Classical Lie superalgebras at infinity

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Abstract We study certain categories of modules over direct limits of classical Lie superalgebras. In many cases these categories are equivalent to similar categories for classical Lie algebras. The functors establishing this equivalence can be used to obtain a new result for representation theory of direct limits of Lie algebras.

1 Introduction

There are several generalizations of simple Lie algebras and superalgebras in the infinite-dimensional case. In this paper, we discuss representations of locally simple Lie algebras, i.e. Lie algebras and superalgebras we consider are the direct limits $\mathfrak{g} = \varinjlim \mathfrak{g}_i$ of finite-dimensional simple Lie algebras (or superalgebras) $\mathfrak{g}_i$. In particular, we are interested in the cases when $\mathfrak{g} = \mathfrak{sl}(\infty), \mathfrak{so}(\infty)$ or $\mathfrak{sp}(\infty)$.

In [5] we tried to define a nice analogue of the category of finite-dimensional modules for $\mathfrak{g}$. The most obvious analogue, the category of integrable modules, is rather difficult to study. Even the problem of classifying simple modules involves infinitely many continuous parameters.

On the other hand, $\mathfrak{g}$ has a very natural class of representations in the tensor powers of the standard and costandard modules. In [2] we give an intrinsic definition of a category $\mathbb{T}_{\mathfrak{g}}$ that contains all such representations. It turns out that $\mathbb{T}_{\mathfrak{g}}$ has many remarkable properties. Although it is not semi-simple, it is a Koszul category in the sense of [1]. That allows one to calculate extensions between simple modules and their injective resolutions. We also prove that the categories $T_{\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{so}(\infty)$ and $\mathfrak{sp}(\infty)$ are equivalent. In the recent preprint [8] the same categories are studied from slightly different point of view.

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The goal of the present paper is to define and study analogues of $\mathbb{T}_{\mathfrak{g}}$ for direct limits of classical Lie superalgebras. As follows from the Kac classification, [4], there are four such superalgebras $\mathfrak{sl}(\infty,\infty)$, $\mathfrak{osp}(\infty,\infty)$, $P(\infty)$ and $Q(\infty)$. We will see that in the first three cases we do not obtain new categories. Namely, $\mathbb{T}_{\mathfrak{sl}(\infty,\infty)}$ is equivalent to $\mathbb{T}_{\mathfrak{sl}(\infty)}$, and $\mathbb{T}_{\mathfrak{osp}(\infty,\infty)}$ and $\mathbb{T}_{P(\infty)}$ are equivalent to $\mathbb{T}_{\mathfrak{o}(\infty)}$. The latter fact can be used to construct a direct equivalence functor $\mathbb{T}_{\mathfrak{o}(\infty)} \to \mathbb{T}_{\mathfrak{sp}(\infty)}$. This result is somewhat surprising, since it appears that these categories are easier than the corresponding categories for finite-dimensional superalgebras. The rather complicated matter of atypical representations disappears after going to direct limits.

In the case of $Q(\infty)$ we obtain a completely new category. It is interesting to study it in detail.

2 Direct limits of classical Lie algebras

2.1 General and special Lie algebras at infinity

Let V and V_* be countable-dimensional vector spaces with non-degenerate pairing $tr: V \otimes V_* \to \mathbb{C}$.

Definition 1 $\mathfrak{gl}(\infty) = V \otimes V_*$ has a natural Lie algebra structure given by

$$[v_1 \otimes u_1, v_2 \otimes u_2] = tr(v_2 \otimes u_1) v_1 \otimes u_2 - tr(v_1 \otimes u_2) v_2 \otimes u_1.$$

Ker$(tr) = \mathfrak{sl}(\infty)$ is a simple Lie subalgebra of $\mathfrak{gl}(\infty)$.

One can also realize $\mathfrak{g}$ as a direct limit

$$\mathfrak{sl}(\infty) = \varinjlim \mathfrak{sl}(n),\ \mathfrak{gl}(\infty) = \varinjlim \mathfrak{gl}(n),$$

and identify $\mathfrak{gl}(\infty)$ with the space of infinite matrices $(a_{ij})_{i,j \in \mathbb{N}}$ with finitely many non-zero entries and $\mathfrak{sl}(\infty)$ with the subalgebra of traceless matrices in $\mathfrak{gl}(\infty)$.

It is clear that V and V_* are simple $\mathfrak{g}$-modules. Furthermore, the classical Schur-Weyl duality works in the infinite-dimensional case.

Theorem 1 (Schur-Weyl duality) *Let* $\mathfrak{g} = \mathfrak{gl}(\infty)$ *or* $\mathfrak{sl}(\infty)$. *Then*

$$V^{\otimes n} = \bigoplus_{|\lambda|=n} V^{\lambda} \otimes Y_{\lambda},\ V_*^{\otimes n} = \bigoplus_{|\lambda|=n} V_*^{\lambda} \otimes Y_{\lambda},$$

where λ *runs the set of partitions of size* n *and* Y_λ *denotes the corresponding irreducible representation of* S_n *and* $V^{\lambda} = \pi_\lambda(V^{\otimes n})$, *where* π_λ *is a Young projector with the Young diagram* λ.

However, the representation of $\mathfrak{g}$ in the space of mixed tensors $V^{\otimes n} \otimes V_*^{\otimes m}$ is not completely reducible in contrast with finite-dimensional case. Indeed, for instance, the exact sequence of $\mathfrak{sl}(\infty)$-modules

$$0 \to \mathfrak{sl}(\infty) \to V \otimes V_* \to \mathbb{C} \to 0$$

does not split. The following result gives a description of the $\mathfrak{g}$-module structure on $V^{\otimes n} \otimes V_*^{\otimes m}$.

Theorem 2 ([6]) *Let $\mathfrak{g} = \mathfrak{gl}(\infty)$ or $\mathfrak{sl}(\infty)$. Then*

$$V^{\otimes n} \otimes V_*^{\otimes m} = \bigoplus_{|\lambda|=n, |\mu|=m} \tilde{V}^{\lambda,\mu} \otimes (Y_\lambda \boxtimes Y_\mu),$$

where each $\tilde{V}^{\lambda,\mu} = V^\lambda \otimes V_*^\mu$ is an indecomposable $\mathfrak{g}$-module with irreducible socle $V^{\lambda,\mu}$ and $Y_\lambda \boxtimes Y_\mu$ is the exterior tensor product of irreducible S_n and S_m-modules.

The socle filtration of $\tilde{V}^{\lambda,\mu}$ is given by

$$\mathrm{soc}^k(\tilde{V}^{\lambda,\mu})/\mathrm{soc}^{k-1}(\tilde{V}^{\lambda,\mu}) = \bigoplus_{|\gamma|=k} N^{\lambda}_{\gamma,\lambda'} N^{\mu}_{\gamma,\mu'} V^{\lambda',\mu'}.$$

Here $N^{\lambda}_{\gamma,\lambda'}$ stand for Littlewood–Richardson coefficients.

The proof is based on the results of Howe, Tan and Willenbring [3] who calculated asymptotic decomposition of mixed tensor products in the finite-dimensional case.

2.2 Orthogonal and symplectic Lie algebras

Assume now that a countable-dimensional vector space V has a non-degenerate symmetric (resp. skew-symmetric) form $\omega : V \otimes V \to \mathbb{C}$.

Definition 2 $\mathfrak{so}(\infty)$ (resp. $\mathfrak{sp}(\infty)$) is the Lie subalgebra of finite rank linear operators in V preserving ω.

One can use identification

$$\mathfrak{so}(\infty) = \Lambda^2(V),\ \mathfrak{sp}(\infty) = S^2(V)$$

given by

$$X_{v \wedge w}(u) = \omega(v,u)w - \omega(u,w)v,\ \forall\, v,w,u \in V. \tag{1}$$

Another way to define $\mathfrak{g}$ is via direct limits

$$\mathfrak{so}(\infty) = \lim_{\rightarrow} \mathfrak{so}(n),\ \mathfrak{sp}(\infty) = \lim_{\rightarrow} \mathfrak{sp}(n).$$

The representation of $\mathfrak{g}$ in the tensor algebra $T(V)$ were also described by Penkov and Styrkas.

Theorem 3 *([6]) Let $\mathfrak{g} = \mathfrak{so}(\infty)$ or $\mathfrak{sp}(\infty)$. Then*

$$V^{\otimes n} = \bigoplus_{|\lambda|=n} \tilde{V}^{\lambda} \otimes Y_\lambda,$$

where each $\tilde{V}^\lambda$ is an indecomposable $\mathfrak{g}$-module with irreducible socle V^λ.

The socle filtration of $\tilde{V}^{\lambda}$ *is given by*

$$\mathrm{soc}^k(\tilde{V}^{\lambda})/\mathrm{soc}^{k-1}(\tilde{V}^{\lambda}) = \bigoplus_{|\gamma|=k} N^{\lambda}_{2\gamma,\lambda'} V^{\lambda'},$$

for $\mathfrak{g} = \mathfrak{so}(\infty)$*, and*

$$\mathrm{soc}^k(\tilde{V}^{\lambda})/\mathrm{soc}^{k-1}(\tilde{V}^{\lambda}) = \bigoplus_{|\gamma|=k} N^{\lambda}_{2\gamma^{\perp},\lambda'} V^{\lambda'},$$

for $\mathfrak{g} = \mathfrak{sp}(\infty)$.

3 The category $\mathbb{T}_{\mathfrak{g}}$

Definition 3 Let $\mathfrak{g} = \mathfrak{gl}(\infty)$ or $\mathfrak{sl}(\infty)$. A subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is a finite corank subalgebra if there exist finite dimensional subspaces $W \subset V$ and $W_* \subset V_*$ such that the restriction of the canonical pairing to $W \otimes W_* \to \mathbb{C}$ is non-degenerate and $\mathfrak{k} \supset \mathfrak{g} \cap (W_*^{\perp} \otimes W^{\perp})$.

If $\mathfrak{g} = \mathfrak{so}(\infty)$ or $\mathfrak{sp}(\infty)$, then $\mathfrak{k} \subset \mathfrak{g}$ is a finite corank subalgebra if there exists a finite dimensional subspace $W \subset V$ such that the restriction of ω on W is non-degenerate and $\mathfrak{k} \supset \Lambda^2(W^{\perp})$ or $S^2(W^{\perp})$ respectively.

We define $\mathbb{T}_{\mathfrak{g}}$ as a full subcategory of $\mathfrak{g}$-modules whose objects M satisfy the following conditions

- M is integrable.
- For every $m \in M$ the annihilator of m in $\mathfrak{g}$ is a finite corank subalgebra.
- M has finite length.

It is not difficult to see that $\mathbb{T}_{\mathfrak{g}}$ is closed under tensor product, hence it is a monoidal category. However, it is not rigid since there is no a reasonable duality functor on $\mathbb{T}_{\mathfrak{g}}$. The following results relate tensor representations of $\mathfrak{g}$ with $\mathbb{T}_{\mathfrak{g}}$.

Theorem 4 ([2])

- *For* $\mathfrak{g} = \mathfrak{gl}(\infty)$ *or* $\mathfrak{sl}(\infty)$ *all (up to isomorphism) simple objects of* $\mathbb{T}_{\mathfrak{g}}$ *are* $V^{\lambda,\mu}$.
- *For* $\mathfrak{g} = \mathfrak{so}(\infty)$ *or* $\mathfrak{sp}(\infty)$ *all (up to isomorphism) simple objects of* $\mathbb{T}_{\mathfrak{g}}$ *are* V^{λ}.
- $\tilde{V}^{\lambda,\mu}$ *(respectively* $\tilde{V}^{\lambda}$*) are all up to isomorphism indecomposable injective in* $\mathbb{T}_{\mathfrak{g}}$.

To prove injectivity of $\tilde{V}^{\lambda,\mu}$ we use the fact (proven in [5]) that for any integrable $\mathfrak{g}$-module M, the integrable part of M^* is injective in the category of integrable modules. From this it is easy to see that the functor Γ from the category of all integrable $\mathfrak{g}$-modules to $\mathbb{T}_{\mathfrak{g}}$ defined by

$$\Gamma(M) = \bigcup M^{\mathfrak{k}}$$

(where the union is taken over all finite corank $\mathfrak{k} \subset \mathfrak{g}$) maps an injective module in the category of integrable modules to an injective module in $\mathbb{T}_{\mathfrak{g}}$. On the other hand, by a direct calculation done in [2]

$$\Gamma(V^{\otimes m}\otimes V_*^{\otimes n})=\bigoplus_{k\leq m,l\leq n}(V^{\otimes k}\otimes V_*^{\otimes l})^{\oplus c(k,l)}.$$

Note that Γ can be considered as a certain version of the Zuckerman functor [9].

There exists a Borel subalgebra $\mathfrak{b}\subset\mathfrak{g}$ such that all simple modules of $\mathbb{T}_{\mathfrak{g}}$ are highest weight modules. For instance, if $\mathfrak{g}=\mathfrak{sl}(\infty)$, considered as the algebra of matrices $(a_{ij})_{i,j\in\mathbb{N}}$, we define a nonstandard total order on $\mathbb{N}$ by $1<3<5<\cdots<6<4<2$ and positive roots $\varepsilon_i-\varepsilon_j$ for all $i<j$. The corresponding infinite "Dynkin diagram" is

$$\circ-\circ-\cdots-\circ-\circ.$$

Lemma 1 *If S is a non-zero quotient of the Verma module and $S\in\mathbb{T}_{\mathfrak{g}}$, then S is simple.*

To show that the the category $\mathbb{T}_{\mathfrak{g}}$ is Koszul we use the following

Lemma 2 ([2])

- *If $\mathfrak{g}=\mathfrak{gl}(\infty)$ or $\mathfrak{sl}(\infty)$, then*

$$\mathrm{Ext}^k(V^{\lambda,\mu},V^{\nu,\kappa})\neq 0$$

implies $|\lambda|-|\nu|=|\mu|-|\kappa|=k$.
- *If $\mathfrak{g}=\mathfrak{so}(\infty)$ or $\mathfrak{sp}(\infty)$, then*

$$\mathrm{Ext}^k(V^{\lambda},V^{\nu})\neq 0$$

implies $|\lambda|-|\nu|=2k$.

Let $T=T(V)$ for $\mathfrak{g}=\mathfrak{so}(\infty)$ or $\mathfrak{sp}(\infty)$ and $T=\bigoplus_{m,n\geq 0}V^{\otimes m}\otimes V_*^{\otimes n}$ for $\mathfrak{g}=\mathfrak{sl}(\infty)$ or $\mathfrak{gl}(\infty)$. For $\mathfrak{g}=\mathfrak{gl}(\infty)$ or $\mathfrak{sl}(\infty)$ we set

$$\mathscr{A}_{\mathfrak{g}}^k=\bigoplus_{m,n\geq 0}\mathrm{Hom}_{\mathfrak{g}}(V^{\otimes m}\otimes V_*^{\otimes n},V^{\otimes m-k}\otimes V_*^{\otimes n-k}),\ \mathscr{A}_{\mathfrak{g}}=\bigoplus_{k\geq 0}\mathscr{A}_{\mathfrak{g}}^k.$$

For $\mathfrak{g}=\mathfrak{so}(\infty)$ or $\mathfrak{sp}(\infty)$ set

$$\mathscr{A}_{\mathfrak{g}}^k=\bigoplus_{n\geq 0}\mathrm{Hom}_{\mathfrak{g}}(V^{\otimes n},V^{\otimes n-2k}),\ \mathscr{A}_{\mathfrak{g}}=\bigoplus_{k\geq 0}\mathscr{A}_{\mathfrak{g}}^k.$$

Note that $\mathscr{A}_{\mathfrak{g}}$ is by definition a graded algebra. It does not have the identity but it is a direct limit of unital algebras.

Theorem 5 ([2])

- *The category $\mathbb{T}_{\mathfrak{g}}$ is antiequivalent to the category $A_{\mathfrak{g}}$-fmod of locally unitary finite-dimensional $A_{\mathfrak{g}}$-modules.*
- *$A_{\mathfrak{g}}$ is a direct limit of Koszul rings.*

For $\mathfrak{g}=\mathfrak{gl}(\infty)$ and $\mathfrak{sl}(\infty)$ the corresponding algebras $\mathscr{A}_{\mathfrak{g}}$ are the same. It is shown in [2] that $\mathscr{A}^0_{\mathfrak{g}}=\bigoplus_{m,n\geq 0}\mathbb{C}[S_n\times S_m]$ and $\mathscr{A}^1_{\mathfrak{g}}$ is generated by contractions.

In this case one can prove that $\mathscr{A}_{\mathfrak{g}}$ is Koszul self-dual, i.e. $(A^!_{\mathfrak{g}})^{op}\simeq\mathscr{A}_{\mathfrak{g}}$. That implies the following formulas for extensions between simple modules

$$\dim\mathrm{Ext}^{\mathrm{k}}(V^{\lambda',\mu'},V^{\lambda,\mu})=\sum_{|\gamma|=k}N^{\lambda}_{\gamma,\lambda'}N^{\mu}_{\gamma^{\perp},\mu'}.$$

It is also shown in [2] that for $\mathfrak{g}=\mathfrak{so}(\infty)$ and $\mathfrak{sp}(\infty)$ $\mathscr{A}^0_{\mathfrak{g}}=\bigoplus_{n\geq 0}\mathbb{C}[S_n]$ and $\mathscr{A}^1_{\mathfrak{g}}$ is generated by contractions. Knowing this it is easy to obtain an isomorphism

$$\mathscr{A}_{\mathfrak{so}(\infty)}\simeq\mathscr{A}_{\mathfrak{sp}(\infty)}.$$

The latter implies an equivalence of abelian categories $\mathbb{T}_{\mathfrak{sp}(\infty)}$ and $\mathbb{T}_{\mathfrak{so}(\infty)}$ by Theorem 5. Under this equivalence V^{λ} goes to $V^{\lambda^{\perp}}$. It is proven in [8] that this is an equivalence of monoidal categories. A different proof of this fact is given in the next section.

4 Superalgebras

4.1 Direct limits of classical Lie superalgebras

Let $V=V_0\oplus V_1$ be a superspace, both V_0 and V_1 are countable-dimensional. Below we consider the following possibilities.

- There is a countable-dimensional V_* and a non-degenerate pairing $str:V\otimes V_*\to\mathbb{C}$. Then we set $\mathfrak{gl}(\infty,\infty)=V\otimes V_*$ and $\mathfrak{sl}(\infty,\infty)=\mathrm{Ker}(str)$ with the commutator defined in the same way as in the purely even case (with the usual sign convention). Note that $\mathfrak{g}=\mathfrak{gl}(\infty,\infty)$ has a $\mathbb{Z}$-grading (compatible with $\mathbb{Z}_2$-grading) $\mathfrak{g}=\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_1$, where

$$\mathfrak{g}_0=V_0\otimes(V_0)_*\oplus(V_1)_*\otimes V_1\simeq(\mathfrak{gl}(\infty))\oplus(\mathfrak{gl}(\infty)),$$
$$\mathfrak{g}_1=V_0\otimes(V_1)_*,\quad \mathfrak{g}_{-1}=(V_0)_*\otimes V_1.$$

  This grading naturally can be restricted to $\mathfrak{sl}(\infty,\infty)$. The Lie superalgebra $\mathfrak{sl}(\infty,\infty)$ is simple since it can be obtained as a direct limit of simple Lie superalgebras.

- We fix an *even* non-degenerate symmetric form $\omega:S^2(V)\to\mathbb{C}$ and define $\mathfrak{osp}(\infty,\infty)$ as the subalgebra of operators in V of finite rank preserving ω. One can identify $\mathfrak{osp}(\infty,\infty)$ with $\Lambda^2(V)$ using (1). In this case

$$\mathfrak{g}_0=\mathfrak{so}(\infty)\oplus\mathfrak{sp}(\infty),\ \mathfrak{g}_1=V_0\otimes V_1.$$

  The Lie superalgebra $\mathfrak{osp}(\infty,\infty)$ is simple because it is isomorphic to a direct limit of finite-dimensional simple Lie superalgebras.

- We fix an *odd* non-degenerate form $\omega : S^2(V) \to \mathbb{C}$ and define the Lie superalgebra $P(\infty)$ as the subalgebra of linear operators of finite rank in V preserving ω. The superalgebra $P(\infty)$ has a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, with

$$\mathfrak{g}_0 = V_0 \otimes V_1 \; \mathfrak{g}_1 = S^2(V_0), \; \mathfrak{g}_{-1} = \Lambda^2(V_1).$$

  Note that $\mathfrak{g}_0$ is isomorphic to $\mathfrak{gl}(\infty)$, V_0 and V_1 are standard and costandard representation of $\mathfrak{g}_0$. Furthermore, $\mathfrak{g}_1$ (resp. $\mathfrak{g}_{-1}$) is identified with $S^2(V_0)$ (resp. $\Lambda^2(V_1)$) by setting

$$X_{u,w}(v) = (u,v)w + (w,v)u,$$

  for any $u, w \in V_0$, $v \in V$ and

$$X_{u,w}(v) = (u,v)w - (w,v)u,$$

  for any $u, w \in V_1$, $v \in V$. Observe that $P(\infty) = \varinjlim P(n)$ is not a simple Lie algebra. Its commutator is a simple ideal $SP(\infty)$ of all traceless matrices in $P(\infty)$.
- Let $J : V \to V$ be an odd operator such that $J^2 = -1$. The Lie superalgebra $Q(\infty)$ is the centralizer of J in $\mathfrak{gl}(\infty, \infty)$. As in the finite-dimensional case $\mathfrak{g}_0 = \mathfrak{gl}(\infty)$ and $\mathfrak{g}_1$ is the adjoint representation of $\mathfrak{g}_0$. Note that $Q(\infty) = \varinjlim Q(n)$ is not simple. It contains a simple ideal $SQ(\infty)$ of odd codimension 1 consisting of operators X such that $str(XJ) = 0$. Note that in contrast with all other cases, $SQ(\infty)$ is the direct limit of $SQ(n)$, but $SQ(n)$ are not simple.

  We leave to the reader the definition of finite corank subalgebras in this case.

4.2 $\mathbb{T}_{\mathfrak{g}}$ *for Lie superalgebras*

Now let $\mathfrak{g}$ denote one of the Lie superalgebras defined in the previous section. Let $\mathbb{T}_{\mathfrak{g}}$ be a full subcategory of $\mathfrak{g}$-modules M satisfying the following three conditions:

(1) M is integrable over $\mathfrak{g}_0$, and therefore over $\mathfrak{g}$,
(2) the annihilator of every vector in M is a finite corank subalgebra in $\mathfrak{g}$,
(3) M has finite length over $\mathfrak{g}_0$.

It is clear that $\mathbb{T}_{\mathfrak{g}}$ is an abelian monoidal category. If $\mathfrak{g} \neq Q(\infty)$, in order to avoid the annoying but not essential parity chasing we allow morphisms which change parity, i. e. the standard functor Π changing the parity is an isomorphism in our category. In fact, it is not difficult to show that if $\mathfrak{g} \neq Q(\infty)$, then

$$\mathbb{T}_{\mathfrak{g}} = \mathbb{T}_{\mathfrak{g}}^+ \oplus \mathbb{T}_{\mathfrak{g}}^-$$

with $\Pi : \mathbb{T}_{\mathfrak{g}}^+ \to \mathbb{T}_{\mathfrak{g}}^-$ defining an equivalence of categories. Threfore, admitting odd isomorphisms in the category $\mathbb{T}_{\mathfrak{g}}$ is the same as studying $\mathbb{T}_{\mathfrak{g}}^+$ instead of $\mathbb{T}_{\mathfrak{g}}$.

4.3 Orthosymplectic superalgebra

Let $\mathfrak{g} = \mathfrak{osp}(\infty,\infty)$. The goal of this subsection is to prove the following

Theorem 6 *The monoidal categories $\mathbb{T}_{\mathfrak{g}}$, $\mathbb{T}_{\mathfrak{sp}(\infty)}$ and $\mathbb{T}_{\mathfrak{so}(\infty)}$ are equivalent.*

We start with studying tensor powers of the standard representation V. If M is a $\mathfrak{g}$-module and $\mathfrak{k} \subset \mathfrak{g}$ is a subalgebra, then $M^{\mathfrak{k}}$ denotes the space of $\mathfrak{k}$-invariants in M.

Lemma 3 (a) $(V^{\otimes n})^{\mathfrak{so}(\infty)} = V_1^{\otimes n}$ *and* $(V^{\otimes n})^{\mathfrak{sp}(\infty)} = V_0^{\otimes n}$.
(b) $V_0^{\otimes n}$ *or* $V_1^{\otimes n}$ *generates* $V^{\otimes n}$ *over* $\mathfrak{g}$.

Proof. We have the obvious isomorphism of $\mathfrak{g}_0$-modules

$$V^{\otimes n} \simeq \bigoplus_{p+q=n} (V_0^{\otimes p} \otimes V_1^{\otimes q})^{\oplus C(n,p)}.$$

The identity

$$(V_0^{\otimes p} \otimes V_1^{\otimes q})^{\mathfrak{so}(\infty)} = (V_0^{\otimes p})^{\mathfrak{so}(\infty)} \otimes V_1^{\otimes q}$$

together with the fact that $(V_0^{\otimes p})^{\mathfrak{so}(\infty)} = 0$ for $p \neq 0$ imply $(V^{\otimes n})^{\mathfrak{so}(\infty)} = V_1^{\otimes n}$. The second statement of (a) is similar.

Now we prove that $V_0^{\otimes n}$ generates $V^{\otimes n}$. Assume that the statement is not true. Define the grading on $V^{\otimes n}$ by setting the degree of a homogeneous indecomposable tensor $u = u_1 \otimes \cdots \otimes u_n$ equal the number of $u_i \in V_1$. Pick up an indecomposable u of minimal degree that does not belong to $U(\mathfrak{g})V_0^{\otimes n}$. Then $k = \deg(u) > 0$ and $u_i \in V_1$ for some i. Pick up $e, e' \in V_0$ such that $(e,e') = 1$, $(e,u_1) = \cdots = (e,u_n) = 0$. Then

$$u = \pm X_{e \wedge u_i}(u_1 \otimes \cdots \otimes u_{i-1} \otimes e' \otimes u_{i+1} \otimes \ldots u_n) + v,$$

for some v of degree $k-2$. Note that $\deg(u_1 \otimes \cdots \otimes u_{i-1} \otimes e' \otimes u_{i+1} \otimes \ldots u_n) = k-1$. Hence both v and $(u_1 \otimes \cdots \otimes u_{i-1} \otimes e' \otimes u_{i+1} \otimes \ldots u_n)$ belong to $U(\mathfrak{g})V_0^{\otimes n}$. Therefore $u \in U(\mathfrak{g})V_0^{\otimes n}$. Contradiction. In the same way one can prove that $V_1^{\otimes n}$ generates $V^{\otimes n}$ over $\mathfrak{g}$.

Let

$$T(V) = \bigoplus_{n \geq 0} V^{\otimes n},\ T^{\geq m}(V) = \bigoplus_{n \geq m} V^{\otimes n},\ T^{\leq m}(V) = \bigoplus_{n \leq m} V^{\otimes n}.$$

A linear operator $X \in \mathrm{End}_k(V)$ is called *bounded* if the there are n and m such that $T^{\geq n}(V) \subset \mathrm{Ker} X$ and $\mathrm{Im} X \subset T^{\leq m}(V)$.

We denote by $\mathscr{A}_{\mathfrak{g}}$ the subalgebra of all bounded operators in $\mathrm{End}_{\mathfrak{g}}(T(V))$. Note that $\mathscr{A}_{\mathfrak{g}} = \bigoplus_{m,n \geq 0} \mathrm{Hom}_{\mathfrak{g}}(V^{\otimes m}, V^{\otimes n})$. By $\mathscr{A}_{\mathfrak{so}(\infty)}$ (resp. $\mathscr{A}_{\mathfrak{sp}(\infty)}$) we denote the algebras of bounded operators in $\mathrm{End}_{\mathfrak{so}(\infty)}(T(V_0))$ (resp. $\mathrm{End}_{\mathfrak{sp}(\infty)}(T(V_0))$).

Lemma 3 (a) implies that there are natural homomorphisms

$$\rho_{\mathfrak{so}} : \mathscr{A}_{\mathfrak{g}} \to \mathscr{A}_{\mathfrak{so}(\infty)},\ \rho_{\mathfrak{sp}} : \mathscr{A}_{\mathfrak{g}} \to \mathscr{A}_{\mathfrak{sp}(\infty)}$$

given by the restriction to $T(V)^{\mathfrak{sp}(\infty)}$ and $T(V)^{\mathfrak{so}(\infty)}$ respectively.

Lemma 4 *Both $\rho_{\mathfrak{so}}$ and $\rho_{\mathfrak{sp}}$ are isomorphisms.*

Proof. Injectivity of $\rho_{\mathfrak{so}}$ and $\rho_{\mathfrak{sp}}$ follows from Lemma 3 (b). To prove surjectivity recall from [2] that $\mathscr{A}_{\mathfrak{so}(\infty)}$ (resp. $\mathscr{A}_{\mathfrak{sp}(\infty)}$) are generated by permutation groups acting on $V_0^{\otimes n}$ (resp. $V_1^{\otimes n}$) and contractions. Both permutations and contraction are defined on $T(V)$ and hence lie in the image of $\rho_{\mathfrak{so}}$ (resp. $\rho_{\mathfrak{sp}}$).

Let λ be a partition and π_λ be the corresponding Young projector in $\mathbb{C}[S_n]$. We define

$$\tilde{V}^\lambda = \pi_\lambda(V^{\otimes n}),\ \tilde{V_0}^\lambda = \pi_\lambda(V_0^{\otimes n}),\ \tilde{V_1}^\lambda = \pi_\lambda(V_1^{\otimes n}).$$

Recall that the socles of $\tilde{V_0}^\lambda$ and $\tilde{V_1}^\lambda$ are simple $\mathfrak{g}_0$-modules. We denote them V_0^λ and V_1^λ respectively.

Lemma 5 (a) *$V_0^\lambda \otimes V_1^\mu$ is a simple $\mathfrak{g}_0$-module.*
(b) *Every simple module in $\mathbb{T}_{\mathfrak{g}_0}$ is isomorphic to $V_0^\lambda \otimes V_1^\mu$ for some partitions λ and μ.*
(c) *$\tilde{V}_0^\lambda \otimes \tilde{V}_1^\mu$ is indecomposable injective in $\mathbb{T}_{\mathfrak{g}_0}$ with socle equal to $V_0^\lambda \otimes V_1^\mu$.*

Proof. (a) can be proven by the standard argument using the Jacobson density theorem. Let $v = \sum_{j=1}^m e_j \otimes f_j \neq 0$ for some linearly independent $e_j \in V_0^\lambda$ and $f_j \in V_1^\lambda$. Let $u = e \otimes f$. Since V_0^λ and V_1^μ are simple there exist $X \in U(\mathfrak{so}(\infty))$ and $Y \in U(\mathfrak{sp}(\infty))$ such that $X(e_1) = e, Y(f_1) = f$ and $X(e_j) = Y(f_j) = 0$ for $j > 1$. Hence $u = X \otimes Y(v)$. Thus any non-zero v generates the whole $V_0^\lambda \otimes V_1^\mu$.

(b) Let $M \in \mathbb{T}_{\mathfrak{g}_0}$ be simple. Then M as an $\mathfrak{so}(\infty)$-module lies in a slightly bigger category $\hat{T}_{\mathfrak{so}(\infty)}$ of modules satisfying (1) and (2). Therefore M contains a subquotient isomorphic to V_0^λ for some λ. Since M is simple $M = V_0^\lambda \otimes W$, where W is some simple module in $\hat{T}_{\mathfrak{sp}(\infty)}$. Hence $W = V_1^\mu$ for some partition μ.

(c) Injectivity of $\tilde{V}_0^\lambda$ in $\mathbb{T}_{\mathfrak{so}(\infty)}$ implies injectivity of $\tilde{V}_0^\lambda \otimes \tilde{V}_1^\mu$ in $\mathbb{T}_{\mathfrak{so}(\infty)}$. By the same reason $\tilde{V}_0^\lambda \otimes \tilde{V}_1^\mu$ is injective in $\mathbb{T}_{\mathfrak{sp}(\infty)}$. For any simple $V_0^{\lambda'} \otimes V_1^{\mu'}$ we have that an exact sequence

$$0 \to \tilde{V}_0^\lambda \otimes \tilde{V}_1^\mu \to M \to V_0^{\lambda'} \otimes V_1^{\mu'} \to 0$$

splits over $\mathfrak{so}(\infty)$ and $\mathfrak{sp}(\infty)$. Hence it splits over $\mathfrak{g}_0$. That proves injectivity of $\tilde{V}_0^\lambda \otimes \tilde{V}_1^\mu$. Irreducibility of the socle follows from the identity

$$\mathrm{Hom}_{\mathfrak{g}_0}(V_0^{\lambda'} \otimes V_1^{\mu'}, \tilde{V}_0^\lambda \otimes \tilde{V}_1^\mu) \simeq \mathrm{Hom}_{\mathfrak{so}(\infty)}(V_0^{\lambda'}, \tilde{V}_0^\lambda) \otimes \mathrm{Hom}_{\mathfrak{sp}(\infty)}(V_1^{\mu'}, \tilde{V}_1^\mu).$$

We define functors $R_{\mathfrak{so}} : \mathbb{T}_{\mathfrak{g}} \to \mathbb{T}_{\mathfrak{so}(\infty)}$ and $R_{\mathfrak{sp}} : \mathbb{T}_{\mathfrak{g}} \to \mathbb{T}_{\mathfrak{sp}(\infty)}$ by

$$R_{\mathfrak{so}}(M) = M^{\mathfrak{sp}(\infty)},\ R_{\mathfrak{sp}}(M) = M^{\mathfrak{so}(\infty)}.$$

Lemma 6 *If $M \in \mathbb{T}_{\mathfrak{g}}$ and $M \neq 0$, then $R_{\mathfrak{so}}(M) \neq 0$ and $R_{\mathfrak{sp}}(M) \neq 0$.*

Proof. Let $L \simeq V_0^\lambda \otimes V_1^\mu$ be a simple submodule in $\mathrm{soc}_{\mathfrak{g}_0}(M)$ with maximal $|\lambda|$. Consider the natural morphism $\theta : L \otimes \mathfrak{g}_1 \to M$ of $\mathfrak{g}_0$-modules given by $\theta(u \otimes X_w) = X_w(u)$. Then $\mathrm{soc}_{\mathfrak{g}_0}(\mathrm{Im}\theta)$ has only constituents $V_0^\kappa \otimes V_1^\nu$ with $|\kappa| < |\lambda|$. Therefore if $u \otimes w \in L$ is such that $u = \pi_\lambda(u_1 \otimes \cdots \otimes u_n)$, $w \in V^{\otimes|\mu|}$ and $e \otimes f \in V_0 \otimes V_1$ is such that $(e, u_i) = 0$ for all $i = 1, \ldots, n$, then

$$\theta(u \otimes w \otimes e \otimes f) = X_{e \wedge f}(u \otimes w) = 0. \tag{2}$$

Pick up $e, e' \in V_0, f, f' \in V_1$ such that $(e, u_i) = (e', u_i) = 0$ for all $i = 1, \ldots, n$, $(e, e') = 1$ and $(f, f') = 0$. Then $X_{f \wedge f'} = [X_{e \wedge f}, X_{e' \wedge f'}]$. By (2) $X_{f \wedge f'}(u \otimes w) = 0$. It is easy to see that $X_{f \wedge f'}$ for all orthogonal f, f' generate $\mathfrak{sp}(\infty)$ we obtain $\mathfrak{sp}(\infty)w = 0$. Hence $\mu = 0$, i.e. $L \subset R_{\mathfrak{so}}(M)$.

The proof that $R_{\mathfrak{sp}}(M) \neq 0$ is similar.

Now we are ready to describe simple objects in $\mathbb{T}_\mathfrak{g}$. Let λ be a partition with $|\lambda| = n$. Choose a Cartan subalgebra $\mathfrak{h}$ such that the roots of $\mathfrak{g}$ are as in $D(\infty, \infty)$. The even roots of $\mathfrak{g}$ are $\{\pm\varepsilon_i \pm \varepsilon_j | i, j > 0, i \neq j\} \cup \{\pm\delta_i \pm \delta_j | i, j > 0\}$ and the odd roots are $\{\pm(\varepsilon_i \pm \delta_j | i, j > 0\}$. Let $\mathfrak{b}$ be the Borel subalgebra defined by the set of positive roots

$$\{\varepsilon_i \pm \varepsilon_j | i < j\} \cup \{\delta_i \pm \delta_j | i < j\} \cup \{2\delta_i | i > 0\} \cup \{\varepsilon_i \pm \delta_j | i, j > 0\}.$$

Let V^λ denote the simple highest weight module with highest weight $\lambda = \sum \lambda_i \varepsilon_i$. We introduce the standard order on weights by setting $\lambda \leq \mu$ if $\mu - \lambda$ is a non-negative integral linear combination of positive roots.

Lemma 7 $V^\lambda \in \mathbb{T}_\mathfrak{g}$. *Any simple object in* $\mathbb{T}_\mathfrak{g}$ *is isomorphic to* V^λ *for some partition* λ.

Proof. Recall that V_0^λ is a highest weight module over $\mathfrak{so}(\infty)$. Hence there exists a unique up to proportionality $v \in V_0^\lambda \subset \tilde{V}^\lambda$ of weight $\lambda = \sum \lambda_i \varepsilon_i$. An easy calculation shows that $\mathfrak{n}(v) = 0$. By Frobenius reciprocity there exists a non-zero homomorphism ψ from the Verma module M^λ to $\tilde{V}^\lambda$. The image of ψ has a unique simple quotient isomorphic to V^λ. Thus $V^\lambda \in \mathbb{T}_\mathfrak{g}$.

Now let $M \in \mathbb{T}_\mathfrak{g}$ be a simple module. Pick up a $\mathfrak{g}_0$-submodule $L \simeq V_0^\lambda$ in $R_{\mathfrak{so}}(M)$ with maximal $|\lambda|$. Then a non-zero $v \in L$ of weight λ is annihilated by $\mathfrak{n}$. Therefore $M \simeq V^\lambda$.

Lemma 8 *Let* $M \in \mathbb{T}_\mathfrak{g}$ *be a non-zero quotient of the Verma module* M^λ *for some partition* λ. *Then* $M \simeq V^\lambda$.

Proof. Suppose that the statement is false. Then there exists $V^\mu \subset M$ with $\mu < \lambda$. Let $v_\mu \in V^\mu$ be a highest vector of weight μ and v_λ be a non-zero vector of weight λ. Then $v_\mu \in U(\mathfrak{n}^-)v_\lambda$ and therefore $v_\mu \in U(\mathfrak{n}^- \cap \mathfrak{g}')v_\lambda$ for some finite-dimensional $\mathfrak{g}' = \mathfrak{osp}(2p, 2q) \subset \mathfrak{g}$. Without loss of generality we may assume $p > 2q + |\mu|$. Therefore the quadratic Casimir element in $U(\mathfrak{g}')$ has the same eigenvalue on v_μ and v_λ. This is impossible since $(\lambda + \rho, \lambda + \rho) \neq (\mu + \rho, \mu + \rho)$ if $p > 2q + |\mu|$, here ρ is the half sum of even positive roots minus the half sum of odd positive roots of $\mathfrak{g}'$.

Lemma 9 *Let*

$$0 \to V^{\lambda} \to M \to V^{\mu} \to 0$$

be a non-split exact sequence in $\mathbb{T}_{\mathfrak{g}}$*, then* $\mu < \lambda$ *in the standard order. Furthermore,* $|\mu| < |\lambda|$.

Proof. If μ and λ are not comparable then a vector of weight μ spans in M a submodule isomorphic to V^{μ} and the sequence splits. Since $M \in \mathbb{T}_{\mathfrak{g}_0}$, M is semisimple over $\mathfrak{h}$, the sequence also splits in the case $\mu = \lambda$. If $\mu > \lambda$, then M is a quotient of the Verma module M^{μ}. By Lemma 8 $M \simeq V^{\mu}$. Therefore the only possibility is $\mu < \lambda$. Note that this implies $|\mu| \le |\lambda|$. We claim that $|\mu| < |\lambda|$. Indeed, let $|\mu| = |\lambda|$. For any module N set N^{+} be the span of all weight spaces of weights $\sum a_i \varepsilon_i + \sum b_j \delta_j$ with $\sum a_i = |\lambda|$. If $\mathfrak{k} \simeq \mathfrak{gl}(\infty)$ be the subalgebra in $\mathfrak{g}$ generated by the roots of the form $\varepsilon_i - \varepsilon_j$, then N^{+} is obviously $\mathfrak{k}$-stable. It is easy to see that $(V^{\lambda})^{+}$ and $(V^{\mu})^{+}$ are simple $\mathfrak{k}$-submodules in the tensor algebra of the standard $\mathfrak{k}$-module. Therefore

$$0 \to (V^{\lambda})^{+} \to M^{+} \to (V^{\mu})^{+} \to 0$$

splits over $\mathfrak{k}$. But then the original sequence must split as well by Lemma 8.

Lemma 10 $\tilde{V}^{\lambda}$ *is injective in* $\mathbb{T}_{\mathfrak{g}}$.

Proof. It is sufficient to prove that for any μ an exact sequence

$$0 \to \tilde{V}^{\lambda} \to M \to V^{\mu} \to 0$$

of modules in $\mathbb{T}_{\mathfrak{g}}$ splits. If $|\mu| \ge |\lambda|$, then the sequence splits by Lemma 9, as $|\mu| \ge |\nu|$ for any simple subquotient V^{ν} in $\tilde{V}^{\lambda}$. Hence we may assume $|\mu| < |\lambda|$.

By Lemma 5 $\tilde{V}^{\lambda}$ is injective in $\mathbb{T}_{\mathfrak{g}_0}$. Therefore the sequence splits over $\mathfrak{g}_0$. Thus, we can write $M = V^{\mu} \oplus \tilde{V}^{\lambda}$ as a $\mathfrak{g}_0$-module. The action of $\mathfrak{g}_1$ is given by $X(u,w) = (Xu, c(u \otimes X) + Xw)$ for any $u \in V^{\mu}, w \in \tilde{V}^{\lambda}$, $X \in \mathfrak{g}_1$ and some $c \in \mathrm{Hom}_{\mathfrak{g}_0}(V^{\mu} \otimes \mathfrak{g}_1, \tilde{V}^{\lambda})$. By Lemma 5

$$\mathrm{soc}_{\mathfrak{g}_0} \tilde{V}^{\lambda} = \bigoplus_{(\nu,\nu')} V_0^{\nu} \otimes V_1^{\nu'}$$

for some set of pairs (ν, ν') such that $|\nu| + |\nu'| = |\lambda|$. Therefore we have

$$c(V_0^{\mu} \otimes \mathfrak{g}_1) \cap \mathrm{soc}_{\mathfrak{g}_0} \tilde{V}^{\lambda} = \bigoplus_{\nu} V_0^{\nu} \otimes V_1,$$

where the summation is taken over some set of partitions ν such that $|\nu| = |\lambda| - 1$. If $u \in V_0^{\mu}$, then

$$c(u \otimes X_{e\wedge f}) = \sum a_{\nu} \pi_{\nu}(u \otimes e) \otimes f,$$

for some $a_{\nu} \in \mathbb{C}$. We claim that in fact all $a_{\nu} = 0$. Indeed, assume $a_{\nu} \neq 0$. Let $u = \pi_{\lambda}(u_1 \otimes \cdots \otimes u_n) \in V_0^{\lambda}$ for some linearly independent isotropic mutually orthogonal $u_1, \ldots, u_n$. For any $e \in V_0$ linearly independent of $u_1, \ldots, u_n$ and any non-zero $f \in V_1$, we have $c(u \otimes X_{e\wedge f}) \neq 0$. But then the annihilator of $u \in M$ is not a finite corank subalgebra, hence M is not in $\mathbb{T}_{\mathfrak{g}}$. Contradiction.

Thus, $c(V_0^\mu \otimes \mathfrak{g}_1) = 0$. Let $v_\mu \in V_0^\mu$ be the highest vector. Then $(v_\mu, 0) \in M$ is $\mathfrak{n}$-invariant. Consider the submodule $N \subset M$ generated by $(v_\mu, 0)$. By Lemma 8 $N \simeq V^\mu$. Therefore the exact sequence splits.

Corollary 1 *The socle of* $\tilde{V}^\lambda$ *coincides with* V^λ.

Proof. Follows from Lemma 6 and 7 since $R_{\mathfrak{so}}(\tilde{V}^\lambda) = \tilde{V}_0^\lambda$.

Corollary 2

$$R_{\mathfrak{so}}(V^\lambda) = V_0^\lambda,\ R_{\mathfrak{sp}}(V^\lambda) = V_1^{\lambda^\perp}. \tag{3}$$

Proof. By Corollary 1 and Lemma 10

$$V^\lambda = \bigcap_{\varphi \in \mathrm{Hom}_{\mathfrak{g}}(\tilde{V}^\lambda, T^{\leq |\lambda|-1}(V))} \mathrm{Ker}\varphi.$$

Therefore Lemma 4 implies the statement.

One can prove as in [2] that $\mathbb{T}_{\mathfrak{g}}$ is antiequvivalent to the category of locally unitary finite-dimensional $\mathscr{A}_{\mathfrak{g}}$- modules. Therefore Lemma 4 implies that $\mathbb{T}_{\mathfrak{osp}(\infty,\infty)}$ and $\mathbb{T}_{\mathfrak{so}(\infty)}$ are equivalent abelian categories.

Define now the functors $S_{\mathfrak{so}} : \mathbb{T}_{\mathfrak{so}(\infty)} \to \mathbb{T}_{\mathfrak{g}}$ and $S_{\mathfrak{sp}} : \mathbb{T}_{\mathfrak{sp}(\infty)} \to \mathbb{T}_{\mathfrak{g}}$ as follows. Let $M \in \mathbb{T}_{\mathfrak{so}(\infty)}$ (resp. $\mathbb{T}_{\mathfrak{sp}(\infty)}$). By $I(M)$ we denote the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M$, where we define the action of $\mathfrak{sp}(\infty)$ (resp. $\mathfrak{so}(\infty)$) on M to be trivial. We set

$$S_{\mathfrak{so}}(M) = I(M)/(\bigcap_{\varphi \in \mathrm{Hom}_{\mathfrak{g}}(I(M), T(V))} \mathrm{Ker}\varphi),$$

respectively

$$S_{\mathfrak{sp}}(M) = I(M)/(\bigcap_{\varphi \in \mathrm{Hom}_{\mathfrak{g}}(I(M), T(V))} \mathrm{Ker}\varphi).$$

Observe that $\mathrm{Hom}_{\mathfrak{g}}(I(M), T^{\geq n}(V)) = \mathrm{Hom}_{\mathfrak{g}_0}(M, T^{\geq n}(V)) = 0$ for sufficiently large n. Thus, $S_{\mathfrak{so}}(M)$ (resp. $S_{\mathfrak{sp}}(M)$) have finite length over $\mathfrak{g}_0$ and hence lie in $\mathbb{T}_{\mathfrak{g}}$. It is not hard to see that $S_{\mathfrak{so}}(M)$ (resp. $S_{\mathfrak{sp}}(M)$) is the maximal quotient of $I(M)$ belonging to $\mathbb{T}_{\mathfrak{g}}$. Hence by Frobenius reciprocity

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{g}}(S_{\mathfrak{so}}(M), N) &\simeq \mathrm{Hom}_{\mathfrak{so}(\infty)}(M, R_{\mathfrak{so}}(N)), \\ \mathrm{Hom}_{\mathfrak{g}}(S_{\mathfrak{sp}}(M), N) &\simeq \mathrm{Hom}_{\mathfrak{sp}(\infty)}(M, R_{\mathfrak{sp}}(N)). \end{aligned} \tag{4}$$

Proposition 1 *The functors* $S_{\mathfrak{so}}$ *(resp.* $S_{\mathfrak{sp}}$*) and* $R_{\mathfrak{so}}$ *(resp.* $R_{\mathfrak{sp}}$*) are mutually inverse equivalences between* $\mathbb{T}_{\mathfrak{so}(\infty)}$ *(resp.* $\mathbb{T}_{\mathfrak{sp}(\infty)}$*) and* $\mathbb{T}_{\mathfrak{g}}$.

Proof. A functor $F : \mathbb{T}_{\mathfrak{g}} \to \mathbb{T}_{\mathfrak{so}(\infty)}$ establishing an equivalence can be taken as a composition of $F_1 : \mathscr{A}_{\mathfrak{g}}$-fmod $\to \mathbb{T}_{\mathfrak{so}(\infty)}$ and $F_2 : \mathbb{T}_{\mathfrak{g}} \to \mathscr{A}_{\mathfrak{g}}$-fmod, where $F_1 = \mathrm{Hom}_{\mathscr{A}_{\mathfrak{g}}}(\cdot, T(V_0))$ and $F_2 = \mathrm{Hom}_{\mathfrak{g}}(\cdot, T(V))$. Lemma 4 implies $R_{\mathfrak{so}} = F_1 \circ F_2$. Since $S_{\mathfrak{so}}$ is left adjoint to $R_{\mathfrak{so}}$ by (4), $S_{\mathfrak{so}}$ must be inverse of $R_{\mathfrak{so}}$ by general nonsense.

The case of $\mathfrak{sp}(\infty)$ is similar.

To prove Theorem 6 we claim the following.

Proposition 2 *The functors $S_{\mathfrak{so}}$ (resp. $S_{\mathfrak{sp}}$) and $R_{\mathfrak{so}}$ (resp. $R_{\mathfrak{sp}}$) are equivalences of monoidal categories.*

The proof of the above Proposition amounts to showing that $R_{\mathfrak{so}}$ and $R_{\mathfrak{sp}}$ preserve tensor products. It is an easy consequence of the following curious fact. We leave its proof to the reader.

Lemma 11 *Let $\mathfrak{k} = \mathfrak{gl}(\infty), \mathfrak{sl}(\infty), \mathfrak{so}(\infty), \mathfrak{sp}(\infty)$. In the category $\mathbb{T}_{\mathfrak{k}}$ the functor of invariants preserves tensor product. In other words, $(M \otimes N)^{\mathfrak{k}} = M^{\mathfrak{k}} \otimes N^{\mathfrak{k}}$.*

4.4 The case of $\mathfrak{gl}(\infty,\infty)$

The case $\mathfrak{g} = \mathfrak{gl}(\infty,\infty)$ is similar to the case of $\mathfrak{osp}(\infty,\infty)$. Therefore we only state the results omitting the proofs. Recall that the even part $\mathfrak{g}_0$ is a direct sum of two copies of $\mathfrak{gl}(\infty)$. Let $\mathfrak{k} = V_0 \otimes (V_0)_*$ and $\mathfrak{l} = V_1 \otimes (V_1)_*$. We define the functors $R_{\mathfrak{k}} : \mathbb{T}_{\mathfrak{g}} \to \mathbb{T}_{\mathfrak{k}}$, $R_{\mathfrak{l}} : \mathbb{T}_{\mathfrak{g}} \to \mathbb{T}_{\mathfrak{l}}$ by $R_{\mathfrak{k}}(M) = M^{\mathfrak{l}}$, $R_{\mathfrak{l}}(M) = M^{\mathfrak{k}}$ and $S : \mathbb{T}_{\mathfrak{gl}(\infty)} \to \mathbb{T}_{\mathfrak{gl}(\infty|\infty)}$ by

$$S(M) = I(M)/(\bigcap_{\varphi \in \mathrm{Hom}(I(M), T(V \oplus V_*))} \mathrm{Ker}\varphi,$$

where $I(M) = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M$.

It is not hard to see that S is left adjoint to $R_{\mathfrak{k}}$ and $R_{\mathfrak{l}}$.

Theorem 7 *The functors $R_{\mathfrak{k}}$ and S establish an equivalence of monoidal categories $\mathbb{T}_{\mathfrak{gl}(\infty,\infty)}$ and $\mathbb{T}_{\mathfrak{gl}(\infty)}$.*

Note that the composition $S \circ R_{\mathfrak{l}} : \mathbb{T}_{\mathfrak{k}} \to \mathbb{T}_{\mathfrak{l}}$ defines an autoequivalence of the category $\mathbb{T}_{\mathfrak{gl}(\infty)}$ that sends $V^{\lambda,\mu} \to V^{\lambda^\perp,\mu^\perp}$.

4.5 The case of $P(\infty)$

It turns out that for $\mathfrak{g} = P(\infty)$ the category $\mathbb{T}_{\mathfrak{g}}$ is also equivalent to $\mathbb{T}_{\mathfrak{so}(\infty)}$ and hence to $\mathbb{T}_{\mathfrak{sp}(\infty)}$ and $\mathbb{T}_{\mathfrak{osp}(\infty|\infty)}$. However, the proof is different in this case.

We claim that any module M in $\mathbb{T}_{\mathfrak{g}}$ can be equipped with $\mathbb{Z}$-grading $M = \bigoplus M^k$ such that $\mathfrak{g}_i M^k \subset M^{k+2i}$. Indeed, note that any simple $\mathfrak{g}_0$-subquotient of M is isomorphic $V_0^{\lambda,\mu}$, and we assign to it degree $|\lambda| - |\mu|$.

Define a functor $R : \mathbb{T}_{\mathfrak{g}} \to \mathbb{T}_{\mathfrak{g}_0}$ by

$$R(M) = M^{\mathfrak{g}_1}.$$

Lemma 12 (a) *For any $M \in \mathbb{T}_{\mathfrak{g}}$, $R(M) \neq 0$.*
(b) *If M is simple then $R(M)$ is simple.*
(c) *If $R(M) \simeq R(L)$ for two simple $M, L \in \mathbb{T}_{\mathfrak{g}}$, then $M \simeq L$.*

Proof. Since M has finite length over $\mathfrak{g}_0$ there exists a maximal k such that $M^k \neq 0$. Then $\mathfrak{g}_1(M^k) \subset M^{k+2} = 0$. That proves (a).

(b) Let k be maximal such that $M^k \neq 0$. If N is a proper submodule in M^k, then $U(\mathfrak{g})N = U(\mathfrak{g}_{-1})N$ is a proper submodule in M since $(U(\mathfrak{g})N)^k = N$. Therefore M^k is a simple $\mathfrak{g}_0$-module. If $R(M)^i \neq 0$ for some $i < k$, then $U(\mathfrak{g})(R(M)^i)$ is a proper non-zero submodule in M. Hence if M is simple, then $R(M) = M^k$ is simple.

(c) It is clear that both M and L are simple quotients of the parabolically induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} R(M)$. But the latter has a unique simple quotient. Hence (c).

Lemma 13 *Let $M \in \mathbb{T}_{\mathfrak{g}}$.*
(a) *If $V_0^\lambda \subset M$ is an embedding of $\mathfrak{g}_0$-modules, then $V_0^\lambda \subset R(M)$.*
(b) *Any simple submodule in $R(M)$ is isomorphic to V_0^λ for some partition λ.*

Proof. (a) Consider the map $\psi : \mathfrak{g}_1 \otimes V_0^\lambda \to M$ defined by $\psi(X \otimes v) = Xv$. Suppose $|\lambda| = k$. Consider linearly independent $u_1, \dots, u_k \in V_0$. Then $u = \pi_\lambda(u_1 \otimes \dots \otimes u_k) \in V_0^\lambda$ is not zero. Moreover, for any $w, v \in V_0$ we have

$$\psi(X_{w,v} \otimes u) = \sum_\nu a_\nu \pi_\nu(w \otimes v \otimes u)$$

for some ν obtained from λ by adding two boxes not in the same column and some $a_\nu \in \mathbb{C}$. Note that if some $a_\nu \neq 0$, then for any w, v such that $u_1, \dots, u_k, v, w$ are linearly independent, we have $\psi(X_{w,v} \otimes u) \neq 0$. But then the annihilator of $v \in V_0^\lambda$ is not of finite corank. Hence $\psi = 0$, i.e. $V_0^\lambda \subset R(M)$.

(b) Let L be a simple $\mathfrak{g}_0$-submodule in $R(M)$. Then $L \simeq \mathrm{soc}_{\mathfrak{g}_0}(V_0^\lambda \otimes V_1^\mu)$. We want to show that $\mu = 0$. Assume the opposite. Consider the $\mathfrak{g}_0$-homorphism $\psi : L \otimes \mathfrak{g}_{-1} \to M$ given by $\psi(w \otimes X) = X(w)$ for all $X \in \mathfrak{g}_{-1}, w \in L$. Since the annihilator of any vector in M has finite corank we have

$$\psi(\mathrm{soc}_{\mathfrak{g}_0}(L \otimes \mathfrak{g}_{-1})) = 0. \tag{5}$$

Let $w \in L$ be of the form

$$w = \pi_\lambda(v_1 \otimes \dots \otimes v_n) \otimes \pi_\mu(u_1 \otimes \dots \otimes u_m),$$

where $v_i \in V_0, u_j \in V_1$ are linearly independent, $n = |\lambda|, m = |\mu|$, $(v_i, u_j) = 0$ for all $i \leq n, j \leq m$. Let $f_1, f_2 \in V_1$ be orthogonal to $v_1, \dots, v_n$ and linearly independent with $u_1, \dots, u_m$. By (5)

$$\psi(w \otimes X_{f_1,f_2}) = X_{f_1,f_2}(w) = 0. \tag{6}$$

Let $e \in V_0$. Then for any $v \in V_0, u \in V_1$ we have

$$\begin{aligned} [X_{e,e}, X_{f_1,f_2}](v) &= 2((f_1,v)(f_2,e) - (f_2,v)(f_1,e))e, \\ [X_{e,e}, X_{f_1,f_2}](u) &= 2(e,u)((f_1,e)f_2 - (f_2,e)f_1). \end{aligned} \tag{7}$$

Pick up $e \in V_0, f_1, f_2 \in V_1$ such that $(e, f_1) = 1, (e, f_2) = 0$, $(e, u_1) = 1$ and $(e, u_i) = 0$ for $i > 1$. Then by (6) $[X_{e,e}, X_{f_1,f_2}](w) = 0$ and by (7)

$$[X_{e,e}, X_{f_1,f_2}](w) = 2\pi_\lambda(v_1 \otimes \dots \otimes v_n) \otimes \pi_\mu(f_2 \otimes u_2 \otimes \dots \otimes u_m) \neq 0.$$

Contradiction.

Corollary 3 *For any simple $M \in \mathbb{T}_{\mathfrak{g}}$, $R(M) \simeq V_0^{\lambda}$ for some λ.*

We use the notation V^{λ} for the simple $M \in \mathbb{T}_{\mathfrak{g}}$ with $R(M) = V_0^{\lambda}$.

We will prove now that $V^{\otimes n}$ is injective in $\mathbb{T}_{\mathfrak{g}}$. Consider the action of S_n on $V^{\otimes n}$ such that an adjacent transposition $\sigma_{i,i+1}$ acts by

$$\sigma_{i,i+1}(v_1 \otimes \cdots \otimes v_n) = (-1)^{p(v_i)p(v_{i+1})} v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n.$$

This action commutes with the action of $\mathfrak{g}$.

Proposition 3 (a) $R(V^{\otimes n}) = (V_0)^{\otimes n}$;
(b) $\mathrm{soc} V^{\otimes n} = U(\mathfrak{g}_0)(V_0^{\otimes n})$;
(c) $\mathrm{End}_{\mathfrak{g}}(V^{\otimes n}) = \mathrm{End}_{\mathfrak{g}_0}(V_0^{\otimes n}) = \mathbb{C}[S_n]$;
(d) $V^{\otimes n} = \bigoplus_{|\lambda|=n} \tilde{V}^{\lambda} \otimes \mathbb{Y}_{\lambda}$, *where $\mathbb{Y}_{\lambda}$ is the irreducible S_n-module associated to λ and $\tilde{V}^{\lambda}$ is an indecomposable module with socle V^{λ}.*

Proof. (a) Consider $V^{\otimes n}$ as a $\mathfrak{g}_0$-module. It splits into indecomposable sumands ${V_0}^{\mu} \otimes {V_1}^{\nu}$ with $|\mu| + |\nu| = n$. Hence the statement follows from Lemma 13(b)

(b) By Lemma 12(a) and (a) each $V_0^{\mu} \subset V_0^{\otimes n}$ generates a simple submodule. Therefore (b) follows from (a).

(c) The restriction map: $\mathrm{End}_{\mathfrak{g}}(V^{\otimes n}) \to \mathrm{End}_{\mathfrak{g}_0}(V_0^{\otimes n}) = \mathbb{C}[S_n]$; is obviously surjective. We claim that the quotient of $V^{\otimes n}$ by the socle can have only simple subquotients V^{μ} for $|\mu| < n$. Indeed, if V^{μ} is a simple subquotient of $V^{\otimes n}$, then V_0^{μ} is a simple $\mathfrak{g}_0$-subquotient of $V^{\otimes n}$. Hence $|\mu| \leq n$. On the other hand, if $|\mu| = n$, then $V_0^{\mu} \subset V_0^{\otimes n} \subset \mathrm{soc} V^{\otimes n}$. Therefore any $\varphi \in \mathrm{End}_{\mathfrak{g}}(V^{\otimes n})$ that kills the socle must be zero. That implies injectivity of the restriction map.

(d) is a consequence of (c) and (b).

Note that (b) also implies

Corollary 4 *If* $\mathrm{Hom}_{\mathfrak{g}}(V^{\otimes n}, V^{\otimes m}) \neq 0$, *then $n - m$ is non-negative even.*

Lemma 14 *Let $M \in \mathbb{T}_{\mathfrak{g}}$, $V_0^{\lambda} \subset R(M)$ generates M, then $M \simeq V^{\lambda}$.*

Proof. Let $|\lambda| = n$. Consider the parabolically induced module $K = U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0 \oplus \mathfrak{g}_1)} V_0^{\lambda}$. We have an isomorphism $K \simeq \Lambda(\mathfrak{g}_{-1}) \otimes V_0^{\lambda}$ of $\mathfrak{g}_0$-modules. Let $N_0 = \mathrm{soc}_{\mathfrak{g}_0}(\mathfrak{g}_{-1} \otimes V_0^{\lambda})$. First, we show that $\mathfrak{g}_1(N_0) = 0$. For this we fix the Borel subalgebra of $\mathfrak{g}_0$ such that all tensor modules are highest weight modules (see [2] and Sect. 3). If $v \in V_0^{\lambda}$ is a highest vector, $Y \in \mathfrak{g}_{-1}$ is a highest vector with respect to the adjoint action of $\mathfrak{g}_0$, then $Y \otimes v$ generates N_0 (as a $\mathfrak{g}_0$-module). Let $X \in \mathfrak{g}_1$, then $X(Y \otimes v) = 1 \otimes [X,Y]v$. By a straightforward check $[X,Y]v = 0$. Thus, $\mathfrak{g}_1(Y \otimes v) = 0$ and hence the whole N_0 is annihilated by $\mathfrak{g}_1$.

Now let $N \subset K$ be the submodule generated by N_0 and $Q = K/N$. Let $\pi : K \to Q$ denote the natural projection. Note that the annihilator of $\pi(1 \otimes v)$ is of finite corank. Since $U(\mathfrak{g})\pi(1 \otimes v) = Q$, Q satisfies (1) and (2). Note that M is a quotient of K. Consider the natural projection $\sigma : K \to M$. If $\sigma(N_0) \neq 0$, then $Z\sigma(1 \otimes v) \neq 0$ for any

$Z \in \mathfrak{g}_{-1}$, that contradicts our assumption $M \in \mathbb{T}_{\mathfrak{g}}$, as $\sigma(1 \otimes v)$ does not have the annihilator of finite corank. Hence $\sigma(N_0) = 0$ and therefore M is a quotient of Q.

Note that although $K \notin \mathbb{T}_{\mathfrak{g}}$, it is still equipped with the $\mathbb{Z}$-grading such that $K^{n-2k} = \Lambda^k(\mathfrak{g}_{-1}) \otimes V_0^{\lambda}$. Hence both N and Q are also graded. We claim that for all μ and $k > 0$

$$\mathrm{Hom}_{\mathfrak{g}_0}(V_0^{\mu}, Q^{n-2k}) = 0. \tag{8}$$

Indeed, N^{n-2k} is generated by $\Lambda^{k-1}(\mathfrak{g}_{-1})(Y \otimes v)$ over $\mathfrak{g}_0$. Any weight vector in $\Lambda^{k-1}(\mathfrak{g}_{-1})(Y \otimes v)$ has weight $\sum a_i \varepsilon_i$ with at least two negative a_i and hence belongs to $\mathrm{soc}_{\mathfrak{g}_0}^{2k-1} K^{n-2k}$. But then

$$N^{n-2k} \subset \mathrm{soc}_{\mathfrak{g}_0}^{2k-1} K^{n-2k}.$$

Therefore we have an embedding of $\mathfrak{g}_0$-modules

$$\mathrm{soc}_{\mathfrak{g}_0} Q^{n-2k} \subset (\mathrm{soc}_{\mathfrak{g}_0}^{2k} K^{n-2k})/N^{n-2k}.$$

All $\mathfrak{g}_0$-simple subquotients of $\mathrm{soc}_{\mathfrak{g}_0}^{2k} K^{n-2k}$ are of the form $\mathrm{soc}_{\mathfrak{g}_0}(V_0^{\mu} \otimes V_1^{\nu})$ with $|\nu| \geq 1$. Hence (8).

Now we can prove that Q is simple and hence isomorphic to V^{λ}. Indeed, it is equivalent to proving that $R(Q) = V_0^{\lambda}$. Suppose the latter is false, i.e. $R(Q)^{n-2k} \neq 0$ for some $k > 0$. By Lemma 13(a) there exists μ such that $V_0^{\mu} \subset R(Q)^{n-2k}$. But this contradicts (8).

Theorem 8 *$\tilde{V}^{\lambda}$ is the injective hull of V^{λ} in $\mathbb{T}_{\mathfrak{g}}$.*

Proof. It suffices to prove that any exact sequence in $\mathbb{T}_{\mathfrak{g}}$ of the form

$$0 \to \tilde{V}^{\lambda} \to M \to V^{\mu} \to 0$$

splits. Since $\tilde{V}^{\lambda}$ is injective in $\mathbb{T}_{\mathfrak{g}_0}$ the sequence splits over $\mathfrak{g}_0$. In particular, we have an embedding $V_0^{\mu} \subset M$ of $\mathfrak{g}_0$-modules. By Lemma 13(a) $V_0^{\mu} \subset R(M)$. By Lemma 14 V_0^{μ} generates $V^{\mu} \subset M$. Hence the statement.

The above theorem and Proposition 3(d) imply

Corollary 5 *$V^{\otimes n}$ is injective in $\mathbb{T}_{\mathfrak{g}}$.*

Recall that $\mathscr{A}_{\mathfrak{g}}$ denotes the subalgebra of all bounded operators in $\mathrm{End}_{\mathfrak{g}}(T(V))$. Theorem 8 implies

Corollary 6 *$\mathbb{T}_{\mathfrak{g}}$ is antiequivalent to the category of locally unitary finite-dimensional modules over $\mathscr{A}_{\mathfrak{g}}$.*

By Corollary 4 we have a $\mathbb{Z}$-grading

$$\mathscr{A}_{\mathfrak{g}} = \bigoplus_{i \geq 0} \mathscr{A}_{\mathfrak{g}}^i, \ \mathscr{A}_{\mathfrak{g}}^i = \bigoplus_{n \geq 0} \mathrm{Hom}_{\mathfrak{g}}(V^{\otimes n}, V^{\otimes n-2i}).$$

By Proposition 3

$$\mathscr{A}_{\mathfrak{g}}^{0} = \bigoplus_{n\geq 0} \mathbb{C}[S_n].$$

For any $1 \leq i < j \leq n$ define $\tau_{ij}^{n} \in \mathrm{Hom}_{\mathfrak{g}}(V^{\otimes n}, V^{\otimes n-2})$ by the formula

$$\tau_{ij}^{n}(v_1 \otimes \cdots \otimes v_n) = (-1)^{s}(v_i, v_j) v_1 \otimes \cdots \otimes \hat{v}_i \otimes \cdots \otimes \hat{v}_j \otimes \cdots \otimes v_n,$$

where $s = (p(v_i)+p(v_j))(p(v_1)+\cdots+p(v_{i-1}))+p(v_j)(p(v_{i+1})+\cdots+p(v_{j-1}))$.

Lemma 15 $\{\tau_{ij}^{n}\}$ *for all* $n > 1$ *and* $1 \leq i < j \leq n$ *form a basis in* $\mathscr{A}_{\mathfrak{g}}^{1}$.

Proof. It is sufficient to show that for a fixed n, the set $\{\tau_{ij}^{n}\}$ for all $1 \leq i < j \leq n$ is a basis in $\mathrm{Hom}_{\mathfrak{g}}(V^{\otimes n}, V^{\otimes n-2})$. Linear independence is straightforward. To prove that $\{\tau_{ij}^{n}\}$ span $\mathrm{Hom}_{\mathfrak{g}}(V^{\otimes n}, V^{\otimes n-2})$ consider the homomorphism

$$\rho : \mathrm{Hom}_{\mathfrak{g}}(V^{\otimes n}, V^{\otimes n-2}) \to \mathrm{Hom}_{\mathfrak{g}_0}((V^{\otimes n})^{n-2}, (V^{\otimes n-2})^{n-2})$$

defined by restriction to the $n-2$-th graded component. Since $(V^{\otimes n-2})^{n-2} = V_0^{\otimes n-2}$ generates the socle of $V^{\otimes n-2}$, this homomorphism is injective. Write

$$(V^{\otimes n})^{n-2} = \bigoplus_{k=1}^{n} M_k,$$

where $M_k = V_0^{\otimes k-1} \otimes V_1 \otimes V_0^{\otimes n-k}$. Any $\psi \in \mathrm{Hom}_{\mathfrak{g}_0}((V^{\otimes n})^{n-2}, V_0^{\otimes n-2})$ can be written in the form $\sum a_{kl}\theta_k^{l}$ where $\theta_k^{l} : M_k \to V_0^{\otimes n-2}$ is the restriction of τ_{kl}^{n} to M_k if $k < l$ or τ_{lk}^{n} if $k > l$. If ψ lies in the image of ρ, then $\psi = \rho(\phi)$ and therefore

$$\psi(X_{f_1,f_2}(v_1 \otimes \cdots \otimes v_n)) = X_{f_1,f_2}\phi(v_1 \otimes \cdots \otimes v_n) = 0 \qquad (9)$$

for any $v_1,\ldots,v_n, f_1, f_2 \in V_0$. Choose $v_1,\ldots,v_n \in V_0, f_1, f_2 \in V_1$ so that $(v_i, f_1) = 0$ for any $i \neq k$, $(v_i, f_2) = 0$ for any $i \neq l$, $(v_k, f_1) = (v_l, f_2) = 1$. Then (9) implies $a_{kl} = a_{lk}$. Therefore $\psi = \rho(\sum_{k<l} a_{kl}\tau_{kl}^{n})$. The result follows now from injectivity of ρ.

Lemma 16 *Let* λ^{+} *(resp.* λ^{-}*) denote the set of all* μ *obtained from* λ *by adding (resp. removing) one box. Then we have the following exact sequence*

$$0 \to \bigoplus_{\nu\in\lambda^{+}} V^{\nu} \to V^{\lambda} \otimes V \to \bigoplus_{\mu\in\lambda^{-}} V^{\mu} \to 0.$$

Proof. Assume $|\lambda| = n$. From embedding $V^{\lambda} \otimes V \subset V^{\otimes n+1}$ we have

$$R(V^{\lambda} \otimes V) = V_0^{\lambda} \otimes V_0 = \bigoplus_{\nu\in\lambda^{+}} V_0^{\nu}.$$

Let M be the submodule in $V^{\lambda} \otimes V$ generated by $R(V^{\lambda} \otimes V)$. Then $M = \bigoplus_{\nu\in\lambda^{+}} V^{\nu}$ by Lemma 14 and Pierri rule. Let $S = (V^{\lambda} \otimes V)/M$ and $\pi : V^{\lambda} \otimes V \to S$ be the natural projection. Then S is generated by $\pi(V_0^\lambda \otimes V_1)$. Moreover, $\pi(V_0^\lambda \otimes V_1) \subset R(S)$. By Lemma 13(b) and [5]

$$\pi(V_0^\lambda \otimes V_1) \subset \bigoplus_{\mu\in\lambda^-} V_0^\mu.$$

To see that

$$\pi(V_0^\lambda \otimes V_1) = \bigoplus_{\mu\in\lambda^-} V_0^\mu$$

observe that the set $\{\tau_{j,n+1}^{n+1} | 1 \le j \le n\}$ spans $\mathrm{Hom}_{\mathfrak{g}_0}(V_0^\lambda \otimes V_1, V_0^{n-1})$. Since $\tau_{j,n+1}^{n+1}(M) = 0$ for all $j \le n$ we have

$$M \cap (V_0^\lambda \otimes V_1) \subset \mathrm{soc}_{\mathfrak{g}_0}(V_0^\lambda \otimes V_1)$$

Now the statement follows from Lemma 14.

Lemma 17

$$\mathrm{soc} V^{\otimes n} = \bigcap_{\varphi\in\mathrm{Hom}_{\mathfrak{g}}(V^{\otimes n}, V^{\otimes n-2})} \mathrm{Ker}\varphi = \bigcap_{1\le i<j\le n} \mathrm{Ker}\tau_{ij}^n.$$

Proof. The inclusion

$$\mathrm{soc} V^{\otimes n} \subset \bigcap_{1\le i<j\le n} \mathrm{Ker}\tau_{ij}^n$$

is trivial since $V_0^{\otimes n} \subset \mathrm{Ker}\tau_{ij}^n$ for all i, j and $\mathrm{soc} V^{\otimes n}$ is generated by $V_0^{\otimes n}$.

We prove equality by induction in n. Let $X_n = \bigcap_{1\le i<j\le n} \mathrm{Ker}\tau_{ij}^n$. By induction assumption we have

$$X_n \subset X_{n-1} \otimes V = (\mathrm{soc} V^{\otimes n-1}) \otimes V.$$

Using the previous lemma one can easily see that there is an exact sequence

$$0 \to \mathrm{soc} V^{\otimes n} \to X_{n-1} \otimes V \to Z \to 0,$$

where Z is a direct sum of some V^μ with $|\mu| = n-2$. By the above exact sequence it is sufficient to check

$$(\mathrm{soc} V^{\otimes n})^{n-2} = \bigcap_{1\le i<j\le n} \mathrm{Ker}\tau_{ij}^n \cap (V^{\otimes n})^{n-2}.$$

Our calculation in the proof of Lemma 15 implies that

$$\bigcap_{1\le i<j\le n} \mathrm{Ker}\tau_{ij}^n \cap (V^{\otimes n})^{n-2} = \mathfrak{g}_{-1} V_0^{\otimes n}.$$

Hence the statement.

Lemma 18 *$\mathscr{A}_{\mathfrak{g}}$ is generated by $\mathscr{A}_{\mathfrak{g}}^0$ and $\mathscr{A}_{\mathfrak{g}}^1$.*

Proof. Let $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(V^{\otimes n}, V^{\otimes n-2k})$ for $k > 1$. Then $\mathrm{soc}V^{\otimes n} \subset \mathrm{Ker}\varphi$. Let $M = V^{\otimes n}/\mathrm{soc}V^{\otimes n}$. By Lemma 17

$$\bigoplus_{1\leq i<j\leq n} \tau^n_{ij} : V^{\otimes n} \to (V^{\otimes n-2})^{\oplus n(n-1)/2}$$

defines the embedding $M \subset (V^{\otimes n-2})^{\oplus n(n-1)/2}$. By injectivity of $V^{\otimes n-2k}$ there exists $\psi \in \mathrm{Hom}_{\mathfrak{g}}((V^{\otimes n-2})^{\oplus n(n-1)/2}, V^{\otimes n-2k})$ such that

$$\varphi = \psi \circ (\bigoplus_{1\leq i<j\leq n} \tau^n_{ij}).$$

Hence

$$\varphi = \sum_{1\leq i<j\leq n} \psi_{ij} \circ \tau^n_{ij}$$

for some $\psi_{ij} \in \mathrm{Hom}_{\mathfrak{g}}(V^{\otimes n-2}, V^{\otimes n-2k})$. Now the statement easily follows by induction in k.

Theorem 9 *The graded algebras $\mathscr{A}_{P(\infty)}$ and $\mathscr{A}_{\mathfrak{so}(\infty)}$ are isomorphic. Hence the categories $\mathbb{T}_{P(\infty)}$, $\mathbb{T}_{\mathfrak{so}(\infty)}$, $\mathbb{T}_{\mathfrak{sp}(\infty)}$ and $\mathbb{T}_{\mathfrak{osp}}$ are all equivalent.*

It is an open problem to construct directly a functor $\mathbb{T}_{P(\infty)} \to \mathbb{T}_{\mathfrak{so}(\infty)}$ that preserves tensor product.

Finally, let us observe that we know very little about finite-dimensional representations of $P(n)$. In particular, characters of simple modules and extensions between simple modules are unknown. On the other hand, in $\mathbb{T}_{P(\infty)}$ these questions are easy to answer. Is it possible to use information about representations of $P(\infty)$ to make some progress in the finite-dimensional case?

5 The case of the queer Lie superalgebra $Q(n)$

5.1 Sergeev duality

In this section we assume $\mathfrak{g} = Q(\infty)$. Let us recall the analogue of Schur–Weyl duality result in this case. It is due to Sergeev [7].

Let H_n be the semidirect product of $\mathbb{C}[S_n]$ and the Clifford algebra Cliff_n with generators $p_i, i = 1, \dots, n$ satisfying the relations

$$p_i^2 = -1, p_i p_j + p_j p_i = 0.$$

Define the action of H_n on $V^{\otimes n}$ as follows

$$s_{i,i+1}(v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n) = (-1)^{p(v_i)p(v_{i+1})}(v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_n),$$

$$p_i(v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_n) = (-1)^{p(v_1)+\cdots+p(v_{i-1})}(v_1 \otimes \cdots \otimes J(v_i) \otimes \cdots \otimes v_n).$$

One can show (see [7]) that $H_n \simeq U_n \otimes \mathrm{Cliff}_n$ for a certain finite-dimensional algebra U_n. The category of finite-dimensional representations of U_n is equivalent to the category of projective representations of S_n. Irreducible representations of U_n are enumerated by strict partitions.

Theorem 10 ([7]) *H_n is the centralizer of $Q(\infty)$ in $V^{\otimes n}$. There is a decomposition*

$$V^{\otimes n} = \bigoplus 2^{-\delta(\lambda)} V^{\lambda} \otimes T_{\lambda},$$

here summation is taken over all strict partitions of size n, T_λ is the irreducible H_n-module corresponding to the strict partition λ, $\delta(\lambda)$ is the parity of the length of λ.

The coefficient $2^{-\delta(\lambda)}$ appears in the case when $\dim \mathrm{End}_{Q(\infty)}(V_\lambda) = (1|1)$. For example, the second tensor power of the standard representation V has a decomposition

$$V \otimes V = S^2(V) \oplus \Lambda^2(V).$$

But $S^2(V) \simeq \Lambda^2(V)$ as $Q(\infty)$-modules because $p_1 p_2(S^2(V)) = \Lambda^2(V)$. There is only one strict partition $\lambda = (2,0,\ldots,0)$ of size 2, $\delta(\lambda) = 1$. The reader will check that H_2 has an irreducible 4-dimensional module (in the category of $\mathbb{Z}_2$-graded modules).

5.2 *The category* $\mathbb{T}_{Q(\infty)}$

Let us consider the mixed tensor module

$$T = \bigoplus_{m,n \geq 0} V^{\otimes n} \otimes V_*^{\otimes m}$$

We claim that T is an injective cogenerator in the category $\mathbb{T}_{Q(\infty)}$.

Lemma 19

$$V^{\otimes n} \otimes V_*^{\otimes m} = \bigoplus_{|\lambda|=n, |\mu|=m} 2^{-max(\delta(\lambda),\delta(\mu))} \tilde{V}^{\lambda,\mu} \otimes (T_\lambda \boxtimes T_\mu),$$

where $\tilde{V}^{\lambda,\mu}$ is an indecomposable injective module in $\mathbb{T}_{\mathfrak{g}}$ with simple socle $V^{\lambda,\mu}$.

Define the graded subalgebra $\mathscr{A}_{Q(\infty)} \subset \mathrm{End}_{Q(\infty)}(T)$ generated by $\bigoplus_{n,m\geq 0} H_n \otimes H_m$ and all contractions in $\mathrm{Hom}_{Q(\infty)}(V^{\otimes n} \otimes V_*^{\otimes m}, V^{\otimes n-1} \otimes V_*^{\otimes m-1})$.

Conjecture Let $\mathfrak{g} = Q(\infty)$.

- $V^{\lambda,\mu}$ (for pairs of strict partitions (λ, μ)) are all up to isomorphism simple objects in $\mathbb{T}_{\mathfrak{g}}$.
- $\tilde{V}^{\lambda,\mu}$ are all up to isomorphism indecomposable injective objects in $\mathbb{T}_{\mathfrak{g}}$.
- $\mathscr{A}_{\mathfrak{g}}$ is a direct limit of self-dual Koszul rings.
- $\mathbb{T}_{\mathfrak{g}}$ is antiequivalent to the category of finite dimensional (locally unitary) $\mathscr{A}_{\mathfrak{g}}$-modules.

- The socle filtration of $\tilde{V}^{\lambda,\mu}$ is given by

$$\mathrm{soc}^k(\tilde{V}^{\lambda,\mu}) = \bigoplus_{|\gamma|=k} R^{\lambda}_{\gamma,\lambda'} R^{\mu}_{\gamma,\mu'} V^{\lambda',\mu'}.$$

  Here $R^{\lambda}_{\gamma,\lambda'}$ stand for the Littlewood–Richardson coefficients for the Lie superalgebra $Q(\infty)$.

Note that an analogue of Howe, Tan, Willenbring result for $Q(N)$ for $N >> 0$ is difficult to get since there is no complete reducibility. On the other hand, even if we had such result, it is unclear how to proceed to ∞, because we "loose" some simple constituents at ∞.

For instance, $SQ(n)$ has a one dimensional center for every n but

$$SQ(\infty) = \lim_{\rightarrow} SQ(n)$$

is simple.

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