Random Vibration of a Nonlinear Autoparametric System

Kristjan Onu, Nishanth Lingala and N. Sri Namachchivaya

Abstract We examine a stochastically forced autoparametric system for its stationary motion and stability. The deterministic form of this system is nearly Hamiltonian (with small dissipation) and exhibits 1:2 resonance and phase-locking. We develop a stochastic averaging technique to achieve a lower dimensional description of the dynamics of this system. Stochastic averaging is possible due to three time scales involved in this problem. Each time scale is fully exploited while averaging. The dimensional reduction techniques developed here consist of a sequence of averaging procedures that are uniquely adapted to study stochastic autoparametric systems. What motivates our analysis is that classical averaging methods fail when the original Hamiltonian system has resonances, because, at these resonances, singularities arise in the lower-dimensional description. At these singularities we introduce *gluing conditions*; these complete the specification of the dynamics of the reduced model. Examination of the reduced Markov process (which takes values on a nonstandard space) yields important results for probability density functions.

1 Introduction

We investigate the random vibrations of a nonlinear autoparametric system of the form

$$\begin{aligned} \ddot{q}_1(t) + \zeta_1 \dot{q}_1(t) + f_1(q_1(t), q_2(t)) &= \xi(t) \\ \ddot{q}_2(t) + \zeta_2 \dot{q}_2(t) + f_2(q_1(t), q_2(t)) &= 0 \end{aligned} \qquad t \geq 0, \tag{1}$$

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where for each time $t > 0$, $(q_1(t), q_2(t))$ represents the generalized coordinates of the system, the constants ζ_1 and ζ_2 are damping coefficients, and $\xi(t)$ is a stationary random process. We are interested in questions of stability of the stochastic system (1), and in the transfer of energy from the forced mode q_1 to the unforced mode q_2. It is well known that, in the presence of $1 : 2$ resonance and periodic excitation, as the intensity of excitation is increased, the excited mode reaches a certain value of amplitude at which saturation takes place and then the energy is transferred to the unforced mode. This may be undesirable, because disturbances affecting one mode may cause unwanted instability in another mode. Our effort is to answer whether the saturation and energy transfer occurs in the presence of noisy input. Towards this goal, we achieve a lower dimensional description of the above system.

The dissipation and random perturbations are assumed to be small. This means that their effect will be visible only over a long time horizon. When the nonlinearities are also assumed small, the dominant part of the dynamics is that of two uncoupled oscillators. In particular, the dynamics of the unperturbed system identify a reduced phase space (the orbit space) on which to carry out stochastic averaging. While the classical theory of stochastic averaging is a natural framework for such a program, the equations of interest contain resonances and bifurcations, which precludes a simple application of classical techniques. In particular, the resonance gives rise to an intermediate scale, and the bifurcations give rise to some non-standard singularities in the orbit space.

The equations of motion considered (1) can model the dynamics of a number of mechanical systems, namely a random excitation of a initially deformed shallow arch, a suspended elastic cable driven by planar excitation, or a water vessel subject to longitudinal wave action. To keep things as simple as possible, we shall consider a very simple system, namely a pendulum hanged from a mass which is attached by a spring to a support (Fig. 1). The mass is randomly excited. For clarity, we use *mass* to refer to the object at the free end of the spring, while the object at the end of the pendulum is referred to as the *bob*. The quantity φ is the angle of the pendulum (with respect to the vertical axis) and the quantity y represents the height of the mass (relative to a rest position defined by the position of the pendulum). The mass is forced according to a stochastic signal $\Xi(t)$. The subscripts here refer to the fact that this is our original physical model. The equations for such a system can be written as

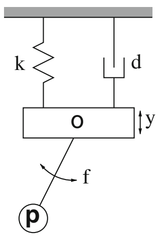


Fig. 1 Schematic of autoparametric system

$$
\begin{aligned}
(m_o + m_p)\ddot{y} + d_o\dot{y} + ky + m_p l(\ddot{\varphi}\sin\varphi + \dot{\varphi}^2\cos\varphi) &= \Xi,\\
m_p l^2\ddot{\varphi} + d_p\dot{\varphi} + m_p l(g + \ddot{y})\sin\varphi &= 0,
\end{aligned} \tag{2}
$$

where $m_o,\ d_o$ and k are the mass, damping and the spring constant of the spring-mass system and $m_p,\ d_p$ and l are the mass, damping and the length of the pendulum. The kinetic and the potential energies of the system are given by

$$
\begin{aligned}
T &= \frac{1}{2}(m_o + m_p)\dot{y}^2 + \frac{1}{2}m_p\, l^2\dot{\varphi}^2 + m_p\, l\dot{y}\,\dot{\varphi}\sin\varphi,\\
U &= m_p\, gl(1 - \cos\varphi) + \frac{1}{2}ky^2.
\end{aligned}
$$

It is clear that the nonlinearities in the equations of motion arise due to the gravitational restoring force and due to the dependence of kinetic energy on the angle φ which leads to inertial coupling between the the two coordinates. It also turns out (we shall use this later) that in the absence of noise and damping, this system is Hamiltonian, so the dynamics of y and φ are governed by the geometry of this Hamiltonian.

The above equations in dimensionless coordinates are

$$
\begin{aligned}
\ddot{\hat{\eta}} + 2\hat{\zeta}_o\dot{\hat{\eta}} + \hat{\eta} + R(\ddot{\hat{\theta}}\sin\hat{\theta} + \dot{\hat{\theta}}^2\cos\hat{\theta}) &= \hat{\xi}(t),\\
R\ddot{\hat{\theta}} + 2R\hat{\zeta}_p\dot{\hat{\theta}} + R(q^2\sin\hat{\theta} + \ddot{\hat{\eta}}\sin\hat{\theta}) &= 0.
\end{aligned}
$$

where

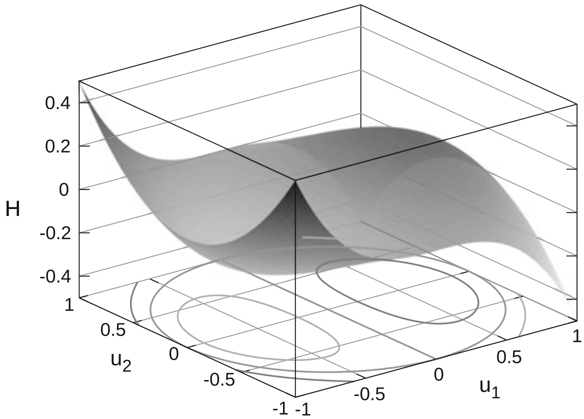
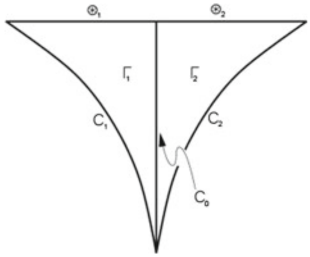


Fig. 2 Surface and contour plots of $K(u_1, u_2)$. $I = 1$

Fig. 3 State space $\mathfrak{M}$ for Y_t^ε



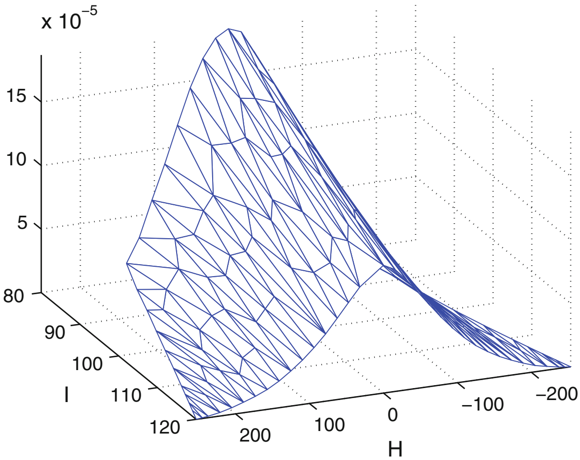


Fig. 4 Probability density by FEM

$$\omega^2 \stackrel{\text{def}}{=} \frac{g}{l}, \quad \omega_o^2 \stackrel{\text{def}}{=} \frac{k}{m_o + m_p}, \quad q \stackrel{\text{def}}{=} \frac{\omega}{\omega_o},$$

$$R \stackrel{\text{def}}{=} \frac{m_p}{m_o + m_p}, \quad \hat{\zeta}_o \stackrel{\text{def}}{=} \frac{d_o}{2\sqrt{k(m_o + m_p)}}, \quad \hat{\zeta}_p \stackrel{\text{def}}{=} \frac{d_p\sqrt{(m_o + m_p)}}{2l^2\sqrt{k}} = \frac{d_p}{2l^2\omega_o}.$$

and where

$$\hat{\xi}(t) = \frac{\hat{\xi}(t/\omega_o)}{kl}, \quad \hat{\eta}(t) = \frac{y(t/\omega_o)}{l}, \quad \hat{\theta}(t) = \varphi(t/\omega_o)$$

for all $t > 0$.

Our interest here is a refined stability analysis near the fixed point $(\hat{\eta}, \hat{\theta}) \equiv 0$ of the unperturbed system. In particular, we are interested in the effect of small random perturbations, so we will let $\hat{\xi}$ be of the form $\hat{\xi} = \varepsilon^2 \nu \xi$, where ξ is a noise process of "unit" variance and ν is some empirical parameter. Our dynamics are most interesting when they are not over-damped, so let $\hat{\zeta}_o$ and $\hat{\zeta}_p$ be of the form $\hat{\zeta}_o = \varepsilon^2 \zeta_o$ and $\zeta_p = \varepsilon^2 \zeta_p$, where ζ_o and ζ_p are some positive constants (this corresponds to letting d_o and d_p be of size ε). Guided by the corresponding analysis for periodic

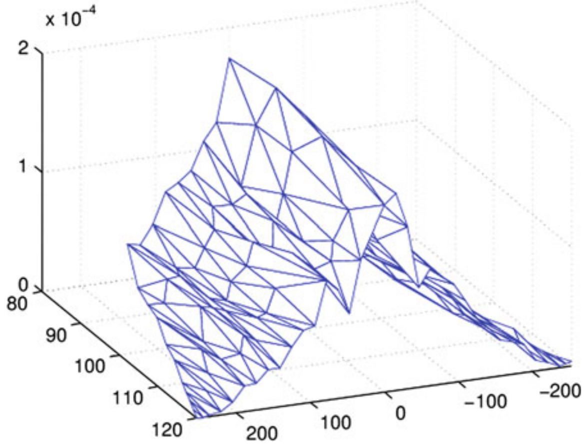


Fig. 5 Probability density by numerical simulation

forcing, we are interested in the behavior when q^2 is very close to $q_o^2 \equiv 1/4$. Let's replace q by $q_o + \varepsilon^2\mu$, where μ is an unfolding parameter. Since we are interested in $\hat{\eta}$ and $\hat{\theta}$ near the fixed point 0, we should look at these quantities on a finer resolution. Namely, let η and θ be defined by

$$\hat{\eta}(t) = \varepsilon\eta(t), \qquad \hat{\theta}(t) = \varepsilon\theta(t)$$

then the dynamics of the system yields

$$\begin{aligned} &\ddot{\eta} + 2\varepsilon^2\zeta_o\dot{\eta} + \eta + R(\ddot{\theta}\sin(\varepsilon\theta) + \varepsilon\dot{\theta}^2\cos(\varepsilon\theta)) = \varepsilon\nu\xi(t), \\ &R\ddot{\theta} + 2\varepsilon^2 R\zeta_p\dot{\theta} + R\left(\left(q_\circ + \varepsilon^2\mu\right)\frac{\sin(\varepsilon\theta)}{\varepsilon} + \ddot{\eta}\sin(\varepsilon\theta)\right) = 0, \end{aligned} \tag{3}$$

where ε is a small scaling parameter, $q_0 = 1/2$ signifying $1:2$ resonance, μ is the parameter representing unfolding from the resonance, R is the ratio of mass of the unforced mode to the total mass.

2 Single Mode Solutions

To clarify some general qualitative effects of noise, let's consider a simple stability analysis using some spectral methods and the first-order linearization. The mass on the spring can move only in the vertical (η) direction and is excited by $\nu\xi$. Assume that the pendulum is locked vertically, i.e. $\theta(t) \equiv 0$. We get the equation

$$\ddot{\eta} + 2\varepsilon^2\zeta_o\dot{\eta} + \eta = \varepsilon\nu\xi.$$

If ξ is white noise we can solve for η explicitly. Its power spectral density is

$$S_\eta(\omega) = \frac{\varepsilon^2 \nu^2 S_0}{(1-\omega^2)^2 + 4\varepsilon^4 \zeta_o^2 \omega^2},$$

where S_0 is the power spectral density of ξ. The peak intensity and the carrying frequency of η are determined by the filter parameter ζ_o.

The stability of the locked mass steady-state oscillation is now obtained by using the first-order approximation of sine and cosine in the dynamics for θ. We get

$$\ddot{\theta} + 2\varepsilon^2 \zeta_p \dot{\theta} + ((q_0 + \varepsilon^2 \mu)^2 + \varepsilon \ddot{\eta})\theta = 0,$$

and the power spectral density of $\ddot{\eta}$ is given by

$$S_{\ddot{\eta}}(\omega) = \frac{\omega^4 \varepsilon^2 \nu^2 S_0}{(1-\omega^2)^2 + 4\varepsilon^4 \zeta_o^2 \omega^2}.$$

The maximal Lyapunov exponent can now be easily calculated and the stability boundary can be obtained in terms of excitation intensity ν and the dissipation coefficients ζ_p. An explicit expression for the maximal Lyapunov exponents of the single mode solution is given by expanding it in ε, we have

$$\lambda_1 \approx \varepsilon^2 \left(-\zeta_p + \frac{1}{8\, q_o^2} S_{\ddot{\eta}}(2\,(q_o + \varepsilon^2 \mu)) \right) \quad \text{and} \quad \lambda_2 = \varepsilon^2 \left(-\zeta_p - \frac{1}{8\, q_o^2} S_{\ddot{\eta}}(2\,(q_o + \varepsilon^2 \mu)) \right).$$

The noise has no effect on the other two exponents; i.e., $\lambda_3 = \lambda_4 = -\varepsilon^2 \zeta_o$.

Since the point $\theta \equiv 0$ is a stable point for the hanging pendulum, the pendulum undergoes small random motion near $\theta \equiv 0$, and all four Lyapunov exponents are negative. However, as we further increase the noise intensity, the top exponent becomes positive when $\nu^2 S_0 = 8\zeta_o^2 \zeta_p$. The system then becomes unstable, and the following question arises.

- Do both the mass spring oscillator and the pendulum undergo random vibrations when the top exponent becomes positive (i.e., $\nu^2\, S_0 > 8\, \zeta_o^2\, \zeta_p$), i.e., does a new coupled-mode “stationary solution” or “stationary density function” appear?

3 Coupled Mode Solutions

Making use of a time-varying symplectic transformation (see [1] for details), we arrive at

$$\dot{x}_t^\varepsilon = \varepsilon b^1(x_t^\varepsilon, t) + \varepsilon^2 b^2(x_t^\varepsilon, t : \zeta, \mu) + \varepsilon \sigma(x_t^\varepsilon, t : \nu)\xi(t), \tag{4}$$

where (x_1, x_2) and (x_3, x_4) are conjugate pairs and can be thought of as the amplitudes of periodic orbits of the dominant dynamics.

The coefficients b^1, b^2, σ are periodic in time. Standard deterministic averaging can be used to average out the effects of rapidly-oscillating periodic coefficients. Let $\mathbb{M}$ be this averaging operator.

Definition 1 *(Time averaging operator)* For a function $\varphi \in C^\infty(\mathbb{R}^4 \times \mathbb{R})$ which is 2π periodic in its last argument, define the time averaging operator $\mathbb{M}$ by

$$(\mathbb{M}\varphi)(x) \equiv \frac{1}{2\pi}\int_0^{2\pi} \varphi(x,t)dt.$$

From the explicit formulas (see [1]) for b^1 (where $q = 1/2$), we see that for $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$,

$$(\mathbb{M}b^1)(x) = \left(-\tfrac{1}{2}x_2x_4,\ \ \tfrac{1}{2}(x_1x_4 - x_2x_3),\ \ \tfrac{1}{4}(x_2^2 - x_4^2),\ \ \tfrac{1}{2}(x_1x_2 + x_3x_4)\right).$$

Then the averaged system $\dot{x}_t = (\mathbb{M}b^1)(x)$ is a 4-dimensional Hamiltonian system with two first integrals K and I in involution.

The Hamiltonian associated with these dynamics is

$$K(x) = \frac{1}{4}x_1(x_4^2 - x_2^2) - \frac{1}{2}x_2x_3x_4. \tag{5}$$

The unperturbed four-dimensional Hamiltonian system

$$\dot{z} = \bar{\nabla}K(z) \tag{6}$$

has *two first integrals in involution*, namely, the Hamiltonian itself (5) and a second constant of motion (momentum variable)

$$I(x) = (x_1^2 + x_3^2) + \frac{1}{2}\,(x_2^2 + x_4^2). \tag{7}$$

The invariant I is functionally independent of K, exists globally and is single valued. Note that the Hamiltonian system's equations remain unchanged when $t \to -t,\ x_1 \to -x_1$ and $x_3 \to -x_3$.

3.1 Dimensional Reduction

Our main analytical tool is a certain method of dimensional reduction of nonlinear systems with symmetries and small noise. As the noise becomes asymptotically small, one can exploit symmetries and a separation of scales to use well-known methods (viz. stochastic averaging) to find an appropriate lower-dimensional description of the system.

In the flow given by (4), the quantities $(K(x), I(x))$ are slow-varying. The variation of $y^\varepsilon_t := (K(x^\varepsilon_t), I(x^\varepsilon_t))$ is given by the following set of equations

$$\dot{y}^\varepsilon_t = \varepsilon F^1(x^\varepsilon_t, t) + \varepsilon^2 F^2(x^\varepsilon_t, t : \zeta, \mu) + \varepsilon G(x^\varepsilon_t, t : \nu)\xi(t), \tag{8}$$

where $F^i_j(x, t) = (b^i(x, t).\nabla)y_j$ and $G_j(x, t) = (g(x, t).\nabla)y_j$.

Since K and I are integrals of motion for $\dot{x}_t = (\mathbb{M}b^1)(x)$, it is clear that $\mathbb{M}F^1(X) = 0$. Thus, to see the fluctuations of K and I, we need to look on a time scale of order $1/\varepsilon^2$. Thus, we make a time rescaling, setting $X^\varepsilon_t \overset{\text{def}}{=} x^\varepsilon_{t/\varepsilon^2}$ and $Y^\varepsilon \overset{\text{def}}{=} y^\varepsilon_{t/\varepsilon^2}$. Then we have

$$\begin{aligned}
\dot{X}^\varepsilon_t &= \frac{1}{\varepsilon} b^1(X^\varepsilon_t, t/\varepsilon^2) + b^2(X^\varepsilon_t, t/\varepsilon^2) + g(X^\varepsilon_t, t/\varepsilon^2)\,\frac{1}{\varepsilon}\xi(t/\varepsilon^2),\\
\dot{Y}^\varepsilon_t &= \frac{1}{\varepsilon} F^1(X^\varepsilon_t, t/\varepsilon^2) + F^2(X^\varepsilon_t, t/\varepsilon^2) + G(X^\varepsilon_t, t/\varepsilon^2)\,\frac{1}{\varepsilon}\xi(t/\varepsilon^2).
\end{aligned} \tag{9}$$

Roughly, our goal is to study (9) and show that as ε tends to zero, the dynamics of $Y^\varepsilon(X^\varepsilon_t)$ tends to a lower-dimensional Markov process and to identify the infinitesimal generator of the limiting law.

There are three time scales. The periodic fluctuations of the coefficients occur over time scales of order ε^2. The effects of drift due to b^1 can be seen on time scales of order ε. The drift and diffusion coefficients of Y^ε_t are of order 1. We perform two averaging steps, one to average ($\mathbb{M}$) the periodic behavior of the coefficients, and one to average (**A**) along the orbits of the Hamiltonian system $\dot{x}_t = (\mathbb{M}b^1)(x)$.

To understand the state space of the slow variable Y^ε_t, we consider the following symplectic transformation (it would also be useful later for simplifying calculations).

3.1.1 Structure of the Unperturbed System: Hamiltonian Structure

$$\begin{aligned}
x_1 &= u_1\cos(2\psi) + u_2\sin(2\psi), & x_3 &= -u_1\sin(2\psi) + u_2\cos(2\psi),\\
x_2 &= \sqrt{2(I - u_1^2 - u_2^2)}\sin\psi, & x_4 &= \sqrt{2(I - u_1^2 - u_2^2)}\cos\psi.
\end{aligned} \tag{10}$$

The conjugate pairs are (u_1, u_2) and (ψ, I). This transformation yields

$$\dot{u}_{1t} = -u_{1t}u_{2t}, \quad \dot{u}_{2t} = \frac{1}{2}(3u_{1t}^2 + u_{2t}^2 - I_t), \quad \dot{\psi}_t = \frac{1}{2}u_{1t}, \quad \dot{I}_t = 0 \tag{11}$$

and the corresponding Hamiltonian is

$$K = \frac{1}{2}u_1\left(I - (u_1^2 + u_2^2)\right). \tag{12}$$

The relation between K, u_1, u_2, and I is illustrated in Fig. 2. Note that this system's equations remain unchanged when $t \to -t$, $u_2 \to -u_2$ and $\psi \to -\psi$. System (11) has four fixed points. They are $(u_1, u_2) = (0, \pm\sqrt{I})$ and $(u_1, u_2) = (\pm\frac{\sqrt{3I}}{3}, 0)$.

The points on the u_1 axis are saddle points and those on the u_2 axis are center fixed points.

3.1.2 State Space of Y_t^ε

The slow variable Y_t^ε evolves on an arrowhead. Let $\mathbf{S} \overset{\text{def}}{=} \{x \in \mathbb{R}^4 : K_* < K(x) < K^*,\ 0 < I(x) < I^*\}$. Then define an equivalence relation $\sim$ on $\mathbb{R}^4$ by identifying $x \sim y$ if x and y are on the same orbit of the hamiltonian flow $\dot{x}_t = (\mathbb{M}b^1)(x)$. Define $\mathfrak{M} \overset{\text{def}}{=} \bar{\mathbf{S}}/\sim$, and endow $\mathfrak{M}$ with the quotient topology defined by $\sim$. If $x \in \bar{\mathbf{S}}$, we let $[x] := \{y \in \bar{\mathbf{S}} :\ y \sim x\}$ be the equivalence class of x. $\pi(x) := [x]$. The slow variable Y_t^ε evolves on $\mathfrak{M} = \bigcup_{i=1}^{2} \Gamma_i \cup \bigcup_{i=0}^{2} [\mathfrak{c}_i] \cup \bigcup_{i=1}^{2} \circledast_i$ where $\mathfrak{c}_i$ are the fixed points, the $\circledast_i$ are closed orbits whose union is $\partial\bar{\mathbf{S}}$, and each Γ_i is the π-image of a maximal open subset of $\mathbb{R}^4$ which does not intersect any of the $[\mathfrak{c}_i]$ or $\circledast_i$. The state space is illustrated in Fig. 3.

3.1.3 $\mathbb{M}$ & A Averaging

If the external noise $\xi(\tau)$ represents mean zero, stationary, independent stochastic processes with the strong mixing property, then roughly, as $\varepsilon \to 0$, $\frac{1}{\varepsilon}\xi(t/\varepsilon^2)$ approaches a white noise process. Khasminskii [2] gave a rigorous proof that a family of processes X_t^ε converges to a diffusion process. The aim here is to make use of this and derive a reduced graph-valued process for the integrals of motion, Y^ε.

We have pointed out that that there are three time-scales involved in our averaging problems. The first step is to average the periodic fluctuations of the coefficients and obtain $\mathbb{M}$-averaged quantities as the precursors to the stochastically averaged drift and diffusion coefficients. Somewhat laborious calculations yield the $\mathbb{M}$-averaged quantities

$$m_i(x) \equiv \left(\mathbb{M}\left(F_1^2 + \mathfrak{f}_1 + \mathfrak{g}_1\right)\right)(x) \qquad \text{and} \qquad a_{ij}(x) \equiv \left(\mathbb{M}\left(\sigma\sigma^T\right)_{ij}\right)(x). \quad (13)$$

These calculations can be simplified by considering the symplectic transformation (10) which provides a convenient geometric structure of the unperturbed integrable Hamiltonian problem. In (K, I, u) coordinates, the drift and diffusion (13) coefficients are

$$\begin{aligned} m_1(K, I, u) &= -(\zeta_o + 2\zeta_p)K - \frac{1}{4}(8\mu + 3I)\,K\frac{u_2}{u_1} + \frac{1}{2}\left(3 + \frac{1}{R}\right)K^2\frac{u_2}{u_1^2}, \\ m_2(K, I, u) &= 2[\sigma^2 S_{\xi\xi}(1) - \zeta_o I + 2(\zeta_o - \zeta_p)K/u_1], \end{aligned} \quad (14)$$

$$
\begin{aligned}
a_{11}(K,I,u) &= \frac{1}{2}\sigma^2 S_{\xi\xi}(1)K^2\frac{1}{u_1^2}, \quad a_{12}(K,I,u) = \sigma^2 S_{\xi\xi}(1)K, \\
a_{22}(K,I,u) &= 2\sigma^2 S_{\xi\xi}(1)(I-2K/u_1).
\end{aligned} \tag{15}
$$

To obtain a limiting generator for the martingale problem, we need an averaging operator where the averaging is done with respect to the invariant measure concentrated on the closed trajectories. Using (14) in the **A**-averaging operator yields on each leaf Γ_i, for $z=(K,I)\in\Gamma_i$,

$$
\begin{aligned}
&\mathfrak{b}^i_j(z) = \frac{1}{T_i(z)}\int_0^{T_i(z)} m_j\,(z,u(t))\,dt, \qquad \mathfrak{a}^i_{jk}(z) = \frac{1}{T_i(z)}\int_0^{T_i(z)} a_{jk}(u(t),K,I)dt, \\
&\mathfrak{b}^i_1(z) = -(\zeta_o+2\zeta_p)K, \qquad \mathfrak{b}^i_2(z) = 2[\sigma^2 S_{\xi\xi}(1)-\zeta_o I] + 4(\zeta_o-\zeta_p)K\frac{\mathscr{I}^1_i}{T_i}, \\
&\mathfrak{a}^i_{11}(z) = \frac{1}{2}\sigma^2 S_{\xi\xi}(1)K^2\frac{\mathscr{I}^2_i}{T_i}, \quad \mathfrak{a}^i_{12}(z) = \sigma^2 S_{\xi\xi}(1)K, \quad \mathfrak{a}^i_{22}(z) = 2\sigma^2 S_{\xi\xi}(1)(I-2K\frac{\mathscr{I}^1_i}{T_i}).
\end{aligned}
$$

Here, $T_i(z)$ is the time period of the Hamiltonian orbit on leaf i with value of K and I given by z and $\mathscr{I}^1_i = \int_0^{T_i}\frac{1}{u_1(t)}dt$ and $\mathscr{I}^2_i = \int_0^{T_i}\frac{1}{u_1^2(t)}dt$.

3.1.4 Generator of the Reduced Markov Process

We want to put these $\mathscr{L}_i$'s together to get a Markov process on $\mathfrak{M}$ with generator $\mathscr{L}^\dagger_{\mathfrak{M}}$ with domain $\mathscr{D}^\dagger_{\mathfrak{M}}$, where $\mathfrak{M}$ has a shape of an arrowhead.

Let us define the drift and diffusion coefficients

$$
\mathfrak{b}_i(z) \equiv \left(\mathbf{A}\left(\mathbb{M}\left(F^2_i+\mathfrak{f}_i+\mathfrak{g}_i\right)\right)\right)(z), \qquad \mathfrak{a}_{ij}(z) \equiv \left(\mathbf{A}\left(\mathbb{M}\left(\sigma\sigma^T\right)_{ij}\right)\right)(z) \tag{16}
$$

for $i,\ j=1,2$ and for all $z\in\mathfrak{M}$, where

$$
\begin{aligned}
&\mathfrak{f}_i(x,t) \equiv \sum_{j=1}^{4}\frac{\partial F^1_i}{\partial x_j}(x,t)\tilde{f}^1_j(x,t), \qquad \tilde{f}^1_i(x,t) \equiv \int_0^t\left\{b^1_i(x,s)-\mathbb{M}_s(b^1_i(x,s))\right\}ds, \\
&\mathfrak{g}_i(x,t) \equiv \int_{-\infty}^{0}\mathbb{E}\left[\frac{\partial G_i}{\partial x_j}(x,t,\xi_t)g_j(x,t+\tau,\xi_{t+\tau})\right]d\tau, \\
&\qquad \left(\sigma\sigma^T\right)_{jk}(x,t) \equiv \int_{-\infty}^{\infty}\mathbb{E}\left[G_j(x,t,\xi_t)G_k(x,t+\tau,\xi_{t+\tau})\right]d\tau.
\end{aligned}
$$

exists uniformly in $x\in\mathbb{R}^4$.

For notational convenience, we also define $f_i \equiv f\big|_{\Gamma_i}$ for all $1\le i\le 2$. From the results of [3] , it is clear the gluing conditions, which we need to specify at the interior edges, solely depend on the diffusion coefficients $\mathfrak{a}^i_{jk}$. To this end, we define $\mathring{\mathfrak{a}}^i_{jk}(z) \equiv \mathfrak{a}^i_{jk}(z)\,T(z)$. The limiting domain for the graph valued process is

$$\mathscr{D}^{\dagger}_{\mathfrak{M}} = \Bigg\{ f \in C(\mathfrak{M}) \cap C^2(\cup_{i=1}^{2} \Gamma_i) : \lim_{z \to (K(\mathfrak{c}_i), I(\mathfrak{c}_i))} (\mathscr{L}_i f_i)(h) \text{ exists } \forall i, \\ \lim_{I \to I^*} (\mathscr{L}_i f_i)(z) = 0 \quad \forall i, \quad \text{and} \quad \sum_{i=1}^{2} \{\pm\} \, (\mathring{\mathfrak{a}}^i_{11} \frac{\partial f_i}{\partial z_1})(\mathfrak{c}_0) = 0 \Bigg\}, \tag{17}$$

where the '+' sign is taken if the coordinate h on the leg Γ_i is greater than 0 (the value of $z_1 (= h)$ at the vertex $\mathfrak{c}_0$) and the '−' sign is taken otherwise. Then for $f \in \mathscr{D}^{\dagger}_{\mathfrak{M}}$, the generator is

$$(\mathscr{L}^{\dagger}_{\mathfrak{M}} f)(z) = \lim_{\substack{z' \to z \\ z \in \Gamma_i}} (\mathscr{L}_i f_i)(z') = \sum_{j=1}^{2} \mathfrak{b}^i_j(z) \frac{\partial f_i}{\partial z_j}(z) + \frac{1}{2} \sum_{j,k=1}^{2} \mathfrak{a}^i_{jk}(z) \frac{\partial^2 f_i}{\partial z_j \partial z_k}(z) \tag{18}$$

for all $z \in \bar{\Gamma}_i$.

The gluing conditions can be derived by determining the asymptotic values of the drift and diffusion coefficients as $K \to 0$. The period is asymptotically equivalent to $T(z) \sim \ln |K|$ as $K \to 0$. This yields $\lim_{K \to 0} \mathring{\mathfrak{b}}^i_1 = 0$. Furthermore,

$$\lim_{K \to 0} \mathring{\mathfrak{a}}^i_{11}(\mathfrak{c}_0) \equiv \lim_{K \to 0} \left(\mathfrak{a}^i_{11}(z) \, T_i(z) \right) = \sigma^2 S_{\xi\xi}(1) \frac{I\sqrt{I}}{3} \geq 0.$$

The values of $\mathring{\mathfrak{b}}^i_2$, $\mathring{\mathfrak{a}}^i_{12}$ and $\mathring{\mathfrak{a}}^i_{22}$ in the limit $K \to 0$ all approach infinity. Hence $-\frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_1} = 0$.

3.2 Fokker–Planck Equation and Stationary Probability Density Function

We turn our attention to producing solutions with the results of stochastic averaging theory presented in the previous section. Specifically, stationary probability density functions are produced. First, the Fokker–Planck equation is derived by taking the adjoint of the reduced generator (18). Then the solutions for the the autoparametric oscillator are obtained by a finite element formulation of the Fokker–Planck problem. Finally, the finite element results are validated with a sample path method.

Finite-element triangulations of the $K - I$ domains are produced using *TRIANGLE*. The domains of the Fokker–Planck equation have boundaries defined by polynomial functions. *TRIANGLE* does not allow specifying such boundaries directly, rather a certain number of points on the boundary must be given. In order to create elements of a specified area, *TRIANGLE* may place additional nodes between points given to it as input. Experience with *TRIANGLE* shows that these problems can be avoided by specifying the number of input points in (inverse) proportion to the requested element area. Specifically, input points are placed by calculating the arc length along the boundary and the spacing between the points is made equal to the length of the side of an equilateral triangle with an area equal to the requested element area. As long as the domain triangulated does not include cusps, this procedure seems to produce triangulation that have none, or few, Steiner points.

Across the gluing edge, the finite element method is formulated carefully so that the solution does not exhibit any singularities. The solutions appear to be continuous across the gluing edge, as expected based on analytic calculations.

As the amplitude of stochastic forcing is varied, the peak of the probability distribution moves to larger values of I while remaining symmetric about the I axis. The latter fact is worth contemplating. Recalling the structure of the Hamiltonian, the outer edge of the domain in the left hand plane corresponds to a sink and the outer edge of the domain in the right hand plane is a valley. As such it seems reasonable to think that as forcing amplitude increases, the peak of the PDF will shift from the left hand plane to the right hand plane, but this is not observed in Figs. 4 or 5. In fact, simply by looking at the form of $\mathfrak{b}_1$ one notices that along the K axis, the drift coefficient tends to center the probability density on the I axis. It is curious that $\mathfrak{b}_1$ does not contain any stochastic effects; whether this is a generic feature for systems in 1:2 resonance remains to be determined.

4 Conclusions

A two degree-of-freedom nonlinear autoparametric vibration absorber with weak quadratic nonlinearities is considered. The averaged nonlinear response of the system in the absence of disspative and random effects is Hamiltonian. A nonstandard method of stochastic averaging is developed to reduce the dimension of a randomly-perturbed four-dimensional integrable Hamiltonian systems with one-to-two resonance. The reduction to a graph valued process was possible due to three time-scales involved in this problem.

The interest of this paper is when the original Hamiltonian system has one-to-two resonances. Hence the averaged nonlinear Hamiltonian system is integrable with both homoclinic and heteroclinic orbits in the phase-space. This gives rise to singularities in the lower-dimensional description. At these singularities, *gluing conditions* were derived, these gluing conditions completing the specification of the dynamics of the reduced model by examining the boundary-layer behavior close to homoclinic and heteroclinic orbits.

In this context it is also important to point to the work in [4] and [5] where they considered fast oscillating random perturbations of dynamical systems with first integrals. Then under suitable regularity and ergodicity conditions it was shown that the evolution of first integrals in an appropriate time scale is given by a diffusion process. The main emphasis in these papers is the mixing properties of fast oscillating random perturbations. The method used in this paper and the assumptions on the noise terms are different, and the presence of one-to-two resonance leads to an interesting limiting generator.

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