Chapter 7
Reasoning About Knowledge in Context

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7.1 Introduction

A major goal of post-cartesian epistemology is to respond to radical skepticism, the view that we know (almost) nothing of what we ordinarily take ourselves to know. The argument most commonly associated with this view is the so-called "argument from ignorance". It starts with the premise that we cannot exclude the possibility that we might be in such alternative worlds as those described by Descartes' *Evil Genius* hypothesis or by Putnam's *Brain in a Vat* scenario. The alleged reason for this inability is that these skeptical hypotheses are designed in such a way that if we were in the skeptical worlds they describe, we'd have exactly the same experiences, memories, beliefs, etc., as those that we actually have, so that for all we know, we might be in these worlds of mass(ive) error. From here, the skeptic's reasoning takes us to the conclusion that we do not know any (or most) of the things that we ordinarily take ourselves to know, for instance, that we have hands, that we are sitting at our desk, etc.

The problem is, of course, that this skeptical conclusion goes against our powerful tendency to think that we do know a lot about many things. In the last few decades, emphasis has been put on the importance, in dealing with this problem, of taking (some notion of) context into consideration when thinking about knowledge and knowledge ascription. This shows in the ever-growing number of discussions on

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so-called "contextualist" approaches as defended by Cohen (1999, 2000), DeRose (1992, 1995, 1999), Heller (1999a,b), Lewis (1979, 1996) and others, as well as the increasing interest in theories of "subject-sensitive" knowledge ascriptions *à la* Hawthorne (2004) and Stanley (2005), and of "assessment-sensitive" ascriptions following MacFarlane (2005), to mention but a few of the available accounts that have context playing a significant epistemological role.

In Sect. 7.2, we present four major epistemological positions on the problem of skepticism, with a view to showing the importance of (different notions of) context in the recent philosophical discussions on knowledge. In Sect. 7.3, we provide the foundations for a general formal framework based on the technical notion of "contextual models", which will make it possible to capture those epistemological positions using one and the same language and semantics, and to investigate the logical connections they are bound to endorse between knowledge and context. Finally, Sect. 7.4 adds to that "static" formalism a "dynamic" formalization of context based on a simplified version of Discourse Representation Theory, which will allow us to account for the apparent variation over the course of a conversation in the epistemic standards of the participants, and to overcome such difficulties as the logical omniscience problem that usually plagues normal modal logical approaches to knowledge.

In short, our purpose is to provide formal tools for exploring reasoning about knowledge in context in both its static and dynamic aspects.

7.2   Highlights of Informal Epistemology

In this section we briefly describe and illustrate four prominent positions on the problem of skepticism which, through the discussions they have given rise to in the recent philosophical literature on knowledge, have significantly contributed to making context an issue of prime epistemological importance.

7.2.1   Anti-skeptical Invariantism

The first position, *anti-skeptical invariantism* as we will call it, which can also be found in the literature under the name "moderate invariantism" or "radical anti-skepticism", has it that many and perhaps most of our ordinary knowledge claims are literally true. This view is endorsed by ordinary language philosopher Austin (1946),[1] who observes that the epistemic standards that the skeptic has in mind

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[1]It is very common in the literature to find this position associated with the name G. E. Moore, and for this reason called "Moorean invariantism". We find it more convenient to refer to Austin instead.

and which require of us the ability to exclude absolutely all possibilities of error, including the most far-fetched ones, are very different from those which govern our everyday knowledge claims and which do not have such stringent requirements. In everyday life, the standards in place are such that as long as we have no reason to think, for example, that we might be brains in vats, we are not required to rule out this possibility to properly count as knowing, say, that we have hands. Then, according to Austin, the only standards that can be legitimate are those that match our ordinary practice of knowledge ascription. This is the case with the standards that prevail in everyday life: we – ordinarily competent speakers – judge many and perhaps most of our everyday knowledge claims to be correct. Not so with the skeptic's standards, which would make it merely impossible to ever use "know" correctly. So, on this view, we do know a lot.

7.2.2 Skeptical Invariantism

The next position, *skeptical invariantism*, or simply *skepticism*, takes us in exactly the opposite direction by claiming that most and perhaps all of our ordinary knowledge claims are literally false. This claim is held by Unger who, in Unger (1971) for instance, proposes relating the word "know" to a class of natural language expressions that are systematically used erroneously, yet in a pragmatically correct way, by competent speakers, and which he labels "absolute terms", as their application admits no degree/exception. The predicate "flat" is one such term: (most of) our ordinary flatness ascriptions are literally false since a surface is flat only if it has absolutely no bumps or other irregularities on it, but no real physical surface can meet this condition, even microscopically. Real surfaces are, however, close enough to being flat given our everyday life interests, goals, presuppositions, etc., for those ascriptions to be justifiedly made, pragmatically speaking. Likewise, "know" too is an absolute term: because we cannot rule out all logical possibilities of error, (most of) our ordinary knowledge ascriptions are literally false; yet, we are pragmatically justified in making them in that we are close enough, with respect to our everyday purposes, interests, etc., to satisfying the conditions for a true ascription. In any case, on this view, we know almost nothing.

7.2.3 Contextualism

A third position, *contextualism* as it is called, aims at overcoming the apparent conflict between skepticism and anti-skepticism by holding that the truth or falsity of knowledge ascriptions depends on the context in which they are made. It is defended by Lewis (1979, 1996) for instance, who claims that for an ascription of the type "*S* knows that *p*" to be true, the possibilities of error that *S* must be able to rule

out are all and only those that are relevant in the context of the ascription, e.g. those attended to by the ascriber. This condition is met when no not-p possibility is contextually relevant. This is the case in everyday contexts for most contingent propositions, e.g. that we have hands, because in these contexts the far-fetched skeptical possibility that we might be handless brains in vats simply is irrelevant, and therefore need not be ruled out. By contrast, in a philosophical context where such skeptical possibilities have been raised and are being attended to, they are relevant and need to be excluded; but we are unable to rule them out. In these contexts it is therefore false that we know that we are not brains in vats and that we have hands. It is true that we know a lot in ordinary contexts, and very little in skeptical contexts. More generally, on the contextualist view, whether or not it is true that one knows something will depend on the focus, interests, stakes, presuppositions, etc., that make up the context of the "attributor", i.e. the person who is attributing/denying knowledge.

7.2.4 Subjectivism

According to *subjectivism* (or *sensitive moderate invariantism,* or *subject-sensitive invariantism* as it is often called), as defended by Hawthorne (2004) or Stanley (2005), such factors as attention, interests, stakes, etc., are considered relevant to the truth of knowledge ascriptions, but only insofar as they make up the context, not of the attributor, but of the "subject", the person who is being attributed/denied knowledge. This is clear from Hawthorne, who insists that the practical importance, for the subject, of being right and not making a mistake is epistemically crucial, since a subject's anxiety can contribute to making certain possibilities of error salient to him, where salience is equated with relevance. These and only these possibilities will have to be ruled out for the subject to truly count as knowing. Thus, one can know more by worrying less. In particular, "the philosopher who worries about being a brain in a vat, etc., will know less than the dullard who doesn't" (Hawthorne 2004, p 167).

The formal framework to be described in the next section will help us understand more precisely where the connections between subjectivism, contextualism, anti-skepticism and skepticism lie.

7.3 Static Formalism

In this section, we first propose a formal framework for reasoning about knowledge in context, and then show how this framework can be used to capture the various epistemological positions described in the previous section.

7.3.1   The Formal Framework

We first describe the "epistemic language" we will be using throughout the paper. We then provide a "contextual semantics" for it that allows four possible definitions of truth for epistemic formulas, and we investigate what usual logical properties of knowledge are preserved in our proposed semantics and how.

7.3.1.1   Syntax

Definition 7.3.1 (Epistemic Language). Let $\mathcal{A}t$ be a set of atomic formulas and J a set of agents. The language we will be using is defined by:

$$\varphi := \ p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_j\varphi$$

where $p \in \mathcal{A}t$ and $j \in J$.

We use the common definitions of $\vee$, $\rightarrow$, and $\leftrightarrow$ in terms of $\neg$ and $\wedge$. The intended reading of the epistemic operator $K_j\varphi$ is "Agent j knows that φ". As a convention, an *epistemic formula* will be any formula containing at least one occurrence of an epistemic operator, and a *non-epistemic formula* any formula that contains none.

   As should be obvious, the epistemic language that we will be using is simply that of standard epistemic logic.

7.3.1.2   Semantics

The semantics, however, will differ from the standard Kripke semantics in that it will include (i) a set $\mathcal{C}$ of *contexts*, and (ii) a function $\mathcal{R}$ of *relevance* determining the worlds that are relevant in each context:

Definition 7.3.2 (Contextual Model). A *contextual model* for the epistemic language is a structure $\mathcal{M} = \langle W, \{\mathcal{K}_j : j \in J\}, \mathcal{C}, \mathcal{R}, V \rangle$ where (i) W is a non-empty set of worlds, (ii) $\mathcal{K}_j \subseteq W \times W$ is a relation of epistemic accessibility (for each $j \in J$), (iii) $\mathcal{C} = \{c_i : i \in I\}$ is a non-empty set of contexts, which may be finite or not, such that $J \subseteq I$, (iv) $\mathcal{R} : \mathcal{C} \rightarrow \wp(W)^W$ is a function of contextual relevance that associates with each context c_i, for each world w, the set of worlds that are relevant in c_i for w, and (v) $V : \mathcal{A}t \rightarrow \wp(W)$ is a valuation associating with each atom p the set of worlds in which p holds.

Remark. A context c_i can be connected with an agent, i.e., when $i \in J$. But nothing prevents us from connecting contexts with groups of agents instead of individual agents, or with conversations, etc. Also, the semantics considered here remains

neutral as to the *nature* of contexts. (A possible modelling inspired by Discourse Representation Theory and whereby contexts can evolve through time over the course of several assertions will be described in Sect. 7.4).

The idea, then, is to relativize truth not only to (a model and) a world as in standard Kripke semantics, but also to a context:

Definition 7.3.3 (Truth). Given a contextual model $\mathcal{M} = \langle W, \{\mathcal{K}_j : j \in J\}, \mathcal{C}, \mathcal{R}, V\rangle$, a state $w \in W$, and a formula φ in the epistemic language, we can define $\mathcal{M}, c, w \models \varphi$ in four possible ways, depending on the clause we choose for epistemic formulas, as follows:

(i) $\mathcal{M}, c, w \models p$ iff $w \in V(p)$
(ii) $\mathcal{M}, c, w \models \neg\varphi$ iff $\mathcal{M}, c, w \not\models \varphi$
(iii) $\mathcal{M}, c, w \models \varphi \wedge \psi$ iff $\mathcal{M}, c, w \models \varphi$ and $\mathcal{M}, c, w \models \psi$
(iv) $\mathcal{M}, c_i, w \models K_j\varphi$ iff for every w', if $\mathcal{K}_j ww'$ and $w' \in \mathcal{R}(c_k)(w)$ then $\mathcal{M}, c_l, w' \models \varphi$ with either one of the following options:

    1.1. $k = l = i$
    1.2. $k = i, l = j$
    2.1. $k = j, l = i$
    2.2. $k = l = j$

To refer specifically to one of the four resulting definitions, we will subscript $\models$ with the appropriate number: $\mathcal{M}, c, w \models_{1.2} \varphi$, $\mathcal{M}, c, w \models_{2.2} \varphi$, etc. We will sometimes group the notions two-by-two, letting $\models_{-.2}$ refer unspecifically to the $\models_{1.2}$ case or the $\models_{2.2}$ case, for instance; and $\models$ will refer indifferently to any one of the four notions.[2]

7.3.1.3   The Properties of Knowledge in Contextual Models

We may wonder if the following common axioms and inference rules are preserved in our contextual models, and if not, what condition(s) must be imposed on the relevance function in order to restore them?

K $\models (K_j\varphi \wedge K_j(\varphi \rightarrow \psi)) \rightarrow K\psi$
RN   If $\models \varphi$ then $\models K_j\varphi$
T $\models K_j\varphi \rightarrow \varphi$
D $\models K_j\varphi \rightarrow \neg K_j\neg\varphi$, or $\models \neg K_j\bot$
4       $\models K_j\varphi \rightarrow K_j K_j\varphi$
5       $\models \neg K_j\varphi \rightarrow K_j\neg K_j\varphi$
B $\models \varphi \rightarrow K_j\neg K_j\neg\varphi$

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[2]We mention all four logically possible options here for the sake of exhaustiveness, although we will not examine all of them. To be precise, option 2.1 will not be relevant to our purpose. See Footnote 6.

To answer this question, let us assume the following convention. In a given contextual model, from the relevance function $\mathcal{R}$ and a context c_k, a new accessibility relation R_{c_k} can be defined by:

$$\forall w. \; \forall w'. \; \mathcal{R}_{c_k} w w' \Leftrightarrow w' \in \mathcal{R}(c_k)(w).$$

Then, clause (iv) for epistemic formulas can be rewritten using the intersection $KK_j^k = KK_j \cap \mathcal{R}_{c_k}$ of the two accessibility relations:

(iv') $\mathcal{M}, c_i, w \models K_j \varphi$ iff for all $w' \in W$, if $KK_j^k w w'$ then $\mathcal{M}, c_l, w' \models \varphi$
(with some conditions on k and l)

Let $\mathbb{M}_n$ be the class of all (unrestricted) contextual models for n agents regardless of the specific choice among options (1.1)–(2.2). It is easy to see that the following proposition holds:

Proposition 7.3.4. *Both the axiom (K) and the necessitation rule* (RN) *are valid with respect to* $\mathbb{M}_n$:

(K) $\mathbb{M}_n \models (K_j \varphi \wedge K_j (\varphi \rightarrow \psi)) \rightarrow K_j \psi$
(RN) *If* $\mathbb{M}_n \models \varphi$ *then* $\mathbb{M}_n \models K_j \varphi$

Things get more complex when we turn to the additional possible properties. We will give only sufficient conditions for preserving these properties in contextual models. Two cases are to be systematically distinguished:

1. A simple case has to do with the (−.1) definitions, which by (iv') amount to:

   (iv'/−.1) $\mathcal{M}, c_i, w \models_{-.1} K_j \varphi$ iff for all $w' \in W$, if $KK_j^k w w'$ then
   $\mathcal{M}, c_i, w' \models_{-.1} \varphi$.

   Here, the only role of context c_i is to set the value of k. The definition is strictly equivalent to that of truth in the standard Kripke model $\mathcal{M}' = \langle W, \{KK_j : j \in J\}, V \rangle$ that corresponds with the contextual model $\mathcal{M} = \langle W, \{KK_j : j \in J\}, \mathcal{C}, \mathcal{R}, V \rangle$, for an operator K_j^k with accessibility relation KK_j^k:
   $\mathcal{M}, c_i, w \models_{-.1} K_j \varphi \Leftrightarrow \mathcal{M}', w \models K_j^k \varphi$.
   Intuitively, contextual models will preserve for K_j, the principles corresponding to the properties of KK_j that are preserved by KK_j^k. To be precise:

Proposition 7.3.5. *The axioms (T), (B), (4), and (5) are (1.1)- and (2.1)-valid in contextual models* $\mathcal{M}$ *whose relations* K_j *and* $\mathcal{R}_{c_k}$ *are all respectively reflexive, symmetric, transitive, and Euclidean:*

(T) $\mathcal{M} \models_{-.1} K_j \varphi \rightarrow \varphi$ $\Leftrightarrow$ KK_j *is reflexive and* $w \in \mathcal{R}(c_k)(w)$
(for all w)

(B) $\mathcal{M} \models_{-.1} \varphi \rightarrow K_j \neg K_j \neg \varphi$ $\Leftrightarrow$ KK_j *is symmetric and* $w \in \mathcal{R}(c_k)(w')$
$\Rightarrow w' \in \mathcal{R}(c_k)(w)$

(4) $\mathcal{M} \models_{-.1} K_j \varphi \rightarrow K_j K_j \varphi$ $\Leftrightarrow$ KK_j *is transitive and* $w \in \mathcal{R}(c_k)(w')$ &
$w' \in \mathcal{R}(c_k)(w'') \Rightarrow w \in \mathcal{R}(c_k)(w'')$

(5) $\mathcal{M} \models_{-.1} \neg K_j \varphi \rightarrow K_j \neg K_j \varphi$ $\Leftrightarrow$ KK_j *is Euclidean and* $w \in \mathcal{R}(c_k)(w')$ &
$w \in \mathcal{R}(c_k)(w'') \Rightarrow w' \in \mathcal{R}(c_k)(w'')$

Making K_j *and* $\mathcal{R}_{c_k}$ *serial is not sufficient to ensure the validity of (D).*

2. The case is more complicated with the $(-.2)$ definitions, which by (iv') again amount to:

(iv'/–.2) $\mathcal{M}, c_i, w \vDash_{-.2} K_j \varphi$ iff for all $w' \in W$, if $KK_j^k ww'$ then $\mathcal{M}, c_j, w' \vDash_{-.2} \varphi$.

The difficulty here has to do with formulas with embedded modalities, since their truth will depend on several contexts. We illustrate this with axioms (T), (4), and (5):

Proposition 7.3.6. *Schema (T) is neither (1.2)-valid nor (2.2)-valid in contextual models $\mathcal{M}$ with reflexive K_j and $\mathcal{R}_{c_k}$ relations. Nevertheless, the following instances of (T) hold in these models:*

- $\mathcal{M}, c_i, w \vDash_{-.2} K_j \varphi \to \varphi$, *for φ a non-epistemic formula;*
- $\mathcal{M}, c_i, w \vDash_{-.2} K_j \varphi \to \varphi$ *if $\mathcal{R}_{c_i} \subseteq \mathcal{R}_{c_j}$, for φ an epistemic formula in disjunctive normal form with no negated epistemic operator in it;*
- $\mathcal{M}, c_i, w \vDash_{-.2} K_j \varphi \to \varphi$ *if $\mathcal{R}_{c_i} = \mathcal{R}_{c_j}$, for φ an epistemic formula in normal disjunctive form with at least one negated epistemic operator.*

Proposition 7.3.7. *Validity of (4):*

- *Schema (4) is (2.2)-valid in contextual models $\mathcal{M}$ with reflexive K_j and $\mathcal{R}_{c_k}$ relations;*
- *Schema (4) is not (1.2)-valid in contextual models $\mathcal{M}$ with transitive K_j and $\mathcal{R}_{c_k}$ relations. Nonetheless, the following instantiation of (4) holds in such models: $\mathcal{M}, c_i, w \vDash_{1.2} K_j \varphi \to K_j K_j \varphi$ if $\mathcal{R}_{c_j} \subseteq \mathcal{R}_{c_i}$.*

Proposition 7.3.8. *Validity of (5):*

- *Schema (5) is (2.2)-valid in contextual models $\mathcal{M}$ with Euclidean K_j and $\mathcal{R}_{c_k}$ relations;*
- *Schema (5) is not (1.2)-valid in contextual models $\mathcal{M}$ with Euclidean K_j and $\mathcal{R}_{c_k}$ relations. However, in such models the following instantiation of (5) holds: $\mathcal{M}, c_i, w \vDash_{1.2} \neg K_j \varphi \to K_j \neg K_j \varphi$ if $\mathcal{R}_{c_j} \subseteq \mathcal{R}_{c_i}$.*

We add a final technical remark:

Proposition 7.3.9 (Reduction to Standard Kripke Semantics). *Definition (1.1) with w-constant relevance function $\mathcal{R}$ – i.e. such that for any context c, $\mathcal{R}(c)$ is constant – reduces to a case of standard Kripkean semantics.*

The proofs of the above propositions are in the Appendix.

7.3.2 The Formal Framework Applied

We now give an epistemological interpretation of our formalism and an application of it in capturing, within a unique framework, the various epistemological positions described in Sect. 7.2.

7.3.2.1 Elements of Epistemological Interpretation

Interpreting the $\mathcal{K}_j$-s

We propose to interpret the accessibility relations $\mathcal{K}_j$ in terms of epistemic indiscernibility, i.e., we have $\mathcal{K}_j ww'$ iff agent j cannot tell w from w' on the (sole) basis of what he knows. If φ holds in a $\mathcal{K}_j$-accessible world thus interpreted, then for all j knows, it might be that φ; in other words, it is epistemically possible that φ.

A question that naturally arises is what kind of relation is the epistemic accessibility relation. Here, our answer is that the $\mathcal{K}_j$ are to be construed as equivalence relations – i.e. reflexive, symmetric, and transitive – thus following the common tendency in the logico-epistemic literature.[3] The main epistemological reason for this is that as a general rule, epistemologists grant the skeptic the premiss that their skeptical worlds are epistemically indiscernible from the actual world, i.e., are exactly the same as the actual world with respect to whatever evidence or information we may have; and *being exactly the same as* is an equivalence relation.

Interpreting $\mathcal{R}$

The relevance function $\mathcal{R}$ allows us to capture the idea of epistemic standards and their contextual variability. Indeed, it makes it possible to represent, for a given situation or world, the set of possible worlds that are relevant relative to a context, where the appropriate context (reference context, agent's context, etc.) depends on which definition we select for $\models$. Given a world w, two contexts c, c' can be associated with two sets of contextually relevant worlds, $\mathcal{R}(c)(w)$ and $\mathcal{R}(c')(w)$. When $\mathcal{R}(c)(w) \subseteq \mathcal{R}(c')(w)$, the truth of an epistemic formula relative to c' will be more difficult to obtain than relative to c. So, each context can be understood as determining via $\mathcal{R}$ a certain level of epistemic requirement. The strengthening – resp. the weakening – of epistemic standards will thus translate, in our framework, as an extension – resp. a restriction – of the set of relevant worlds.

Epistemic Accessibility and Contextual Relevance

The set of contextually relevant worlds for an agent j in a world w cannot be strictly included in the set of epistemically possible worlds for j. For j could know that φ, hence having his accessibility relation $\mathcal{K}_j$ restricted to φ-worlds, yet also know that another agent k does not know that φ when some $\neg\varphi$-world is accessible by $\mathcal{K}_k$; this requires that j be able to *consider* some of these $\neg\varphi$-worlds although they are

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[3]See for instance the reference handbook (Fagin et al. 1995) on epistemic logic by Fagin et al. Dissenting views do nonetheless exist, as expressed by Hintikka in 1962, and more recently by Stalnaker in 2006.

not accessible to him. So, letting $\mathcal{K}(j)(w) = \{w' : w' \in W \ \& \ \mathcal{K}_j ww'\}$ be the set of worlds that are epistemically accessible to agent j, what we must not have is this: $\mathcal{R}(c_j)(w) \subsetneq \mathcal{K}(j)(w)$.

Interpreting the c_i-s

Crucial to the epistemological use we want to make of our formal framework is the distinction between the subject and the attributor of knowledge. In both cases, however, we are dealing with an agent. In one case, it is the agent i whom knowledge of a proposition φ is being attributed to; in the other case, it is the agent j who attributes knowledge of φ to i. An agent can also attribute knowledge of a proposition to himself in the first person, as in "I know this-or-that". In this case, he is both knowledge attributor and knowing subject.

To account for these various cases in our framework, we adopt the following conventions. In an evaluation of the form $\mathcal{M}, c_i, w \models K_j \varphi$, (i) at the most general level, context c_i will be called the "context of reference", and c_j the "context of agent j"; (ii) for $i \in I$, on the one hand, agent i will be associated with the "attributor" and c_i referred to as "attributor i's context", and on the other hand, agent j will be associated with the "subject" and c_j referred to as "subject j's context"; and (iii) when considering $\mathcal{R}(c_k)(w)$, we shall refer to c_k as the "attributor's context" or as the "subject's context" depending on whether $k = i$ or $k = j$.

7.3.2.2   Epistemological Interpretation of the $\models$-s

We can now consider the various definitions of $\models$ and connect them with the various epistemological positions mentioned earlier. As we will see, one of the major advantages of our formalism is that it shows what answer each of these positions can give to a problem often underestimated in the epistemological literature, viz. that of embedded epistemic operators.

A first observation is that there seems to be no noticeable difference between definition (1.1) and (1.2) as far as their epistemological interpretation is concerned. The former:

$$\mathcal{M}, c_i, w \models_{1.1} K_j \varphi \text{ iff for every } w', \text{ if } \mathcal{K}_j ww' \text{ and } w' \in \mathcal{R}(c_i)(w) \text{ then } \mathcal{M}, c_i, w' \models \varphi$$

says, basically, that I (= attributor) can truly say that you (= subject) know that φ when I can truly say, against my epistemic standards, that given your evidence, you know that φ; and the latter definition:

$$\mathcal{M}, c_i, w \models_{1.2} K_j \varphi \text{ iff for every } w', \text{ if } \mathcal{K}_j ww' \text{ and } w' \in \mathcal{R}(c_i)(w) \text{ then } \mathcal{M}, c_j, w' \models \varphi$$

says, basically, that I (= attributor) can truly say that you (= subject) know that φ when you can truly say, against my epistemic standards, that given your evidence,

you know that φ. On both definitions, whether a world is relevant or not depends on the attributor's context.

However, an important difference shows up between the two definitions when we turn to formulas with embedded occurrences of epistemic operators, e.g. formulas like $K_1 K_2 \ldots K_m \varphi$.

Definition (1.1) as Invariantism

Regarding embedded epistemic operators, definition (1.1) yields:

$$\mathcal{M}, c_1, w \models_{1.1} K_2 K_3 \varphi \text{ iff for every } w', w'', \text{ if } \mathcal{K}_2 ww', \mathcal{K}_3 w'w'', w' \in \mathcal{R}(c_1)(w) \text{ and}$$
$$w'' \in \mathcal{R}(c_1)(w'), \text{ then } \mathcal{M}, c_1, w'' \models_{1.1} \varphi.$$

which amounts to saying this: When attributor 1 says that subject 2 knows that subject 3 knows that φ, for 1's attribution to be true, it is always exactly the same standards as 2 that 3 must satisfy, that is to say, those in place in attributor 1's context. We propose to associate this definition with the two (insensitive) invariantist positions distinguished in Sect. 7.2, viz. skepticism and anti-skepticism. Both hold that the standards for making a true knowledge attribution are the same always and everywhere, regardless of who is attributing and who is being attributed knowledge. Simply, the former holds that those standards are too demanding for any such attribution to ever come out true, while the latter says they are lax enough to make (most of our) everyday knowledge attributions true.

The difference can be expressed formally in our framework by putting different constraints on the relevance function $\mathcal{R}$:

- For skepticism, the constraint that $\mathcal{R}(c_i)(w) = W$, for any i and any w. This means that whatever the attributor's context, the corresponding relevance set is always the entire set of all logically possible worlds, including, of course, such far-fetched worlds as those described by the Evil Genius or the Brain in a Vat hypotheses, which cannot be eliminated on the basis of our limited epistemic capabilities. In assuming this constraint, the skeptic make-believes that she is a god, and that people can reason on other people's knowledge only if they are gods themselves. Skeptical epistemic logic is epistemic logic for divine agents.
- For anti-skepticism, the constraint that for any i and any w, $\mathcal{R}(c_i)(w) = W*$, for some proper subset $W*$ of W, seems to be a minimum requirement, which nonetheless makes it less impossible for non-divine epistemic agents like us to truly claim knowledge. At least in some cases, all contextually relevant possibilities of error can be excluded. A further requirement – given here informally as a first approximation – will have to be that $W*$ be a set of epistemically accessible worlds where most propositions we ordinarily think we have knowledge of are true (e.g. that we have hands).

As should be clear, either one of these constraints will ensure that the set of epistemically relevant worlds is constant across contexts, that is, for any w, i, and j, $R(c_i)(w) = R(c_j)(w)$, which justifies applying to them the label "invariantism".[4]

Definition (1.2) as Contextualism

In contrast with definition (1.1), with embedded epistemic operators, definition (1.2) yields:

$$\mathcal{M}, c_1, w \models_{1.2} K_2 K_3 \varphi \text{ iff for every } w', w'', \text{ if } \mathcal{K}_2 w w', \mathcal{K}_3 w' w'', w' \in \mathcal{R}(c_1)(w) \text{ and}$$
$$w'' \in \mathcal{R}(c_2)(w'), \text{ then } \mathcal{M}, c_3, w'' \models_{1.2} \varphi.$$

This entails that if an attributor 1 says that subject 2 knows that subject 3 knows that φ, for 1's attribution to be true, the standards that 2 must satisfy for 1's attribution to be true will be those in place in 1's context, while those that 3 must satisfy for it to be true that he knows φ will be those in place in subject 2's context, not attributor 1's context; and the two sets of standards might well be different in their requirements. This, in our opinion, is what we may and must expect from the behavior of genuine contextualist agents (who assume themselves to be such): a contextualist agent ought to reason about other agents' knowledge in the light of her own standards, but she also ought to be aware that the other agents do and ought to do the same too. We therefore suggest associating definition (1.2) with (genuine, self-assumed, coherent) contextualism.[5] To capture formally the contextualist idea that the views of the skeptic and the anti-skeptic are not incompatible, we can simply put on $\mathcal{R}$ the softer constraint that $\mathcal{R}(c_i)(w) \subseteq W$, so that when c_i is a philosophical context, $\mathcal{R}$ yields the set of all logically possible worlds, and when it is an everyday context, $\mathcal{R}$ yields a proper subset of those worlds – preferentially with epistemically possible worlds where most of what we ordinarily think we know is true.

Interestingly, identifying contextualism with definition (1.2) in this way shows that contextualism renders a relatively uncontroversial epistemic principle truly problematic, viz. the "veridicality principle" whereby knowledge requires truth. As Proposition 7.3.6 indicates, the formal version of this principle (schema (T) $K\varphi \to \varphi$) is (1.2)-valid in contextual models with reflexive contextualized

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[4]Note in passing that they make the epistemic relevance set constant across *worlds* too, differing in this respect from "non-absolutist", "circumstance-sensitive" forms of invariantism *à la* Dretske or Nozick, not treated here.

[5]Here, some proponents of contextualism might disagree, as they would be willing to maintain that contextualism is true despite most people lacking awareness of this fact, and being semantically blind to the context-dependence of knowledge ascriptions. This is how, for instance, DeRose seems to conceive of the position. In Lihoreau and Rebuschi (2009), we insist on the distinction between these two construals of the contextualist stance and explore their respective bearings on the issue of epistemic factivity.

accessibility relations $\mathcal{K}_i^k$ only when φ is non-epistemic. This restriction of the implication from knowledge to truth to non-epistemic "facts" is totally in line with the spirit of contextualism for which there are no such things as "epistemic facts". "Know" does not relate to things like $knowledge_1, knowledge_2, \ldots$ that would exist objectively. Since one can count as knowing with respect to one attributor, yet as not knowing with respect to another, the contextualist's concept of knowledge cannot be descriptive, but only purely evaluative. So, if knowledge implies truth, it can only be non-epistemic truth. This is a consequence of contextualism that our proposed formal framework makes clearly salient.

Definition (2.2) as Subjectivism

According to Proposition 7.3.6, we get similar restricted veridicality with definition (2.2.):

$$\mathcal{M}, c_i, w \models_{2.2} K_j \varphi \text{ iff for every } w', \text{ if } \mathcal{K}_j ww' \text{ and } w' \in \mathcal{R}(c_j)(w) \text{ then } \mathcal{M}, c_j, w' \models \varphi$$

This definition, however, is very different from the contextualist spirit of definition (1.2), and closer in our opinion to the spirit of subjectivism. What it says is that only the subject, j, matters. When we attribute knowledge of a proposition to a subject, it is the standards in place in his context, not ours, that matter for the truth or falsity of our attribution. His context is also that in which he himself settles on the truth of the proposition whose knowledge we attribute to him. This means that according to this definition (unlike contextualist definition (1.2)), there are epistemic facts: it is the subject's knowledge that varies from context to context, not merely the truth of our attributions of knowledge to him. Depending on what is at stake in his context, a subject can possess, lack, or lose possession of his knowledge. This is in line with the characterization we gave earlier of subjectivism.

Now, although (not) knowing something is an epistemic *fact*, someone A's knowing that someone else B knows something p does not entail that B knows p. For suppose it is true that A knows that B knows p. Then, A must somehow satisfy the standards in place in his own context for knowing that B knows p. Does this mean that B thereby knows p? No, because nothing in subjectivism prohibits A priori that the standards in place in B's context be no more demanding than those in place in A's context. So, if subjectivism is true, veridicality cannot extend to one's knowledge of someone else's knowledge. This is exactly what definition (2.2) says, and constitutes further motivation for associating subjectivism with it.[6]

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[6]As mentioned in Footnote 2, for our purposes, we do not need definition (2.1):

$$\mathcal{M}, c_i, w \models_{2.1} K_i \varphi \text{ iff for every } w', \text{ if } \mathcal{K}_j ww' \text{ and } w' \in \mathcal{R}(c_j)(w) \text{ then } \mathcal{M}, c_i, w' \models \varphi$$

which says that I (attributor) can truly say that you (=subject) know that φ when I can truly say, against *your* epistemic standards, that given your evidence, you know that φ. It might, however, prove useful if we augmented the non-modal part of the language with indexical expressions in

7.4 Dynamic Formalism

7.4.1 The Main Idea

The notion of context in the previous section was left totally unspecified: we simply took it as a point c_i in a set $\mathcal{C}$. In this section we propose a dynamic formalization of this notion, inspired by the semantics of discourse for natural languages.

The semantics of discourse, in particular *Discourse Representation Theory* (DRT), was invented by Kamp (1981) to account for semantic phenomena specifically connected with discourse dynamics (as opposed to sentence dynamics) and not explainable in standard, say Montagovian, analysis: anaphora resolution, donkey sentences, etc.[7] The "dynamic turn" in formal semantics has led to the creation of other formalisms too, like DPL (*Dynamic Predicate Logic*; see Groenendijk and Stokhof (1991)) where the semantic value of an utterance is treated as a program modifying a context.

DRT builds an intermediate representational level between language and model, made of *discourse representation structures* (DRSs). Each DRS constitutes both (i) the context of interpretation and (ii) the update of an already given DRS by this interpretation – yielding a new DRS. What matters here is the idea of introducing a representational level produced by the interpretation, and which contains "syntactic traces" of the various assertions made in a discourse.

7.4.2 Pseudo-DRT for a Propositional Language

For our purposes we will not need such a complex formalism as in DRT. We will not need a universe (due to the lack of individual variables) – except if we want to specify features of extralinguistic context like the speaker, the place, etc., and in what follows we will simply dispense with such features. However, we will need and make use of a set of conditions preceded with a label (in the spirit of Geurts and Maier (2003)).

As usual, a DRS (i.e. a context) will be represented by a box. For instance, the DRS produced by the interpretation of the following discourse:

φ_1; if φ_1 then agent i knows that φ_2; therefore i knows that φ_2

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order to account for such knowledge ascriptions as "So-and-so knows that I am here" or "I know that you are there".

[7]For an overview, see van Eijk (2005); for a more complete presentation, see Kamp and Reyle (1993).

will be represented by:

$$c = \boxed{\begin{array}{c|c} A & \varphi_1, \ \varphi_1 \to K_i\varphi_2, \ K_i\varphi_2 \end{array}}$$

where the label A is meant to indicate that the relevant context was produced by means of a sequence of assertions. (We do not introduce sub-DRSs since in a propositional language the question of accessibility between universes does not arise.)

This context will be interpreted against a "proto-context", that is to say, a set of presuppositions consisting exclusively of literals, i.e. atoms or negations of atoms (of the form "Agent i is not a brain in a vat", or "Agent j's vision is reliable"). This proto-context will be represented by a box labelled with P:

$$c^\star = \boxed{\begin{array}{c|c} P & h_1, h_2, \ldots, h_n \end{array}}.$$

Several definitions are in order here.

Definition 7.4.1. A *discourse* in language $\mathcal{L}$ is a finite (ordered) sequence of formulas of $\mathcal{L}$:

$$D = \langle \varphi_1, \ \ldots, \ \varphi_p \rangle$$

Definition 7.4.2. A *(discourse) context* for $\mathcal{L}$ is a pair consisting of a label X and a set Σ of formulas of $\mathcal{L}$: $c = \langle X, \Sigma \rangle$. Notation: $\boxed{\begin{array}{c|c} X & \Sigma \end{array}}$.

- An *assertion context* is a context labelled with A: $\boxed{\begin{array}{c|c} A & \varphi_1, \ldots, \varphi_m \end{array}}$.
- A *proto-context*, or *presupposition context*, is a context labelled with P: $\boxed{\begin{array}{c|c} P & h_1, h_2, \ldots, h_n \end{array}}$, each of the formulas h_i being a literal.

To refer to the labels, formulas, and atoms involved in a context, we use the following conventions:

Notation 7.4.3. Writing conventions:

- The label of a context c is written $Lab(c)$ – i.e. $Lab\left(\boxed{\begin{array}{c|c} X & \Sigma \end{array}}\right) = X$;
- The set of formulas of a context c is written $Fo(c)$ – i.e. $Fo\left(\boxed{\begin{array}{c|c} X & \Sigma \end{array}}\right) = \Sigma$;
- The set of atoms making up the formulas of a context c is written $At(c)$: $At(c) \in \wp(\mathcal{A}t)$.

Definition 7.4.4. The *agglomeration* of a formula φ with a context c, written $c + \varphi$, is a binary function on $\mathcal{C} \times \mathcal{L}$ taking its values in $\mathcal{C}$ and defined as follows:

$$\boxed{\begin{array}{c|c} X & \varphi_1, \ldots, \varphi_l \end{array}} + \psi = \boxed{\begin{array}{c|c} X & \varphi_1, \ldots, \varphi_l, \ \psi \end{array}}.$$

The foregoing definition entails that if a formula is already in a context, its agglomeration does not modify this context.

Definition 7.4.5. The *representation* of a discourse $D = \langle \varphi_1, \ldots, \varphi_p \rangle$ relative to a context c, $R(c, D)$, is a finite sequence of assertion contexts $\langle c_0, c_1, \ldots, c_p \rangle$ formed by successive agglomeration of the formulas of D, i.e. such that:

- If $c = \boxed{\begin{array}{c|c} X & \varphi_1, \ldots, \varphi_l \end{array}}$, then $c_0 = \boxed{\begin{array}{c|c} A & \varphi_1, \ldots, \varphi_l \end{array}}$;
- For each index $i \in \{0, \ldots, p-1\}$, we have: $c_{i+1} = c_i + \varphi_{i+1}$.

Now that we can represent discourses by a set of markers (the formulas of the context) representing the various assertions of a discourse, we must consider the semantic interpretation of contexts.

Definition 7.4.6. The *semantic value* $[\![c]\!]$ of a context c in a Kripke model $\mathcal{M} = \langle W, \mathcal{K}, V \rangle$ is the set of possible worlds compatible with the formulas of c:

If $c = \boxed{\begin{array}{c|c} X & \varphi_1, \ldots, \varphi_l \end{array}}$, then $[\![c]\!] = \{w : w \in W \ \& \ \mathcal{M}, w \models (\varphi_1 \wedge \ldots \wedge \varphi_l)\}$.

What remains to be done is to introduce a dynamic component at the level of proto-contexts. A proto-context must enable regimentation of the presuppositions of a discourse, that is, of those statements whose truth is not put into question and which are not even made explicit in the discourse context. The discourse itself can make a presupposition explicit or put it into question, and thereby modify the proto-context.

Definition 7.4.7. The *fusion* of two contexts is a partial binary function $\# : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ such that, for any pair $\langle c_1, c_2 \rangle$ where $c_1 = \boxed{\begin{array}{c|c} P & h_1, h_2, \ldots, h_n \end{array}}$ is a proto-context and $c_2 = \boxed{\begin{array}{c|c} A & \varphi_1, \varphi_2, \ldots, \varphi_m \end{array}}$ is an assertion context:

$$c_1 \# c_2 = \boxed{\begin{array}{c|c} P & h_1, h_2, \ldots, h_n \end{array}} \# \boxed{\begin{array}{c|c} A & \varphi_1, \varphi_2, \ldots, \varphi_m \end{array}} = \boxed{\begin{array}{c|c} P & h_{i_1}, h_{i_2}, \ldots, h_{i_k} \end{array}}$$

with:

- $Fo(c_1 \# c_2) \subseteq Fo(c_1)$;
- $At(c_1) \backslash At(c_1 \# c_2) = At(c_1) \cap At(c_2)$.

In other words, fusion removes from the proto-context all those literals that are atoms or negations of atoms included in the assertion context. Based on this definition, we can then consider a new sequence of contexts resulting from the analysis of a discourse, viz. the sequence of proto-contexts which parallels the representation of the discourse:

Definition 7.4.8. Given a proto-context $c^\star$ and a discourse $D = \langle \varphi_1, \ldots, \varphi_p \rangle$ interpreted relative to an initial context c with representation $R(c, D) = \langle c_0, c_1, \ldots, c_p \rangle$, we build the *history* of proto-context $c^\star$, written $H(c^\star, c, D)$, consisting of a sequence of proto-contexts $\langle c_0^\star, c_1^\star, \ldots, c_p^\star \rangle$ such that:

- $c_0^\star = c^\star \# c$;
- For every index $i \in \{0, \ldots, p-1\}$, we have: $c_{i+1}^\star = c_i^\star \# c_{i+1}$.

A history of proto-contexts thus explains the progressive modification of the set of presuppositions by removing the literals that are made explicit (or whose negation is made explicit) in the discourse.

We can associate with a history of proto-contexts a (*w*-constant) *relevance function* such that:

$$\forall w : \mathcal{R}(c_i^\star)(w) = [\![c_i^\star]\!].$$

This function allows the set of contextually relevant worlds to evolve over the course of the interpretation of a discourse.

7.4.3 Application

The static formalism described in Sect. 7.3 runs into a problem faced by all systems of normal modal logic and having to do with the logical omniscience that follows from accepting axiom K – the epistemic closure principle – and the necessitation rule.

The DRT-based semantics just described allows us to overcome these difficulties. The effects of the necessitation rule can indeed be bypassed if we suppose that each assertion modifies the evaluation context. That is to say, although we do have $K\varphi_1$, $K(\varphi_1 \rightarrow \varphi_2)$, $K\varphi_2$ relative to a constant context, this is no longer the case when the context evolves over a sequence of assertions.

We propose evaluating $K\varphi_1$ relative to an initial empty context, $c = \boxed{A \quad }$; we then evaluate the next formula $K(\varphi_1 \rightarrow \varphi_2)$ relative to the new context produced by the agglomeration of $K\varphi_1$, viz.: $c_1 = \boxed{A \mid K\varphi_1}$. This generates a third context, $c_2 = \boxed{A \mid K\varphi_1,\ K(\varphi_1 \rightarrow \varphi_2)}$, relative to which the conclusion $K\varphi_2$ can be false, depending on the effect of the first two assertions on the history of the initial proto-context.

7.4.3.1 Example

To make things simple, let us assume an epistemic language with only one operator K and a unique relation $\mathcal{K}$. Given a presupposition context containing $h_1 = \neg r$, read as "The agent is not a brain in a vat", the initial proto-context then is: $c^\star = \boxed{P \mid \neg r, h_2, \ldots, h_n}$, and corresponds with ordinary, rather lax epistemic standards, quite unlike those of the skeptic. Let us analyze the following (well-known) piece of discourse:

- φ_1. The agent knows that he has two hands.
- φ_2. If the agent knows that he has two hands then he knows that he is not a brain in a vat.
- φ_3. Therefore, the agent knows that he is not a brain in a vat.

Letting p stand for the atom expressing that the agent has two hands, we get the following discourse: $D = \langle Kp, K(p \rightarrow \neg r), K\neg r \rangle$. To interpret it, we suppose that neither p nor $\neg p$ is part of the initial presuppositions ($p \notin At(c^\star)$). In the following figure, the two sequences generated by D are in two parallel columns:

D	Representation of D	Proto-context history
		$c^\star = \boxed{P \;\mid\; \neg r, h_2, \ldots, h_n}$
	$c_0 = c = \boxed{A \mid }$	$c_0^\star = \boxed{P \;\mid\; \neg r, h_2, \ldots, h_n}$
$\varphi_1.Kp$	$c_1 = \boxed{A \mid Kp}$	$c_1^\star = \boxed{P \;\mid\; \neg r, h_2, \ldots, h_n}$
$\varphi_2.K(p \rightarrow \neg r)$	$c_2 = \boxed{A \mid Kp, K(p \rightarrow \neg r)}$	$c_2^\star = \boxed{P \;\mid\; h_2, \ldots, h_n}$
$\varphi_3.K\neg r$	$c_3 = \boxed{A \mid Kp, K(p \rightarrow \neg r), K\neg r}$	$c_3^\star = \boxed{P \;\mid\; h_2, \ldots, h_n}$

As mentioned above, we can put the history of the proto-context to use to define a relevance function, thereby obtaining a constant function on $c^\star$, $c_0^\star$, and $c_1^\star$, whose co-domain (the relevant worlds) extends from $c_2^\star$:

$$\mathcal{R}(c^\star) = \mathcal{R}(c_0^\star) = \mathcal{R}(c_1^\star) = [\![c^\star]\!] \subsetneqq [\![c_2^\star]\!] = \mathcal{R}(c_2^\star) = \mathcal{R}(c_3^\star).$$

Each formula φ_i of D is interpreted against the previous context of the representation, c_{i-1}. Let us suppose that the interpretation of the first two assertions is true in a given world w. Then:

$$\mathcal{M}, c_0, w \models Kp \qquad \text{i.e.:} \qquad \forall w' \in \mathcal{R}(c_0^\star)(w)(= [\![c^\star]\!]) : \mathcal{K}w, w'$$
$$\Rightarrow \mathcal{M}, c_0, w' \models p$$
$$\mathcal{M}, c_1, w \models K(p \rightarrow \neg r) \qquad \text{i.e.:} \qquad \forall w' \in \mathcal{R}(c_1^\star)(w)(= [\![c^\star]\!]) : \mathcal{K}w, w'$$
$$\Rightarrow \mathcal{M}, c_1, w' \models (p \rightarrow \neg r)$$

It follows that *relative to c_1*, the formula $K\neg r$ is true in w. But it is relative to c_2 that it is evaluated, whose class of relevant worlds is a proper extension of $[\![c^\star]\!]$:

$$\mathcal{M}, c_2, w \models K\neg r \quad \Leftrightarrow \quad \forall w'' \in \mathcal{R}(c_2^\star)(w)(= [\![c_2^\star]\!]) : \mathcal{K}w, w'' \Rightarrow \mathcal{M}, c_2, w'' \models \neg r.$$

So, for $K\neg\varphi$ to fail to hold in w, all we need is a world that is $\mathcal{K}$-accessible from w, where the agent is a brain in a vat, and which is relevant in context c_2. And in this context, unlike in c_1, there can be one such world, as the following example shows.
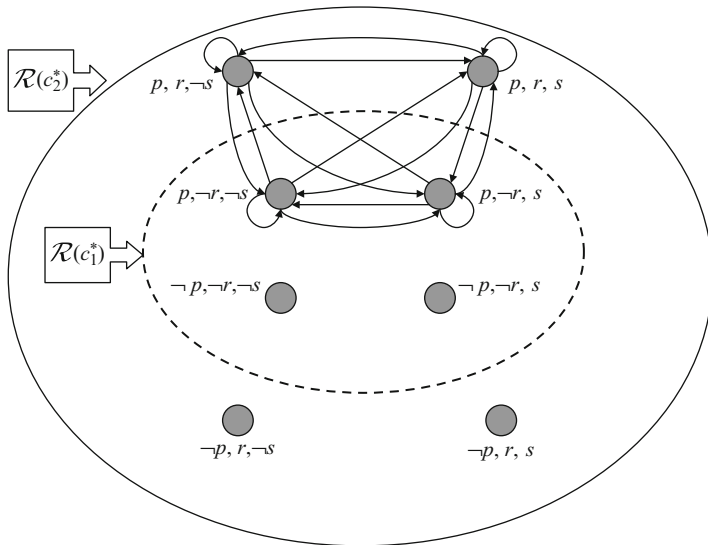
7.4.4   Discourse Contexts and Belief Bases

The dynamic approach just proposed is akin to the well-known AGM model of belief revision proposed in Alchourrón et al. (1985) (for a recent overview, see van Ditmarsch et al. (2008)). We chose not to make use of the latter approach because it is not sensitive enough to the syntax of formulas, and is therefore too *static*.

Belief bases in the AGM model are deductively closed: this results in a level of idealization that is too high to deal with certain epistemological issues (see Hansson (2003)), as we inherit *ipso facto* the problems of logical omniscience. By contrast, the DRT-inspired discourse contexts are finite – and even very limited – sets of formulas: only those formulas that directly represent assertions are introduced in an assertion context. A formula being in a context therefore does not imply that, say, all disjunctions containing that formula are in that context. Unlike belief bases, discourse contexts are not deductively closed.

The set of formulas of a given context c, $Fo(c)$, nonetheless coincides with its deductive closure ($Cn(c)$) *as far as evaluation is concerned*. We can then consider describing agglomeration as an *expansion*, and fusion of a proto-context with an assertion context as a *contraction* of the proto-context.

Belief revision theories could also be of relevance at the level of assertion contexts when a discourse generates an inconsistency that calls for a revision of the DRS. Several connections can therefore be drawn between the approach developed in this paper and the dominant approaches to doxastic dynamics in artificial intelligence.

7.5 Conclusion

In this paper we have laid out the foundations of a formal framework that uses the tools of epistemic logic to advance epistemological analysis.

Epistemologically speaking, the application of our framework to capturing various philosophical positions about knowledge will have to be extended so as to account for two sorts of positions: "assessment-sensitive" positions à la MacFarlane (2005), for whom epistemic standards vary with the context of the person who evaluates a knowledge ascription for truth or falsity; and "circumstance-sensitive" positions as those advanced by Dretske or Nozick, for whom epistemic standards vary not with any context whatsoever, but with the world with respect to which the subject's epistemic position is being evaluated. This is left for future work.

Logically speaking, the very framework of contextual models requires further exploration. Here we have sketched a "deviant" two-dimensional semantics that augments the usual possible world structure with a relevance function. A step further would be to develop a syntax to match those models by introducing context operators of the type $[c_i]$ and $\langle c_i \rangle$, which would make it possible to refer explicitly to context-dependence directly in the object-language.[8] Combining such operators with contextualized epistemic operators would allow us to capture various epistemological positions within the same contextual model, and therefore to account for the logical behavior of, say, a contextualist agent reasoning about the knowledge of a skeptical agent reasoning about an anti-skeptical agent. The contextual model framework could also be extended along further lines, e.g.:

- By exploring different axiom systems (S4, T, etc.) for defining the epistemic operators K_i;
- By adding an *awareness* operator, or a notion of *similarity* between worlds, or any other modification that might block epistemic closure at the "static" level of the framework (as required by positions like Dretske's or Nozick's);
- By adding operators for belief and justification (possibly in the line of Artemov and Nogina (2005));
- By adding appropriate alethic modalities to allow for the treatment of counterfactual epistemic statements like "Had Mary seen Paul at the party, she'd know";
- By adding indexical symbols to the non-modal fragment of the language so as to account for statements like "Mary knows I'm in Paris" or "I know you're there".

This too is left for future work.

Generally speaking, studying how our epistemic language can be modified and enriched should allow us to provide a finer-grained modelling of the various positions that can be found in the epistemological literature within our framework of contextual models.

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[8]Work in this vein can be found in our paper on "contextual epistemic logic" (Rebuschi and Lihoreau 2008).

Appendix: Proofs

Proposition 7.3.5 *The axioms (T), (B), (4), and (5) are (1.1)- and (2.1)-valid in contextual models $\mathcal{M}$ whose relations K_j and $\mathcal{R}_{c_k}$ are all respectively reflexive, symmetric, transitive, and Euclidean:*

(T)	$\mathcal{M} \vDash_{-.1} K_j\varphi \to \varphi)$	$\Leftrightarrow$	KK_j *is reflexive and* $w \in \mathcal{R}(c_k)(w)$ *(for all* w*)*
(B)	$\mathcal{M} \vDash_{-.1} \varphi \to K_j\neg K_j\neg\varphi$	$\Leftrightarrow$	KK_j *is symmetric and* $w \in \mathcal{R}(c_k)(w')$
			$\Rightarrow w' \in \mathcal{R}(c_k)(w)$
(4)	$\mathcal{M} \vDash_{-.1} K_j\varphi \to K_j K_j\varphi$	$\Leftrightarrow$	KK_j *is transitive and* $w \in \mathcal{R}(c_k)(w')$ *&*
			$w' \in \mathcal{R}(c_k)(w'') \Rightarrow w \in \mathcal{R}(c_k)(w'')$
(5)	$\mathcal{M} \vDash_{-.1} \neg K_j\varphi \to K_j\neg K_j\varphi$	$\Leftrightarrow$	KK_j *is Euclidean and* $w \in \mathcal{R}(c_k)(w')$ *&*
			$w \in \mathcal{R}(c_k)(w'') \Rightarrow w' \in \mathcal{R}(c_k)(w'')$

Making K_j and $\mathcal{R}_{c_k}$ serial is not sufficient to ensure the validity of (D).

Proof.

- Suppose that for any j and k, KK_j and $\mathcal{R}_{c_k}$ are reflexive. Since the intersection of two reflexive relations is itself reflexive, the intersection KK_j^k of KK_j and $\mathcal{R}_{c_k}$ must be reflexive. Therefore, $\mathcal{M}', w \vDash K_j^k\varphi \to \varphi$, which amounts to $\mathcal{M}, c_i, w \vDash_{-.1} K_j\varphi \to \varphi$.
- As to axioms (B), (4) and (5), we can likewise simply observe that symmetry, transitivity and Euclideanness are preserved by the intersection of two relations.
- As to axiom (D), however, seriality is not preserved in this way. For instance, the two relations $S = \{\langle a,b\rangle, \langle b,b\rangle\}$ and $T = \{\langle a,b\rangle, \langle b,a\rangle\}$ are both serial, yet their intersection boils down to $\{\langle a,b\rangle\}$.

Proposition 7.3.6. *Schema (T) is neither (1.2)-valid nor (2.2)-valid in contextual models $\mathcal{M}$ with reflexive K_j and $\mathcal{R}_{c_k}$ relations. Nevertheless, the following instances of (T) hold in these models:*

- *$\mathcal{M}, c_i, w \vDash_{-.2} K_j\varphi \to \varphi$, for φ a non-epistemic formula;*
- *$\mathcal{M}, c_i, w \vDash_{-.2} K_j\varphi \to \varphi$ if $\mathcal{R}_{c_i} \subseteq \mathcal{R}_{c_j}$, for φ an epistemic formula in disjunctive normal form with no negated epistemic operator in it;*
- *$\mathcal{M}, c_i, w \vDash_{-.2} K_j\varphi \to \varphi$ if $\mathcal{R}_{c_i} = \mathcal{R}_{c_j}$, for φ an epistemic formula in normal disjunctive form with at least one negated epistemic operator.*

Proof.

- (T) is not (-.2)-valid: Let $\mathcal{M}$ be a contextual model with reflexive relations, c_i a context, and w a world s.t. for any given formula φ, $\mathcal{M}, c_i, w \vDash_{-.2} K_j\varphi$. Then, for every w' s.t. $KK_j^k ww'$, $\mathcal{M}, c_j, w' \vDash_{-.2} \varphi$. Since KK_j^k is reflexive, in particular $\mathcal{M}, c_j, w \vDash_{-.2} \varphi$. This does not imply that $\mathcal{M}, c_i, w \vDash_{-.2} \varphi$: for instance, if φ is of the form $K_m\psi$, there can be a world w_0 s.t. $\mathcal{R}_{c_i} ww_0, \neg\mathcal{R}_{c_j} ww_0$ where ψ is not satisfied.
- Under the same assumptions, take φ to be non-epistemic. Then, from clauses (i)–(iii) for $\vDash$ in contextual models, the value of φ is independent of context. So, $\mathcal{M}, c_j, w \vDash_{-.2} \varphi$ entails $\mathcal{M}, c_i, w \vDash_{-.2} \varphi$. Therefore, (T)'s instantiation with non-epistemic formulas is (-.2)-valid.

- With the same assumptions again, let us further suppose (1) φ to be in normal disjunctive form with no negation of an epistemic operator, and (2) that $\mathcal{R}_{c_i} \subseteq \mathcal{R}_{c_j}$. Then, φ is a disjunction of formulas: $\varphi_h = K_{h_1}\psi_{h_1} \wedge K_{h_m}\psi_{h_m} \wedge \psi_{h_{m+1}} \wedge \ldots \wedge \psi_{h_{m+n}}$, for $\psi_{h_{m+1}}, \ldots, \psi_{h_{m+n}}$ non-epistemic formulas. When $\mathcal{M}, c_j, w \vDash_{-.2} \varphi$, we have two cases:

  1. The truth of $\varphi = \varphi_1 \vee \ldots \vee \varphi_k$ rests on a sub-formula φ_h with no epistemic component. In this case, $\mathcal{M}, c_j, w \vDash_{-.2} \varphi_h$ implies that $\mathcal{M}, c_i, w \vDash_{-.2} \varphi_h$, hence that $\mathcal{M}, c_i, w \vDash_{-.2} \varphi$.
  2. The truth of $\varphi = \varphi_1 \vee \ldots \vee \varphi_k$ rests on a sub-formula φ_h with epistemic components, hence on the truth of the sub-formulas $K_{h_1}\psi_{h_1}, \ldots, K_{h_m}\psi_{h_m}$ of φ_h. Therefore, for each index $e \in \{h_1, \ldots, h_m\}$, we have $\mathcal{M}, c_j, w \vDash_{-.2} K_e\psi_e$; in other words, for all w', if $KK_e^k ww'$ then $\mathcal{M}, c_e, w' \vDash_{-.2} \psi_e$. We (provisionally) distinguish the two (-.2) definitions:
     - 2.2: $\mathcal{M}, c_j, w \vDash_{2.2} K_e\psi_e$ equates: for all w', if $KK_e^e ww'$ then $\mathcal{M}, c_e, w' \vDash_{2.2} \psi_e$; in other words, the truth condition of $K_e\psi_e$ is independent of context c_j; so, we also have: $\mathcal{M}, c_i, w \vDash_{2.2} K_e\psi_e$.
     - 1.2: $\mathcal{M}, c_j, w \vDash_{1.2} K_e\psi_e$ equates: for all w', if $KK_e^j ww'$ then $\mathcal{M}, c_e, w' \vDash_{1.2} \psi_e$; here, the truth condition for $K_e\psi_e$ depends on context c_j. But we have assumed that $\mathcal{R}_{c_i} \subseteq \mathcal{R}_{c_j}$, which entails that $\mathcal{R}_{c_i} \cap KK_e \subseteq \mathcal{R}_{c_j} \cap KK_e$, i.e. $KK_e^i \subseteq KK_e^j$. Therefore, for all w', if $KK_e^i ww'$ then $KK_e^j ww'$, hence $\mathcal{M}, c_e, w' \vDash_{1.2} \psi_e$. This amounts to $\mathcal{M}, c_i, w \vDash_{1.2} K_e\psi_e$.

  We have thus shown that for each of the epistemic components $K_e\psi_e$ of φ_h, $\mathcal{M}, c_i, w \vDash_{-.2} K_e\psi_e$; for the non-epistemic components $\psi_{h_{m+1}}, \ldots, \psi_{h_{m+n}}$, truth is independent of context and transposes from c_j to c_j. Hence, we have established that φ_h, $\mathcal{M}, c_i, w \vDash_{-.2} \varphi_h$, and therefore that $\mathcal{M}, c_i, w \vDash_{-.2} \varphi$. Conclusion: we have $\mathcal{M}, c_i, w \vDash_{-.2} K_j\varphi \to \varphi$.

- The case of formulas $\varphi = \varphi_1 \vee \ldots \vee \varphi_k$ whose truth rests on conjunctive components φ_i including sub-formulas of the type $\neg K_e\psi_e$ requires more than the mere inclusion of the sets of contextually relevant worlds. Assuming the identity of these sets guarantees the transition of truth from context to context. The proof is similar to the previous one.

Proposition 7.3.7. *Validity of (4):*

- *Schema (4) is (2.2)-valid in contextual models $\mathcal{M}$ with reflexive K_j and $\mathcal{R}_{c_k}$ relations;*
- *Schema (4) is not (1.2)-valid in contextual models $\mathcal{M}$ with transitive K_j and $\mathcal{R}_{c_k}$ relations. Nonetheless, the following instantiation of (4) holds in such models: $\mathcal{M}, c_i, w \vDash_{1.2} K_j\varphi \to K_j K_j\varphi$ if $\mathcal{R}_{c_j} \subseteq \mathcal{R}_{c_i}$.*

Proof.

(2.2): Let $\mathcal{M}$ be a contextual model with transitive relations, c_i a context, w a world, and φ a formula. We assume that (a) $\mathcal{M}, c_i, w \vDash_{2.2} K_j\varphi$, and, *ad*

absurdum, that (b) $\mathcal{M}, c_i, w \nvDash_{2.2} K_j K_j \varphi$. From (a), it follows that for any w', if $KK_j^j ww'$ then $\mathcal{M}, c_j, w \vDash_{2.2} \varphi$. From (b), it follows that there are two worlds w_1, w_2, s.t. $KK_j^j ww_1$, $KK_j^j w_1 w_2$, and (c) $\mathcal{M}, c_j, w_2 \nvDash_{2.2} \varphi$. Since the relations are supposed to be transitive, we have $KK_j^j ww_2$; and therefore by (a), we get $\mathcal{M}, c_j, w_2 \vDash_{2.2} \varphi$, which directly contradicts (c). Conclusion: $\mathcal{M}, c_i, w \vDash_{2.2} K_j \varphi \rightarrow K_j K_j \varphi$.

(1.2): Let $\mathcal{M}$ be a contextual model with transitive relations, c_i and c_j two contexts s.t. $\mathcal{R}_{c_j} \subseteq \mathcal{R}_{c_i}$, w a world, and φ a formula. We assume that (a) $\mathcal{M}, c_i, w \vDash_{1.2} K_j \varphi$, and *ad absurdum* that (b) $\mathcal{M}, c_i, w \nvDash_{1.2} K_j K_j \varphi$. From (a), it follows that for all w', if $KK_j^i ww'$ then $\mathcal{M}, c_j, w \vDash_{1.2} \varphi$. From (b), it follows that there are two worlds w_1, w_2 s.t. $KK_j^i ww_1$, $KK_j^j w_1 w_2$, and (c) $\mathcal{M}, c_j, w_2 \nvDash_{1.2} \varphi$. Since $\mathcal{R}_{c_j} \subseteq \mathcal{R}_{c_i}$, we may infer $KK_j^i w_1 w_2$ from $KK_j^j w_1 w_2$. Now, since the relations are supposed to be transitive, from $KK_j^i ww_1$ and $KK_j^i w_1 w_2$ we may infer $KK_j^i ww_2$; therefore, by (a), we have $\mathcal{M}, c_j, w_2 \vDash_{1.2} \varphi$, which contradicts (c). Conclusion: $\mathcal{M}, c_i, w \vDash_{1.2} K_j \varphi \rightarrow K_j K_j \varphi$.

Proposition 7.3.8. *Validity of (5):*

- *Schema (5) is (2.2)-valid in contextual models $\mathcal{M}$ with Euclidean K_j and $\mathcal{R}_{c_k}$ relations;*
- *Schema (5) is not (1.2)-valid in contextual models $\mathcal{M}$ with Euclidean K_j and $\mathcal{R}_{c_k}$ relations. However, in such models the following instantiation of (5) holds: $\mathcal{M}, c_i, w \vDash_{1.2} \neg K_j \varphi \rightarrow K_j \neg K_j \varphi$ if $\mathcal{R}_{c_j} \subseteq \mathcal{R}_{c_i}$.*

Proof.
The proof is similar to that of (4). We assume that we have both $\mathcal{M}, c_i, w \vDash_{-.2} \neg K_j \varphi$ and $\mathcal{M}, c_i, w \vDash_{-.2} \neg K_j \neg K_j \varphi$, which leads to contradictory requirements on a world w_0, the first assumption requiring that $\mathcal{M}, c_j, w_0 \nvDash_{-.2} \varphi$, the second implying that $\mathcal{M}, c_j, w_0 \nvDash_{-.2} \varphi$ given the Euclideanness of KK_j^j for (2.2), and that of KK_j^i for (1.2).

Proposition 7.3.9. *Definition (1.1) with w-constant relevance function $\mathcal{R}$, i.e. s.t. for any context c, $\mathcal{R}(c)$ is constant, reduces to a case of standard Kripkean semantics.*

Proof.
Suppose the relevance function $\mathcal{R}$ in a contextual model $\mathcal{M} = \langle W, \{KK_j : j \in I\}, \mathcal{C}, \mathcal{R}, V \rangle$ is w-constant, and consider a particular context c_i. We build a standard Kripke model from $\mathcal{M}$, $\mathcal{M}^{c_i} = \langle W^{c_i}, \{KK_j^{c_i} : j \in I\}, \mathcal{C}^{c_i}, \mathcal{R}^{c_i}, V \rangle$, where $W^{c_i} = \mathcal{R}(c_i)$, and $KK_j^{c_i}$ is the restriction of KK_j to $\mathcal{R}(c_i)$. Then, for any world $w \in \mathcal{R}(c_i)$:

(i) if α is an atom: $\mathcal{M}, c_i, w \vDash_{1.1} \alpha \Leftrightarrow w \in V(\alpha) \Leftrightarrow \mathcal{M}^{c_i}, w \vDash \alpha$

For points (ii) and (iv) below, we accept the recurrence hypothesis: $\mathcal{M}, c_i, w \vDash_{1.1} \varphi \Leftrightarrow \mathcal{M}^{c_i}, w \vDash \varphi$, and for point (iii), the corresponding hypotheses with φ_1 and φ_2.

(ii) $\mathcal{M}, c_i, w \vDash_{1.1} \neg\varphi \Leftrightarrow \mathcal{M}, c_i, w \nvDash_{1.1} \varphi \Leftrightarrow \mathcal{M}^{c_i}, w \nvDash \varphi \Leftrightarrow \mathcal{M}^{c_i}, w \vDash \neg\varphi$

(iii) $\mathcal{M}, c_i, w \vDash_{1.1} \varphi_1 \vee \varphi_2 \Leftrightarrow \mathcal{M}, c_i, w \vDash_{1.1} \varphi_1$ or $\mathcal{M}, c_i, w \vDash_{1.1} \varphi_2 \Leftrightarrow$
$\mathcal{M}^{c_i}, w \vDash \varphi_1$ or $\mathcal{M}^{c_i}, w \vDash \varphi_2 \Leftrightarrow \mathcal{M}^{c_i}, w \vDash \varphi_1 \vee \varphi_2$

(iv) $\mathcal{M}, c_i, w \vDash_{1.1} K_j\varphi \Leftrightarrow$ for all $w' \in W$ s.t. $KK_j ww'$, if $w' \in \mathcal{R}(c_i)$ $[= W^{c_i}]$
then $\mathcal{M}, c_i, w' \vDash_{1.1} \varphi \Leftrightarrow$ for all $w' \in W^{c_i}$ s.t. $KK_j ww'$, $\mathcal{M}^{c_i}, w' \vDash \varphi \Leftrightarrow$
$\mathcal{M}^{c_i}, w \vDash K_j\varphi$.

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