ is a euclidian length of velocity v.

Theorem 1.6 (Jacobi, Shasles) *Tangent lines to the geodesic curve on the quadric $u \in R^n$ (where u belongs to the family of the confocal quadrics U) are tangent to the $n-1$ different confocal quadrics $(u_1, ...u_{n-1}) \in U$. The set $(u_1, ...u_{n-1})$ is the same for all points of the initial curve.*

Due to this theorem in two-dimensional case the tangent line to each points of trajectory of the billiard is tangent to an ellipse or a hyperbola confocal to the family of quadrics that define the boundry of Ω. In other terms this billiard system has two integrals: velocity $|v|$ and the parameter of the family Λ. Fixing $|v|$ restricts the system onto three-dimensional surface $Q^3 \in M^4$.

Definition 1.6 Let $(M_1^4, \omega_1, f_1, g_1)$ and $(M_2^4, \omega_2, f_2, g_2)$ be two integrable Hamiltonian systems on the simplectic manifolds M_1^4 and M_2^4 with integrals f_1, g_1 and f_2, g_2, respectively. Consider restriction of these systems onto the "isoenergetic" manifolds $Q_1^3 = \{\xi \in M_1^4 \mid f_1(\xi) = c_1\}$ and $Q_2^3 = \{\xi \in M_2^4 \mid f_2(\xi) = c_2\}$. Two Liouville foliations on the Q_1^3 and Q_2^3 are Liouville equivalent iff there exists a diffeomorphism that sends the leaves of the first foliation to those of the second one (with two rather technical restrictions on the orientation) [4].

According to the Liouville theorem the manifold Q^3 is foliated into two-dimentional tori (see [4, Sect. 1.5]) and singular leaves. Consider the base of this Liouville foliation. It is a one-dimensional graph W, which is called rough molecule. The "atoms" describing the corresponding neighborhoods of the singular leaves are the vertices of the graph W. In the classical billiards the following atoms appear: A, A^*, B, C_2, D_1 [4].

All these atoms except atom A^* can be described as the cartesian product of the two-dimensional atom and one-dimensional circle S^1. For A this two-dimensional atom is a disk D^2, for atom B it is a neighborhood of the bouquet of two circles and for C_2 this two-dimensional atom is a neighborhood of two intercecting (in two points) circles on the plane. For the atom D_1 we should take a neighborhood of the three circles intersecting in two points. Finally, the atom A^* is similar to the atom B: we start with the bouguet of two circles. The only difference is that when the neighbourhood of the bouquet is multiplied by S^1 it should be cut in one place and then twisted. As a result we obtain three-dimensional manifold with boundary consisting of two (instead of three in the case of B) two-dimensional tori.

Rough molecule W does not describe the topology of the Liouville foliation completely because it does not have information about the gluing of the singular leaves. To save the information we can choose basis in the fundamental group of the boundary tori of all atoms (according to the proper set of rules, see [4, 18, 20, 21]

for the list) and write transformation matrices called the gluing matrices. These matrices are dependent on the choice of the basis, but it's possible to calculate a set of numbers, which are both independent from the choice of basis and at the same time allow to write gluing matrices after fixing the basis. This numbers are called marks. Together with the graph they form marked molecule, which is an invariant of the Liouville equivalence.

Dragovich and Radnovich calculated these marks for some billiard systems [12] in the domain bounded by confocal quadrics. The work was continued by Fokicheva [15, 16], who not only completed calculation, but also found that some of the original results by Dragovich and Radnovich contained errors. Fokicheva also did the calculations for so called "covering billiards". The latter notion was introduced by Oshemkov and Kudryavtseva

Let us start with the classification of all possible domains Ω. We call domain regular if its boundary doesn't contain horizontal line segments and singular otherwise.

Definition 1.7 Domain Ω bounded by quadrics from the family (1.8) is called *equivalent* to domain Ω' bounded by quadrics from the same family (1.8) iff Ω' obtained from Ω by symmetries via axes and/or continuous change of parameter Λ with the only condition $\Lambda \neq b$.

For all pairs of real numbers $a > b > 0$ there are 7 classes of equivalence of the regular compact domains Ω and 6 classes of singular domains. This is the list of regular domains:

Domain and its notation	Boundary
Ω_1	Ellipse
$\Omega_{1.1.1}$	Ellipse and hyperbola
$\Omega_{1.1.2}$	Ellipse and two hyperboles
$\Omega_{1.2}$	Two ellipses
$\Omega_{1.2.1}$	Two ellipses and two hyperbolas
$\Omega_{1.2.2}$	Two ellipses and two hyperbolas
$\Omega_{1.2.3}$	Two ellipses and two hyperbolas

This is the list of singular domains:

Domain and its notation	Boundary
ω_1	Ellipse and horizontal line
ω_2	Ellipse, horizontal line and hyperbola
ω_3	Ellipse, two hyperbolas and horizontal line
ω_4	Two ellipses, one segment of hyperbola and one horizontal segment
ω_5	Two ellipses and two segments of the horizontal line
ω_6	Two ellipses, one hyperbolic segment and one horizontal segment inside domain

Theorem 1.7 (Fokicheva [15, 16]) *For all 1-connected domains the isoenergetic manifold Q^3 is homeomorphic to sphere S^3, for domain bounded by two ellipses this manifold is homeomorphic to $S^1 \times S^2$*

For the domains which boundary is smooth this fact was mentioned in the article [25]. Consider now the integral value $\Lambda < b$ and the trajectory corresponding to it. This trajectory (or its continuation) is a tangent to the ellipse of the family (1.8) with the parameter Λ and lies on the Liouville torus on the lower edge of the molecule. The trajectory corresponding to the integral value $\Lambda > b$ (or its continuation) is a tangent to the hyperbola and lies on the upper edge of the molecule. The trajectory corresponding to the irregular value $\Lambda = b$ (or its continuation) must go through the focuses of the family. The critical circles on the corresponding leaf lie along the horizontal line. The number V of such circles coincides with the number of the horizontal segments inside the domain. If $V \neq 0$ then the neighborhood of such leaf is a saddle atom. Otherwise this is a torus and the molecule of such system is A–A.

Theorem 1.8 (Fokicheva [15, 16]) *The marks and the saddle atom describing the topology of the billiard system in regular domain are written in the following table. The molecule describing the topology of the billiard system in domain $\Omega_{1.2.1}$ is A–A, with marks $r = 1$ and $\varepsilon = \pm 1$.*

Domain	V	Saddle atom	Lower edges	Upper edges	n
Ω_1	1	B	$r = 0\ \varepsilon = 1$	$r = 0\ \varepsilon = 1$	± 1
$\Omega_{1.1.1}$	1	A^*	$r = 0\ \varepsilon = 1$	$r = 0\ \varepsilon = 1$	± 1
$\Omega_{1.1.2}$	1	B	$r = 0\ \varepsilon = 1$	$r = \infty\ \varepsilon = \pm 1$	
$\Omega_{1.2}$	2	C_2	$r = \infty\ \varepsilon = \pm 1$	$r = 0\ \varepsilon = 1$	
$\Omega_{1.2.1}$	0	–	–	–	
$\Omega_{1.2.2}$	1	B	$r = \infty\ \varepsilon = \pm 1$	$r = 0\ \varepsilon = 1$	
$\Omega_{1.2.3}$	2	D_1	$r = \infty\ \varepsilon = \pm 1$	$r = 0\ \varepsilon = 1$	

The following theorem completes the discription of the topology of billiard systems in confocal quadrics.

Theorem 1.9 (Fokicheva [15, 16]) *The molecule describing the topology of the billiard system in singular domain (except the domain ω_6) is A–A, $r = 0$, $\varepsilon = 1$. For ω_6 the molecule coincides with the molecule for the system in the domain $\Omega_{1.2.2}$.*

The billiard systems admit the following generalization. Consider k copies of the domain $\Omega_{1.2}$ (this domain is bounded by two ellipses) and make a cut along the lower segment of the coordinate line Oy. Then glue cuts by the following rule: the left edge of the cut on the i-th copy is glued to the right edge of the cut on the $i + 1$-th copy. This domain is called Δ_k. If we glue the rest of the edges of the cut together we get the domain Δ'_k.

For the first time the problem of the describing this billiard system was proposed by Oshemkov and Kudryavtseva. Fokicheva showed that the isoenergetic manifold Q^3 for the system in the domain Δ_k is homeomorphic to $S^1 \times S^2$. She also computed that the saddle atom describing the topology of the billiard system in Δ_k is A_{2k-1} [30], marks are same as marks for the molecule for domain $\Omega_{1.2}$. The isoenergetic manifold Q^3 for the system in the domain Δ'_k is homeomorphic to S^3. The saddle atom describing the topology of the billiard system in Δ'_k is D_{2k-1} [30], marks are like marks in the molecule for domain $\Omega_{1.2.3}$.

1.5 Bertrand's Manifolds and Their Properties

Consider the movement of a particle (moving point) in a central potential field on a surface of bounded revolution, that is on a manifold $S \approx (a, b) \times S^1$ with the metric of revolution $ds^2 = dr^2 + f^2(r)d\varphi^2$ in polar coordinates $(r, \varphi mod 2\pi)$ for some arbitrary smooth function $f(r)$. Denote the potential by $V(r)$. The system is Hamiltonian with hamiltonian $H = \frac{p_r^2}{2} + \frac{p_\varphi^2}{2f^2(r)} + V(r)$. This construction appears naturally in many physical and mechanical problems.

Originally this problem was formulated by Bertrand in 1873: *if $S = R^2$ and all the bounded trajectories of the particle are closed (regardless of initial conditions) what can be said about the potential $V(r)$?* This problem was solved by Bertrand himself. Later it was generalized as follows: *consider a class $\mathscr{P}$ of central potentials with certain properties (that is all the potentials yielding closed trajectories for a given class of initial conditions) defined on the surface S; the problem is to find all the pairs $(f \in C^\infty, V \in \mathscr{P})$, i.e. classify all manifolds of revolution allowing a potential of the chosen class to exist and to describe all the corresponding potentials.*

Definition 1.8 Manifolds of revolution equipped with a central potential of class $\mathscr{P}$ are called *Bertrand's $\mathscr{P}-$manifolds*.

For Bertrand's manifolds with such $f(r)$ that $f'(r) \neq 0\ \forall r \in (a, b)$ the latter problem was completely solved by Fedoseev et al. in the work [13] which generalizes results obtained by Bertrand [2], Santoprete [37], Darboux, Libman and others.

In their work Fedoseev et al. considered the following classes of potentials:

Definition 1.9 Let $V(r)$ be a central potential on the surface S. It is called *closing*, if

($\exists$) there exists a nonsingular bounded noncircular orbit γ in S;
($\forall$) every nonsingular bounded orbit in S is closed.

Potential $V(r)$ is called *locally closing*, if

$(\exists)^{loc}$ there exists a strongly stable circular orbit $\{r_0\} \times S^1$ in S;
$(\forall)^{loc}$ for every strongly stable circular orbit $\{r_0\} \times S^1$ in S there exists an $\varepsilon > 0$, that every nonsingular bounded orbit in $[r_0 - \varepsilon, r_0 + \varepsilon] \times S^1$ with kinetic

moment in $(K_0 - \varepsilon, K_0 + \varepsilon)$, is closed; K_0 is the kinetic moment value for the corresponding circular trajectory.

Potential $V(r)$ is called *semi-locally closing*, if the conditions $(\exists)$, $(\forall)^{loc}$ are satisfied as well as the following:

$(\forall)$sloc every nonsingular bounded orbit in $U = [a', b'] \times S^1$ with kinetic moment value equal to $\hat{K}$ is closed, where $a' := \inf r|_\gamma$, $b' := \sup r|_\gamma$, γ is the bounded orbit existing due to $(\exists)$, $\hat{K}$ it's kinetic moment value.

Potential $V(r)$ is called *strongly* (*weakly*) *closing*, if the condition $(\forall)^{loc}$ is satisfied (it's analog for every orbitally stable circular orbit) and the following condition is satisfied: every circle $\{r\} \times S^1$ is strongly (orbitally) stable circular orbit.

The following theorem by Fedoseev et al. gives an explicit solution to the stated above generalized Bertrand problem on surfaces (manifolds) of revolution without "equators" (i.e. points $x \in (a, b)$ such that $f'(x) = 0$).

Theorem 1.10 (Fedoseev et al. [13]) *Consider a manifold of revolution $S \approx (a, b) \times S^1$ with the metric $ds^2 = dr^2 + f^2(r)d\varphi^2$ in polar coordinates $(r, \varphi mod 2\pi)$ and $f' \neq 0$ on (a, b). Then*

(a) the above defined classes of potentials coincide (therefore from now on we call the potentials in question "closing" meaning potentials of all the defined types);

(b) if there exists such a $\xi \in Q_{>0}$ that the following equality $-f'^2(r) + f(r)f'(r)' = -\xi^2$ holds for every $r \in (a, b)$ on the corresponding surface (Bertrand's manifold of type I) that there exists exactly two closing potentials and they are of the form $V_i(r) = (-1)^i A|\theta(r)|^{2-i^2}/i + B$, $i = 1, 2$, where $A > 0$, B are arbitrary constants, $\theta(r) = -\frac{f'(r)}{f(r)}$;

(c) if for every $\xi \in Q_{>0}$ the equality $-f'^2(r) + f(r)f'(r)' = -\xi^2$ is not tautological, there exists a smooth function $\theta = \theta(r)$, $\theta(r) \neq 0$ on (a, b) such that in the coordinates $(\theta, \varphi mod 2\pi)$ the metric can be written as $ds^2 = \frac{d\theta^2}{(\theta^2 + c - t\theta^{-2})^2} + \frac{d\varphi^2}{\mu^2(\theta^2 + c - t\theta^{-2})}$ for some constants $c \in R, t \in R \setminus \{0\}, \mu \in Q_{>0}$ and there also exists exactly one closing potential on the corresponding surface (Bertrand's manifold of type II) which is of the form $V_2(r) = \frac{A}{2\theta(r)} + B$, $A, B \in R$.

Bertrand's manifolds of type I can be described explicitly. They are the rational cones and rational coverings of sphere and hyperbolic plane, that is

$$f(r) = \xi f_c(r - r_0) := \begin{cases} \pm\xi(r - r_0), & c = 0, \\ \frac{\xi}{\sqrt{c}} \sin(\sqrt{c}(r - r_0)), & c > 0, \\ \pm\frac{\xi}{\sqrt{-c}} \mathrm{sh}(\sqrt{-c}(r - r_0)), & c < 0, \end{cases} \tag{1.11}$$

Type II Bertrand's manifolds are classified with the pair of parameters (c, t); the parameter μ is irrelevant to the geometry of the manifold. The following rigor definition follows from the Theorem 1.10:

Definition 1.10 Bertrand's manifold is a manifold $S_{\mu,c,t} \approx \cup_{k=1}^{k_{c,t}} I_{k,c,t} \times S^1$, $I_{k,c,t} \subset (-\infty, 0)$, with coordinates θ, $\varphi mod 2\pi$, and the metric of revolution

$$ds^2_{\mu,c,t} = \frac{d\theta^2}{(\theta^2 + c - t\theta^{-2})^2} + \frac{d\varphi^2}{\mu^2(\theta^2 + c - t\theta^{-2})}. \tag{1.12}$$

where $c, t \in R$, $\mu > 0$. The manifold consist of $k_{c,t}$ connected components. The component corresponding to $k = 1$, is called main, corresponding to $k = 2$—additional. The additional component exists only if $t < 0$. Manifolds with $t = 0$ are called type I manifolds, with $t \neq 0$—type II.

Not all the type II Bertrand's manifolds are real *surfaces* of revolution embedded in R^3. The following result holds:

Theorem 1.11 (Fedoseev et al. [13])

1. *Additional component is never realized as a surface of revolution in* R^3*;*
2. *Main component is realized completely as a surface of revolution for the following values of the parameters* (c, t) *and only for them:* $\{t = 0, c \geq 0, \mu \geq 1\} \cup \{t < 0, c = -2\sqrt{-t}, \mu \geq 2\} \cup \{t < 0, c \geq 0, \mu \geq 1\} \cup \{t < 0, -2\sqrt{-t} < c < 0, \mu \in (1, \tilde{\mu})\} \cup \{t < 0, -2\sqrt{-t} < c < 0, \mu \in [2, \infty)\}$*, where* $\tilde{\mu}$ *is a real positive root of the equation* $-256t + 192tx^2 + (27c^2 + 60t)x^4 + 4tx^6 = 0$.

Partial realization of the main component of type II Bertrand's manifolds was also completely studied by the authors.

Movement in a closing potential field on a Bertrand's manifold is an integrable Hamiltonian system with an additional integral p_φ. Therefore a classical hamiltonian analysis can be performed as well as the construction of Fomenko-Zishang invariants of Liouville's equivalence. This was done by Zagryadsky et al. in [13].

It appears that Bertrand's systems are a simple and natural example of Hamiltonian systems with non-compact atoms. Therefore those systems are a good testing case for the non-compact analog to the theory of Fomenko-Zishang invariants.

It is also possible to generalize this problem further to the "Hamiltonian" study of movement on an arbitrary surface of revolution. Some results concerning the connections between the properties of the function $f(r)$ and the bifurcation diagram, the image of the moment map, atoms and molecules were obtained by Zagryadsky et al.

Bertrand's problem for the manifolds with equators is still under consideration as well as the problem on pseudo-Riemannian manifolds of revolution. On those manifolds the study of dynamics and topology of Liouville foliation is also a natural and promising problem.

1.6 Lie Algebras with Generic Coadjoint Orbits of Dimension Two

The Euler top is a textbook example of integrable system in classical mechanics. One of the reasons for its popularity is the simplicity of the analysis of its dynamics. Hamiltonian representation on dual space to Lie algebra $so(3)$ allows to easily see the effects of small perturbations of "stable rotations", for example. In this case we don't need to write the explicit solution in terms of elliptic functions. Instead we can just look at the intersection of two-dimensional symplectic leaves and level surface of system's Hamiltonian. The intersection is one-dimensional and is an integral trajectory of a system or a disjoint union of such trajectories.

The simplicity is mainly due to the fact that in case of Poisson structure with symplectic leaves of dimension two the restriction of every Hamiltonian system on such leaf is integrable as it needs only one integral that is Hamiltonian itself. Therefore the same approach for let's say dynamical analysis holds for two-dimensional leaves in case of a linear Poisson manifold of higher dimension. This posses a natural question: *what are the Possion brackets with generic symplectic leaves of dimension two or, in other terms, what are the Lie algebras with generic coadjoint orbits of dimension two?* This question was formulated in [8] by Bolsinov et al. The complete answer to the question is given by Konyaev in [27] (see also [38] for similar question for complex Lie algebras in terms of homogeneous spaces). It turns out that it is possible to classify up to isomorphism all real Lie algebras with generic coadjoint orbits of dimension two.

Definition 1.11 We call Poisson bracket with Poisson tensor of rank less or equal to 2 *decomposable* if there exist two vector fields v and w such that Poisson tensor equals to $v \wedge w$.

From the properties of Schouten bracket immediately follows that wedge product of two vector fields defines Poisson bivector iff the distribution spanned by these fields is integrable. This construction gives a lot of simple examples of polynomial bivector fields. For example, if v and w are both linear then their wedge product defines quadratic Poisson bracket.

In linear case the decomposable brackets define a dual space to the series of Lie algebras in the form of semidirect sum $R +_\rho R^n$ via representation ρ. These Lie algebras are solvable. The vector fields, that define their Lie-Poisson tensor can be chosen in the form of one linear field and one constant vector field, that also commute and everywhere independent. As both fields are complete, that is their trajectories exist for all times $-\infty < t < \infty$, and tangent to the symplectic leaves, it can be shown that all the generic leaves are diffeomorphic to the two-dimensional plane R^2.

Main tool for the classification of Lie algebras with generic coadjoint orbits of dimension two is the following result by Konyaev, concerning linear vector field.

Theorem 1.12 (Konyaev [27]) *Consider a pair of linear vector fields v and w on affine space R^n, that are everywhere dependent, i.e. $v \wedge w = 0$. Than at least one of the following is true:*

- *$v = \lambda w$, where λ is a constant,*
- *$v = l(x)a$, $w = m(x)a$, where a is a constant vector and $l(x), m(x)$ are linear functions*

The following theorem provides a complete description of the Lie algebras with generic coadjoint orbits of dimension two. It should be noted that the first infinite series of Lie algebras is a result of the first part of the previous theorem, while the exceptional cases are central extensions of three-dimensional Lie algebras that are not isomorphic to the first infinite series.

Theorem 1.13 (Konyaev [27]) *Up to the direct sum with commutative Lie algebra of orbitrary dimension there exists one infinite series of real Lie algebras with generic coadjoint orbits of dimension two and six exeptional Lie algebras. The exeptional Lie algabras are pairwise non-isomorphic and are not isomorphic to any Lie algebra from the infinite series:*
(1) Semidirect sums $R +_\rho R^n$
(2) Three-dimensional simple Lie algebra $so(3)$
(3) Three-dimensional simple Lie algebra $sl(2)$
(4) Four-dimensional solvable Lie algebra $A_{4,8}$. In the special basis e_1, e_2, e_3, e_4 the commutative relations for this Lie algebra have the form (given only non-zero commutators):

$$[e_2, e_3] = e_1, [e_2, e_4] = e_2, [e_3, e_4] = -e_3$$

(5) Four-dimensional solvable Lie algebra $A_{4,10}$. In special basis e_1, e_2, e_3, e_4 the commutative relations for this Lie algebra have the form (given only non-zero commutators):

$$[e_2, e_3] = e_1, [e_2, e_4] = -e_3, [e_3, e_4] = e_2$$

(6) Five-dimensional solvable Lie algebra $A_{5,3}$. In special basis e_1, e_2, e_3, e_4, e_5 the commutative relations for this Lie algebra have the form (given only non-zero commutators):

$$[e_3, e_4] = e_5, [e_3, e_5] = e_1, [e_4, e_5] = e_3$$

(7) Six-dimensional nilpotent Lie algebra $A_{6,3}$. In special basis $e_1, e_2, e_3, e_4, e_5, e_6$ the commutative relations for this Lie algebra have the form (given only non-zero commutators):

$$[e_1, e_2] = e_6, [e_1, e_3] = e_4, [e_2, e_3] = e_5$$

To complete the classification one needs a theorem, that discribes Lie algebras from the infinite series $R +_\rho R^n$ up to the isomorphism.

Definition 1.12 We call two linear operators P and P' equivalent iff for some non-zero constant μ operators P and $\mu P'$ are adjoint, that is have the same Jordan normal forms.

Definition 1.13 We call two linear representations ρ and ρ' of R in $gl(R^n)$ equivalent iff for any $v \in R$ and $v \neq 0$ operators $\rho(v)$ and $\rho'(v)$ are equivalent.

Theorem 1.14 (Konyaev [27]) *Consider a pair of Lie algebras $R+_\rho R^n$ and $R+_{\rho'} R^n$. They are isomorphic iff the representations ρ and ρ' are equivalent.*

In other words the set of semidirect sums is in bijection with the set of equivalence classes of operators.

In [28, 36] the invariants for the exceptional Lie algebras are given. For the infinite series the invariants are the first integrals of some linear vector fields. They have also been computed see [6, 7]. Recently Konyaev found that these invariant admit a simpler description.

Acknowledgments This work was supported by the Government grant of the Russian Federation for support of research projects implemented by leading scientists, in the Federal State Budget Educational Institution of Higher Professional Education Lomonosov Moscow State University under the agreement No. 11.G34.31.0054.

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