Chapter 6
Interpolating Control—Output Feedback Case

6.1 Problem Formulation

Consider the problem of regulating to the origin the following uncertain and/or time-varying linear discrete-time system, described by the input-output relationship,

$$\begin{aligned} &y(k+1) + E_1 y(k) + E_2 y(k-1) + \cdots + E_s y(k-s+1) \\ &\quad = N_1 u(k) + N_2 u(k-1) + \cdots + N_r u(k-r+1) + w(k) \end{aligned} \tag{6.1}$$

where $y(k) \in \mathbb{R}^p$, $u(k) \in \mathbb{R}^m$ and $w(k) \in \mathbb{R}^p$ are respectively, the output, the input and the disturbance vector. The matrices $E_i \in \mathbb{R}^{p\times p}$, $i = 1, 2, \ldots, s$ and $N_j \in \mathbb{R}^{p\times m}$, $j = 1, 2, \ldots, r$.

For simplicity, it is assumed that $s = r$. The matrices E_i and N_i, $i = 1, 2, \ldots, s$ satisfy,

$$\begin{bmatrix} E_1 & E_2 & \ldots & E_s \\ N_1 & N_2 & \ldots & N_s \end{bmatrix} = \sum_{j=1}^{q} \alpha_j(k) \begin{bmatrix} E_1^{(j)} & E_2^{(j)} & \ldots & E_s^{(j)} \\ N_1^{(j)} & N_2^{(j)} & \ldots & N_s^{(j)} \end{bmatrix} \tag{6.2}$$

where $\alpha_j(k) \geq 0$ and $\sum_{j=1}^{q} \alpha_j(k) = 1$ and

$$\begin{bmatrix} E_1^{(j)} & E_2^{(j)} & \ldots & E_s^{(j)} \\ N_1^{(j)} & N_2^{(j)} & \ldots & N_s^{(j)} \end{bmatrix}, \quad j = 1, 2, \ldots, q$$

are the extreme realizations of the polytopic model (6.2).

The output, control and disturbance vectors are subject to the following bounded polytopic constraints,

$$\begin{cases} y(k) \in Y, & Y = \{y \in \mathbb{R}^p : F_y y \leq g_y\}, \\ u(k) \in U, & U = \{u \in \mathbb{R}^m : F_u u \leq g_u\}, \\ w(k) \in W, & W = \{w \in \mathbb{R}^p : F_w w \leq g_w\}, \end{cases} \tag{6.3}$$

where the matrices F_y, F_u, F_w and the vectors g_y, g_u, g_w are assumed to be constant with $g_y > 0$, $g_u > 0$ and $g_w > 0$.

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6.2 Output Feedback—Nominal Case

In this section, we consider the case when the matrices E_i and N_i for $i = 1, 2, \dots, s$ are known and fixed. The case when E_i and N_i for $i = 1, 2, \dots, s$ are uncertain and/or time-varying will be treated in the next section.

A state space representation will be constructed along the lines of [126]. All the steps of the construction are detailed such that the presentation are self contained. The state of the system is chosen as follows,

$$x(k) = \begin{bmatrix} x_1(k)^T & x_2(k)^T & \dots & x_s(k)^T \end{bmatrix}^T \tag{6.4}$$

where

$$\begin{cases} x_1(k) = y(k) \\ x_2(k) = -E_s x_1(k-1) + N_s u(k-1) \\ x_3(k) = -E_{s-1} x_1(k-1) + x_2(k-1) + N_{s-1} u(k-1) \\ x_4(k) = -E_{s-2} x_1(k-1) + x_3(k-1) + N_{s-2} u(k-1) \\ \quad\vdots \\ x_s(k) = -E_2 x_1(k-1) + x_{s-1}(k-1) + N_2 u(k-1) \end{cases} \tag{6.5}$$

The components of the state vector can be interpreted exclusively in terms of the input and output vectors as,

$$\begin{aligned} x_2(k) &= -E_s y(k-1) + N_s u(k-1) \\ x_3(k) &= -E_{s-1} y(k-1) - E_s y(k-2) + N_{s-1} u(k-1) + N_s u(k-2) \\ &\vdots \\ x_s(k) &= -E_2 y(k-1) - E_3 y(k-2) - \dots - E_s y(k-s+1) \\ &\quad + N_2 u(k-1) + N_3 u(k-2) + \dots + N_s u(k-s+1) \end{aligned}$$

It holds that,

$$\begin{aligned} y(k+1) = &-E_1 y(k) - E_2 y(k-1) - \dots - E_s y(k-s+1) \\ &+ N_1 u(k) + N_2 u(k-1) + \dots + N_s u(k-s+1) + w(k) \end{aligned}$$

or, equivalently

$$x_1(k+1) = -E_1 x_1(k) + x_s(k) + N_1 u(k) + w(k)$$

The state space model is then defined in a compact form as follows,

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + Dw(k) \\ y(k) = Cx(k) \end{cases} \tag{6.6}$$

where

$$A = \begin{bmatrix} -E_1 & 0 & 0 & \dots & 0 & I \\ -E_s & 0 & 0 & \dots & 0 & 0 \\ -E_{s-1} & I & 0 & \dots & 0 & 0 \\ -E_{s-2} & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -E_2 & 0 & 0 & \dots & I & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} N_1 \\ N_s \\ N_{s-1} \\ N_{s-2} \\ \vdots \\ N_2 \end{bmatrix}, \qquad D = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$C = \begin{bmatrix} I & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Clearly, the realization (6.6) is *minimal* in the single-input single-output case. However, in the multi-input multi-output case, this realization might not be minimal, as shown in the following example.

Consider the following single-input multi-output linear discrete-time system,

$$\begin{aligned} y(k+1) - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} y(k) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} y(k-1) \\ = \begin{bmatrix} 0.5 \\ 2 \end{bmatrix} u(k) + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u(k-1) + w(k) \end{aligned} \tag{6.7}$$

Using the construction (6.4), (6.5), the state space model is given as,

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + Dw(k) \\ y(k) = Cx(k) \end{cases}$$

where

$$A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ -1.5 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

This realization is not minimal, since it unnecessarily replicates the common poles of the denominator in the input-output description. There exists minimal state space realization like,

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Define

$$z(k) = \begin{bmatrix} y(k)^T & \dots & y(k-s+1)^T & u(k-1)^T & \dots & u(k-s+1)^T \end{bmatrix}^T \tag{6.8}$$

Using (6.5), the state $x(k)$ is expressed through $z(k)$ as,

$$x(k) = Tz(k) \tag{6.9}$$

where $T = [T_1 \;\; T_2]$ and

$$T_1 = \begin{bmatrix} I & 0 & 0 & \dots & 0 \\ 0 & -E_s & 0 & \dots & 0 \\ 0 & -E_{s-1} & -E_s & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & -E_2 & -E_3 & \dots & -E_s \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ N_s & 0 & 0 & \dots & 0 \\ N_{s-1} & N_s & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N_2 & N_3 & N_4 & \dots & N_s \end{bmatrix}$$

Hence, it becomes obvious that at any time instant k, the state vector is available exclusively though measured input and output variables and their past values.

Using (6.3), (6.5), it follows that the state constraints are $x_i \in X_i$, where X_i are given as,

$$\begin{cases} X_1 = Y, \\ X_2 = E_s(-X_1) \oplus N_s U, \\ X_i = E_{s+2-i}(-X_1) \oplus X_{i-1} \oplus N_{s+2-i} U, \quad \forall i = 3, \dots, s \end{cases} \tag{6.10}$$

Example 6.1 Consider the following discrete-time system,

$$y(k+1) - 2y(k) + y(k-1) = 0.5u(k) + 0.5u(k-1) + w(k) \tag{6.11}$$

The constraints are,

$$-5 \le y(k) \le 5, \qquad -5 \le u(k) \le 5, \qquad -0.1 \le w(k) \le 0.1$$

Using the construction (6.4), (6.5), the state space model is given as,

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + Dw(k) \\ y(k) = Cx(k) \end{cases}$$

where

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$x(k)$ is available though the measured input, output and their past values as,

$$x(k) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0.5 \end{bmatrix} \begin{bmatrix} y(k) \\ y(k-1) \\ u(k-1) \end{bmatrix}$$

Using (6.10), the constraints on the state are,

$$-5 \le x_1 \le 5, \qquad -7.5 \le x_2 \le 7.5$$

The local controller is chosen as an LQ controller with the following weighting matrices,

$$Q = C^T C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad R = 0.1$$

giving the state feedback gain,

$$K = \begin{bmatrix} -2.3548 & -1.3895 \end{bmatrix}$$

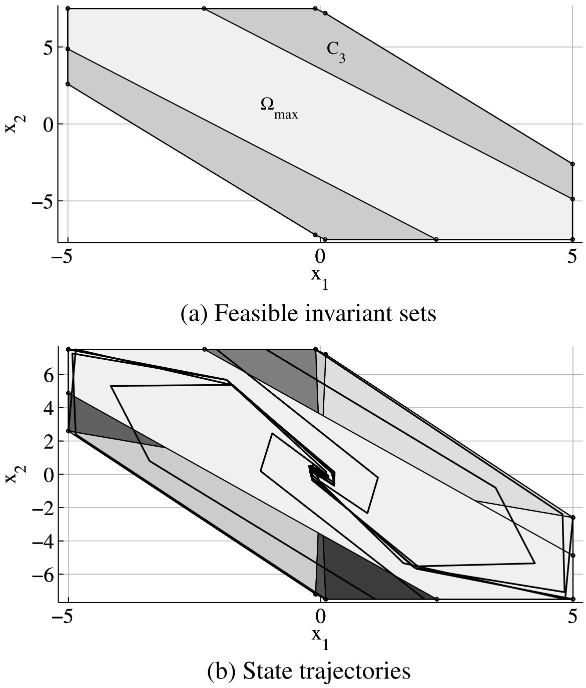


Fig. 6.1 Feasible invariant sets and state trajectories for Example 6.1

This example will use Algorithm 5.1 in Sect. 5.2, where vertex control is a global controller. Using Procedure 2.2 and Procedure 2.3, the sets $\Omega_{\max}$ and C_N with $N = 3$ are found and shown in Fig. 6.1(a). Note that $C_3 = C_4$ is the maximal invariant set for system (6.11). Figure 6.1(b) presents state trajectories for different initial conditions and realizations of $w(k)$.

The set of vertices of C_N is given by the matrix $V(C_N)$ below, together with the control matrix U_v,

$$V(C_N) = \begin{bmatrix} -5 & -0.1 & 5 & 0.1 & -0.1 & -5 & 0.1 & 5 \\ 7.5 & 7.5 & -2.6 & 7.2 & -7.2 & 2.6 & -7.5 & -7.5 \end{bmatrix},$$
$$U_v = \begin{bmatrix} -5 & -5 & -5 & -4.9 & 5 & 5 & 5 & 4.9 \end{bmatrix}$$

$\Omega_{\max}$ is presented in minimal normalized half-space representation as,

$$\Omega_{\max} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 1.0000 & 0 \\ 0 & 1.0000 \\ -1.0000 & 0 \\ 0 & -1.0000 \\ -0.8612 & -0.5082 \\ 0.8612 & 0.5082 \end{bmatrix} x \le \begin{bmatrix} 5.0000 \\ 7.5000 \\ 5.0000 \\ 7.5000 \\ 1.8287 \\ 1.8287 \end{bmatrix} \right\}$$

For the initial condition $x(0) = [-0.1000 \;\; 7.5000]^T$, Fig. 6.2 shows the output and input trajectories as functions of time.

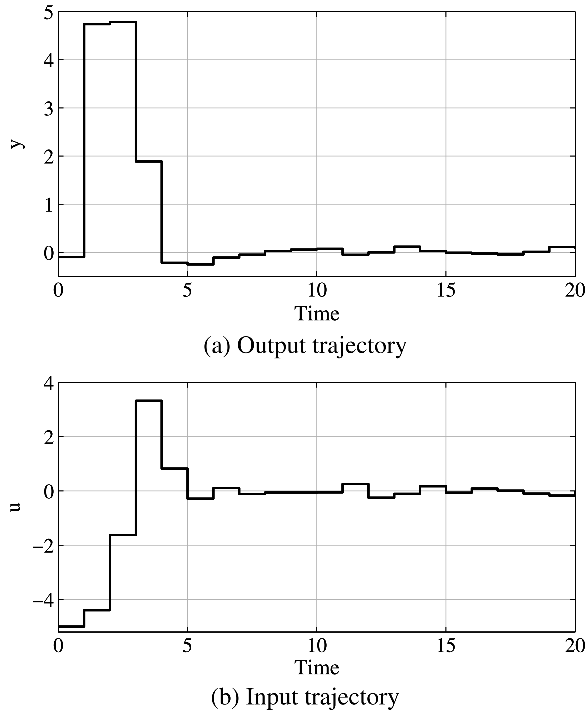


Fig. 6.2 Output and input trajectories of the closed loop system for Example 6.1

The interpolating coefficient and the realization of $w(k)$ as functions of time are depicted in Fig. 6.3. As expected, the interpolating coefficient, i.e. the Lyapunov function is positive and non-increasing.

As a comparison, we present a solution based on the well-known steady state Kalman filter. Figure 6.4 shows the output trajectories for the constrained output feedback approach (solid) and for the Kalman filter + constrained state feedback approach (dashed).

The *Kalman* function of Matlab 2011b was used for designing the Kalman filter. The process noise is a white noise with an uniform distribution and no measurement noise was considered. The disturbance w is a random number with an uniform distribution, $w_l \leq w \leq w_u$ where $w_l = -0.1$ and $w_u = 0.1$. The variance of w is given as,

$$C_w = \frac{(w_u - w_l + 1)^2 - 1}{12} = 0.0367$$

The estimator gain of the Kalman filter is obtained as,

$$L = [2 \quad -1]^T$$

The Kalman filter is used to estimate the state of and then this estimation is used to close the loop with the interpolating controller. In contrast to the output feedback approach, where the state is exact with respect to the measurement, in the Kalman

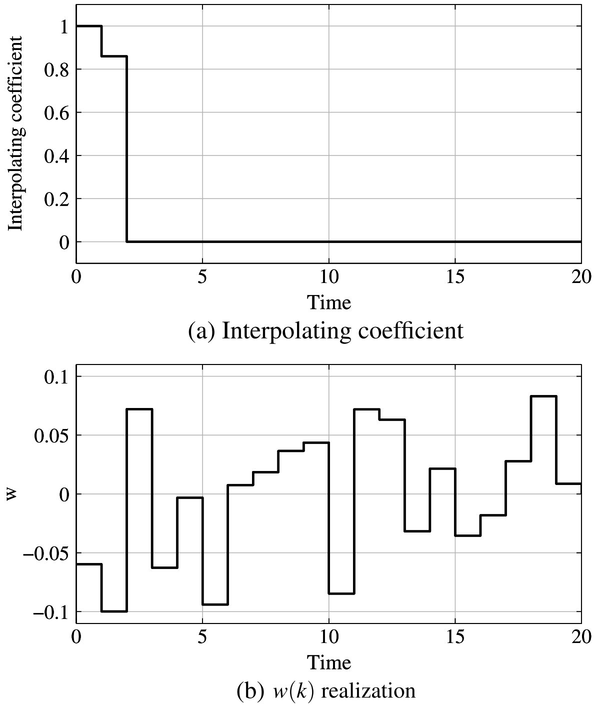


Fig. 6.3 Interpolating coefficient and realization of $w(k)$ for Example 6.1

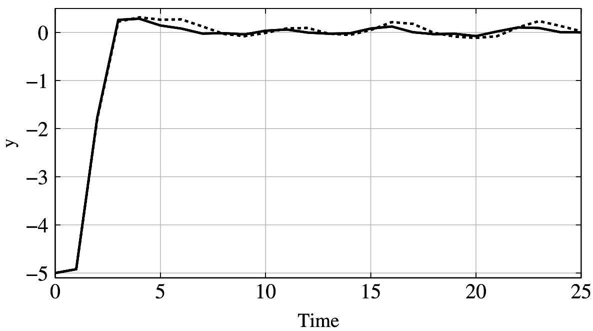


Fig. 6.4 Output trajectories for our approach (*solid*) and for the Kalman filter based approach (*dashed*) for Example 6.1

filter approach, an extra level of uncertainty is introduced, since the real state is unknown. Thus there is no guarantee that the constraints are satisfied in the transitory stage. This constraint violation effect is shown in Fig. 6.5.

6.3 Output Feedback—Robust Case

A weakness of the approach in Sect. 6.2 is that the state measurement is available if and only if the parameters of the system are known. For uncertain and/or time-varying system, that is not the case. In this section, we provide another method

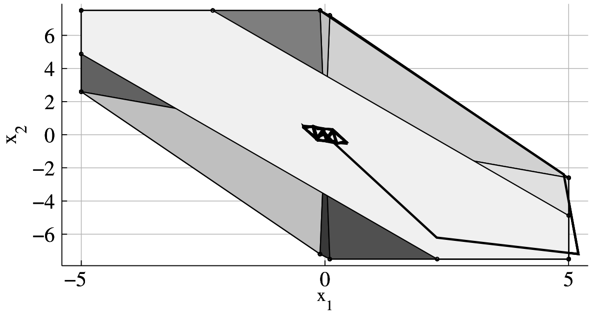


Fig. 6.5 Constraint violation for the Kalman filter based approach for Example 6.1

for constructing the state variables, that do not use the information of the system parameters. The price to be paid is that the realization is in general *non-minimal* even in the single-input single-output case.

Based on the measured plant input, output and their past measured values, the state of the system (6.1) is chosen as,

$$x(k) = \begin{bmatrix} y(k)^T & \dots & y(k-s+1)^T & u(k-1)^T & \dots & u(k-s+1)^T \end{bmatrix}^T \tag{6.12}$$

The state space model is then defined as follows,

$$\begin{cases} x(k+1) = A(k)x(k) + B(k)u(k) + Dw(k) \\ y(k) = Cx(k) \end{cases} \tag{6.13}$$

where

$$A(k) = \begin{bmatrix} -E_1(k) & -E_2(k) & \dots & -E_s(k) & N_2(k) & \dots & N_{s-1}(k) & N_s(k) \\ I & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & O & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & O & 0 & \dots & I & 0 \end{bmatrix}$$

$$B(k) = \begin{bmatrix} N_1(k)^T & 0 & 0 & \dots & 0 & I & 0 & \dots & 0 \end{bmatrix}^T$$

$$D = \begin{bmatrix} I & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}^T$$

$$C = \begin{bmatrix} I & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Using (6.2), it follows that matrices $A(k)$ and $B(k)$ belong to a polytopic set,

$$(A, B) \in \text{Conv}\{(A_1, B_1), (A_2, B_2), \dots, (A_q, B_q)\} \tag{6.14}$$

where the vertices (A_i, B_i), $i = 1, 2, \dots, q$ are obtained from the vertices of (6.2).

Although the obtained representation is non-minimal, it has the merit that the original output-feedback problem for the uncertain and/or time-varying plant has been transformed into a state-feedback problem where the matrices A and B lie in the polytope defined by (6.14) without any additional uncertainty. Clearly, any state-feedback control which is designed for the representation (6.13) in the form $u = Kx$ can be translated into a dynamic output-feedback controller.

Using (6.3), it follows that $x(k) \in X \subset \mathbb{R}^{s\times(p+m)}$, where the set X is given by,

$$X = \underbrace{Y \times Y \times \cdots \times Y}_{\text{s times}} \times \underbrace{U \times U \times \cdots \times U}_{\text{s times}}$$

Example 6.2 Consider the following transfer function,

$$P(s) = \frac{k_1 s + 1}{s(s + k_2)} \tag{6.15}$$

where $k_1 = 0.787$, $0.1 \le k_2 \le 3$. Using a sampling time of 0.1 and Euler's first order approximation for the derivative, the following input-output relationship is obtained,

$$\begin{aligned} y(k+1) - (2 - 0.1k_2)y(k) + (1 - 0.1k_2)y(k-1) \\ = 0.1k_1 u(k) + (0.01 - 0.1k_2)u(k-1) + w(k) \end{aligned} \tag{6.16}$$

The signal $w(k)$ is added to represent the process noise with $-0.01 \le w \le 0.01$. The constraints on output and input are,

$$-10 \le y \le 10, \qquad -5 \le u \le 5$$

The state $x(k)$ is constructed as follows,

$$x(k) = \begin{bmatrix} y(k) & y(k-1) & u(k-1) \end{bmatrix}^T$$

Hence, the state space model is given by,

$$\begin{cases} x(k+1) = A(k)x(k) + Bu(k) + Dw(k) \\ y(k) = Cx(k) \end{cases}$$

where

$$A(k) = \begin{bmatrix} (2 - 0.1k_2) & -(1 - 0.1k_2) & (0.01 - 0.1k_1) \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.1k_1 \\ 0 \\ 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

Using the polytopic uncertainty description, one obtains,

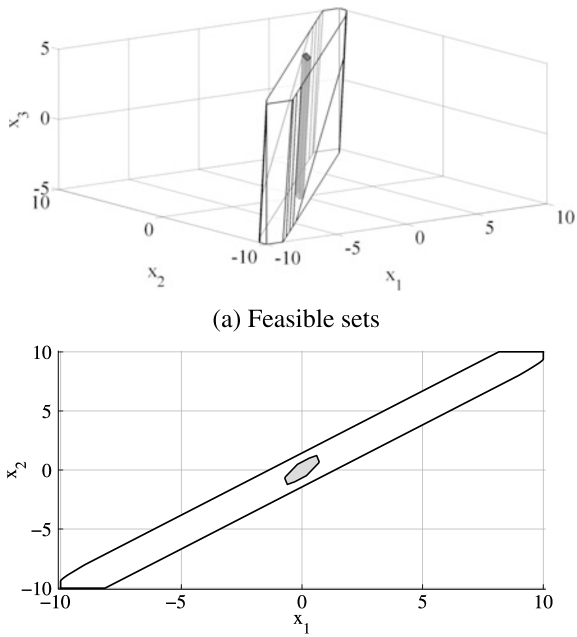
$$A(k) = \alpha(k)A_1 + \big(1 - \alpha(k)\big)A_2$$

where

$$A_1 = \begin{bmatrix} 1.99 & -0.99 & -0.0687 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1.7 & -0.7 & -0.0687 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

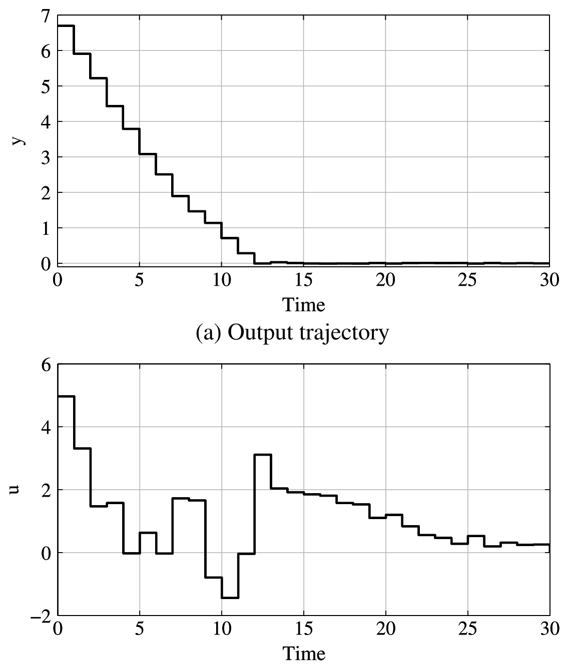
At each time instant $0 \le \alpha(k) \le 1$ and $-0.01 \le w(k) \le 0.01$ are uniformly distributed pseudo-random numbers. This example will use Algorithm 5.1 with a global saturated controller. For this purpose, two controllers have been designed

Fig. 6.6 Feasible invariant sets for Example 6.2



(b) Projection onto (x_1, x_2) space

Fig. 6.7 Output and input trajectories of the closed loop system for Example 6.2



(b) Input trajectory

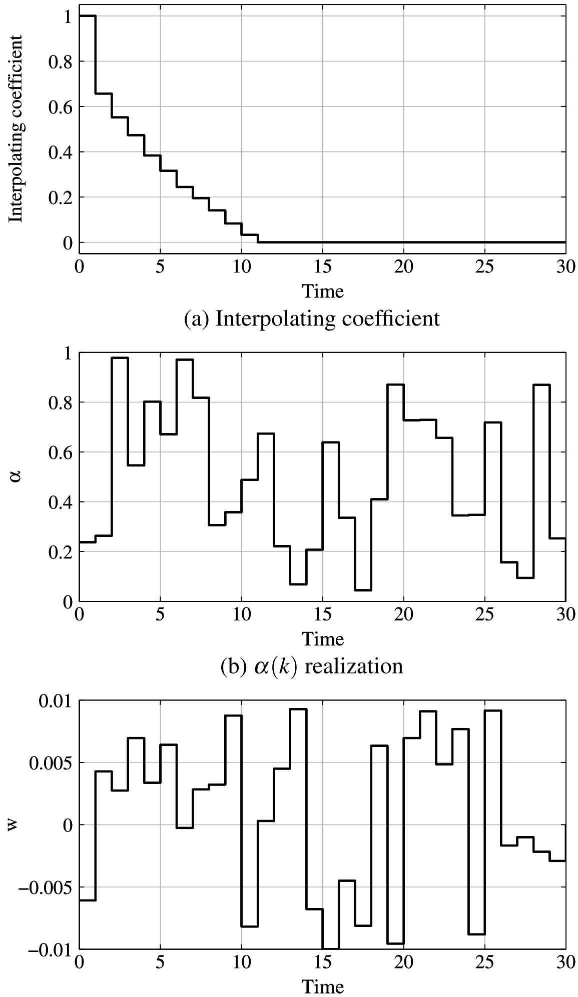


Fig. 6.8 Interpolating coefficient and realizations of $\alpha(k)$ and $w(k)$ for Example 6.2

- The local linear controller $u(k) = Kx(k)$ for the performance is chosen as,

$$K = [-22.7252 \quad 10.7369 \quad 0.8729]$$

- The global saturated controller $u(k) = \text{sat}(K_s x(k))$ for the domain of attraction,

$$K_s = [-4.8069 \quad 4.5625 \quad 0.3365]$$

It is worth noticing that $u(k) = Kx(k)$ and $u(k) = \text{sat}(K_s x(k))$ can be described in the output-feedback form as,

$$K(z) = \frac{-22.7894 + 10.7369z^{-1}}{1 - 0.8729z^{-1}}$$

and respectively

$$K_s(z) = \text{sat}\left(\frac{-4.8069 + 4.5625z^{-1}}{1 - 0.3365z^{-1}}\right)$$

Overall the control scheme is described by a second order plant and two first order controllers, which provide a reduced order solution for the stabilization problem.

Using Procedure 2.2 and Procedure 2.4 and corresponding to the control laws $u(k) = Kx(k)$ and $u(k) = \text{sat}(K_s x(k))$, the maximal robustly invariant sets $\Omega_{\max}$ (white) and Ω_s (black) are computed and depicted in Fig. 6.6(a). Figure 6.6(b) presents the projection of the sets $\Omega_{\max}$ and Ω_s onto the (x_1, x_2) state space.

For the initial condition $x(0) = [6.6970 \;\; 7.7760 \;\; 5.0000]^T$, Fig. 6.7 presents the output and input trajectories as functions of time.

Finally, Fig. 6.8 shows the interpolating coefficient, the realizations of $\alpha(k)$ and $w(k)$ as functions of time.